A Note on Conditional Exponential Moments and Onsager-Machlup Functionals

Larry A. Shepp
University of Pennsylvania

Ofer Zeitouni

Follow this and additional works at: http://repository.upenn.edu/statistics_papers

Part of the Probability Commons

Recommended Citation

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/statistics_papers/326
For more information, please contact repository@pobox.upenn.edu.
A Note on Conditional Exponential Moments and Onsager-Machlup Functionals

Abstract
It is proven that, for any deterministic $L^2[0,1]$ function $\phi(t)$,

$$E(\exp^\int_0^1 \phi(t)dw_t \mid \|w\| < \varepsilon) \to 1 \text{ as } \varepsilon \to 0,$$

where $w_t$ is a standard Brownian motion and $\|\cdot\|$ is any "reasonable" norm on $C_0[0,1]$. Applications to the computation of Onsager-Machlup functionals are pointed out.

Keywords
Gaussian norms, Onsager-Machlup, correlation inequalities

Disciplines
Probability

This journal article is available at ScholarlyCommons: http://repository.upenn.edu/statistics_papers/326
A NOTE ON CONDITIONAL EXPONENTIAL MOMENTS AND ONSAGER-MACHLUP FUNCTIONALS

BY LARRY A. SHEPP AND OFER ZEITOUNI

AT & T Bell Laboratories and Technion

It is proven that, for any deterministic \(L^2[0,1]\) function \(\phi(t)\),

\[
E\left(\exp \int_0^1 \phi(t) \, dw_t \bigg| \|w\| < \varepsilon\right) \to 1 \quad \text{as } \varepsilon \to 0,
\]

where \(w_t\) is a standard Brownian motion and \(\| \cdot \|\) is any "reasonable" norm on \(C_0[0,1]\). Applications to the computation of Onsager-Machlup functionals are pointed out.

Let \(\phi \in L^2[0,1]\) be a given deterministic function. Let \(w_t\) be a standard Brownian motion, and define \(I(\phi) \equiv \int_0^1 \phi_t \, dw_t\). Denote by \(\| \cdot \|\) the supremum norm on \([0,1]\), and by \(\| \cdot \|_2\) the \(L^2\) norm on \([0,1]\). In [2], it is shown that if for a given deterministic path \(\phi_t, t \in [0,1]\), one has that

\[
E(\exp I(\phi) | \|w\| < \varepsilon) \to 1 \quad \text{as } \varepsilon \to 0,
\]

then the Onsager-Machlup functional computation for \(\phi_t = \int_0^t \phi _s \, dt\) (cf. [3], chapter 6, and [2]) follows, that is, if \(x_t\) is a diffusion satisfying

\[
dx_t = f(x_t) \, dt + dw_t,
\]

with \(f\) bounded and having two continuous bounded derivatives, then

\[
\frac{\text{Prob}\{\|x - \phi\| < \varepsilon\}}{\text{Prob}\{\|w\| < \varepsilon\}} \to \exp \left(-\frac{1}{2} \int_0^1 \left(\phi_t - f(\phi_t)\right)^2 + f'(\phi_t) \, dt\right) \quad \text{as } \varepsilon \to 0.
\]

Further applications to construction of the skeleton of Wiener functionals follow and will be described elsewhere.

In [2], (1) is proven for \(\phi \in C^\alpha[0,1]\), for any \(\alpha > 0\). The proof, which uses analytic methods, breaks down in the case of \(\phi \in L^2[0,1]\) and, moreover, is quite complex. In this note, we bring a new simple proof to (1), based on different methods, which allows for \(\phi \in L^2[0,1]\).

Denote by \(\| \cdot \|\) any norm on \([0,1]\) which dominates the \(L^1\) norm, that is, for all measurable \(\psi\) on \([0,1]\),

\[
\int_0^1 |\psi| \, dt \leq K\|\psi\|.
\]

In particular, we wish to consider the case of the supremum norm. We claim the following.

Received November 1990.

1The work of this author was partially done while he visited AT & T Bell Laboratories, Murray Hill, New Jersey.

AMS 1980 subject classifications. Primary 60G15, 60F10; secondary 76J95.

Key words and phrases. Gaussian norms, Onsager-Machlup, correlation inequalities.
THEOREM. Assume that, for any deterministic constant c,

\[ E(\exp cw_1 \mid \|w\| < \varepsilon) \to 1 \quad \text{as} \quad \varepsilon \to 0. \]

Then

\[ E(\exp I(\phi) \mid \|w\| < \varepsilon) \to 1 \quad \text{as} \quad \varepsilon \to 0. \]

\[ E(\exp I(\phi)) \mid \|w\| < \varepsilon \] 

Remark. (2) and (3) are trivially satisfied for “reasonable” norms, such as \( L^p \) norms, supremum norm and so on.

Proof of the theorem. First, by Jensen’s inequality, using the fact that the conditioning is symmetric and that \( I(-\phi) = -I(\phi) \),

\[ E(\exp I(\phi) \mid \|w\| < \varepsilon) \geq 1. \]

Let \( \phi^\delta \) be a \( C^1[0,1] \) function such that \( \|\phi - \phi^\delta\|_2 < \delta \). Note that by an integration by parts,

\[ E(\exp(I(\phi^\delta)) \mid \|w\| < \varepsilon) = E\left(\exp\left(\int_0^1 \omega_1 \phi'(t) \, dt\right) \mid \|w\| < \varepsilon\right) \to 1 \quad \text{as} \quad \varepsilon \to 0, \]

where the last limit follows by (2) and (3). Using now Schwartz’s inequality, one has that

\[ E(\exp I(\phi) \mid \|w\| < \varepsilon) \leq E^{1/2}(\exp 2I(\phi^\delta)) \mid \|w\| < \varepsilon) \]

\[ \times E^{1/2}(\exp 2(I(\phi) - I(\phi^\delta)) \mid \|w\| < \varepsilon). \]

Due to (6), the first term in the right-hand side of (7) converges to 1, and due to (5), the theorem will hold true for any \( \phi \in L^2[0,1] \) once we show that, for any \( \psi \in L^2[0,1] \),

\[ E(\exp I(\psi) \mid \|w\| < \varepsilon) \leq E(\exp I(\psi)). \]

Indeed, using the fact that \( I(\psi) \) is a Gaussian random variable with zero mean and variance \( \sigma^2 = \|\psi\|^2_2 \), one has

\[ E(\exp I(\psi)) = \frac{2}{\sqrt{2\pi} \sigma} \int_0^\infty \exp \left( -\frac{x^2}{2\sigma^2} \right) dx \to 1 \quad \text{as} \quad \sigma \to 0. \]

Using now \( \psi = 2(\phi - \phi^\delta) \), noting that \( \sigma^2 \leq \delta^2 \to 0 \) as \( \delta \to 0 \) and substituting (8) and (9) into (6),

\[ \lim_{\varepsilon \to 0} E(\exp I(\phi) \mid \|w\| < \varepsilon) \leq k(\delta), \]

where \( k(\delta) \to 0 \) as \( \delta \to 0 \), which together with (5) proves the theorem.

To prove (8), we will actually prove that, for any \( \eta > 0 \) and \( \varepsilon > 0 \),

\[ Pr(|I(\psi)| < \eta \mid \|w\| < \varepsilon) \geq Pr(|I(\psi)| < \eta), \]

from which (8) follows immediately.
To see (10), let $e_i$ denote an orthonormal base for $L_2^2[0, 1]$ such that $e_1 = \psi$. Let $x_i = \int_0^1 e_i(t) \, dw_i$, and denote by $x$ the infinite vector whose components are $x_i$. Note that the $x_i$ are standard, i.i.d. normal random variables. By Theorem 2.1 of [1], one has that for any convex, symmetric set $C_k \subset R^k$,

$$
Pr(|x_1| < \delta, (x_1, \ldots, x_k) \in C_k) \geq Pr(|x_1| < \delta) Pr((x_1, \ldots, x_k) \in C_k).
$$

Let $C$ be a convex, symmetric set in $R^n$, and denote by $C_k$ its projection on $R^k$. Clearly, $C_k$ is convex and symmetric and therefore (11) applies to it. Using dominated convergence, one concludes that

$$
Pr(|x_1| < \delta, x \in C) \geq Pr(|x_1| < \delta) Pr(x \in C).
$$

Therefore, choosing $C = \{x \mid \|x\| < \delta\}$, and noting that $C$ is both symmetric and convex, we get from (12) that (10) holds true. Inequality (8), and therefore the theorem, follow. □

**Remark 1.** Note that (10) is not a direct consequence of the FKG inequalities, except in the particular case when $\|\cdot\|$ is the $L^2$ norm. In other cases, the particular properties of elliptically contoured measures are needed for Theorem 2.1 of [1] to hold true, and then the properties of the Gaussian measure are needed to translate that into a statement concerning conditional probabilities as in (10).

**Remark 2.** One could ask whether (4) holds true for functionals which are more complex than the Wiener integral. For an $L^2$ norm conditioning, it follows by the FKG inequalities that, for a class of symmetric and convex functionals, the answer is positive. This approach may be extended for some classes of functionals to other norm conditionings. To use the method presented previously, one would need the extension of Theorem 2.1 of [1] to the situation where the set $\{|x_1| < \delta\}$ is replaced by a general convex and symmetric set. This is stated as an open conjecture in [1] and, to our knowledge, is still an open problem.

**REFERENCES**


**AT & T Bell Laboratories**

600 Mountain Avenue

Room 2C 374

Murray Hill, New Jersey 07974-2070

**DEPARTMENT OF ELECTRICAL ENGINEERING**

**TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY**

**TECHNION CITY, HAIFA 32000 ISRAEL**