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Abstract

Let E be an infinite-dimensional vector space carrying a Gaussian measure μ with mean 0 and a measurable norm q . Let $F(t) := \mu(q \leq t)$. By a result of Borell, F is logarithmically concave. But we show that F may have infinitely many local maxima for norms $q = \sup_n |f_n|/a_n$ where f_n are independent standard normal variables. We also consider Hilbertian norms $q = (\sum b_n f_{2n}^2)^{1/2}$ with $b_n > 0, \sum b_n < \infty$. Then as $t \downarrow 0$ we can have $F(t) \downarrow 0$ as rapidly as desired, or as slowly as any function which is $o(t_n)$ for all n . For $b_n = 1/n^2$ and in a few closely related cases, we find the exact asymptotic behavior of F at 0. For more general b_n we find inequalities bounding F between limits which are not too far apart.

Keywords

Gaussian processes, seminorms, measure of small balls, lower tail distribution

Disciplines

Probability

ON THE LOWER TAIL OF GAUSSIAN SEMINORMS

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Let E be an infinite-dimensional vector space carrying a Gaussian measure μ with mean 0 and a measurable norm q . Let $F(t) := \mu(q \leq t)$. By a result of Borell, F is logarithmically concave. But we show that F' may have infinitely many local maxima for norms $q = \sup_n |f_n|/a_n$ where f_n are independent standard normal variables. We also consider Hilbertian norms $q = (\sum b_n f_n^2)^{1/2}$ with $b_n > 0$, $\sum b_n < \infty$. Then as $t \downarrow 0$ we can have $F(t) \downarrow 0$ as rapidly as desired, or as slowly as any function which is $o(t^n)$ for all n . For $b_n = 1/n^2$ and in a few closely related cases, we find the exact asymptotic behavior of F at 0. For more general b_n we find inequalities bounding F between limits which are not too far apart.

1. Introduction. Let $\eta = (\eta_j)$ be a sequence of independent Gaussian, mean 0, variance 1, random variables in all of this paper. We shall then study the distribution of

$$q(\eta) \quad \text{or} \quad q(\eta - a)$$

where $q : \mathbb{R}^\infty \rightarrow \overline{\mathbb{R}}_+ = [0, \infty]$ is a seminorm and $a \in \mathbb{R}^\infty$. In particular we shall study the behavior of $P(q(\eta) \leq t)$ as $t \rightarrow 0$.

In Section 3 we study *supremum norms*, that is, seminorms of the following form:

$$(1.1) \quad q(x) = \sup_n \{|x_n|/a_n\} \quad \forall x = (x_n) \in \mathbb{R}^\infty$$

where (a_n) is a given sequence of positive numbers.

In Section 4 and Section 5 we study *Hilbertian norms*, that is, a seminorm of the following form:

$$(1.2) \quad q(x) = \left\{ \sum_{n=1}^\infty \tau_n^2 x_n^2 \right\}^{1/2} \quad \forall x = (x_n) \in \mathbb{R}^\infty$$

where (τ_n) is a given sequence of positive numbers.

The setting above actually covers the following general case: Let E be a locally convex space and μ a Gaussian Radon probability on E , with mean 0; that is, μ is a Radon probability on E , whose finite dimensional marginals all are Gaussian and have mean 0. In particular if $x' \in E'$ ($=$ the topological dual of E), then x' has a Gaussian distribution, when x' is considered as a random variable on $(E, \mathcal{B}, \mu)$. Hence we have $E' \subseteq L^2(\mu)$, and so we may consider the L^2 -closure of E' , which we denote H' . Then H' is a Hilbert space and its dual H may be identified with a

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subspace of E in the following manner:

$$\begin{aligned} H &= \{x \in E \mid \varphi \rightarrow \varphi(x) \text{ is } L^2\text{-continuous on } H'\} \\ &= \{x \in E \mid \exists K > 0 : |\langle x', x \rangle|^2 \leq K \int_E \langle x', y \rangle^2 \mu(dy) \forall x' \in E'\} \end{aligned}$$

H is the reproducing kernel Hilbert space (RKHS) of μ , and we define the Hilbert norm, $\|\cdot\|$, in H by

$$\|x\| = \sup\{|\langle x', x \rangle| : x' \in E', \int_E \langle x', y \rangle^2 \mu(dy) \leq 1\}.$$

From [5] we have that $L^2(\mu)$ and H are separable, and we can find biorthonormal bases $\{f_j\} \subseteq E'$ and $\{e_j\} \subseteq H$ for H' and H , satisfying the following Karhunen-Loève expansion:

$$(1.3) \quad f_1, f_2, \dots \text{ are independent Gaussian, mean 0, variance 1, random variables on the probability space } (E, \mathfrak{B}, \mu).$$

$$(1.4) \quad \langle f_j, e_i \rangle = \delta_{ij}$$

$$(1.5) \quad x = \sum_{j=1}^{\infty} \langle f_j, x \rangle e_j \quad \text{for } \mu - \text{a.a. } x \in E^{\infty}$$

$$(1.6) \quad H = \{x \in E \mid \sum_{j=1}^{\infty} \langle f_j, x \rangle^2 < \infty\}$$

$$(1.7) \quad \|x\| = \left\{ \sum_{j=1}^{\infty} \langle f_j, x \rangle^2 \right\}^{\frac{1}{2}}, \quad x \in H$$

So the study of a seminorm $r : E \rightarrow \overline{\mathbb{R}}_+$ reduces to the study of the seminorm

$$q(t) = r(\sum_1^{\infty} t_j e_j)$$

on $\mathbb{R}^{\infty}$ (we put $q = \infty$ if the sum diverges).

Our original case is a special example taking $E = \mathbb{R}^{\infty}$ and μ equal to the infinite product of $N(0, 1)$. In this case we may take f_j to be the projection on the j th coordinate and e_i to be the i th unit vector, and we have

$$H = l^2 = \{x \in \mathbb{R}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\},$$

$$\|x\| = \left\{ \sum_{n=1}^{\infty} |x_n|^2 \right\}^{\frac{1}{2}},$$

and the series in (1.5) converges for all $x \in \mathbb{R}^{\infty}$.

If q is a Borel measurable seminorm on $\mathbb{R}^{\infty}$ we define (μ is a given measure, and H and $\|\cdot\|$ are defined as above)

$$\|q\| = \sup\{q(x) \mid x \in H, \|x\| \leq 1\}.$$

Then we have (Kallianpur [8], Borell [3], Marcus and Shepp [9])

THEOREM 1.1. *The two probabilities $P(q(\eta) < \infty)$ and $P(q(\eta) = 0)$ are 0 or 1, and $q(\eta) < \infty$ a.s. implies $\|q\| < \infty$.*

Moreover if $q(\eta) < \infty$ a.s. then

$$\lim_{t \rightarrow \infty} t^{-2} \log P(q(\eta) > t) = -\frac{1}{2} \|q\|^{-2}.$$

If $\|q\| = 0$, then $q(\eta) = \text{constant}$ a.s.

This theorem settles the behavior of the upper tail of the distribution function of q . Notice that q may be constant a.s. without being 0 a.s., e.g.

$$q(x) = \limsup_{n \rightarrow \infty} |x_n| (2 \log n)^{-\frac{1}{2}}.$$

Then $q(\eta) = 1$ a.s., q is a seminorm and $\|q\| = 0$. If

$$q(x) = \limsup_{n \rightarrow \infty} |x_n|$$

then q is a seminorm with $\|q\| = 0$ and $q(\eta) = \infty$ a.s.

We shall mainly be concerned with seminorms of the form (1.1) or (1.2). These seminorms satisfy the following:

$$(1.8) \quad q(x) = \sup_n q(x_1, \dots, x_n, 0, 0, \dots) \quad \forall x,$$

in contrast to the examples above. Note that (1.8) implies

$$(1.9) \quad q \text{ is lower semicontinuous on } \mathbb{R}^\infty, \text{ and so in particular Borel measurable.}$$

$$(1.10) \quad q(x_1, \dots, x_n, 0, 0, \dots) \leq q(x_1, \dots, x_m, 0, 0, \dots) \quad \forall m \geq n.$$

In [2] Borell introduces the class of 0-convex measures. A Radon probability μ on the locally convex space E is 0-convex if μ satisfies

$$(1.11) \quad \mu_*(\lambda A + (1 - \lambda)B) > \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for all $0 < \lambda < 1$ and all Borel sets A and B . Here μ_* denotes the inner measure generated by μ . Borell proves in [2] that μ is 0-convex if and only if all finite dimensional marginals are 0-convex and in [3] he proves that a probability μ on $\mathbb{R}^n$ is 0-convex, if and only if μ is concentrated on some affine subspace L of $\mathbb{R}^n$ with

$$\mu \ll \lambda_L \quad \text{and} \quad \log \left(\frac{d\mu}{d\lambda_L} \right) \quad \text{concave}$$

where λ_L is Lebesgue measure on L . In particular,

$$(1.12) \quad \text{any Gaussian measure is 0-convex.}$$

Using this one easily establishes a conjecture of Marcus and Shepp (see [9], page 435) on the number of jumps of the distribution of q (see also Cirel'son [6]). Suppose that $q : \mathbb{R}^\infty \rightarrow \mathbb{R}_+$ is a Borel measurable seminorm with $q(\eta) < \infty$ a.s. and put

$$F(t) = P(q(\eta) \leq t) \quad t \geq 0,$$

$$C(q) = \inf\{t \geq 0 | F(t) > 0\}.$$

Then from (1.11) (with $A = \{q(x) \leq t\}$, $B = \{q(x) \leq s\}$) we find that

$$(1.13) \quad \log F(t) \quad \text{is concave:} \quad \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_- = [-\infty, 0].$$

Now a concave function is absolutely continuous on the interior of the set where it is finite. So we have

THEOREM 1.2. *If $q(\eta) < \infty$ a.s., then F admits right and left derivatives everywhere except possibly at $t = C(q)$.*

Moreover $F'(t)$ exists except possibly at countably many points t where F' has a jump downwards. Moreover $F''(t)$ exists Lebesgue-a.e. and

$$\begin{aligned} F(t) &= p + \int_0^t F'(s) ds & \text{for } t \geq C(q) \\ &= 0 & \text{for } t < C(q) \end{aligned}$$

where $p = P(q(\eta) = C(q))$.

Hence, apart from a possible jump at $C(q)$, F is absolutely continuous. If $C(q) > 0$, then we shall see in Example 3.2 that $p = P(q(\eta) = C(q))$ may take any value in $[0, 1]$.

If $C(q) = 0$, then by Theorem 1.1 the jump p is either 0 or 1, the latter case occurs if and only if $q = 0$ a.s.

Finally let $B_q(a, r)$ denote the closed q -ball with center at a and radius r :

$$B_q(a, r) = \{x \in \mathbb{R}^\infty \mid q(x - a) \leq r\}.$$

2. The measure of a translated ball. Let q be a Borel measurable seminorm: $\mathbb{R}^\infty \rightarrow \overline{\mathbb{R}}_+$, then we put $\pi_n x = (x_1, \dots, x_n, 0, 0, \dots)$ and

$$\begin{aligned} q_n(x) &= q(\pi_n x) & \forall x \in \mathbb{R}^\infty, \\ q_n^*(x) &= \sup\{|\sum_{j=1}^n x_j y_j| : q_n(y) \leq 1\} & \forall x \in \mathbb{R}^\infty. \end{aligned}$$

Note that $q_n^*(x)$ is everywhere finite if and only if $q_n(x) = 0$ implies $\pi_n x = 0$. Obviously we have

$$(2.1) \quad |\sum_{j=1}^n x_j y_j| \leq q_n(x) q_n^*(y) \quad \text{if } q_n(x) < \infty$$

(with the usual convention: $0 \cdot \infty = 0$). Note that if q satisfies (1.8) then by (1.10) we have

$$(2.2) \quad q_n(x) \uparrow q(x) \quad \forall x.$$

THEOREM 2.1. Let q be a Borel measurable seminorm on $\mathbb{R}^\infty$ with $q(\eta) < \infty$ a.s. Then we have

$$(2.1.1) \quad P(q(\eta - a) \leq t) \leq P(q(\eta) \leq t) \quad \forall t \geq 0, \forall a \in \mathbb{R}^\infty.$$

Moreover if q satisfies (1.8) and $a \in l^2$, then

$$(2.1.2) \quad \exp\left(-\frac{1}{2}\|a\|^2\right)F(t) \leq F(t, a) \leq \exp\left(-\frac{1}{2}\|\pi_n a\|^2 + tq_n^*(a)\right)F(t)$$

for all $n \geq 1$ and all $t \geq 0$. Here $\|\cdot\|$ is the usual norm on l^2 and

$$F(t) = P(q(\eta) \leq t), \quad F(t, a) = P(q(\eta - a) \leq t).$$

REMARK. (2.1.1) and (2.1.2) show that $F(t)$ and $F(t, a)$ are of the same order of magnitude as $t \rightarrow 0$ for $a \in l^2$. If $q_n(x) = 0$ implies $\pi_n x = 0$ then we have

$$F(t, a) \sim \exp\left(-\frac{1}{2}\|a\|^2\right)F(t) \quad \text{as } t \rightarrow 0$$

for $a \in l^2$.

PROOF. (2.1.1): Let $a \in \mathbb{R}^\infty$ and $t \geq 0$ be given, then we put $K = B_q(0, t)$ and $\alpha = F(t, a)$. Then K is convex closed and symmetric and

$$F(t, a) = \mu(K + a) = \alpha$$

where μ is the probability law of η on $\mathbb{R}^\infty$. Let

$$M = \{x \in \mathbb{R}^\infty \mid \mu(K + x) \geq \alpha\}.$$

Then M is symmetric, by symmetry of K and μ , and M is convex by (1.11). Moreover $a \in M$ hence $0 = \frac{1}{2}a + \frac{1}{2}(-a) \in M$. That is

$$F(t) = \mu(K) \geq \alpha = \mu(K + a) = F(t, a)$$

and (2.1.1) is proved.

(2.1.2): Let μ be the probability law of η , that is μ is the infinite product of $N(0, 1)$ and let μ_a be the probability law of $\eta - a$, that is μ shifted by a . If $a \in l^2$, then μ_a is absolutely continuous with respect to μ and

$$\mu_a(dx) = e^{-\frac{1}{2}\|a\|^2 - \langle x, a \rangle} \mu(dx)$$

where $\langle x, a \rangle = \sum_j x_j a_j$, whenever the sum converges, which it does μ -a.s. if $a \in l^2$. Hence

$$F(t, a) = \mu_a(q \leq t) = \int_{\{q \leq t\}} e^{-\frac{1}{2}\|a\|^2 - \langle x, a \rangle} \mu(dx)$$

and since μ and q are symmetric, the Cauchy-Schwarz inequality gives:

$$\begin{aligned} F(t) &= \mu(q \leq t) \leq \left\{ \int_{\{q \leq t\}} e^{-\langle x, a \rangle} \mu(dx) \right\}^{\frac{1}{2}} \left\{ \int_{\{q \leq t\}} e^{\langle x, a \rangle} \mu(dx) \right\}^{\frac{1}{2}} \\ &= \int_{\{q \leq t\}} e^{-\langle x, a \rangle} \mu(dx). \end{aligned}$$

So we have $F(t, a) \geq e^{-\frac{1}{2}\|a\|^2} F(t)$.

Let $a \in l^2$ and let $n \geq 1$ be given and fixed. Then we put $b = \pi_n a$ and $c = a - b$, and we find

$$\begin{aligned} F(t, a) &= \int_{\{q \leq t\}} e^{-\frac{1}{2}\|a\|^2 - \langle x, a \rangle} \mu(dx) \\ &= e^{-\frac{1}{2}\|b\|^2} \int_{\{q \leq t\}} e^{-\langle x, b \rangle - \langle x, c \rangle - \frac{1}{2}\|c\|^2} \mu(dx) \end{aligned}$$

and since $q_n(x) \leq q(x)$ we have (cf. (2.1))

$$|\langle x, b \rangle| = |\sum_1^n x_j a_j| \leq q_n(x) q_n^*(a) \leq t q_n^*(a)$$

for $x \in \{q \leq t\}$. Hence we find

$$\begin{aligned} F(t, a) &\leq \exp\left(-\frac{1}{2}\|b\|^2 + t q_n^*(a)\right) \int_{\{q \leq t\}} e^{-\frac{1}{2}\|c\|^2 - \langle x, c \rangle} \mu(dx) \\ &= \exp\left(-\frac{1}{2}\|b\|^2 + t q_n^*(a)\right) F(t, c) \\ &\leq \exp\left(-\frac{1}{2}\|b\|^2 + t q_n^*(a)\right) F(t) \end{aligned}$$

since $F(t, c) \leq F(t)$ by (2.1.1).

COROLLARY 2.2. Let q be a Borel measurable seminorm with $q(\eta) < \infty$ a.s. If $F = \{x \in \mathbb{R}^\infty | q(x) < \infty\}$ is q -separable, that is, if

$$\forall \varepsilon > 0, \exists (x_j) \subseteq F \text{ so that } F \subseteq \bigcup_{j=1}^{\infty} B_q(x_j, \varepsilon),$$

then $C(q) = 0$, that is, $F(t) > 0 \quad \forall t > 0$.

PROOF. If $C(q) > 0$, then by (2.1.1) we have

$$P(\eta \in B_q(x, \varepsilon)) = 0 \quad \forall x \in \mathbb{R}^\infty \quad \forall \varepsilon < C(q).$$

But then separability of F implies $P(\eta \in F) = 0$, which contradicts $q(\eta) < \infty$ a.s.

EXAMPLE 2.3. Let q be defined by

$$q(x) = \left\{ \sum_{j=1}^{\infty} \sigma_j^2 x_j^2 \right\}^{\frac{1}{2}}$$

where $\sum_j \sigma_j^2 < \infty$. Then $q(\eta) < \infty$ a.s. and

$$q_n^*(x) = \left\{ \sum_{j=1}^n \sigma_j^{-2} x_j^2 \right\}^{\frac{1}{2}}.$$

Taking $a_n = \sigma_n^{-1} e_n$ (e_n is the n th unit vector) gives

$$\|a_n\| = \|\pi_n a_n\| = \sigma_n^{-1}; \quad q_n^*(a_n) = \sigma_n^{-2}.$$

So we have

$$(2.3) \quad \exp\left(-\frac{1}{2} \sigma_n^{-2}\right) F(t) \leq F(t, a_n) \leq \exp\left(-\left(\frac{1}{2} - t\right) \sigma_n^{-2}\right) F(t).$$

In particular we have

$$(2.4) \quad \sum_{n=1}^{\infty} \{-\log F(t, a_n)\}^{-1} < \infty \quad \forall 0 \leq t < \frac{1}{2}.$$

Note that if $0 \leq t < (\frac{1}{2})^{\frac{1}{2}}$, then the balls $B_q(a_n, t)$, $n = 1, 2, \dots$ are mutually disjoint and so

$$(2.5) \quad \sum_n F(t, a_n) < \infty \quad \forall 0 \leq t < \left(\frac{1}{2}\right)^{\frac{1}{2}},$$

which is much weaker than (2.4).

3. Sup-norms. We shall now consider seminorms of the form (1.1), so let $a_n > 0$ for all $n \geq 1$, and let

$$(3.1) \quad q(x) = \sup_n |x_n| / a_n.$$

Let Φ be the standard normal distribution function on $\mathbb{R}$, and put

$$R(t) = 2(1 - \Phi(t)) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_t^{\infty} e^{-\frac{1}{2}x^2} dx.$$

Then we have the following elementary inequality:

$$(3.2) \quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (1+t)^{-1} e^{-\frac{1}{2}t^2} \leq R(t) \leq \frac{4}{3} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (1+t)^{-1} e^{-\frac{1}{2}t^2} \quad \forall t \geq 0$$

and if $F(t) = P(q(\eta) \leq t)$, then

$$(3.3) \quad F(t) = \prod_{j=1}^{\infty} (1 - R(ta_j)).$$

Hence $F(t) > 0$ if and only if $\sum_{j=1}^{\infty} R(ta_j) < \infty$, that is if and only if

$$\sum_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1+ta_n} < \infty.$$

Now we note that this sum converges for all $t > t_0$ if and only if

$$\sum_{n=1}^{\infty} \exp\left(-\frac{1}{2}t^2a_n^2\right) < \infty \quad \forall t > t_0.$$

So if we define

$$(3.4) \quad C_0(a) = \inf\left\{t > 0 \mid \sum_{n=1}^{\infty} \exp\left(-\frac{1}{2}t^2a_n^2\right) < \infty\right\},$$

then $C_0(a) = C(q)$, in other words:

$$(3.5) \quad C_0(a) = \inf\{t > 0 \mid F(t) > 0\};$$

and from (3.2) and (3.3) we deduce:

$$(3.6) \quad F(t) \leq \exp\left\{-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1+ta_n}\right\}$$

Now suppose that $a_n \geq a > 0 \quad \forall n \geq 1$, then by use of the inequality:

$$1-x \geq \exp\left(\frac{x}{y} \log(1-y)\right) \quad \forall 0 \leq x \leq y \leq 1$$

we find (put $x = R(ta_n)$ and $y = R(ta)$):

$$(3.7) \quad F(t) \geq \exp\left\{-\psi(t) \sum_{n=1}^{\infty} \frac{4 \exp\left(-\frac{1}{2}t^2a_n^2\right)}{3(1+ta_n)}\right\}$$

where ψ is given by

$$(3.8) \quad \psi(t) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} R(ta)^{-1} \log(1-R(ta)).$$

It is easily checked that

$$(3.9) \quad \psi(t) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \log \frac{1}{t} \quad \text{as } t \rightarrow 0.$$

So the estimates in (3.6) and (3.7) are fairly close together. Summarizing these observations we have proved:

THEOREM 3.1. *Let q be given by (3.1), and let C_0 be given by (3.4). Then we have*

$$(3.1.1) \quad q(\eta) < \infty \quad \text{a.s. if and only if } C_0(a) < \infty.$$

$$(3.1.2) \quad C_0(a) = \inf\{t \mid F(t) > 0\}.$$

$$(3.1.3) \quad F(t) \leq \exp\left\{-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1+a_nt}\right\}.$$

If $a_n \geq a > 0$ and ψ is given by (3.8), then $\psi(t) \sim -(\frac{2}{\pi})^{\frac{1}{2}} \log t$ as $t \rightarrow 0$, and

$$(3.1.4) \quad F(t) \geq \exp \left\{ -\psi(t) \sum_{n=1}^{\infty} \frac{4 \exp(-\frac{1}{2} t^2 a_n)}{3(1 + a_n t)} \right\}$$

where F is the distribution function of $q(\eta)$.

EXAMPLE 3.2. Let $\alpha \geq 0$ and $\beta > 0$; then we consider the sequence $a_1 = a_2 = \beta$ and

$$(3.10) \quad a_n = (2\beta^2(\log n + \alpha \log \log n))^{\frac{1}{2}} \quad n \geq 3.$$

Then we have

$$\sum_{n=3}^{\infty} \exp(-\frac{1}{2} t^2 a_n^2) = \sum_{n=3}^{\infty} n^{-(\beta t)^2} (\log n)^{-\alpha(\beta t)^2}.$$

So we have $C_0(a) = 1/\beta$. Put $\gamma = 1/\beta$, then

$$(\log n)^{\frac{1}{2}} \leq 1 + \gamma a_n \leq (3 + 2\alpha^{\frac{1}{2}})(\log n)^{\frac{1}{2}} \quad \forall n \geq 3;$$

hence we find

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\exp(-\frac{1}{2} a_n^2 \gamma^2)}{1 + \gamma a_n} &\leq e^{-\frac{1}{2}} + \sum_{n=3}^{\infty} n^{-1} (\log n)^{-\alpha - \frac{1}{2}} \\ &\geq e^{-\frac{1}{2}} + (3 + 2\alpha^{\frac{1}{2}})^{-1} \sum_{n=3}^{\infty} n^{-1} (\log n)^{-\alpha - \frac{1}{2}}. \end{aligned}$$

If

$$s(\alpha) = \sum_{n=3}^{\infty} n^{-1} (\log n)^{-\alpha - \frac{1}{2}}$$

then (3.1.3) and (3.1.4) give

$$k \exp(-ms(\alpha)) \leq F(\gamma) \leq K \exp(-M(3 + 2\alpha^{\frac{1}{2}})^{-1} s(\alpha))$$

where k, m, K and M are positive finite constants not depending on α . So if $\alpha \leq \frac{1}{2}$ then $F(\gamma) = 0$, and if $\alpha > \frac{1}{2}$, then $F(\gamma) > 0$.

If $\alpha \downarrow \frac{1}{2}$, then $s(\alpha) \rightarrow \infty$ so $F(\gamma) \rightarrow 0$, and if $\alpha \rightarrow \infty$ then $s(\alpha) \rightarrow 0$ so $F(\gamma) \rightarrow 1$. Hence F can have a jump of any size $p \in [0, 1[$ at any point $\gamma \in]0, \infty[$.

However, from (3.3) it follows that $F(t) < 1$ for all $t > 0$. So F cannot have a jump of size 1, when q is a sup-norm. Also since q is a norm (i.e., $q(x) = 0$ implies $x = 0$), $C(q) > 0$, so F cannot have a jump at 0.

Now let

$$q_N(x) = \max_{1 \leq j \leq N} \{|x_j|/a_j\}$$

with a_n defined by (3.10). If $0 < b_2 < \gamma < b_1$ then

$$F(b_1) - F(b_2) \geq F(\gamma)$$

and since $q_N \rightarrow q$ and b_1 and b_2 are continuity points of F we can achieve the following lemma by taking α sufficiently large:

LEMMA 3.3. Let $0 < b_2 < b_1$ and $\epsilon > 0$, then there exist $a_1, \dots, a_N > 0$ so that

$$F_N(b_1) - F_N(b_2) > 1 - \epsilon,$$

where F_N is the distribution function of

$$\max_{1 \leq j \leq N} \{|\eta_j|/a_j\}.$$

THEOREM 3.4. Let $\{b_j\}$ be any strictly decreasing sequence of positive numbers. Then there exist sequences $\{a_j\}$ and $\{m_j\}$ so that

$$(3.4.1) \quad b_{j+1} < m_j < b_j \quad \forall j \geq 1,$$

$$(3.4.2) \quad F(b_j) - F(m_j) \geq 2(b_j - m_j) \quad \forall j \geq 1,$$

$$(3.4.3) \quad F(m_j) \leq m_j - b_{j+1} \quad \forall j \geq 1$$

where F is the distribution function of

$$q(\eta) = \sup_n \{|\eta_n|/a_n\}.$$

In particular F has a mode in each of the intervals: $]b_{j+1}, b_{j-1}[$, $j = 2, 3, \dots$, in spite of the log-concavity of F .

PROOF. Let $\mathcal{F}$ denote the set of distribution functions of random variables of the form

$$Q = \max_{1 \leq j \leq N} \{|\eta_j|/a_j\}$$

with $N \geq 1$ and $a_1, \dots, a_N$ positive. Then any infinite product of distributions from $\mathcal{F}$ is the distribution function of $q(\eta)$ for some sup-norm q of the form (3.1).

The distribution F will be an infinite product

$$F(x) = \prod_{j=1}^{\infty} F_j(x)$$

where $F_j \in \mathcal{F}$. The F_j and m_j are defined inductively by:

$$(i) \quad F_j(b_j) - F_j(m_j) \geq 4(b_j - m_j) \prod_{i=1}^{j-1} F_i(b_j)^{-1} \quad \forall j \geq 1$$

$$(ii) \quad F_j(b_j) - F_j(m_j) \geq p_j \quad \forall j \geq 1$$

$$(iii) \quad F_j(b_j) - F_j(m_j) \geq 1 - (m_j - b_{j+1}) \quad \forall j \geq 1$$

where (p_j) is any fixed sequence with $0 < p_j < 1$ and

$$\prod_{j=1}^{\infty} p_j = \frac{1}{2}.$$

First m_1 is chosen so that $m_1 \in]b_2, b_1[$ and $4(b_1 - m_1) < 1$, then we choose $F_1 \in \mathcal{F}$ by Lemma 3.3, such that

$$F_1(b_1) - F_1(m_1) > \max\{4(b_1 - m_1), p_1, 1 - (m_1 - b_2)\}.$$

Then (i)–(iii) are satisfied for $j = 1$.

If $F_1, \dots, F_n$ and $m_1, \dots, m_n$ are constructed, then we choose $m_{n+1} \in]b_{n+2}, b_{n+1}[$, so that

$$4(b_{n+1} - m_{n+1}) < \prod_{i=1}^n F_i(b_{n+1})$$

(note that $F(t) > 0 \quad \forall t > 0 \quad \forall F \in \mathcal{F}$). Then by Lemma 3.3 we can choose $F_{n+1} \in \mathcal{F}$, so that (i)—(iii) holds for $j = n + 1$.

Now we note that (i)—(iii) imply

$$(iv) \quad \prod_{i=j+1}^n F_i(b_j) \geq \prod_{i=j+1}^n F_i(b_i) \geq \prod_{i=1}^\infty p_i = \frac{1}{2} \quad \forall n \geq j + 1,$$

$$(v) \quad F_j(m_j) \leq 1 - (F_j(b_j) - F_j(m_j)) \leq m_j - b_{j+1}.$$

Now we put

$$F(x) = \prod_{j=1}^\infty F_j(x), \quad G_n(x) = \prod_{j=1}^n F_j(x)$$

Then by (i) and (iv) we have for $j \leq n$:

$$\begin{aligned} G_n(b_j) - G_n(m_j) &\geq (G_j(b_j) - G_j(m_j)) \prod_{i=j+1}^n F_i(b_j) \\ &\geq \frac{1}{2} (G_j(b_j) - G_j(m_j)) \\ &\geq \frac{1}{2} G_{j-1}(b_j) (F_j(b_j) - F_j(m_j)) \\ &\geq 2(b_j - m_j). \end{aligned}$$

So we see that F satisfies (3.4.2), and since $F \leq F_j$, it follows from (v) that F satisfies (3.4.3).

Since $F(b_j) > 0$ it follows from Theorem 1.2 that F is absolutely continuous on $]b, \infty[$, where $b = \lim_{n \rightarrow \infty} b_n$. Now (3.4.2) implies that $F'(x) \geq 2$ for some $x \in]m_j, b_j[$ and (3.4.3) implies that $F'(x) \leq 1$ for some $x \in]b_{j+1}, m_j[$. That is, F has at least one mode in each of the intervals $]b_{j+1}, b_{j-1}[$ for $j \geq 2$.

THEOREM 3.5. *Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be increasing, then there exist positive numbers $\{a_j\}$, so that*

$$0 < F(t) \leq f(t) \quad \forall 0 < t \leq 1$$

where F is the distribution function of

$$Q = \sup_n \{|\eta_n|/a_n\}.$$

PROOF. Let $\{p_n\}$ be defined by

$$p_0 = f\left(\frac{1}{2}\right), \quad p_n = f(2^{-n-1})/f(2^{-n}) \quad \text{for } n \geq 1$$

and let $\mathcal{F}$ be defined as in the proof of Theorem 3.4, then we can find $F_n \in \mathcal{F}$ so that

$$F_n(2^{-n+1}) - F_n(2^{-n}) \geq \max\{1 - 2^{-n}, 1 - p_n\}.$$

Let $F = \prod_{j=0}^\infty F_j$, then F is the distribution of some $Q = q(\eta)$, where q is a sup-norm. Moreover if $2^{-n-1} \leq t \leq 2^{-n}$ ($n \geq 0$), then

$$\begin{aligned} F(t) &\leq F(2^{-n}) \leq \prod_{j=0}^n F_j(2^{-n}) \leq \prod_{j=0}^n F_j(2^{-j}) \\ &\leq \prod_{j=0}^n (1 - (F_j(2^{-j+1}) - F_j(2^{-j}))) \leq \prod_{j=0}^n p_j \\ &= f(2^{-n-1}) \leq f(t) \end{aligned}$$

and

$$\begin{aligned} F(t) &\geq F(2^{-n-1}) \geq \prod_{j=0}^{n+1} F_j(2^{-n-1}) \prod_{j=n+2}^{\infty} F_j(2^{-j+1}) \\ &\geq \prod_{j=0}^{n+1} F_j(2^{-n-1}) \prod_{j=n+2}^{\infty} (1 - 2^{-j}) > 0 \end{aligned}$$

and the theorem is proved.

4. Hilbertian norms. We shall in this section study Hilbertian norms, that is, norms of the form (1.2). Before proceeding we shall assume that τ_n is given by $\tau_n = \tau(n)$, where $\tau : [1, \infty[\rightarrow \mathbb{R}_+$ satisfies

$$(4.1) \quad \tau \quad \text{is decreasing,} \quad \tau(t) > 0 \quad \forall t \geq 1$$

$$(4.2) \quad \tau(t) \leq t^{-\frac{1}{2}} \quad \forall t \geq 1$$

$$(4.3) \quad \int_1^{\infty} \tau(t)^2 dt < \infty.$$

And we shall consider the norm

$$(4.4) \quad q(x) = \left\{ \sum_{n=1}^{\infty} \tau(n)^2 x_n^2 \right\}^{\frac{1}{2}} \quad \forall x = (x_n) \in \mathbb{R}^{\infty}.$$

Note that $\sum \tau_n^2 < \infty$ by (4.1) and (4.3), so $Q = q(\eta)$ is finite a.s. Now let F_n and F^n be the two marginals:

$$F_n(t) = P\left(\sum_{j=1}^n \tau(j)^2 \eta_j^2 \leq t^2\right)$$

$$F^n(t) = P\left(\sum_{n+1}^{\infty} \tau(j)^2 \eta_j^2 \leq t^2\right).$$

If $B_n(t)$ denotes the euclidean ball of radius t centered at the origin, then we have

$$(4.5) \quad F_n(t) = (2\pi)^{-n/2} \prod_{j=1}^n \tau(j)^{-1} \int_{B_n(t)} \exp\left(-\frac{1}{2} \sum_{j=1}^n \tau(j)^{-2} x_j^2\right) dx$$

$$(4.6) \quad F_n(s) F^n\left((t^2 - s^2)^{\frac{1}{2}}\right) \leq F(t) \leq F_n(t) \quad \forall 0 \leq s \leq t,$$

since $Q^2 = Q_n^2 + R_n^2$ where

$$Q_n^2 = \sum_{j=1}^n \tau(j)^2 \eta_j^2, \quad R_n^2 = \sum_{n+1}^{\infty} \tau(j)^2 \eta_j^2$$

and Q_n and R_n are independent.

THEOREM 4.1. *Let q be the seminorm given by (4.4), where τ satisfies (4.1)—(4.3). Let*

$$(4.1.1) \quad \varphi(t) = t^{-\frac{1}{2}} \tau(t)^{-1} \quad \text{for } t \geq 1.$$

If F is the distribution function of $Q = q(\eta)$, then there exists $A_1 > 0$, so that

$$(4.1.2) \quad F(t) \leq A_1 \exp\left\{\int_1^x \log \varphi(y) dy + \log \varphi(x) + (x-1) \log t\right\}$$

for all $x \geq 1$ and all $t \in [0, 1]$.

REMARK. In applications of (4.1.2) one should try to minimize the right-hand side in x for t fixed. That is, take $x \geq 1$ to be a suitable solution to

$$\log \varphi(x) + \varphi'(x)/\varphi(x) + \log t = 0.$$

Ignoring the middle term one reasonable choice is $\varphi(x) = 1/t$ or $x = \varphi^{-1}(1/t)$.

PROOF. Let V_n be the volume of the n -dimensional unit ball. Then by Stirling's formula we have

$$(4.7) \quad V_n = \Gamma\left(\frac{1}{2}\right)^n \Gamma\left(1 + \frac{1}{2}n\right)^{-1} = a_1 (2\pi)^{\frac{1}{2}n} n^{-\frac{1}{2}(n+1)} e^{n/2} e^{-\theta/n}$$

where $a_1 = \pi^{-\frac{1}{2}} \doteq 0.56$ and $0 < \theta < \frac{1}{6}$. Hence by (4.5) and (4.6) we have

$$(4.8) \quad F(t) \leq a_1 \exp\left\{-\sum_1^n \log \tau(j) - \frac{1}{2}(n+1)\log n + \frac{1}{2}n + n \log t\right\}$$

for all $n \geq 1$ and all $t \geq 0$. Since $f = -\log \tau$ is increasing we have

$$(4.9) \quad f(1) + \int_1^n f(y) dy \leq \sum_1^n f(j) \leq f(n) + \int_1^n f(y) dy.$$

So we have for the exponent in (4.8):

$$\begin{aligned} & -\sum_1^n \log \tau(j) - \frac{1}{2}(n+1)\log n + \frac{1}{2}n + n \log t \\ & \leq -\int_1^n \log \tau(y) dy - \log \tau(n) - \frac{1}{2}\int_1^n \log y dy + \frac{1}{2} - \frac{1}{2}\log n + n \log t \\ & = \int_1^n \log \varphi(y) dy - \log \tau(n) - \frac{1}{2}\log n + n \log t + \frac{1}{2}. \end{aligned}$$

Now we note that $\varphi(y) \geq 1$ for $y \geq 1$ by (4.3). So if $n \leq x \leq n+1$ we have $x \leq 2n$ and we find

$$\begin{aligned} & \int_1^n \log \varphi(y) dy \leq \int_1^x \log \varphi(y) dy, \\ & -\log \tau(n) - \frac{1}{2}\log n \leq -\log \tau(x) - \frac{1}{2}\log x + \frac{1}{2}\log 2 = \log \varphi(x) + \frac{1}{2}\log 2, \\ & n \log t \leq (x-1)\log t \quad \text{for } 0 < t \leq 1. \end{aligned}$$

Inserting this in (4.8) gives

$$F(t) \leq A_1 \exp\left\{\int_1^x \log \varphi(y) dy + \log \varphi(x) + (x-1)\log t\right\}$$

where

$$(4.10) \quad A_1 = a_1 (2e)^{\frac{1}{2}} = (2e/\pi)^{\frac{1}{2}} \doteq 1.32.$$

THEOREM 4.2. Let q be the seminorm given by (4.4), where τ satisfies (4.1)–(4.3). Suppose in addition that τ satisfies

$$(4.2.1) \quad \log \tau(x) \quad \text{is convex}$$

$$(4.2.2) \quad \varphi(x) = x^{-\frac{1}{2}} \tau(x)^{-1} \quad \text{increases to } +\infty \quad \text{on } [1, \infty].$$

If F is the distribution of $Q = q(\eta)$, then for some constant A_2 we have

$$(4.2.3) \quad F(t) \leq A_2 x^{-\frac{1}{2}} \tau(x)^{-\frac{1}{2}} e^{-H(x-1)} \quad \text{if } x \geq 1 \quad \text{and } \varphi(x) \leq t^{-1}$$

where H is defined by

$$(4.2.4) \quad H(x) = \int_1^x \frac{t\varphi'(t)}{\varphi(t)} dt \quad \text{for } x \geq 1.$$

REMARK. Again in applications of (4.2.3) we have to choose an appropriate x . One possible choice is $\varphi(x) = 1/t$ or $x = \varphi^{-1}(1/t)$.

PROOF. When $f = -\log \tau$ is increasing and concave one may improve (4.9) to (4.11)

$$\frac{1}{2}(f(n+1) - f(2)) + f(1) + \int_1^n f(y) dy \leq \sum_1^n f(j) \leq \frac{1}{2}(f(1) + \frac{1}{2}f(n)) + \int_1^n f(y) dy$$

by estimating the integral over $[j-1, j]$ by the area of two trapezoids:

$$\frac{1}{2}(f(j) + f(j-1)) \leq \int_{j-1}^j f(y) dy \leq \frac{1}{2}(f(j) - f'(j) + f(j))$$

and noting that since f' is decreasing we have

$$\sum_{j=2}^n f'(j) \geq \int_2^{n+1} f'(x) dx = f(n+1) - f(2).$$

Using this we can estimate the exponent in (4.8) by

$$\begin{aligned} & -\sum_{j=1}^n \log \tau(j) - \frac{1}{2}(n+1) \log n + \frac{1}{2}n + n \log t \\ & \leq \int_1^n \log \tau(y) dy - \frac{1}{2}\{\log \tau(n) + \log \tau(1) + \int_1^n \log y dy + \log n - 1\} + n \log t \\ & = \int_1^n \log \varphi(y) dy - \frac{1}{2}\log(n\tau(n)) + n \log t + \frac{1}{2} - \frac{1}{2}\log \tau(1). \end{aligned}$$

By partial integration we find

$$\begin{aligned} \int_1^n \log \varphi(y) dy &= n \log \varphi(n) - \log \varphi(1) - \int_1^n \frac{t\varphi'(t)}{\varphi(t)} dt \\ &= n \log \varphi(n) + \log \tau(1) - H(n) \end{aligned}$$

since $\varphi(1) = \tau(1)^{-1}$. So we have

$$(4.12) \quad F(t) \leq a_2 \exp\left\{-H(n) + n \log(t\varphi(n)) - \frac{1}{2}\log(n\tau(n))\right\}$$

where

$$a_2 = a_1 \exp\left\{\frac{1}{2} + \frac{1}{2}\log \tau(1)\right\} \leq (e/\pi)^{\frac{1}{2}}$$

since $\log \tau(1) \leq 0$. Now suppose that $n \leq x \leq n+1$ ($n \geq 1$). Then $H(n) \geq H(x-1)$ since H is increasing by (4.2.2) and if $\varphi(x) \leq 1/t$, then

$$\log(t\varphi(n)) \leq \log(t\varphi(x)) \leq 0$$

and finally since $n \geq \frac{1}{2}x$

$$\begin{aligned} \log n + \log \tau(n) &\geq \log x + \log \tau(x) + \log(n/x) \\ &\geq \log(x\tau(x)) - \log 2. \end{aligned}$$

Inserting this in (4.12) gives

$$F(t) \leq A_2 x^{-\frac{1}{2}} \tau(x)^{-\frac{1}{2}} e^{-H(x-1)}$$

where

$$(4.13) \quad A_2 = a_2 2^{\frac{1}{2}} \leq (2e/\pi)^{\frac{1}{2}} \doteq 1.32,$$

proving the theorem.

THEOREM 4.3. Let q be given by (4.4) where τ satisfies (4.1)–(4.3), and let

$$(4.3.1) \quad \psi(x) = \int_x^\infty \tau(t)^2 dt \quad \text{for } x \geq 1.$$

If F is the distribution of $Q = q(\eta)$, then for some $B_1 > 0$ we have
(4.3.2)

$$F(t) \geq B_1 \left\{ 1 - \frac{\psi(x)}{t^2 - s^2} \right\} \exp \left\{ \int_1^x \log \varphi(y) dy - \frac{1}{2} \log x + (x+1) \log s - \frac{s^2}{2\tau(x)^2} \right\}$$

for all $x \geq 1$ and all $0 \leq s \leq t \leq 1$. Here φ is defined as above in (4.1.1).

PROOF. From (4.5) and (4.7) we deduce the following lower bound of F_n :

$$(4.14) \quad F_n(t) \geq b_1 \exp \left\{ -\sum_1^n \log \tau(j) - \frac{1}{2}(n+1) \log n + \frac{1}{2}n + n \log t - \frac{t^2}{2\tau(n)^2} \right\}$$

since we have

$$\frac{1}{2} \sum_{j=1}^n \tau(j)^{-2} x_j^2 \leq \frac{t^2}{2\tau(n)^2} \quad \text{for all } x \in B_n(t).$$

Here $b_1 = \pi^{-\frac{1}{2}} e^{-1/6}$.

Since $\log \tau(1) \leq 0$ we find from (4.9)

$$\begin{aligned} & -\sum_1^n \log \tau(n) - \frac{1}{2}(n+1) \log n + \frac{1}{2}n + n \log t - \frac{1}{2}t^2 \tau(n)^{-2} \\ & \geq -\int_1^n \log \tau(y) dy - \frac{1}{2} \int_1^n \log y dy + \frac{1}{2} - \frac{1}{2} \log n + n \log t - \frac{1}{2}t^2 \tau(n)^{-2} \\ & = \int_1^n \log \varphi(y) dy - \frac{1}{2} \log n - \frac{1}{2}t^2 \tau(n)^{-2} + \frac{1}{2} + n \log t \\ & \geq \int_1^x \log \varphi(y) dy - \frac{1}{2} \log x - \frac{1}{2}t^2 \tau(x)^{-2} + \frac{1}{2} - \frac{1}{2} \log 2 + (x+1) \log t \end{aligned}$$

for $n-1 < x \leq n$, $n \geq 2$ and $0 \leq t \leq 1$. So we have

$$(4.15) \quad F_n(t) \geq b_2 \exp \left\{ \int_1^x \log \varphi(y) dy - \frac{1}{2} \log x - \frac{1}{2}t^2 \tau(x)^{-2} + (x+1) \log t \right\}$$

for $n \geq 2$, $0 \leq t \leq 1$ and $n-1 < x \leq n$, where $b_2 = b_1(e/2)^{\frac{1}{2}}$.

Let $R_n^2 = \sum_{j=1}^n \eta_j^2 \tau(j)^2$. Then by Chebyshev's inequality we have

$$\begin{aligned} F^n(u) &= 1 - P(R_n^2 > u^2) \geq 1 - u^{-2} E R_n^2 \\ &= 1 - u^{-2} \sum_{j=1}^n \tau(j)^2 \geq 1 - u^{-2} \int_x^\infty \tau(y)^2 dy \\ &= 1 - u^{-2} \psi(x) \end{aligned}$$

whenever $n-1 < x \leq n$, $n \geq 2$. So by (4.6) and (4.15) we have

$$F(t) \geq B_1 \left\{ 1 - \frac{\psi(x)}{t^2 - s^2} \right\} \exp \left\{ \int_1^x \log \varphi(y) dy - \frac{1}{2} \log x + (x+1) \log s - \frac{s^2}{2\tau(x)^2} \right\}$$

for $0 \leq s \leq t \leq 1$ and $x \geq 1$, where B_1 and b_2 are given by the equation

$$(4.16) \quad B_1 = b_2 = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{3}} = 0.56.$$

THEOREM 4.4. Let q be given by (4.4) where τ satisfies (4.1)–(4.3) and (4.2.1)–(4.2.2). Let φ , ψ and H be given as above:

$$\varphi(x) = x^{-\frac{1}{2}}\tau(x)^{-1} \quad \text{for } x \geq 1$$

$$\psi(x) = \int_x^\infty \tau(y)^2 dy \quad \text{for } x \geq 1$$

$$H(x) = \int_1^x \frac{t\varphi'(t)}{\varphi(t)} dt \quad \text{for } x \geq 1$$

If F is the distribution of $Q = q(\eta)$, then for some $B_2 > 0$, we have

$$(4.4.1) \quad F(t) \geq B_2 x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} \left\{ 1 - \frac{\psi(x)}{t^2 - s^2} \right\} \exp \left\{ -H(x+1) - \frac{s^2}{2\tau(x)^2} \right\}$$

whenever $x \geq 1$ and $1/\varphi(x) \leq s \leq t \leq 1$.

PROOF. Using (4.11) we have for the exponent in (4.14):

$$\begin{aligned} & -\sum_1^n \log \tau(j) - \frac{1}{2}(n+1)\log n + \frac{1}{2}n + n \log t - \frac{1}{2}t^2\tau(n)^{-2} \\ & \geq -\int_1^n \log \tau(y) dy - \frac{1}{2}\log \tau(n+1) + \frac{1}{2}\log \tau(2) - \frac{1}{2}\int_1^n \log y dy \\ & \quad + \frac{1}{2} - \frac{1}{2}\log n + n \log t - \frac{1}{2}t^2\tau(n)^{-2} \\ & = \int_1^n \log \varphi(y) dy - \frac{1}{2}\log(n\tau(n+1)) + n \log t - \frac{1}{2}t^2\tau(n)^{-2} + \frac{1}{2} + \frac{1}{2}\log \tau(2) \\ & = -H(n) + n \log(t\varphi(n)) - \frac{1}{2}\log(n\tau(n+1)) - \frac{1}{2}t^2\tau(n)^{-2} + \alpha \end{aligned}$$

where we have used the equality:

$$\int_1^n \log \varphi(y) dy = -H(n) + n \log \varphi(n) + \log \tau(1)$$

and where $\alpha = \frac{1}{2} + \frac{1}{2}\log \tau(2) + \log \tau(1)$.

If $n-1 \leq x \leq n$, $n \geq 2$ and $\varphi(x) \geq 1/t$, then

$$\begin{aligned} & -H(n) \geq -H(x+1), \\ & \log(t\varphi(n)) \geq \log(t\varphi(x)) \geq 0, \\ & -\frac{1}{2}\log(n\tau(n+1)) = -\frac{1}{2}\log x + \frac{1}{2}\log(x/n) - \frac{1}{2}\log \tau(n+1) \\ & \geq -\frac{1}{2}\log(x\tau(x)) - \frac{1}{2}\log 2, \end{aligned}$$

so we find as before

$$F(t) \geq B_2 x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} \left\{ 1 - \frac{\psi(x)}{t^2 - s^2} \right\} \exp \left\{ -H(x+1) - \frac{s^2}{2\tau(x)^2} \right\}$$

where

$$(4.17) \quad B_2 = b_1\tau(1)^{-1}\left(\frac{1}{2}e\tau(2)\right)^{\frac{1}{2}}.$$

EXAMPLE 4.5. If $\tau(x) = x^{-\alpha} (\alpha > \frac{1}{2})$, then we have

$$(4.5.1) \quad F(t) \leq A t^{\rho(1-\alpha)} \exp\left(-\left(\alpha - \frac{1}{2}\right)t^{-2\rho}\right)$$

$$(4.5.2) \quad F(t) \geq B t^{\rho(3-\alpha)} \exp(-\alpha(1+\rho)t^{-2\rho})$$

where $\rho = (2\alpha - 1)^{-1}$ and A and B are positive constants.

In this case the functions φ , ψ and H take the form:

$$\varphi(x) = x^{\alpha - \frac{1}{2}}, \quad \psi(x) = \rho x^{1-2\alpha}, \quad H(x) = \left(\alpha - \frac{1}{2}\right)(x - 1).$$

Then $\varphi(t^{-2\rho}) = t^{-1}$, so putting $x = t^{-2\rho}$ in (4.2.3) gives (4.5.1).

Putting $x = s^{-2\rho}$ in (4.4.1) gives

$$(4.18) \quad F(t) \geq B_2 s^{\rho(1-\alpha)} \left\{ 1 - \frac{\rho s^2}{t^2 - s^2} \right\} \exp(-\alpha s^{-2\rho})$$

for $0 \leq s \leq t \leq 1$. Now we take

$$s = t \left(\frac{1 - \frac{1}{2}t^{2\rho}}{\rho + 1} \right)^{\frac{1}{2}}.$$

Then we have

$$1 - \frac{\rho s^2}{t^2 - s^2} = \frac{t^2 - (\rho + 1)s^2}{t^2 - s^2} = \frac{t^{2\rho}(1 + \rho)}{2\rho + t^{2\rho}} \geq \frac{1}{2}t^{2\rho}$$

$$\begin{aligned} s^{-2\rho} &= (\rho + 1)^\rho \left(1 - \frac{1}{2}t^{2\rho}\right)^{-\rho} t^{-2\rho} \\ &\leq (\rho + 1)^\rho \left(1 + \frac{1}{2}t^{2\rho}\rho 2^{\rho+1}\right) t^{-2\rho} \\ &= (\rho + 1)^\rho t^{-2\rho} + \text{constant} \end{aligned}$$

where we in the last inequality used:

$$\left(1 - \frac{1}{2}t^{2\rho}\right)^{-\rho} = 1 + \frac{1}{2}t^{2\rho}\rho\xi^{-\rho-1} \leq 1 + \frac{1}{2}t^{2\rho}\rho 2^{\rho+1}$$

where $\frac{1}{2} \leq 1 - \frac{1}{2}t^{2\rho} \leq \xi \leq 1$. Inserting all this in (4.18) gives (4.5.2).

EXAMPLE 4.6. Let $\tau(x) = x^{-\frac{1}{2}}(1 + \log x)^{-1}$; then we have

$$(4.6.1) \quad F(t) \leq A \exp(-te^{t^{-1}-1}),$$

$$(4.6.2) \quad F(t) \geq B \exp\left(-\left(\frac{1}{2} + 3t^2\right)e^{t^{-2}+1}\right)$$

where A and B are positive constants.

In this case we have

$$\begin{aligned} \varphi(x) &= 1 + \log x, \\ \psi(x) &= (1 + \log x)^{-1}, \\ H(x) &= \int_1^x \frac{dt}{1 + \log t}, \end{aligned}$$

and since

$$\begin{aligned}\frac{d}{dt} \frac{t}{1 + \log t} &= \frac{\log t}{(1 + \log t)^2} \leq \frac{1}{1 + \log t} \quad \forall t \geq 0, \\ \frac{d}{dt} \frac{t}{\log t - 1} &= \frac{\log t - 2}{(\log t - 1)^2} > \frac{1}{1 + \log t} \quad \forall t \geq e^3,\end{aligned}$$

we have

$$\begin{aligned}H(x) &\geq \frac{x}{1 + \log x} - 1 \\ H(x) &\leq H(e^3) + \frac{x}{\log x - 1} - \frac{e^3}{2} \quad \forall x \geq e^3.\end{aligned}$$

Let us choose $x = e^{1/t-1}$ in (4.2.3); then we find

$$\begin{aligned}x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} &= \exp\left(-1/(4t) + \frac{1}{4} - \frac{1}{2}\log t\right), \\ H(x-1) &\geq \frac{x-1}{1 + \log x} - 1 = te^{t^{-1}-1} - t - 1,\end{aligned}$$

and since $t \leq 1/(4t) + \frac{1}{2}\log t$ for t sufficiently small (4.6.1) follows.

Let us then choose $x = e^{1/s} - 1$ in (4.4.1); then it is easily checked that $\varphi(x) \geq s^{-1}$, and we have

$$\begin{aligned}x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} &= (e^{1/s} - 1)^{-\frac{1}{4}}\varphi(x)^{\frac{1}{2}} \geq e^{-1/(4s)}, \\ \psi(x) &= \varphi(x)^{-1} \leq s, \\ H(x+1) &\leq K + se^{1/s}/(1-s) \\ \frac{1}{2}s^2\tau(x)^{-2} &\leq \frac{1}{2}s^2\tau(x+1)^{-2} = \frac{1}{2}(1+s)^2e^{1/s}.\end{aligned}$$

Then we choose $s = t^2/(1+t^2)$, and we find

$$\begin{aligned}x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} &\geq \exp\left(-\frac{1+t^2}{4t^2}\right), \\ 1 - \frac{\psi(x)}{t^2 - s^2} &\geq \frac{t^2 - s^2 - s}{t^2 - s^2} = \frac{t^4}{1+t^4+t^2} \geq \frac{t^4}{3} \quad t \leq 1, \\ -H(x+1) &\geq -K - t^2e^{t^{-2}+1}, \\ -\frac{1}{2}s^2\tau(x)^{-2} &\geq -\frac{1}{2}(1+t^2)^2e^{t^{-2}+1} = -\left(\frac{1}{2} + \frac{1}{2}t^4 + t^2\right)e^{t^{-2}+1}.\end{aligned}$$

And since

$$-4\log t + \frac{1}{2}t^4e^{t^{-2}+1} + \frac{1+t^2}{4t^2} \leq t^2e^{t^{-2}+1}$$

for t sufficiently small, (4.6.2) follows by inserting the inequalities above in (4.4.1).

EXAMPLE 4.7. Let $\tau(x) = x^{-\frac{1}{2}}e^{-x}$; then we have

$$(4.7.1) \quad F(t) \leq At^{-\frac{3}{2}} \left(\log \frac{1}{t} \right)^{-\frac{1}{4}} \exp \left(-\frac{1}{2} \left(\log \frac{1}{t} \right)^2 \right),$$

$$(4.7.2) \quad F(t) \geq Bt^{1+\log 2} \left(\log \frac{1}{t} \right)^{-\frac{1}{4}} \exp \left(-\frac{1}{2} \left(\log \frac{1}{t} \right)^2 \right),$$

where A and B are positive constants.

In this case we have:

$$\varphi(x) = e^x, \quad H(x) = \frac{1}{2}(x+1)(x-1),$$

$$\psi(x) = \int_x^\infty \frac{e^{-2t}}{t} dt \leq \frac{e^{-2x}}{2}.$$

Choosing $x = \log(1/t)$ in (4.2.3) gives

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} = \left(\log \frac{1}{t} \right)^{-\frac{1}{4}} t^{-\frac{1}{2}},$$

$$-H(x-1) = -\frac{1}{2}x(x-2) = -\frac{1}{2} \left(\log \frac{1}{t} \right)^2 - \log t.$$

Now (4.7.1) follows by use of (4.2.3).

Choosing $x = \log(1/s)$ and $s = \frac{1}{2}t$ gives

$$\begin{aligned} x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} &= \left(\log \frac{1}{t} + \log 2 \right)^{-\frac{1}{4}} 2^{\frac{1}{2}} t^{-\frac{1}{2}} \geq kt^{-\frac{1}{2}} \left(\log \frac{1}{t} \right)^{-\frac{1}{4}} \\ -H(x+1) &= -\frac{1}{2}x(x+2) = -\frac{1}{2} \left(\log \frac{1}{t} + \log 2 \right)^2 + \log t \\ &= -\frac{1}{2} \left(\log \frac{1}{t} \right)^2 - \frac{1}{2}(\log 2)^2 + (1 + \log 2)\log t \\ -\frac{1}{2}s^2\tau(x)^{-2} &= -\frac{1}{2}s^2xe^{2x} = -\frac{1}{2}\log \frac{1}{t} - \frac{1}{2}\log 2 \\ 1 - \frac{\psi(x)}{t^2 - s^2} &\geq 1 - \frac{e^{-2x}}{2(t^2 - s^2)} = 1 - \frac{s^2}{2(t^2 - s^2)} = \frac{5}{6}. \end{aligned}$$

Now (4.7.2) follows from (4.4.1).

EXAMPLE 4.8. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be increasing and with $\lim_{t \rightarrow 0} f(t) = 0$. Then there exists a function $\tau: [1, \infty[\rightarrow \mathbb{R}_+$ satisfying (4.1)–(4.3) and such that

$$(4.8.1) \quad F(t) \leq f(t) \quad \forall 0 \leq t \leq 1/e$$

where F is the distribution of $Q = \{\sum_{j=1}^\infty \eta_j^2 \tau(j)^2\}$

Let $p(t) = [1/(et)]$ (here $[x]$ denote the integer part of x) and let $n_0 = 1$ and for $p \geq 1$:

$$n_p = 1 + \left[-2 \log f \left(\frac{1}{e(p+1)} \right) - 2 \log(e(p+1)) \right].$$

We assume that $0 < f(t) \leq 1$ for $0 < t \leq 1$, $f(t)/t$ increases and $\lim_{t \rightarrow 0} f(t)/t = 0$, which is possible by substituting f by $tf(t)$. Let $\tau(1) = 1$, and

$$\tau(t) = p^{-1}n_p^{-\frac{1}{2}} \quad \text{for } n_{p-1} < t \leq n_p.$$

Then $n_p \leq n_{p+1}$ for all $p \geq 0$ and $n_p \rightarrow \infty$, since $f(x)/x$ increases and tends to 0 as $x \rightarrow 0$; (4.1)–(4.3) are easily checked, and

$$\varphi(t) = t^{-\frac{1}{2}}p(n_p)^{\frac{1}{2}} \quad \text{for } n_{p-1} < t \leq n_p.$$

So

$$\begin{aligned} \int_{n_{p-1}}^{n_p} \log \varphi(t) dt &= (n_p - n_{p-1}) \log(p(n_p)^{\frac{1}{2}}) - \frac{1}{2}(n_p(\log n_p - 1) - n_{p-1}(\log n_{p-1} - 1)) \\ &\leq (n_p - n_{p-1}) \log(p(n_p)^{\frac{1}{2}}) - (n_p - n_{p-1}) \left(\log(n_p)^{\frac{1}{2}} - \frac{1}{2} \right) \\ &= (n_p - n_{p-1}) \left(\log p + \frac{1}{2} \right), \end{aligned}$$

and

$$\int_1^n \log \varphi(t) dt \leq (n_p - 1) \left(\log p + \frac{1}{2} \right).$$

Inserting this in the exponent of (4.1.2) with $x = n_p$ we get

$$F(t) \leq A_1 \exp \left(n_p \left(\log p + \frac{1}{2} + \log t \right) - \frac{1}{2} - \log t \right).$$

Taking $p = p(t)$ gives $p \leq (et)^{-1}$, so

$$\log p + \frac{1}{2} + \log t \leq -\frac{1}{2}$$

and $(1/e(p+1)) \leq t$ gives

$$-\frac{1}{2}n_p \leq \log f(t) + \log t$$

so

$$F(t) \leq e^{-\frac{1}{2}}A_1 f(t) \leq f(t)$$

since $e^{-\frac{1}{2}}A_1 = (2/\pi)^{\frac{1}{2}} \leq 1$ (cf. (4.10)).

EXAMPLE 4.9. Let $g :]0, 1] \rightarrow \mathbb{R}_+$ be an increasing function with $g(1) < 1$ and

$$g(t) = 0(t^n) \quad \text{as } t \rightarrow 0 \quad \forall n \geq 1.$$

Then there exists a function $\tau : [1, \infty[\rightarrow \mathbb{R}_+$ satisfying (4.1)–(4.3), and such that

$$(4.9.1) \quad F(t) \geq g(t) \quad \forall t \in [0, 1]$$

where F is the distribution of $Q = \{\sum_1^\infty \eta_j^2 \tau(j)^2\}^{\frac{1}{2}}$.

There exist constants $A_n > 0$, so that

$$g(t) \leq A_n t^{n+3} \quad \forall 0 < t \leq 1, \forall n \geq 1.$$

Hence if $B_n = \log A_n$ we have

$$\log g(t) \leq B_n + (n+3)\log t \quad \forall t \in [0, 1] \forall n \geq 1.$$

Let $\tau_0^2 = 1$ and put

$$\alpha_n = e^{-B_n-1} 2^{-n-7} (n+1)^{-\frac{1}{2}} \quad \forall n \geq 1.$$

Then we define τ_n^2 inductively by

$$\tau_n^2 = \min\{\alpha_n, 2^{-n}\tau_0^2, 2^{-n+1}\tau_1^2, \dots, 2^{-1}\tau_{n-1}^2\}$$

for $n \geq 1$, and we put

$$\tau(t) = \tau_n \quad \text{for } n < t \leq n+1 \quad \text{and } n \geq 0.$$

Then τ satisfies (4.1)–(4.3), and we have

$$\psi(n) = \int_n^\infty \tau(t)^2 dt = \sum_{j=n}^\infty \tau_j^2 \leq \sum_{j=n}^\infty 2^{-(j-n)} \tau_n^2 = 2\tau_n^2.$$

Let $n \geq 1$ and $4\psi(n) \leq t^2 \leq 4\psi(n-1)$, then we shall apply (4.3.2) with $x = n$ and $s = \frac{1}{2}t$. The exponent in (4.3.2) gives ($s^2 \leq \psi(n-1) \leq 2\tau_{n-1}^2$)

$$\begin{aligned} & \int_1^n \log \varphi(y) dy - \frac{1}{2} \log n + (n+1) \log s - \frac{1}{2} s^2 \tau_{n-1}^{-2} \\ & \geq -\frac{1}{2} \log n + (n+1) \log t - (n+1) \log 2 - 1 \\ & = (B_{n-1} + (n+3) \log t) - 2 \log t - B_{n-1} - \frac{1}{2} \log n - \log 2^{n+1} - 1 \\ & \geq \log g(t) - \log t^2 + \log \alpha_{n-1} + 5 \log 2 \\ & \geq \log g(t) + 2 \log 2 \end{aligned}$$

since $t^2 \leq 4\psi(n-1) \leq 8\tau_{n-1}^2 \leq 8\alpha_{n-1}$. The factor in (4.3.2) gives

$$1 - \frac{\psi(n)}{t^2 - s^2} = 1 - \frac{\psi(n)}{3s^2} \geq \frac{2}{3}$$

since $\psi(n) \leq s^2$. Now since $B_1 = 0$, $56 \geq \frac{1}{2}$ we have

$$F(t) \geq B_1(8/3)g(t) \geq g(t)$$

for $t \in [0, 1]$ and $t \leq 2(\psi(0))^{\frac{1}{2}}$. However $\psi(0) \geq \tau_0^2 = 1$, so (4.9.1) holds.

5. Exact distributions. We shall now give some cases where the distribution of Q can be given in an exact form for certain Hilbertian norms q . Note that (3.3) gives the exact distribution for sup-norms. Let q and Q be given by

$$(5.1) \quad q(x) = \left\{ \sum_{j=1}^n (x_{2j-1}^2 + x_{2j}^2) / (2\lambda_j) \right\}^{\frac{1}{2}},$$

$$(5.2) \quad Q^2 = q(\eta)^2 = \sum_{j=1}^\infty (\eta_{2j-1}^2 + \eta_{2j}^2) / (2\lambda_j).$$

Let Q_n^2 denote the n th partial sum in (5.2). Since $(\eta_{2j-1}^2 + \eta_{2j}^2)/(2\lambda_j)$ is exponentially distributed with parameter λ_j , we have (see, e.g., [7], page 40)

$$(5.3) \quad P(Q_n \leq x) = 1 - \sum_{j=1}^n A_j^n e^{-\lambda_j x^2} \quad \forall x \geq 0$$

if $\lambda_i \neq \lambda_j \forall i \neq j$, and where A_j^n is defined by

$$A_j^n = \prod_{k=1; k \neq j}^n (1 - \lambda_j/\lambda_k)^{-1} \quad \text{for } j = 1, \dots, n.$$

Now let

$$(5.4) \quad A_j = \prod_{k \neq j} (1 - \lambda_j/\lambda_k)^{-1} = \prod_{k \neq j} \frac{\lambda_k}{\lambda_k - \lambda_j}$$

and assume that (λ_j) satisfies:

$$(5.5) \quad 0 < \lambda_1 < \lambda_2 < \dots,$$

$$(5.6) \quad \sum_1^\infty \lambda_j^{-1} < \infty,$$

$$(5.7) \quad \sum_1^\infty |A_j| e^{-\lambda_j x} < \infty \quad \forall x > 0.$$

Then we have for $j \leq n$

$$|A_j^n| = |A_j| \pi_{k=n+1}^\infty (1 - \lambda_j/\lambda_k) \leq |A_j|.$$

So by (5.3), (5.7) and the dominated convergence theorem we deduce:

$$(5.8) \quad P(Q \leq x) = 1 - \sum_{j=1}^\infty A_j e^{-\lambda_j x^2} \quad \forall x > 0.$$

And if we assume, in addition to (5.5)–(5.7), that we have

$$(5.9) \quad \sum_{j=1}^\infty \lambda_j |A_j| e^{-\lambda_j x} < \infty \quad \forall x > 0,$$

then the density, f , of Q is given by

$$(5.10) \quad f(t) = 2t \sum_{j=1}^\infty \lambda_j A_j e^{-\lambda_j t^2}.$$

So under (5.5)–(5.7) the formulae (5.8) gives the distribution of Q defined by (5.2), and under (5.9) the density of Q is given by (5.10). Note that $\text{sign } A_j = (-1)^{j-1}$, so the series in (5.8) and (5.10) are alternating.

In order to use (5.8) and (5.10) we should be able to find A_j . One way is the following: suppose that we have given a product formula

$$\varphi(x) = \prod_{k=1}^\infty (1 - \psi(x)/\lambda_k)$$

where φ and ψ are differentiable. Let x_j be a solution to $\psi(x_j) = \lambda_j$; then $\varphi(x_j) = 0$ and for $x \neq x_j$

$$\prod_{k \neq j} (1 - \psi(x)/\lambda_k) = -\lambda_j \frac{\varphi(x) - \varphi(x_j)}{x - x_j} \left\{ \frac{\psi(x) - \psi(x_j)}{x - x_j} \right\}^{-1}.$$

Letting $x \rightarrow x_j$ gives

$$(5.11) \quad A_j = \prod_{k \neq j} (1 - \lambda_j/\lambda_k)^{-1} = -\frac{\psi'(x_j)}{\lambda_j \varphi'(x_j)}.$$

The series (5.8) and (5.10) will in general be divergent or at least slowly convergent at $x = 0$. But the Poisson summation formulae

$$(5.12) \quad \sum_{n=-\infty}^\infty \cos(\gamma n) \hat{f}(xn) = \frac{2\pi}{x} \sum_{k=-\infty}^\infty f\left(\frac{\gamma + 2k\pi}{x}\right)$$

(f is an even density, $\hat{f}$ its Fourier transform) may in certain cases be used to transform the sums (5.8) and (5.10) into sums which are rapidly convergent for small x (see, e.g., [7], page 630 for the validity of (5.12)).

From the product formula (see [1], page 255)

$$\frac{\sin \pi x}{\pi x} = \prod_{k=1}^\infty (1 - x^2/k^2),$$

we find by (5.11):

$$\prod_{k \neq j} (1 - k^2/j^2)^{-1} = 2(-1)^{j-1}.$$

So if $\lambda_j = j^2$ we have

$$(5.13) \quad P(Q \leq x) = 1 - \sum_{j=1}^{\infty} 2(-1)^{j-1} e^{-j^2 x^2} = \sum_{j=-\infty}^{\infty} (-1)^j e^{-j^2 x^2}.$$

If we put $f =$ the normal density in (5.12) we get

$$(5.14) \quad \sum_{n=-\infty}^{\infty} \cos(ny) e^{-\frac{1}{2} x^2 n^2} = \frac{(2\pi)^{\frac{1}{2}}}{x} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(y + 2k\pi)^2}{2x^2}\right).$$

Putting $y = \pi$ and $x = t2^{\frac{1}{2}}$ gives

$$\sum_{n=-\infty}^{\infty} (-1)^n e^{-t n^2} = \frac{2\pi^{\frac{1}{2}}}{t} \sum_{k=0}^{\infty} \exp\left(-\frac{(2k+1)^2 \pi^2}{4t^2}\right).$$

The term for $k = 0$ is clearly dominant for small t , so we have

THEOREM 5.1. *Let*

$$Q = \left\{ \sum_{j=1}^{\infty} (\eta_{2j-1}^2 + \eta_{2j}^2) / (2j^2) \right\}^{\frac{1}{2}}.$$

Then we have

$$(5.1.1) \quad P(Q \leq t) = \frac{2\pi^{\frac{1}{2}}}{t} \sum_{k=0}^{\infty} \exp\left(-\frac{(2k+1)^2 \pi^2}{4t^2}\right) \quad \forall t > 0,$$

$$(5.1.2) \quad P(Q \leq t) \sim \frac{2\pi^{\frac{1}{2}}}{t} \exp\left(-\frac{\pi^2}{4t^2}\right) \quad \text{as } t \rightarrow 0.$$

From (5.11) and the product formulae (see [1], page 255):

$$\cos\left(\frac{1}{2} \pi x\right) = \prod_{k=1}^{\infty} (1 - x^2 / (2k-1)^2),$$

we find

$$\prod_{k \neq j} (1 - (2j-1)^2 / (2k-1)^2)^{-1} = \frac{4}{\pi} (-1)^{j-1} (2j-1)^{-1}.$$

So if $\lambda_j = (2j-1)^2$, then the density of Q is given by (cf. (5.10)):

$$\begin{aligned} f(t) &= \frac{8t}{\pi} \sum_{j=1}^{\infty} (2j-1) (-1)^{j-1} \exp(-(2j-1)^2 t^2) \\ &= \frac{4t}{\pi} \sum_{j=-\infty}^{\infty} (2j-1) (-1)^{j-1} \exp(-(2j-1)^2 t^2). \end{aligned}$$

Differentiating (5.14) with respect to y gives

$$\sum_{n=-\infty}^{\infty} n \sin(ny) e^{-\frac{1}{2} x^2 n^2} = \frac{(2\pi)^{\frac{1}{2}}}{x} \sum_{k=-\infty}^{\infty} \frac{y + 2k\pi}{x^2} \exp\left(-\frac{(y + 2k\pi)^2}{2x^2}\right),$$

so putting $y = \pi/2$ and $x = t2^{\frac{1}{2}}$ gives

$$\begin{aligned} f(t) &= \frac{4t}{\pi} \sum_{n=-\infty}^{\infty} n \sin\left(\frac{1}{2}n\pi\right) e^{-t^2 n^2} \\ &= \frac{4}{\pi^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} \frac{(4k+1)\pi}{4t^2} \exp\left(-\frac{(4k+1)^2 \pi^2}{16t^2}\right) \\ &= \frac{\pi^{\frac{1}{2}}}{t^2} \left\{ \sum_{k=0}^{\infty} (4k+1) \exp\left(-\frac{(4k+1)^2 \pi^2}{16t^2}\right) \right. \\ &\quad \left. - \sum_{k=1}^{\infty} (4k-1) \exp\left(-\frac{(4k-1)^2 \pi^2}{16t^2}\right) \right\} \\ &= \frac{\pi^{\frac{1}{2}}}{t^2} \sum_{j=0}^{\infty} (2j+1)(-1)^j \exp\left(-\frac{(2j+1)^2 \pi^2}{16t^2}\right). \end{aligned}$$

For small t the term for $j = 0$ dominates the others. So we have

$$f(t) \sim \frac{\pi^{\frac{1}{2}}}{t^2} \exp\left(-\left(\frac{\pi}{4t}\right)^2\right) \quad \text{as } t \rightarrow 0.$$

Hence $F(t) = \int_0^t f(s) ds$, and satisfies

$$F(t) \sim \int_0^t \frac{\pi^{\frac{1}{2}}}{s^2} \exp\left(-\left(\frac{\pi}{4s}\right)^2\right) ds \quad \text{as } t \rightarrow 0$$

by l'Hospital's rule; but

$$\int_0^t \frac{\pi^{\frac{1}{2}}}{s^2} \exp\left(-\left(\frac{\pi}{4s}\right)^2\right) ds = 4 \left(1 - \Phi\left(\frac{\pi}{t8^{\frac{1}{2}}}\right)\right)$$

and as $t \downarrow 0$,

$$1 - \Phi\left(\frac{\pi}{t8^{\frac{1}{2}}}\right) \sim (t\pi^{-1}8^{\frac{1}{2}})(2\pi)^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{16t^2}\right),$$

that is,

$$F(t) \sim 8t\pi^{-3/2} \exp\left(-\frac{\pi^2}{16t^2}\right).$$

And we have proved:

THEOREM 5.2. *Let*

$$Q = \left\{ \sum_{j=1}^{\infty} \frac{\eta_{2j-1}^2 + \eta_{2j}^2}{2(2j-1)^2} \right\}^{\frac{1}{2}}.$$

and let f denote the density of Q . Then we have

$$(5.2.1) \quad f(t) = \frac{\pi^{\frac{1}{2}}}{t^2} \sum_{j=0}^{\infty} (2j+1)(-1)^j \exp\left(-\frac{(2j+1)^2 \pi^2}{16t^2}\right),$$

$$(5.2.2) \quad P(Q \leq t) \sim 8t\pi^{-3/2} \exp\left(-\frac{\pi^2}{16t^2}\right) \quad \text{as } t \rightarrow 0.$$

THEOREM 5.3. *Let*

$$Q = \left\{ \sum_{j=1}^{\infty} j^{-2} \eta_j^2 \right\}^{\frac{1}{2}}.$$

Then we have

$$(5.3.1) \quad P(Q \leq t) \leq t^{-1} (2\pi)^{\frac{1}{2}} (1 + \varepsilon_1(t)) \exp\left(-\frac{\pi^2}{8t^2}\right),$$

$$(5.3.2) \quad P(Q \leq t) \geq 4t\pi^{-3/2} 2^{\frac{1}{2}} (1 + \varepsilon_2(t)) \exp\left(-\frac{\pi^2}{8t^2}\right)$$

where $\varepsilon_j(t) \rightarrow_{t \rightarrow 0} 0$ for $j = 1, 2$.

PROOF. Let Q_1 and Q_2 be the random variables defined in Theorem 5.1 and Theorem 5.2 respectively. Then

$$Q_1/2^{\frac{1}{2}} \leq Q \leq 2^{\frac{1}{2}} Q_2,$$

so

$$P(Q \leq t) \leq P(Q_1 \leq 2^{\frac{1}{2}} t),$$

$$P(Q \leq t) \geq P(Q_2 \geq t/2^{\frac{1}{2}}),$$

and the theorem follows from (5.1.2) and (5.2.2).

Added in proof. We thank David Siegmund for calling our attention to a paper of T. W. Anderson and D. A. Darling (*Ann. Math. Statist.* **23** 191–212), where they give an exact series for the distribution of Q from Theorem 5.3, from which the exact behavior at $t = 0$ can be read off (Anderson and Darling, page 202).

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