1986

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On the Number of Crossings of Empirical Distribution Functions

Abstract
Let $F$ and $G$ be two continuous distribution functions that cross at a finite number of points $-\infty \leq t_1 < \cdots < t_k \leq \infty$. We study the limiting behavior of the number of times the empirical distribution function $G_n$ crosses $F$ and the number of times $G_n$ crosses $F_n$. It is shown that these variables can be represented, as $n \to \infty$, as the sum of $k$ independent geometric random variables whose distributions depend on $F$ and $G$ only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. The technique involves approximating $F_n(t)$ and $G_n(t)$ locally by Poisson processes and using renewal-theoretic arguments. The implication of the results to an algorithm for determining stochastic dominance in finance is discussed.

Keywords
asymptotic distribution, boundary crossing probability, geometric distribution, Poisson process, renewal theory, stochastic dominance algorithm, Weiner-Hopf technique

Disciplines
Probability

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ON THE NUMBER OF CROSSINGS OF EMPIRICAL DISTRIBUTION FUNCTIONS

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Let $F$ and $G$ be two continuous distribution functions that cross at a finite number of points $-\infty < t_1 < \cdots < t_k < \infty$. We study the limiting behavior of the number of times the empirical distribution function $G_n$ crosses $F$ and the number of times $G_n$ crosses $F_n$. It is shown that these variables can be represented, as $n \to \infty$, as the sum of $k$ independent geometric random variables whose distributions depend on $F$ and $G$ only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. The technique involves approximating $F_n(t)$ and $G_n(t)$ locally by Poisson processes and using renewal-theoretic arguments. The implication of the results to an algorithm for determining stochastic dominance in finance is discussed.

1. Introduction. Consider two given continuous distribution functions $F$ and $G$, and let $F_n$ and $G_n$ be the corresponding empirical distribution functions (edfs) based on $n$ independent and identically distributed (i.i.d.) random variables (rv's). Throughout $F_n$ and $G_n$ are assumed to be independent. Let $K(n)$ be the number of times $G_n(t)$ equals $F(t)$, and let $L(n)$ be the number of intervals for which the graph of $G_n$ equals the graph of $F_n$. The distributions of $K(n)$ and $L(n)$ have been studied when $F \equiv G$, and it is known [see Dwass (1961)] that

$$
\lim_{n \to \infty} P\{K(n) < (2nt)^{1/2}\} = \lim_{n \to \infty} P\{L(n) < (4nt)^{1/2}\} = 1 - e^{-t}.
$$

We consider the situation where $F \neq G$ but $F$ crosses $G$ at a finite number of points $-\infty < t_1 < \cdots < t_k < \infty$. We were led to this problem while investigating certain types of stochastic dominance procedures that arise in finance.

We study the limiting behavior of $K(n)$, $L(n)$, and $L^*(n)$ = the number of times $G_n(t)$ strictly crosses $F_n(t)$ (see Section 2 for definition of strict crossing). It is shown that these variables can be represented, as $n \to \infty$, as the sum of $k$ independent rv's each of which is the number of crossings in a local neighborhood around $t_i$, $i = 1, \ldots, k$. The distributions of these $k$ rv's are related to geometric rv's whose distributions depend on $F$ and $G$ only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. Furthermore, only a finite number of order statistics around $t_i$ play a role in determining these crossings. The results are obtained under the assumption that $F$ and $G$ are continuously differentiable in a neighborhood of $t_i$ with $F'(t_i) \neq G'(t_i)$, $i = 1, \ldots, k$.

The paper is organized as follows. The main results on the limiting distributions and an outline of the proofs are given in Section 2. Details of the proofs are

Received July 1984; revised December 1984.

1 Research supported by NSF Grant No. MCS-83-01793.

AMS 1980 subject classifications. Primary 60G17; secondary 60E05.

Key words and phrases. Asymptotic distribution, boundary crossing probability, geometric distribution, Poisson process, renewal theory, stochastic dominance algorithm, Wiener–Hopf technique.

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deferred to Section 4 where we also derive a number of interesting lemmas concerning the local behavior and boundary crossing probabilities of empirical distributions. In Section 3 we discuss the implications of our results to an algorithm for determining stochastic dominance in finance. Our results suggest the need for using smoothed edfs in the algorithm.

Throughout the paper, we assume that both \( F \) and \( G \) are defined on \([0, 1]\) and that \( F(t) = t \). This presents no loss of generality since \( K(n) \), \( L(n) \), and \( L^*(n) \) are invariant under monotone transformations of the underlying rv’s that determine \( F_n(t) \) and \( G_n(t) \). To simplify matters we assume in addition that \( 0 < G(t) < 1 \) for \( 0 < t < 1 \) and that \( G(0) = 1 - G(1) = 0 \).

2. Limiting distributions. Let \( \{U_j\}_{j=1}^\infty \) denote a sequence of i.i.d. uniform \((0, 1)\) rv’s, i.e., \( U_j \sim F(t) = t \), \( 0 \leq t \leq 1 \). Similarly let \( \{V_j\}_{j=1}^\infty \) be i.i.d. rv’s with \( V_j \sim G(t) \), \( 0 \leq t \leq 1 \). \( G(t) \) equals \( t \) at \( k \) points \( 0 = t_1 < t_2 < \cdots < t_k = 1 \). Given \( \delta > 0 \), consider the \( \delta \)-neighborhood of \( t_i \) defined by \( B(t_i; \delta) = (t_i - \delta, t_i + \delta) \cap [0, 1] \). We assume throughout that \( G(t) \) is continuously differentiable in \( B(t_i; \delta) \) for some \( \delta > 0 \) with \( G'(t_i) \neq 1 \), \( i = 1, \ldots, k \).

\( F_n \) and \( G_n \) denote the edf’s based respectively on \( \{U_j\}_{j=1}^n \) and \( \{V_j\}_{j=1}^n \). We assume that \( F_n \) and \( G_n \) are independent. \( K(n) \) is the number of times \( G_n(t) \) equals \( t \). The points \( t = 0 \) and \( t = 1 \) are included in the computation of \( K(n) \). \( L(n) \) is the number of intervals for which the graph of \( G_n \) equals the graph of \( F_n \). \( L^*(n) \) is the number of times \( G_n(t) \) strictly crosses \( F_n(t) \). To define a strict crossing, let \( Z_1 < \cdots < Z_{2n} \) be the \( 2n \) ordered values of \( \{U_j\}_{j=1}^n \) and \( \{V_j\}_{j=1}^n \). A strict crossing occurs at \( Z_j \) if either (i) \( F_n(Z_{j-2}) < G_n(Z_{j-2}) \) and \( F_n(Z_j) > G_n(Z_j) \) or (ii) \( G_n(Z_{j-2}) < F_n(Z_{j-2}) \) and \( G_n(Z_j) > F_n(Z_j) \). Again for the sake of consistency, we include the points \( t = 0 \) and \( t = 1 \) in the computation of \( L^*(n) \).

It follows from the Glivenko–Cantelli theorem that, with probability \( \to 1 \) as \( n \to \infty \), \( G_n(t) \) does not cross \( t \) or \( F_n(t) \) outside of \( \bigcup_{i=1}^k B(t_i; \delta) \) for every sufficiently small \( \delta > 0 \). Now let \( \delta > 0 \) be small enough so that \( B(t_i; \delta) \), \( i = 1, \ldots, k \), are nonoverlapping. Let \( K_i(n) \), \( L_i(n) \), and \( L_{ij}(n) \) be the corresponding values of \( K(n) \), \( L(n) \), and \( L^*(n) \) restricted to the interval \( B(t_i; \delta) \). \( \theta(u) \) denotes a Poisson process with rate \( \alpha \), \( \to_p \) denotes convergence in probability, and \( \to_d \) denotes convergence in distribution. Let \( \alpha_i = G'(t_i) \) and if \( \alpha_i > 1 \), let \( \theta(\alpha_i) < 1 \) be the solution of

\[
\theta(\alpha_i)e^{-\theta(\alpha_i)} = \alpha_i e^{-\alpha_i}.
\]

Theorem 1. As \( n \to \infty \), \( K_i(n) \to_p K_i \), \( i = 1, \ldots, k \), where the \( K_i \)’s are independent and

\[
P(K_i = j) = \begin{cases} 
(1 - \alpha_i)^{j-1}, & j = 1, 2, \ldots \text{ if } \alpha_i < 1, \\
[1 - \theta(\alpha_i)] \theta^{j-1}(\alpha_i), & j = 1, 2, \ldots \text{ if } \alpha_i > 1, t_i = 0 \text{ or } 1, \\
[\alpha_i + \theta(\alpha_i) - 2]/(\alpha_i - 1), & j = 0 \text{ if } \alpha_i > 1, 0 < t_i < 1, \\
[1 - \theta(\alpha_i)]^2 \theta^{j-1}(\alpha_i)/(\alpha_i - 1), & j = 1, 2, \ldots \text{ if } \alpha_i > 1, 0 < t_i < 1.
\end{cases}
\]
REMARK. We in fact show that \( K_i(n) \rightarrow_c K_i \) in the case \( \alpha_i < 1 \) and in the case \( \alpha_i > 1, \ t_i = 0 \) or 1. When \( \alpha_i > 1 \) and \( 0 < t_i < 1 \), we can show only that conditionally \( (K_i(n)|K_i(n) \geq 1) \rightarrow_c K_i|K_i \geq 1 \).

Let \( \rho(\alpha_i) = \min(\alpha_i, 1/\alpha_i) \) and let
\[
\pi(\alpha_i) = 1 - \frac{[1 - \rho(\alpha_i)]}{[1 + \rho(\alpha_i)]}.
\]

(2.2)

THEOREM 2. As \( n \rightarrow \infty \), \( L_i(n) \rightarrow_c L_i \), \( i = 1, \ldots, k \), where \( L_i \)'s are independent and
\[
P\{L_i = j + 1\} = \frac{[1 - \pi(\alpha_i)]\pi^j(\alpha_i)}, \quad j = 0, 1, \ldots .
\]

THEOREM 3. As \( n \rightarrow \infty \), \( L_i^*(n) \rightarrow_c L_i^* \), \( i = 1, \ldots, k \), where the \( L_i^* \)'s are independent and
\[
P\{L_i^* = j + 1\} = \frac{[1 - \rho(\alpha_i)]\rho^j(\alpha_i)}, \quad j = 0, 1, \ldots .
\]

As the proof is rather lengthy, we defer the details to Section 4 and give only an outline of the proof here. The following two sequences \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) will be used extensively in the proofs in Section 4:

(i) \( \{\varepsilon_n\} \) is any sequence such that as \( n \rightarrow \infty \)
\[
\varepsilon_n \rightarrow 0 \quad \text{but} \quad n^{1/2}\varepsilon_n \rightarrow \infty ;
\]

(ii) \( \{\delta_n\} \) is any sequence such that as \( n \rightarrow \infty \)
\[
n^{1/2}\delta_n \rightarrow 0 \quad \text{but} \quad c_n = n\delta_n \rightarrow \infty .
\]

Since \( [G_n(t) - G(t)] = o_p(\varepsilon_n) \) and \( G(t_i) = t_i \), it can be seen that with probability \( 1 \) as \( n \rightarrow \infty \), all the crossings occur inside the shrinking neighborhoods \( B(t_i; \varepsilon_n) \). Within these shrinking neighborhoods, we show that all the crossings in fact occur in a \( \delta_n \)-subinterval. Further, within these \( \delta_n \)-subintervals, say \( [T_{i,n}, T_{i,n} + \delta_n] \), we approximate \( n[G_n(t) - G_n(T_{i,n})] \) and \( n[F_n(t) - F_n(T_{i,n})] \) by Poisson processes \( M_n(\cdot) \) and \( M_n^{*}(\cdot) \) with rates \( \alpha_i \) and \( 1 \), respectively. Finding the distributions of \( L_i(n) \) and \( L_i^*(n) \) then reduces to finding the distributions of \( L_i = \text{number of times} \ M_i(u) = M_n(u) \) for all \( u \geq 0 \) and \( L_i^* = \text{number of times} \ M_i^*(u) \) strictly crosses \( M_n(u) \) for all \( u \geq 0 \). Similarly, if \( \alpha_i < 1 \) or \( \alpha_i > 1 \) and \( t_i = 0 \) or 1, \( K_i(n) \) reduces to \( K_i = \text{number of times} \ M_n(u) = M_i(u) \) for all \( u \geq 0 \). The case \( \alpha_i > 1 \) and \( 0 < t_i < 1 \) is more complicated because in this case when \( G_n(t) \) first crosses \( F(t) \) from below, it overshoots \( F(t) \). The distribution of \( K_i(n) \) depends on the distribution of this random overshoot. However if \( K_i(n) \geq 1 \), i.e., \( G_n(t) \) crosses \( F(t) \) again from above, say at \( T_{i,n} \), so that \( G_n(T_{i,n}) = F(T_{i,n}) \), we can approximate \( n[G_n(t) - G_n(T_{i,n})] \) by \( M_n(h) \) and use the same arguments as in the case \( \alpha_i > 1 \) and \( t_i = 0 \) to get the results. We use renewal-theoretic arguments and the application of Wiener–Hopf techniques to obtain the distributions of \( K_i, L_i, L_i^* \) and to calculate \( \lim_{n \rightarrow \infty} P(K_i(n) \geq 1) \) when \( \alpha_i > 1 \) and \( 0 < t_i < 1 \).

En route, we derive the following results on the boundary crossing probabilities of Poisson processes. We show that for \( \alpha > 1 \)
\[
P\{M_n(u) = u \text{ for some } u > 0\} = \theta(\alpha)
\]
and

$$\lim_{u \to \infty} P[M(u) = a + u \text{ for some } u > 0] = \frac{[1 - \theta(a)]}{(a - 1)}.$$ 

These two results may already be known in the literature. Even if this is the case, our method of proof based on Lemma 4.4 in Section 4 should be of independent interest.

3. A stochastic dominance problem. We were led to the rv’s $K(n)$, $L(n)$, and $L^*(n)$ while investigating certain types of stochastic dominance procedures that arise in finance. In this section, we briefly describe the implication of the results in Section 2 to an algorithm proposed by Bawa, Lindenberg, and Rafsky (1979) for determining stochastic dominance. We start with some background on the problem.

Decision-making under uncertainty may be viewed as choices between alternative probability distributions of returns. In the stochastic dominance (SD) approach to this problem, one restricts an individual’s utility function to a certain reasonable class and obtains the admissible set of alternatives for this case [see Whitmore and Findlay (1978) and references therein]. The first-order SD (SD₁) admissible set is valid for all decision makers with nondecreasing utility functions, the second-order (SD₂) admissible set for the subset of risk-averse individuals, and the third-order SD (SD₃) admissible set for further subset with decreasing absolute risk aversion.

The determination of the SD-admissible sets involves pairwise comparisons of probability distributions. For example, if $F$ and $G$ denote the distributions of two alternatives, it is known that $F$ dominates $G$ in the SD₂ sense if and only if $\int_{-\infty}^{t} [F(u) - G(u)] \, du \leq 0$ for all $t$ with strict inequality for some $t$, and in the SD₃-sense if and only if $\int_{-\infty}^{t} \int_{-\infty}^{\infty} [F(u) - G(u)] \, du \, dv \leq 0$ for all $t$ with strict inequality for some $t$ (Bawa, 1975). In practice, the distributions $F$ and $G$ are unknown and are estimated by the edf’s $F_n$ and $G_n$ based on past economic data. The SD-admissible sets are then determined by comparing the distributions $F_n$ and $G_n$ at the $2n$ jump points. However, there are typically a large number of such distributions and it can be rather time-consuming to determine the SD-admissible sets.

Recently Bawa, Lindenberg, and Rafsky (1979) introduced an efficient algorithm for determining SD-admissible sets. Among other things, the algorithm exploits a zero-crossing property which observes that the point $t_0$ where $F_n$ crosses $G_n$ is a point of either local minima or maxima for $H_n(t) = \int_{-\infty}^{t} [F_n(u) - G_n(u)] \, du$. To determine SD₂-dominance therefore it suffices to look at the $L^*(n)$ points where $F_n(t)$ crosses $G_n(t)$. A similar property is used for SD₃-comparisons.

At first glance one may speculate that if $F$ crosses $G$ at $k$ points (and typically $k$ will be small), $L^*(n)$ will be close to $k$ for large $n$ so that the number of computations does not grow with $n$. However, our results in Section 2 show that the value of $L^*(n)$ can be rather large even if $k$ is small. For example, let $F(t) = \Phi(t/\sigma)$ and $G(t) = \Phi(t/\sigma_2)$ where $\Phi(t)$ is the standard normal df, and $\sigma_2 < \sigma_1$. Then $F$ crosses $G$ at $t = \pm \infty$ and at $t = 0$, and $G'(t)/F'(t) = 0$ at
\[ t = \pm \infty \text{ and } \sigma_2 = \sigma_0 / \sigma_1 \text{ at } t = 0. \] From Theorem 3 in Section 2 we note that if \( \sigma_2 \) is close to \( \sigma_1 \), \( L^*(n) \) can take on large values. For example,
\[
E(L^*) = \sigma_1 / (\sigma_1 - \sigma_2),
\]
which \( \to \infty \) as \( \sigma_2 \to \sigma_1 \).

It is possible to construct smoothed edf's for which the number of crossings converges in probability to \( k \) as \( n \to \infty \). The nearest-neighbor or kernel type approaches to density estimation can be used to obtain such smoothed edf's. This suggests that one may gain by smoothing the edf's before using them in the algorithm of Bawa et al. (1979) to determine SD-admissible sets. However, to be useful, these estimates need to be computationally simple.

4. Auxiliary results and proofs. Let \( \{N_j\}_{j=1}^{\infty} \) be a sequence of Poisson rv's which are independent of the underlying rv's \( \{U_j\} \) and \( \{V_j\} \) with \( E[N_j] = j \). Let \( \{\epsilon_n\} \) and \( \{\delta_n\} \) be defined as in (2.3) and (2.4) in Section 2 with \( c_n = n \delta_n \). We use the notation \( I(A) \) to denote the indicator function of a set \( A \). We start with some results on the relationships between the edf's and Poisson processes. The following result is well known [see, for example, Kac (1949)].

**Lemma 4.1.** For \( u \in [0, n] \), \( \sum_{j=1}^{N_n} I(U_j \leq u/n) \) is a Poisson process with rate 1.

Next fix \( t_0 \) and \( \alpha \), where \( G(t_0) = t_0, 0 \leq t_0 \leq 1, \) and \( \alpha = G'(t_0) \neq 1 \). Let \( \{t_n\} \), \( 0 < t_n < 1 \), be any sequence for which \( t_n \to t_0 \) as \( n \to \infty \). For \( t \in [t_n, t_n + \delta_n] \) we can approximate \( n\left[ G_n(t) - G_n(t_n) \right] \) by a Poisson process \( M_n \).

**Lemma 4.2.** There exists a Poisson process \( M_n \) such that
\[
\lim_{n \to \infty} P \left( \sup_{t_n \leq t \leq t_n + \delta_n} |n\left[ G_n(t) - G_n(t_n) \right] - M_n(n(t-t_n))| > 0 \right) = 0.
\]

**Proof.** Let \( W_j = G(V_j) \) where \( V_j \sim G \). Consider \( M_n(u) = \sum_{j=1}^{N_n} I(t_n < W_j \leq t_n + \alpha u/n) \). It follows from Lemma 4.1 that \( M_n(u) \) is in fact a Poisson process with rate \( \alpha \) on \([0, c_n]\). So
\[
P \left( \sup_{t_n \leq t \leq t_n + \delta_n} \left| \sum_{j=1}^{N_n} I(t_n < V_j \leq t) - \sum_{j=1}^{N_n} I(t_n < W_j \leq t_n + \alpha(t-t_n)) \right| > 0 \right)
\leq \left[ G(t_n + \delta_n) - G(t_n) \right] E[N_n - n]
\quad + P \left( \sup_{t_n \leq t \leq t_n + \delta_n} \left| \sum_{j=1}^{N_n} \left[ I(G(t_n) < W_j \leq G(t)) \right] - I(t_n \leq W_j \leq t_n + \alpha(t-t_n)) \right| > 0 \right).
\]
The first term \( \to 0 \) as \( n \to \infty \) since \( G(t_n + \delta_n) - G(t_n) \sim \delta_n G'(t_0) \) and \( E|N_n - n| \leq \sqrt{E(N_n - n)^2} = n^{1/2} \). To show that the second term \( \to 0 \), note that
\[
\sup_{|t - t_n| \leq \delta} |G(t) - G(t_n) - \alpha(t - t_n)| = o(\delta) \text{ as } \delta \to 0.
\]
Bound this \( o(\delta) \) term by \( \delta \cdot \beta(\delta) \) where \( \beta(\delta) \to 0 \) monotonically as \( \delta \to 0 \). Now choose \( \delta_n \) with \( c_n \leq \sqrt{1/\beta(n^{-1/2})} \). Then the second term is bounded by \( \delta_n \beta(\delta_n) \beta(n^{-1/2}) \leq \beta(n^{-1/2}) \to 0 \) as \( n \to \infty \). \( \square \)

Next we need some results on the boundary crossing probabilities of Poisson processes. It is known that for \( \alpha < 1 \)
\[
(4.1) \quad P\{M_\alpha(u) = u \text{ for some } u > 0\} = 1 - P\{M_\alpha(u) < u \text{ for all } u > 0\} = \alpha.
\]

The second equality follows from Theorem 1 in Chapter 3 of Takacs (1967). To get the first equality, note from the strong law of large numbers that \( M_\alpha(u)/u \to \alpha < 1 \) as \( u \to \infty \). Hence if \( M_\alpha(u) \geq u \) for some \( u \), it must cross the line \( u \) from above again, say at \( u^* \), so that \( M_\alpha(u^*) = u^* \). The following lemma gives the probability for \( \alpha > 1 \). Let \( \theta(\alpha) < 1 \) satisfy \( \theta(\alpha)e^{-\theta(\alpha)} = \alpha e^{-\alpha} \).

**Lemma 4.3.** For \( \alpha > 1 \),
\[
P\{M_\alpha(u) = u \text{ for some } u > 0\} = \theta(\alpha).
\]

Before proving Lemma 4.3, we calculate
\[
(4.2) \quad f(z) = P\{M_\alpha(u) + z > u \text{ for all } u > 0\}.
\]

Let \( \{X_j\}_{j=1}^\infty \) be i.i.d. standard exponential rv’s, and let
\[
P_n(\alpha, t) = P\{X_1 + \cdots + X_j \leq \alpha(j - 1) + t, j = 1, \ldots, n\}.
\]

It can be checked that
\[
f(z) = P_\infty(\alpha, az).
\]

**Lemma 4.4.**

\[
(i) \quad P_n(\alpha, t) = \begin{cases} 
0 & \text{if } \alpha(j - 1) + t \leq 0 \text{ for some } j \leq N, \\
1 - \sum_{j=0}^{n-1} e^{-ja^{-1}t}(ja + t)^{j-1}/j! & \text{otherwise}.
\end{cases}
\]

\[
(ii) \quad P_\infty(\alpha, t) = \begin{cases} 
0 & \text{if } \alpha \leq 1, \\
1 - e^{-R(1-\theta(\alpha)/\alpha)} & \text{if } \alpha > 1.
\end{cases}
\]

**Proof.** The easiest proof we know is by induction. Let \( B_0 = 1 \) and for \( j \geq 1 \),
\[
B_j = \int_0^t dx_1 \int_0^{\alpha + t - x_1} dx_2 \cdots \int_0^{(j-1)\alpha + t - x_1 -\cdots - x_{j-1}} dx_j.
\]
Then if $P_n = P_n(\alpha, t)$,

$$P_n = \int_0^t dx_1 \int_0^{\alpha + t - x_1} dx_2 \cdots \int_0^{(n-1)\alpha + t - x_1 - \cdots - x_{n-1}} dx_n e^{-x_1 - \cdots - x_n}$$

$$= P_{n-1} - B_{n-1} e^{-(n-1)\alpha - t}.$$ 

We show that $B_n = t(n\alpha + t)^{n-1}/n!$ which immediately gives the expression for $P_n$. Integrating $B_n$ over $x_n$, it can be shown that

$$\sum_{j=0}^n B_{n-j} [(j-1)\alpha]^{j}/j! = [(n-1)\alpha + t]^{n}/n!.$$ 

Now Abel's identity (Riordan, 1968) gives

$$x^{-1}(x + y + n\beta)^n = \sum_{k=0}^n \binom{n}{k} (x + k\beta)^{k-1} [y + (n - k)\beta]^{n-k}.$$ 

Setting $x = t$, $y = -\alpha$, $\beta = \alpha$, and $k = n - j$ in (4.4) we get

$$\sum_{j=0}^n \frac{t}{(n-j)!} (t + (n-j)\alpha)^{n-j-1} \frac{[(j-1)\alpha]^j}{j!} = \frac{t + (n-1)\alpha}{n!},$$

i.e., $B_{n-j} = t[(n-j)\alpha + t]^{n-j-1}/(n-j)!$ as claimed. Now as $n \to \infty$, $P_n \to P_\infty$ where

$$P_\infty = 1 - e^{-t} \sum_{j=0}^\infty \frac{(t/\alpha)(j + t/\alpha)^{j-1}(ae^{-\alpha})^j}{j!}.$$ 

To show $P_\infty$ is of the form claimed in (ii), let $y = -n[1 - 1/\theta(\alpha)]$, $\beta = 1$ in (4.4) and let $n \to \infty$. We get

$$e^{x\theta(\alpha)} = x \sum_{k=0}^\infty [\theta(\alpha)]^{-k} (x + k)^{k-1} e^{-\theta(\alpha)/k}.$$ 

Setting $x = t/\alpha$ gives the desired result. □

**Proof of Lemma 4.3.** Let $T$ first $u > 0$: $M_\alpha(u) > u$. Since $\alpha > 1$, $P(T < \infty) = 1$ by the law of large numbers. Let $Z = M_\alpha(T) - T$, the amount of overshoot at $T$, and let $M_\alpha(u)$ be a Poisson process independent of $Z$. Then

$$P(M_\alpha(u) \neq u \text{ for all } u > 0) = P(M_\alpha(u) > u \text{ for all } u > T)$$

$$= P(M_\alpha(u) + Z > u \text{ for all } u > 0)$$

$$= Ef(Z)$$

from the definition of $f(z)$ in (4.2). The distribution of $Z$ can be obtained from the Wiener–Hopf results in Feller (1971). Let $Y_i = (X_i/\alpha) - 1$ where the $X_i$'s are i.i.d. standard exponential. Then $Z$ can be represented as the ladder height variable for the process $S_n = Y_1 + \cdots + Y_n$, $S_0 = 0$. The results on page 405 of Feller (1971) show, after some algebraic manipulations, that $Z$ has density

$$h(z) = ae^{-(a-\theta(\alpha))(1-z)}, \quad 0 < z < 1.$$
From this we get

\[ E_f(Z) = \int_0^\infty P_{\infty}(\alpha, az)h(z) \, dz \]

\[ = 1 - \theta(\alpha). \]

Lemma 4.5. For \( \alpha > 1 \),

\[ \lim_{a \to \infty} P\{M_a(u) = a + u \text{ for some } u > 0\} = \frac{[1 - \theta(\alpha)]}{(\alpha - 1)}. \]

Proof. For fixed \( \alpha > 0 \), let \( T_a = \text{first } u > 0: M_a(u) > a + u \), and let \( W_a = M_a(T_a) - (a + T_a) \). Then by the same arguments as in the proof of Lemma 4.3,

\[ P\{M_a(u) \neq a + u \text{ for all } u > 0\} = E_f(W_a). \]

To obtain the limiting distribution of \( W_a \) as \( a \to \infty \), once again let \( Y_i = X_i/\alpha - 1 \), \( X_i \)'s i.i.d. standard exponential. Note that \( W_a \) has the same distribution as the amount of overshoot for the process \( S_n = Y_1 + \cdots + Y_n \), with \( S_0 = 0 \), when it first exceeds \( a \). Consider now the related renewal process \( S'_n = Z_1 + \cdots + Z_n \) with \( S'_0 = 0 \) where the \( Z_i \)'s are the ladder height variables that we considered in the proof of Lemma 4.3 for the process \( S_n \). \( W_a \) also has the same distribution as the amount of overshoot for the \( S'_n \) process when it first exceeds \( a \). Since the \( Z_i \)'s are nonnegative, we can apply the results in Feller (1971, page 369) to get the limiting distribution of \( W_a \) as \( a \to \infty \) as

\[ \lim_{a \to \infty} P\{W_a \leq z\} = \mu^{-1}\int_0^z [1 - H(x)] \, dx \]

where \( H(x) = \int_0^x h(y) \, dy \), \( h \) is given by (4.5), and \( \mu = \int_0^1 [1 - H(x)] \, dx \). From this we see that the limiting distribution has density

\[ w(z) = \left[ \frac{\alpha}{(\alpha - 1)} \right](1 - e^{-[a - \theta(\alpha)(1 - z)]}, \quad 0 < z < 1. \]

Since \( f(z) = P_{\infty}(\alpha, az) \) is bounded and continuous, we get

\[ \lim_{a \to \infty} E_f(W_a) = \int_0^1 P_{\infty}(\alpha, az)w(z) \, dz \]

\[ = 1 - \frac{[1 - \theta(\alpha)]}{(\alpha - 1)}. \]

The result now follows from (4.6). □

We study the crossings of \( F_n \) and \( G_n \) in several steps. We start with the crossings of \( F_n(t) \) and \( t_0 + \alpha(t - t_0) \), the line through \( t_0 \) with slope \( \alpha \). Here \( t_0 \) and \( \alpha \) are fixed with \( 0 \leq t_0 \leq 1 \) and \( 0 \leq \alpha \neq 1 \). Let \( \nu_n(t_0) \) be the first time \( F_n(t) \) crosses \( t_0 + \alpha(t - t_0) \), i.e.,

\[ \nu_n(t_0) = \begin{cases} 
0 & \text{if } t_0 = 0, \\
\text{first } t > 0: F_n(t) \leq t_0 + \alpha(t - t_0) & \text{if } t_0 > 0, \alpha > 1, \\
\text{first } t > 0: F_n(t) > t_0 + \alpha(t - t_0) & \text{if } t_0 > 0, \alpha < 1. 
\end{cases} \]

Similarly define \( \nu_n(t_0) \) as the last time \( F_n(t) \) crosses \( t_0 + \alpha(t - t_0) \) with \( \nu_n(1) = 1. \)
From the fact that $1 - F_n(1 - t)$ has the same distribution as $F_n(t)$, it follows that

$$
(4.10) \quad P\left( \tau_n^l(t_0) \leq t_0 + x \right) = P\left( \tau_n^l(1 - t_0) \geq 1 - t_0 - x \right).
$$

This observation will prove useful later. Consider the shrinking neighborhood $B(t_0; \epsilon_n)$. Since $F_n(t) = F(t) + O_p(n^{-1/2})$, it can be shown using Chebychev’s inequality that for $\alpha > 1$

$$
(4.11) \quad \lim_{n \to \infty} P\left( F_n(t) \leq t_0 + \alpha(t - t_0) \text{ for all } t \in \left[ t_0 + \epsilon_n, 1 \right] \right) = 1
$$

and

$$
\lim_{n \to \infty} P\left( F_n(t) \geq t_0 + \alpha(t - t_0) \text{ for all } t \in \left[ 0, t_0 - \epsilon_n \right] \right) = 1.
$$

For $\alpha < 1$, (4.11) holds with the inequalities reversed. Therefore

$$
(4.12) \quad \lim_{n \to \infty} P\left( \left[ \tau_n^l(t_0), \tau_n^l(t_0) \right] \subset B(t_0; \epsilon_n) \right) = 1.
$$

We now show that

$$
(4.13) \quad \lim_{n \to \infty} P\left( \tau_n^l(t_0) - \tau_n^l(t_0) \leq \delta_n \right) = 1,
$$

i.e., all the crossings occur in a $\delta_n$-interval. This means that at most $c_n$ order statistics around $t_1$ are involved in the crossings. Since this is true for any $c_n = n\delta_n \to \infty$ with $n^{1/2}\delta_n \to 0$, there is only a finite number of them.

**Lemma 4.6.**

(i) \hspace{1cm} \lim_{n \to \infty} P\left( F_n(t) < t_0 + \alpha(t - t_0) \text{ for all } t \geq \tau_n^l(t_0) + \delta_n \right) = 1 \quad \text{for } \alpha > 1.

(ii) \hspace{1cm} \lim_{n \to \infty} P\left( F_n(t) > t_0 + \alpha(t - t_0) \text{ for all } t \geq \tau_n^l(t_0) + \delta_n \right) = 1 \quad \text{for } \alpha < 1.

**Proof.** Consider first the case $\alpha > 1$ for which $F_n(\tau_n^l(t_0)) = t_0 + \alpha(\tau_n^l(t_0) - t_0)$. So

$$
P\left( F_n(t) > t_0 + \alpha(t - t_0) \text{ for some } t > \tau_n^l(t_0) + \delta_n \right) \leq P\left( F_n(t) - F_n(\tau_n^l(t_0) + \delta_n) > \alpha(t - \tau_n^l(t_0) - \delta_n) \right)
$$

$$
+ P\left( F_n(\tau_n^l(t_0) + \delta_n) - F_n(\tau_n^l(t_0)) \right) > \delta_n(\alpha - 1)/2 \text{ for some } t > \tau_n^l(t_0) + \delta_n.
$$

(4.14)

It follows from the strong Markov property of $F_n$ that $F_n(\tau_n^l(t_0) + \delta_n) - F_n(\tau_n^l(t_0))$ has the same distribution as $F_n(\delta_n)$. So by Chebychev’s inequality, the second term in (4.14) is less than $4\delta_n(1 - \delta_n)/(n\delta_n^2(\alpha - 1)^2)$ which $\to 0$ as $n \to \infty$. To show that the first term in (4.14) $\to 0$, consider the Poisson variables
\(\{N_n\}\). Since \(P(N_n \geq n) \to \frac{1}{2}\) as \(n \to \infty\),
\[
\lim_{n \to \infty} P\left( F_n(t) - F_n(\tau_n(t_0) + \delta_n) > \alpha(t - \tau_n(t_0) - \delta_n) 
+ \delta_n(a - 1)/2 \text{ for some } t \geq \tau_n(t_0) + \delta_n \right) 
\leq \lim_{n \to \infty} 2P\left( \sum_{j=1}^{N_n} I(\tau_n(t_0) + \delta_n < U_j \leq t) - n\alpha(t - \tau_n(t_0) - \delta_n) 
> c_n(a - 1)/2 \text{ for some } t > \tau_n(t_0) + \delta_n \right) 
\leq \lim_{n \to \infty} 2P\left( \sup_{0 < u < \infty} M_n(u) - au > c_n(a - 1)/2 \right),
\]
which equals 0 from the following argument. Let
\[g_\alpha(a) = P\left( \sup_{0 < u < \infty} M_n(u) - au > a \right).
\]
If \(\alpha' < \alpha\),
\[g_\alpha(a) = g_\alpha(\alpha') E g_\alpha(a - \alpha' - Z),
\]
where \(Z\) is the amount by which \(M_n(u)\) overshoots the line \(\alpha' + au\) at the point of first crossing. Let \(g_\alpha = \lim_{\alpha \to -\infty} g_\alpha(a)\). By letting \(\alpha \to \infty\) and then \(\alpha' \to \infty\) in (4.15), we get \(g_\alpha = g_\alpha^2\), i.e., \(g_\alpha = 0\) or 1. But \(g_\alpha(a)\) is monotone decreasing in \(a\) and from (4.1) we see that \(g_\alpha(0) = 1/\alpha < 1\). Hence \(g_\alpha = 0\).

When \(\alpha < 1\), \(F_n\) overshoots the line \(t_0 + \alpha(t - t_0)\) at the time of crossing \(\tau_n(t_0)\). However, this overshoot is less than \(1/n\) and so
\[
P\left( F_n(t) < t_0 + \alpha(t - t_0) \text{ for some } t \geq \tau_n(t_0) + \delta_n \right) 
\leq P\left( F_n(t) - F_n(\tau_n(t_0) + \delta_n) < \alpha(t - \tau_n(t_0) - \delta_n) 
+ \delta_n(a - 1)/2 \text{ for some } t \geq \tau_n(t_0) + \delta_n \right) 
+ P\left( F_n(\tau_n(t_0) + \delta_n) - F_n(\tau_n(t_0)) < \delta_n(\alpha + 1)/2 - 1/n \right).
\]
The second term in (4.16) \(\to 0\) by Chebychev's inequality. The first term \(\to 0\) by an argument analogous to that for \(\alpha > 1\) but now based on Lemma 4.3. □

Next consider the crossings of \(F_n(t)\) and \(h(t)\) in \(B(t_0; \delta)\) when \(h(t_0) = t_0\) and \(h(t)\) is continuously differentiable in \(B(t_0; \delta)\) with \(0 \leq h'(t_0) \neq 1\). Let \(v_n(t_0)\) and \(v_n'(t_0)\) respectively be the first and last crossings defined as in (4.9). It can once again be shown that
\[
\lim_{n \to \infty} P\left( [v_n(t_0), \nu_n(t_0)] \subset B(t_0; \varepsilon_n) \right) = 1.
\]
Similar arguments as in Lemma 4.6 can be used to obtain the following.

**Lemma 4.7.**
\[
\lim_{n \to \infty} P\left( \nu_n(t_0) - v_n'(t_0) \leq \delta_n \right) = 1.
\]
PROOF. We will consider just the case $h'(t_0) > 1$ since the other case is similar. There exists $a$ with $1 < a < h'(t_0)$ such that $h(t) > h(v_n(t_0)) + a(t - v_n(t_0))$ for $v_n(t_0) \leq t \leq t_0 + \delta$. Since $F_n(v_n(t_0)) = h(v_n(t_0))$,

$$
\lim_{n \to \infty} P\{F_n(t) < h(t) \text{ for all } t \geq v_n(t_0) + \delta_n\} \\
\geq \lim_{n \to \infty} P\{F_n(t) - F_n(v_n(t_0)) < a(t - v_n(t_0)) \text{ for all } t \geq v_n(t_0) + \delta_n\},
$$

which can be shown to equal one by using arguments similar to those in Lemma 4.6. □

Next consider the edf $G_n(t)$ with $G(t_0) = t_0$ and $G(t)$ continuously differentiable in $B(t_0, \delta)$. Let $\mu_n(t_0)$ and $\mu_n'(t_0)$ be respectively the first and last crossings of $G_n(t)$ and the line $t_0 + \beta(t - t_0), \beta \neq G'(t_0), \text{ in } B(t_0, \delta)$. Again we can show that

$$
\lim_{n \to \infty} P\{[\mu_n(t_0), \mu_n'(t_0)] \subset B(t_0; \epsilon_n)\} = 1.
$$

**Lemma 4.8.**

$$
\lim_{n \to \infty} P\{\mu_n'(t_0) - \mu_n'(t_0) \leq \delta_n\} = 1.
$$

**Proof.** Let $v_n'(t_0)$ and $v_n'(t_0)$ be the first and last times $F_n(t)$ crosses $t_0 + \beta(G^{-1}(t) - t_0)$ in $B(t_0, \delta)$. Since $G_n(t)$ has the same distribution as $F_n(G(t))$, $(v_n'(t_0), v_n'(t_0))$ has the same distribution as $(G(\mu_n'(t_0)), G(\mu_n'(t_0)))$. The result now follows from Lemma 4.7 and the fact that $G(\mu_n'(t_0)) - G(\mu_n'(t_0))$ is of the same order as $(\mu_n'(t_0) - \mu_n'(t_0))$. □

We can now complete the proofs of the results in Section 2.

**Proof of Theorem 1.** We consider three separate cases.

**Case (i) $\alpha_i = G'(t_i) < 1$.** Let $\mu_n(t_i)$ be the first time $G_n(t)$ equals $t$ in $B(t_i, \delta)$. Note that $\mu_n(t_i)$ exists with probability $\to 1$ as $n \to \infty$. By Lemma 4.8, we can restrict attention to the number of times $G_n(t)$ equals $t$ in $[\mu_n(t_i), \mu_n(t_i) + \delta_n]$. In this interval, we can use Lemma 4.2 to approximate $G_n(t) - \mu_n(t_i) \equiv G_n(t) - \mu_n(t_i)$ by a Poisson process $M_{\alpha_i}$. Therefore, with probability $\to 1$ as $n \to \infty$, $K_i(n)$ is the same as the number of times $M_{\alpha_i}(u)$ equals $u$ in $[0, c_n]$. Since $c_n \to \infty$ as $n \to \infty$ and $\alpha_i < 1$, it can be shown that the probability $M_{\alpha_i}(u)$ equals $u$ for some $u \in (c_n, \infty)$ tends to 0 as $n \to \infty$. So we may as well consider $K_i = \text{number of times } M_{\alpha_i}(u) = u$ for all $u \geq 0$. Note that $K_i \geq 1$ since $M_{\alpha_i}(0) = 0$. From (4.1),

$$
P(K_i = 1) = P\{M_{\alpha_i}(u) \neq u \text{ for all } u > 0\} \\
= 1 - \alpha_i.
$$

From the strong Markov property and the stationarity of the increments of
\( M_{\alpha_i}(u) \), it can be seen that
\[
P\{K_i = j + 1\} = (1 - \alpha_i)\alpha_i^j, \quad j = 0, 1, \ldots .
\]

**Case (ii):** \( \alpha_i = G_i'(t_i) > 1, \ t_i = 0 \) or 1. It suffices to consider the case \( t_i = 0 \) since the case \( t_i = 1 \) can be obtained from this by considering the process \( 1 - G_n(1 - t) \). By Lemma 4.8, we restrict attention to the crossings in \([0, \delta_n]\). By Lemma 4.2, with probability \( \to 1 \) as \( n \to \infty \), this is the same as the number of times \( M_{\alpha_i}(u) \) equals \( u \) in \([0, c_n]\). Again since \( c_n \to \infty \) we may as well consider \( K_i = \) number of times \( M_{\alpha_i}(u) \) equals \( u \) for all \( u \geq 0 \). By Lemma 4.3,
\[
P\{K_i = j + 1\} = [1 - \theta(\alpha_i)]\theta^j(\alpha_i), \quad j = 0, 1, \ldots .
\]

**Case (iii):** \( \alpha_i = G_i'(t_i) > 1, \ 0 < t_i < 1 \). Let \( \mu_i^j(t) \) be the first \( t \in B(t_i; \delta_n) \) such that \( G_n(t) \). By Lemma 4.8, we can restrict attention to the number of crossings in \([\mu_i^j(t_i), \mu_i^j(t_i) + \delta_n]\). The calculation in this case is a bit more involved because \( G_n(t) \) overshadows \( t \) at \( \mu_i^j(t_i) \). If we condition on \( \{K_i(n) \geq 1\} \) so that there is a first \( t^* \in B(t_i; \delta) \) at which \( G_n(t^*) = t^* \), we can use the same arguments as in case (ii) to get
\[
P\{K_i = j + 1 | K_i \geq 1\} = [1 - \theta(\alpha_i)]\theta^j(\alpha_i), \quad j = 0, 1, \ldots .
\]

We now compute \( \lim_{n \to \infty} P(K_i(n) \geq 1) \). Let
\[
R_n = n\left[G_n(\mu_i^j(t_i)) - \mu_i^j(t_i)\right].
\]

By approximating \( n[G_n(t) - G_n(\mu_i^j(t_i))] \) for \( t \in [\mu_i^j(t_i), \mu_i^j(t_i) + \delta_n] \) by the Poisson process \( M_{\alpha_i} \), we see that
\[
\lim_{n \to \infty} P(K_i(n) = 0) = \lim_{n \to \infty} P\{M_{\alpha_i}(u) + R_n > u \text{ for all } u > 0\}
\]
\[
= \lim_{n \to \infty} E[f(R_n)],
\]

where \( f(\cdot) \) is given by (4.2). Let \( V_{1:n} < \cdots < V_{n:n} \) denote the order-statistics corresponding to \( \{V_j\}_{j=1}^n \) where \( V_j \sim G \). Let \( J_n = \text{first } j \geq n(t_i - \varepsilon_{j/n}) \): \( V_{j:n} \geq j/n \). Then \( R_n = J_n - nV_{j:n} \). We know that \( V_{j:n} \) has the same distribution as \( G\left(U_{j:n}\right) \) where \( U_{j:n} \) is the order statistic from the uniform distribution. Recall also that \( \mu_i^j(t_i) \in B(t_i; \varepsilon_n) \) and that in this interval \( |G(t) - t_i + \alpha_i(t - t_i)| = o(\varepsilon_n) \). By using these facts, it can be shown that the limiting distribution, as \( n \to \infty \), of \( R_n \) is the same as the limiting distribution, as \( a \to \infty \), of \( W_a = M_{\alpha_i}(T_a) - (a + T_a) \) where \( T_a = \text{first } u > 0 \): \( M_{\alpha_i}(u) > a + u \). It then follows from (4.6) and Lemma 4.5 that
\[
\lim_{n \to \infty} P(K_i(n) \geq 1) = \lim_{a \to \infty} P\{M_{\alpha_i}(u) = a + u \text{ for some } u > 0\}
\]
\[
= [1 - \theta(\alpha_i)]/(\alpha_i - 1).
\]
\[\square\]
PROOFS OF THEOREMS 2 AND 3. Let \( \alpha_i = G(t_i) \) and let \( \mu_n'(t_i) \) be the first 
\( t \in B(t_i, \delta) \) at which \( G_n(t) \) equals \( F_n(t) \). For \( \alpha_i > 1, \)
\[
P\{ G_n(t) > F_n(t) \text{ for all } t \in (\mu_n'(t_i) + \delta, t_i + \delta) \} 
\geq P\{ G_n(t) - G_n(\mu_n'(t_i)) > (t - \mu_n'(t_i))(1 + \alpha_i)/2 
\text{ for all } t \in (\mu_n'(t_i) + \delta, t_i + \delta) \} 
\times P\{ F_n(t) - F_n(\mu_n'(t_i)) < (t - \mu_n'(t_i))(1 + \alpha_i)/2 
\text{ for all } t \in (\mu_n'(t_i) + \delta, t_i + \delta) \} 
\rightarrow 1 \text{ as } n \rightarrow \infty \text{ by Corollary 2. Similarly for } \alpha_i < 1, \]
\[
\lim_{n \rightarrow \infty} P\{ G_n(t) < F_n(t) \text{ for all } t \in (\mu_n'(t_i) + \delta, t_i + \delta) \} = 1.
\]
Therefore we need to consider only the number of crossings in \( [\mu_n'(t_i), \mu_n'(t_i) + \delta] \). 
In this interval we approximate \( G_n(t) - G_n(\mu_n'(t_i)) \) and \( F_n(t) - F_n(\mu_n'(t_i)) \) by \( M_{\alpha_i} \) and \( M_i \). 

Therefore, with probability \( \rightarrow 1 \) as \( n \rightarrow \infty \), \( L_i(n) \) is the same as \( L_i \) as the number of intervals for which the graph of \( M_{\alpha_i}(u) \) equals the graph of \( M_i(u) \) for all \( u \geq 0 \). Again we note that \( L_i \geq 1 \) because \( M_{\alpha_i}(0) = M_i(0) = 0 \). To compute \( \pi(\alpha_i) \) for \( \alpha_i < 1 \), let \( T = \text{first } u > 0 \) \( M_{\alpha_i}(u) \neq M_i(u) \), and let
\[
\pi(\alpha_i) = P\{ M_{\alpha_i}(u) = M_i(u) \text{ for some } u > T \}.
\]
Then by the strong Markov property and stationarity of the increments of \( M_{\alpha_i}(u) - M_i(u), \)
\[
P\{ L_i = j + 1 \} = [1 - \pi(\alpha_i)]^j \pi(\alpha_i), \quad j = 0, 1, \ldots
\]
To calculate \( \pi(\alpha_i) \) first suppose \( \alpha_i < 1 \). Then \( 1 - \pi(\alpha_i) = P\{ M_{\alpha_i}(u) < M_i(u) \} \) for all \( u \geq T \). Let \( \{X_i\}_{i=1}^\infty \) and \( \{Y_i\}_{i=1}^\infty \) be mutually independent i.i.d. exponential rv's with means 1 and \( 1/\alpha_i \), respectively, and let \( S_n = \sum_{i=1}^n (X_i - Y_i) \). Then
\[
P\{ M_{\alpha_i}(u) < M_i(u) \text{ for all } u \geq T \} = P\{ S_1 < 0 \}
\times P\{ M_{\alpha_i}(u) \leq M_i(u) \text{ for all } u \geq 0 \}
= P\{ S_1 < 0 \} P\{ \max S_n \leq 0 \}.
\]
If \( H \) denotes the distribution of \( S_1 \), from Eq. (5.9) in Feller (1971, page 410) we get
\[
(4.17) \quad P\{ \max_n S_n \leq 0 \} = \alpha \cdot \kappa(\alpha),
\]
where \( \kappa(\alpha) \) solves the equation \( \int_{-\infty}^\infty e^{\kappa(\alpha) y} dH(y) = 1 \). This gives \( \kappa(\alpha) = \alpha^{-1} - 1 \). Also
\[
(4.18) \quad P\{ S_1 < 0 \} = 1/(1 + \alpha).
\]
Combining these results, we get
\[
1 - \pi(\alpha_i) = (1 - \alpha)/(1 + \alpha).
\]
The result for $\alpha_i > 1$ can be obtained from the fact the number of intervals for which the graph of $M_{\alpha_i}(u)$ equals that of $M_1(u)$ for all $u \geq 0$ has the same distribution as the number of intervals for which the graph of $M_i(u)$ equals that of $M_{1/\alpha_i}(u)$ for all $u \geq 0$. Therefore

$$1 - \pi(\alpha_i) = \frac{[1 - \rho(\alpha_i)]}{[1 + \rho(\alpha_i)]},$$

where

$$\rho(\alpha_i) = \min(\alpha_i, 1/\alpha_i).$$

To prove Theorem 3, observe that with probability $\rightarrow 1$ as $n \to \infty$, $L^*_n$ is the same as $L^*_n = \text{the number of times } M_{\alpha_i}(u) \text{ crosses } M_1(u) \text{ for all } u \geq 0$. Note from (4.17) that for $\alpha_i < 1$

$$P\{M_{\alpha_i}(u) \leq M_1(u) \text{ for all } u \geq 0\} = P\{\max_n S_n \leq 0\} = 1 - \alpha_i.$$ 

Similarly for $\alpha_i > 1$,

$$P\{M_{\alpha_i}(u) \geq M_1(u) \text{ for all } u \geq 0\} = 1 - 1/\alpha_i.$$ 

Using this, it can be shown that

$$P\{L^*_n = j + 1\} = [1 - \rho(\alpha_i)]\rho'(\alpha_i).$$

\[\square\]

**Acknowledgment.** V. N. Nair is grateful to Colin Mallows for valuable discussions.

**REFERENCES**


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