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On the Integral of the Absolute Value of the Pinned Wiener Process

Abstract
Let \( W_t = W_t, 0 \leq t \leq 1 \), be the pinned Wiener process and let \( \xi = \int_0^1 |W_t| \). We show that the Laplace transform of \( \xi \), \( \phi(s) = Ee^{-\xi s} \), satisfies

\[
\int_0^\infty e^{-us} \phi(2\sqrt{s^3/2}) s^{-1/2} ds = -\sqrt{\pi} \frac{Ai(u)}{Ai'(u)}
\]

where \( Ai \) is Airy’s function. Using (\#), we find a simple recurrence for the moments, \( E\xi^n \) (which seem to be difficult to calculate by direct or by other techniques) namely \( E\xi^n = e_n \sqrt{\pi} (36 \sqrt{2})^{-n} / \Gamma(3n+1/2) \) where \( e_0 = 1, g_k = \Gamma(3k+1/2) / \Gamma(k+1/2) \) and for \( n \geq 1 \),

\[
e_n = g_n + \sum_{k=1}^{n-1} e_{n-k} (n-k) (6k+1) / (6k-1) g_k.
\]

Keywords
airy, moments, Kac’s method, Karhunen-Loeve

Disciplines
Probability

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ON THE INTEGRAL OF THE ABSOLUTE VALUE OF THE PINNED WIENER PROCESS

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Let $\tilde{W} = \tilde{W}_t$, $0 \leq t \leq 1$, be the pinned Wiener process and let $\xi = \int_0^1 |\tilde{W}_t| dt$. We show that the Laplace transform of $\xi$, $\phi(s) = Ee^{-st}$ satisfies

\begin{equation}
(\ast) \quad \int_0^\infty e^{-ws}(\sqrt{2}s^{1/2})\pi^{-1/2}ds = -\sqrt{\pi} Ai(u)/Ai'(u)
\end{equation}

where $Ai$ is Airy’s function. Using (\ast), we find a simple recurrence for the moments, $E\xi^n$ (which seem to be difficult to calculate by direct or by other techniques) namely $E\xi^n = e_n\sqrt{\pi}(36\sqrt{2})^{-n/2}/\Gamma\left(\frac{3n+1}{2}\right)$ where $e_0 = 1$, $e_n = \Gamma(3k + \frac{1}{2})/\Gamma(k + \frac{1}{2})$ and for $n \geq 1$,

$$e_n = e_n + \sum_{k=1}^{n} e_{n-k} \left(\binom{n}{k}\frac{6k+1}{6k-1}g_k\right).$$

1. Introduction. The pinned Wiener process $\tilde{W}_t$, $0 \leq t \leq 1$, is obtained by conditioning a standard Wiener process $W_t$, $0 \leq t \leq 1$, to pass through zero at $t = 1$. It is clear from the fact that $\tilde{W}$ is Gaussian with mean zero and covariance

\begin{equation}
E\tilde{W}_s \tilde{W}_t = \min(s, t) - st, \quad 0 \leq s, t \leq 1
\end{equation}

that $E\int_0^1 |\tilde{W}_t| dt = \int_0^1 E|\tilde{W}_s| ds = \sqrt{\pi}/(4\sqrt{2})$, but higher moments of $\xi = \int_0^1 |\tilde{W}_t| dt$

are awkward and unwieldy to obtain directly, and are of some interest in certain problems in random walk arising in empirical distribution theory.

Kac’s formula for

\begin{equation}
u(x) = Ex \int_0^\infty e^{-st-\int_0^t v(X_s)ds} f(X_t) dt,
\end{equation}

where $X_t$ is a time-homogeneous Markov process starting at $x$ at $s = 0$, is a natural tool to find the distribution of random variables of the form (1.2). However, there is difficulty with a direct use of (1.3) in this case, because although $X = \tilde{W}$ is a Markov process, it is not time-homogeneous. Although Kac’s formula has an extension to non-time-homogeneous processes $X$, the formula involves partial rather than ordinary differential equations and so is awkward. Here we use Kac’s technique in a novel way, starting with a time-homogeneous process (namely the Wiener process) and introducing conditioning by allowing $f(x)$ to be a $\delta$-function at $x = 0$, to obtain a formula for $\tilde{W}$ in place of $\tilde{W}$.

Using the above technique described in detail in Section 2 and solving the resulting ordinary differential equation, we obtain, implicitly,

\begin{equation}
\phi(s) = Ee^{-st}
\end{equation}

via

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where $Ai$ is the usual Airy function [2]. Further, inverting the Laplace transform on $u$ in (1.5), we can obtain $\phi(\sqrt{2}s^{3/2})s^{-1/2}$. Hence, in principle at least, we know $\phi(s)$, which is the Laplace transform of the density $f_t$ of $\xi$, and which could then be used (in principle) to determine $f_t$ by a second inverse Laplace transform. A remarkably similar (but note the ratio on the right is inverted) implicit double Laplace transform, viz.,

\[
\int_0^\infty e^{-us} \phi(s^{1/2}) \, ds = \psi'(u)/\psi(u),
\]

(with $\psi$ a parabolic cylinder function) was indeed numerically inverted in [3] but the present case with $s^{3/2}$ in (1.5) appears to be more difficult to treat numerically. (The next paper in this issue, by S. O. Rice "The integral of the absolute value of the pinned Wiener process - calculation of its probability density by numerical integration" performs this numerical inversion of (1.5).)

The moments $E \xi^n$ can be read off from (1.5). Define

\[
e_n = E \xi^n \Gamma \left( \frac{3n + 1}{2} \right) (36\sqrt{2})^{-n}/\Gamma \left( \frac{1}{2} \right).
\]

By comparing asymptotic expansions of both sides of (1.5) as $u \to \infty$, using [1, page 448], we obtain that for $n \geq 1$,

\[
e_n = \frac{\Gamma \left( \frac{3n + 1}{2} \right)}{\Gamma \left( \frac{n + 1}{2} \right)} + \sum_{k=1}^{n-1} e_{n-k} \left( \frac{6k + 1}{6k - 1} \right)^{n-k} \Gamma \left( \frac{k + 1}{2} \right) \Gamma \left( \frac{3n + 1}{2} \right).
\]

This gives the results in Table 1 for $n \leq 5$. Of course for $n \to \infty$, $E \xi^n \to \infty$. Indeed,

\[
E \xi^n \simeq \frac{\Gamma \left( \frac{3n + 1}{2} \right) (36\sqrt{2})^{-n}}{\Gamma \left( \frac{n + 1}{2} \right) \Gamma \left( \frac{3n + 1}{2} \right)}
\]

from (1.7) and the fact that the sum in (1.8) is nonnegative, so that $E \xi^n \geq n^{n/2} \text{ const.}$.

<table>
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<th>$n$</th>
<th>$E \xi^n$</th>
<th>$E \xi^n$</th>
<th>$(E \xi^n)^{1/n}$</th>
<th>$e_n9^{-n}$</th>
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<td>1.0000</td>
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<td>0.4030</td>
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</tr>
<tr>
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<td>$\frac{101}{892} \sqrt{\frac{\pi}{2}}$</td>
<td>0.0155</td>
<td>0.4343</td>
<td>101.7.5^2.2^3</td>
</tr>
</tbody>
</table>
which is of course not surprising since $\xi$ is unbounded.

The technique may also be applied to integrals

\begin{equation}
\xi_\beta = \int_0^1 |\tilde W_s|^\beta \, ds
\end{equation}

for any $\beta \neq 1$, but except for $\beta = 2$, the function playing the role of the Airy function in (1.5) has apparently not been studied. The case $\beta = 2$ is interesting because of the comparison of the present technique with the Karhunen-Loève series technique. Both techniques are discussed in detail in Section 3. It is remarkable that in this seemingly simpler case, no simple recurrence for the moments of $\xi_\beta$ can apparently be obtained.

2. Proof of (1.5). We begin by using Kac’s formula [2, page 54] for the Wiener process $X$ starting at $x$. The expectation (1.3), for $f$ bounded and of compact support and $V \geq 0$, is the unique bounded solution to

\begin{equation}
-\frac{1}{2} u''(x) + (\alpha + V(x))u(x) = f(x).
\end{equation}

Taking $V(x) = |x|$, let $\phi(x), \psi(x)$ be two solutions of the homogeneous equation corresponding to (2.1) with zero right-hand side, with $\phi$ bounded at $+\infty, \psi$ bounded at $-\infty$ and

\begin{equation}
\phi \psi' - \psi \phi' = 2.
\end{equation}

Then the Green operator applied to $f$,

\begin{equation}
u(x) = \phi(x) \int_{-\infty}^x \psi(u)f(u) \, du + \psi(x) \int_x^\infty \phi(u)f(u) \, du
\end{equation}

is the solution to (2.1). Since Airy’s functions $Ai$ and $Bi$ [1, page 446] satisfy $g'' = xg$, and $Ai(x)$ is bounded at $x = +\infty$, we have

\begin{equation}
\phi(x) = d_0 Ai(2^{1/3}(x + \alpha)); \quad x \geq 0
\end{equation}

\begin{equation}
\phi(x) = d_1 Ai(2^{1/3}(-x + \alpha)) + d_2 Bi(2^{1/3}(-x + \alpha)); \quad x \leq 0.
\end{equation}

By symmetry,

\begin{equation}
\psi(x) = \phi(-x), \quad -\infty < x < \infty.
\end{equation}

Because of (2.2) and the fact that $\phi(0^+) = \phi(0^-), \phi'(0^+) = \phi'(0^-)$, we easily determine $d_0$, $d_1$, and $d_2$, and obtain

\begin{equation}
d_0 = -\frac{2^{-1/3}}{Ai(2^{1/3}\alpha)Ai'(2^{1/3}\alpha)}.
\end{equation}

Setting $x = 0$ in (1.3) and (2.3) we obtain

\begin{equation}
E \int_0^\infty e^{-\lambda t} \int_{0}^{W_2 ds} f(W_t) \, dt = \phi(0) \int_{-\infty}^0 \psi f + \psi(0) \int_0^\infty \phi f
\end{equation}

since when $x = 0$, $X_t$ becomes the ordinary standard Wiener process $W$ starting at $x = 0$. In order to obtain the conditioned, or pinned, Wiener process $\tilde W$, we choose

\begin{equation}
f(x) = \frac{\sqrt{2\pi}}{2\epsilon} \chi(|x| < \epsilon)
\end{equation}

where $\chi = \chi(|x| < \epsilon)$ is either one or zero depending on whether $|x| < \epsilon$ or not, and allow $\epsilon \downarrow 0$ in (2.7). On the right side we get

\begin{equation}
\sqrt{2\pi} \phi(0)\psi(0) = \sqrt{2\pi} d_0 Ai(2^{1/3}\alpha) = \sqrt{2\pi} 2^{-1/3} Ai(2^{1/3}\alpha)/(-Ai'(2^{1/3}\alpha))
\end{equation}
from (2.5) and (2.6). On the left side of (2.7) we get

\[ \lim_{t \to 0} \int_0^\infty e^{-at} e^{-\int_0^t W_s \, ds} \frac{X(|W_t| < \epsilon)}{P(|W_t| < \epsilon)} \frac{P(|W_t| < \epsilon)}{\sqrt{2\pi \epsilon \sqrt{t}}} \, dt. \]  

(2.10)

Since the ratio

\[ \frac{P(|W_t| < \epsilon)}{\frac{2\epsilon}{\sqrt{2\pi \epsilon \sqrt{t}}}} \leq 1 \]

(2.11)

tends (boundedly) to 1 as \( \epsilon \downarrow 0 \), we may pass to the limit in (2.10) to obtain, with (2.9)

\[ \int_0^\infty e^{-at} E\left[ e^{-\int_0^t W_s \, ds} \mid W_t = 0 \right] \frac{dt}{\sqrt{t}} = \sqrt{2\pi} 2^{-1/3} Ai(2^{1/3} \alpha) \]

(2.12)

Now we observe that \( W_s, 0 \leq s \leq t \), is the same as \( \sqrt{t} \cdot \hat{W}_s, 0 \leq s \leq 1 \) for a fixed Wiener process \( \hat{W}_t, 0 \leq t \leq 1 \), so that for each \( t \)

\[ E\left[ e^{-\int_0^t \hat{W}_s \, ds} \mid \hat{W}_t = 0 \right] = E\left[ e^{-t^{1/3} \int_0^1 \hat{W}_s \, ds} \mid \hat{W}_0 = 0 \right] = Ee^{-t^{1/3} \int_0^1 \hat{W}_s \, ds} \]

(2.13)

using the definition of \( \hat{W}_t \) as \( \hat{W}_t \) conditioned by \( \hat{W}_0 = 0 \). Setting \( \xi = \int_0^1 \hat{W}_s \) as in (1.4), and \( t = 2^{1/3} s, u = 2^{1/3} \alpha \) we obtain (1.5). Note in (1.5) the factor \( s^{-1/2} \) which appears because of the conditioning or pinning procedure.

3. The case \( \beta = 2 \) in (1.9). For \( \xi = \int_0^1 \hat{W}_s^2 \) we give two methods of attack to determine

\[ \phi_2(s) = Ee^{-\xi s} = Ee^{-\int_0^1 \hat{W}_s \, ds}. \]

(3.1)

First we use the present technique (1.3) with \( V(x) = \frac{1}{2} x^2 \). The differential equation (2.1) now becomes the parabolic cylinder equation,

\[ -\frac{1}{2} u''(x) + \left( \alpha + \frac{1}{8} x^2 \right) u(x) = f(x) \]

(3.2)

which has the unique bounded solution (2.3) with

\[ \phi(x) = d_0 D_\alpha(x), \quad \psi(x) = d_0 D_{-\alpha}(-x) \]

(3.3)

where \( D_\alpha \) is the parabolic cylinder function [4, page 91–94], and

\[ \rho = -\frac{1}{2} - 2\alpha, \quad d_0 = \frac{1}{-D_{\alpha}(0)D'_{\alpha}(0)}. \]

(3.4)

Taking \( x = 0 \) as in (2.7) and \( f \) as in (2.8) and using the argument in (2.9)–(2.13), we easily obtain [5], for \( \alpha \geq 0 \),

\[ \int_0^\infty e^{-\rho t} \phi_2 \left( \frac{1}{8} t^2 \right) \frac{dt}{\sqrt{t}} = \sqrt{2\pi} \frac{D_\alpha(0)}{D'_{\alpha}(0)} = \sqrt{2\pi} \frac{\Gamma \left( \alpha + \frac{3}{4} \right)}{\Gamma \left( \alpha + \frac{1}{4} \right)} \]

(3.5)

from which \( \phi_2 \) can (at least in principle) be determined. Note that the analogue of the moment recurrence (1.7)–(1.8) fails for \( E\xi_2^2 \) because there is apparently no simple asymptotic expansion for the right hand side of (3.5), \( \Gamma (\alpha + \frac{3}{4}) / \Gamma (\alpha + \frac{1}{4}) \), corresponding to that in [2, page 448] for the right side of (1.5), \( Ai(u)/A_1(u) \).
The second approach, based on $L^2$ expansions, shows that the implicit equation (3.5) may actually be explicitly solved for $\phi_2$, namely

$$\phi_2\left(\frac{\lambda}{2}\right) = E e^{-\lambda/2} \xi_2 = \left(\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2}.$$  

(3.6)

It is in fact easily checked that if (3.6) is substituted into (3.5) then an identity is obtained. To derive (3.6) from the $L^2$-expansion, note that

$$\phi_0(t) = 1, \quad \phi_n(t) = \sqrt{2} \cos n\pi t, \quad 0 \leq t \leq 1, \quad n = 1, 2, \ldots$$

(3.7)

is a complete orthonormal family in $L^2[0, 1]$. Thus from [5, page 324], if $\eta_0, \eta_1, \ldots$ is a standard normal sequence,

$$W_t = \sum_{n=0}^{\infty} \eta_n \int_0^t \phi_n, \quad 0 \leq t \leq 1$$

(3.8)

is a standard Wiener process. Note that $W_1 = \eta_0$ so that

$$\tilde{W}_t = W_t - tW_1 = \sum_{n=1}^{\infty} \eta_n \int_0^t \phi_n$$

(3.9)

is a pinned Wiener process [5, page 330], where the last sum omits $n = 0$. We have chosen the family $\phi_n$ so that not only are the $\phi_n$ orthonormal but also $\int_0^t \phi_n$ is an orthogonal family in $L^2[0, 1]$ (this is the only such family with this property). Thus by the Bessel-Parseval identity,

$$\int_0^1 \tilde{W}_t = \sum_{n=1}^{\infty} \eta_n^2 \int_0^1 \left(\int_0^t \phi_n^2\right) = \sum_{n=1}^{\infty} \sinh^2 \phi_n$$

(3.10)

Since $\eta_1, \eta_2, \ldots$ are standard normal,

$$\phi_2\left(\frac{\lambda}{2}\right) = E e^{-\lambda/2} \tilde{W}_1 = \prod_{n=1}^{\infty} E e^{-\phi_n^2 (2n\pi)^2}$$

(3.11)

$$= \prod_{n=1}^{\infty} \frac{1}{1 + \frac{\lambda}{(2n\pi)^2}}^{1/2}$$

$$= \left(\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2}$$

by the well-known product formula for the sinh function, which proves (3.6). Of course, the moments of $\xi^2$ can now be obtained by repeated differentiation at zero of $\phi_2$. Further, a somewhat complicated quadratic recurrence for $E\xi^2$ may be obtained from (3.11) by, for example, using the fact that

$$\phi_2\left(\frac{\lambda}{2}\right)^2 \frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}} = 1$$

(3.12)

since $(\sinh\sqrt{\lambda})/\sqrt{\lambda}$ has a simple power series.

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