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Covariant Master Theory for Novel Galilean Invariant Models and Massive Gravity

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Abstract

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Covariant master theory for novel Galilean invariant models and massive gravity

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Coupling the Galileons to a curved background has been a tradeoff between maintaining second order equations of motion and the Galilean shift symmetries, and allowing the background metric to be dynamical. We propose a construction which can achieve all three for a novel class of Galilean invariant models by coupling a scalar with the Galilean symmetry to a massive graviton. This generalizes the brane construction for Galileons by adding to the brane a dynamical metric (nonuniversally) interacting with the Galileon field. Alternatively, it can be thought of as an extension of the ghost-free massive gravity, or as a massive graviton-Galileon scalar-tensor theory. In the decoupling limit of these theories, new kinds of Galileon invariant interactions arise between the scalar and the longitudinal mode of the graviton. These have higher order equations of motion and infinite powers of the field, yet are ghost free.

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I. INTRODUCTION

The *Galileons* [1,2] are higher derivative scalar field theories with many interesting and important properties, including second order equations of motion and novel nonlinearly realized shift symmetries. Originally formulated in flat space, it is not straightforward to couple the Galileons to a curved background. Implementing a universal coupling by a naíve replacement of partial derivatives by covariant derivatives results in a theory with higher order equations of motion for the metric. It is possible to add nonminimal couplings to restore the second order equations of motion [3,4], however, doing so inevitably results in terms that violate the Galileon shift symmetries.

The brane construction of Ref. [5] is an illuminating way to build the Galileons.[1] One imagines that our spacetime is a 3-brane floating in some higher dimensional bulk spacetime, and the brane-bending modes then become the Galileons. By extending this construction to curved bulks and branes, it becomes possible to couple the Galileons to different background geometries while preserving generalized versions of the Galileon shift symmetries, which are now associated with isometries of the bulk [7–10]. The second order equations of motion are also preserved in this approach. However, the metric is a fixed background and is not dynamical. Making it dynamical corresponds to turning on a zero mode for the bulk metric, which breaks the isometries and hence the Galileon symmetry [11].

On a parallel front, in recent years it has become possible to construct ghost-free [de Rham-Gabadadze-Tolley (dRGT)] theories of massive gravity [12,13] (see Ref. [14] for a theory review and [15] for phenomenology review). These theories can be interpreted as the theory of a 3-brane embedded in a 3 + 1 dimensional bulk (i.e., a spacetime-filling embedding), in which a dynamical metric is put on the brane [16], and the brane worldvolume action takes the form given in Ref. [13]. The brane-bending modes become pure gauge modes, or Stückelberg fields, which in the presence of interaction terms mixing the dynamical bulk metric with the induced metric give the graviton a mass. Interestingly, the Galileon terms emerge in ghost-free massive gravity in the decoupling limit [12,17].

In this paper, we combine elements of the brane construction for Galileons and massive gravity to yield a novel theory that couples a multiplet of scalar fields π^I (where I is a flavor index running from 1 to N) to a metric $g_{\mu\nu}$ in a way that possesses all three desirable features: no extra propagating degrees of freedom, a Galileon symmetry, and dynamics for the metric. We should stress that what we mean by a "Galileon" in this work is a generic scalar field π^I, nontrivially transforming under the field-space Galilean invariance of the theory

$$\pi^I \to \pi^I + \omega^I_\mu x^\mu, \tag{1.1}$$

and propagating no extra degrees of freedom than those of a free field. In particular, π does not necessarily have to interact with itself or other fields through the five standard Galileon terms [2] (or their multifield generalizations [18–20]). Moreover, as we show below, a certain high-energy ("decoupling") limit of the theory is

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[1]For a complementary construction of these theories as Wess-Zumino terms, see Ref. [6].

described by peculiar scalar interactions, which significantly differ from the standard Galileon interactions, and yet their defining properties—ghost freedom and Galileon symmetry—are retained.

In light of the findings of Refs. [3,4], it is not surprising that a theory that can simultaneously achieve these properties is characterized by very special, nonuniversal couplings of the scalars π^I to the dynamical metric. In particular, as already noted above, the absence of higher time derivatives in the equations of motion (or, equivalently, extra propagating ghost degrees of freedom) in theories with π^I coupled to a massless graviton inevitably results in the breaking of Galileon symmetries of the flat space theory. Based on the requirements of Galilean invariance (which arises from nonlinearly realized broken higher-dimensional Poincaré symmetries and is therefore automatic in brane constructions) and ghost freedom, we argue below that the metric that can naturally couple to the scalars π^I describes a massive spin-2 field. The essential degrees of freedom on top of the N scalars π^I in our construction therefore are the five polarizations of a ghost-free massive graviton.

Formally one can take different points of view towards the theory discussed below. On one hand, one can view it as a certain generalization of ghost-free massive gravity, consistently interacting with a set of scalars π^I with the field-space Galilean invariance (1.1).[2] Alternatively, one can imagine a four-dimensional effective field theory obtained by extending the brane construction of the Galileons to allow for an additional intrinsic metric describing dynamical gravity. Starting with the two noninteracting sectors consisting of (a) the Galileons (obtained via invariants formed from the induced metric in the standard way) and (b) dynamical gravity, one can ask whether it is possible to construct well-defined mixings/interactions between these sectors, which would not lead to extra propagating degrees of freedom. As we argue below, one is quite uniquely led to a generalization of the dRGT theories. The by-product of this construction is that the dynamical metric on the brane will inevitably describe a massive graviton, nontrivially transforming under the Galileon symmetries.

Interestingly enough, although governed by similar global symmetries, not all interactions of π^I with the longitudinal scalar mode of the graviton in this theory fall into the categories of standard Galileons or multi-Galileons found in Refs. [2,18–20]. Unlike the usual Galileon terms and their multifield generalizations (of which there are a finite number for a finite number of fields), these interactions consist of an infinite series of terms with higher derivatives and lead to higher order equations. Yet, these theories are degenerate in the higher derivatives, and only two pieces of initial data per field are required to set the dynamics of the system, so in fact the theory is still free of extra propagating (potentially ghostly) degrees of freedom, albeit in a way that differs from the standard Galileon interactions (which have purely second order equations).

The paper is organized as follows. We start with reviewing the brane construction of Galileons in Sec. II. Sections III and IV describe the ghost-free actions for the induced and internal metrics that one can write within this framework, including a generalization of the recently proposed zero-derivative interactions characterizing ghost-free massive gravity. In Sec. V, we introduce the basic model with a flat bulk and comment on its symmetries, while Secs. VI and VII deal with the decoupling limit of the simplest version of the theory and show how, at least in this limit, nonpropagation of extra degrees of freedom is achieved despite the presence of higher derivative interactions. Finally, we conclude in Sec. VIII.

II. THE GENERAL CONSTRUCTION

We begin by presenting the general case of our construction. We will work in arbitrary dimension to start, and later specialize to the four-dimensional case of interest. The theory is that of a $(d-1)$-brane with worldvolume coordinates x^μ moving in a D-dimensional background, $D \geq d$, with coordinates X^A and a fixed background metric $G_{AB}(X)$. The dynamical variables include the brane embedding functions $X^A(x)$, D functions of the worldvolume coordinates x^μ.

We may construct the induced metric $\bar{g}_{\mu\nu}(x)$ via

$$\bar{g}_{\mu\nu}(x) = \frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu} G_{AB}(X(x)). \tag{2.1}$$

In addition to the induced metric, there are other geometric quantities associated with the embedding, such as the extrinsic curvatures and the twist connection (see Appendix A of Ref. [20] for a complete description of these quantities).

We would like the action on the worldvolume to be invariant under reparametrizations of the brane $x^\mu \to x^\mu - \xi^\mu(x)$, under which the embedding functions are scalars,

$$\delta_g X^A = \xi^\mu \partial_\mu X^A. \tag{2.2}$$

The induced metric (2.1) transforms as a tensor under these gauge transformations,

$$\delta_g \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} = \xi^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \partial_\mu \xi^\lambda \bar{g}_{\lambda\nu} + \partial_\nu \xi^\lambda \bar{g}_{\mu\lambda}. \tag{2.3}$$

Gauge invariance requires that the action be written as a diffeomorphism scalar, F, of the induced metric $\bar{g}_{\mu\nu}$, its covariant derivatives $\bar{\nabla}_\mu$, its curvature $\bar{R}^\rho{}_{\sigma\mu\nu}$, and the other induced quantities such as intrinsic curvature and twist (which we denote with ellipses),

[2] Another generalization of dRGT massive gravity by the "quasidilaton," a scalar realizing a new global symmetry in the theory, has recently been considered in Ref. [21].

$$S = \int d^d x \sqrt{-\bar{g}} F(\bar{g}_{\mu\nu}, \bar{\nabla}_\mu, \bar{R}^\rho{}_{\sigma\mu\nu}, \ldots). \quad (2.4)$$

In addition to the gauge symmetry of reparametrization invariance, there can also be global symmetries. If the bulk metric has a Killing vector $K^A(X)$ satisfying the Killing equation

$$K^C \partial_C G_{AB} + \partial_A K^C G_{CB} + \partial_B K^C G_{AC} = 0, \quad (2.5)$$

then the action will have a global symmetry under which the embedding scalars X^A shift,

$$\delta_K X^A = K^A(X). \quad (2.6)$$

The induced metric (2.1) and other induced quantities and therefore the general action (2.4) are invariant under (2.6).

We may completely fix the reparametrization freedom (2.2) by fixing the unitary gauge

$$X^\mu(x) = x^\mu, \qquad X^I(x) \equiv \pi^I(x). \quad (2.7)$$

In this gauge, the worldvolume coordinates of the brane are identified with the first d of the bulk coordinates. The remaining unfixed fields, $\pi^I(x)$, $I = 1\ldots N$ where $N = D - d$ is the codimension of the brane measure the transverse position of the brane. The gauge fixed action is an action solely for π,

$$S_{\bar{g}} = \int d^d x \sqrt{-\bar{g}} F(\bar{g}_{\mu\nu}, \bar{\nabla}_\mu, \bar{R}^\alpha{}_{\beta\mu\nu}, \ldots)|_{X^\mu = x^\mu, X^I = \pi^I}. \quad (2.8)$$

The form of the global symmetries (2.6) is altered once the gauge is fixed because the gauge choice (2.7) is not generally preserved by the global symmetry. The change induced by K^A is $\delta_K x^\mu = K^\mu(x, \pi)$, $\delta_K \pi^I = K^I(x, \pi)$, so to maintain the gauge (2.7), we must simultaneously perform a compensating gauge transformation with the gauge parameter

$$\xi^\mu_{\text{comp}} = -K^\mu(x, \pi). \quad (2.9)$$

The combined symmetry acting on the fields π^I is now

$$(\delta_K + \delta_{\text{comp}})\pi^I = -K^\mu(x, \pi)\partial_\mu \pi^I + K^I(x, \pi), \quad (2.10)$$

and is a global symmetry of the gauge fixed action (2.8).

In addition to the induced metric (2.1), we now introduce an additional worldvolume metric $g_{\mu\nu}(x)$ onto the brane. We demand that this obey the same transformation laws as $\bar{g}_{\mu\nu}$, and so we declare that it is invariant under the global symmetries (2.6), but transforms as a tensor under (2.2),

$$\delta_K g_{\mu\nu} = 0, \quad (2.11)$$

$$\delta_g g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda}. \quad (2.12)$$

We are now free to add to the action terms which are diffeomorphism scalars constructed from the intrinsic metric $g_{\mu\nu}$ and its associated covariant derivative and curvature,

$$S_g = \int d^d x \sqrt{-g} F(g_{\mu\nu}, \nabla_\mu, R^\alpha{}_{\beta\mu\nu}), \quad (2.13)$$

as well as terms that mix the intrinsic metric with the induced metric and other quantities,[3]

$$S_{\text{mix}} = \int d^d x \sqrt{-g} F(g_{\mu\nu}, \nabla_\mu, R^\alpha{}_{\beta\mu\nu}, \bar{g}_{\mu\nu}, \bar{\nabla}_\mu, \bar{R}^\alpha{}_{\beta\mu\nu}, \ldots). \quad (2.14)$$

Once the unitary gauge (2.7) is fixed, the fundamental fields of the theory are $g_{\mu\nu}$ and π^I, and the global symmetries act as

$$\delta g_{\mu\nu} = -K^\lambda(x, \pi)\partial_\lambda g_{\mu\nu} - \partial_\mu[K^\lambda(x, \pi)]g_{\lambda\nu} - \partial_\nu[K^\lambda(x, \pi)]g_{\mu\lambda}, \quad (2.15)$$

$$\delta\pi^I = -K^\mu(x, \pi)\partial_\mu \pi^I + K^I(x, \pi). \quad (2.16)$$

(Here, we must act by the derivatives on the argument of $\pi(x)$ within K^μ, i.e., $\partial_\mu[K^\lambda(x, \pi)] = \partial_\mu K^\lambda + \frac{\partial K^\lambda}{\partial \pi^I}\partial_\mu \pi^I$.) Note that in the unitary gauge the intrinsic metric $g_{\mu\nu}$ transforms nontrivially under the global symmetry, due to the compensating gauge transformation (2.9). Under this (in general nonlinear) transformation, the induced metric (2.1) transforms in precisely the same way as the intrinsic metric (2.15),

$$\delta\bar{g}_{\mu\nu} = -K^\lambda(x, \pi)\partial_\lambda \bar{g}_{\mu\nu} - \partial_\mu[K^\lambda(x, \pi)]\bar{g}_{\lambda\nu} - \partial_\nu[K^\lambda(x, \pi)]\bar{g}_{\mu\lambda}. \quad (2.17)$$

This construction allows us to have scalars with Galileon-like shift symmetries given by (2.16) coupled to a dynamical metric, which now carries a nontrivial transformation (2.15) under the Galilean symmetries. It remains to ensure the final desired property—that the action is free of ghosts.

III. GHOST-FREE ACTIONS

If, as for the original Galileons, the actions are to be free from extra Boulware-Deser—like [22] degrees of freedom, they must take a specific form. For the part of the action S_g depending only on the dynamical metric $g_{\mu\nu}$, we know that the only possibilities giving second order equations of motion are the Einstein-Hilbert term, the cosmological constant, and the higher Lovelock invariants [23,24] if the brane has dimension $d > 4$,

[3]We have chosen $\sqrt{-g}$ to be the universal measure factor for each term in this part of the action. This entails no loss of generality, since this choice may be traded for $\sqrt{-\bar{g}}$, or even a geometric mean such as $(-g)^{1/4}(-\bar{g})^{1/4}$, by pulling out or absorbing powers of factors such as $\det(g^{-1}\bar{g})$ into F. However, whatever choice is made the absence of the ghost should be investigated, as we will do in subsequent sections.

$$S_g = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}[-2\Lambda + R[g] + \cdots]. \quad (3.1)$$

For the term $S_{\bar{g}}$ depending only on the induced metric $\bar{g}_{\mu\nu}$, the possibilities are the Lovelock terms of $\bar{g}$, as well as the boundary terms associated with Lovelock terms in the bulk, as detailed in Ref. [20]. For codimension 1, these are the Myers boundary terms [25]. In the unitary gauge (2.7), these lead to the Galileon terms for the single π field [5]. For higher codimension, the surface terms are more limited and difficult to catalog [26,27]. In the unitary gauge they lead to the multifield Galileons [20]. In all cases, the leading term is the Dirac-Born-Infeld (DBI) term for the induced metric, which contains the kinetic term for the π^I fields, and so we write this part of the action as

$$S_{\bar{g}} = -T\int d^dx\sqrt{-\bar{g}} + \cdots, \quad (3.2)$$

where T is a constant of mass dimension d, and the ellipses denote the possible higher-order Lovelock and boundary terms.

For the mixed terms, it is not immediately obvious what the most general ghost-free terms are. However, if we restrict to terms depending only on $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, with no higher derivatives, we can take a clue from the dRGT theory [12,13] of massive gravity and the related models of bigravity [28], all of which have been shown to be ghost free [29–35]. These models contain interaction terms between two metrics—the second metric is a fixed fiducial metric in the case of massive gravity and a dynamical second metric in the case of bigravity. In this paper we will choose the form of the interactions to be the same, but the second metric will be the induced metric (2.1) containing the π^I degrees of freedom. The interactions can be constructed through the tensor $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\lambda}\bar{g}_{\lambda\nu}}$, in terms of which the relevant piece of the action is given as follows:

$$S_{\rm mix} = -\frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\frac{m^2}{4}\sum_{n=2}^{4}\alpha_n S_n(\mathcal{K}), \quad (3.3)$$

where $S_n(M)$, $0 \le n \le d$ for a $d \times d$ matrix $M^\mu{}_\nu$, are the elementary symmetric polynomials[4]

$$S_n(M) = M^{[\mu_1}_{\mu_1}\ldots M^{\mu_n]}{}_{\mu_n}, \quad (3.4)$$

and $\sqrt{g^{-1}\bar{g}}$ is the matrix square root of the matrix $g^{\mu\sigma}\bar{g}_{\sigma\nu}$. The $\alpha_{3,4}$ are free coefficients, while $\alpha_2 = -8$ is required for the correct normalization of the graviton mass. It is often convenient to work in terms of the expanded action

$$S_{\rm mix} = -\frac{M_{\rm Pl}^2}{2}\int d^dx\sqrt{-g}\frac{m^2}{4}\sum_{n=0}^{d}\beta_n S_n(\sqrt{g^{-1}\bar{g}}), \quad (3.5)$$

where β_n can be expressed in terms of the two free parameters $\alpha_{3,4}$. We will use the latter representation of the mixing terms below. Note that the $n = 0$ and $n = d$ terms in the latter sum are redundant, since these reproduce the cosmological constant $\sqrt{-g}$ and DBI term $\sqrt{-\bar{g}}$, respectively.

IV. CODIMENSION ZERO: MASSIVE GRAVITY

Our construction contains ghost-free dRGT massive gravity as a special case. When the codimension is zero, we are embedding a d-dimensional worldvolume into a bulk space of the same dimension. The fixed bulk metric $G_{\mu\nu}$ therefore has the same dimension as the brane metric.

In unitary gauge there are no π fields, and the induced metric is the bulk metric,

$$\bar{g}_{\mu\nu}(x) = G_{\mu\nu}(x), \quad (4.1)$$

so that the global symmetries are the Killing vectors $\xi^\mu(x)$ of $G_{\mu\nu}(x)$: $\xi^\lambda\partial_\lambda G_{\mu\nu} + \partial_\mu\xi^\lambda G_{\lambda\nu} + \partial_\nu\xi^\lambda G_{\mu\lambda} = 0$. The intrinsic metric then transforms linearly as a tensor,

$$\delta g_{\mu\nu} = \xi^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu\xi^\lambda g_{\lambda\nu} + \partial_\nu\xi^\lambda g_{\mu\lambda}. \quad (4.2)$$

The action $S_{\bar{g}}$ contains no dynamical variables and can be dropped. If the bulk metric is flat, $G_{\mu\nu} = \eta_{\mu\nu}$, the action $S_g + S_{\rm mix}$ is precisely the Lorentz invariant dRGT massive gravity of Refs. [12,13] in the unitary gauge. Further, the Poincaré invariance of these actions comes from (4.2). For a general bulk metric, the theory is that of massive gravity with a general reference metric [30], and the global symmetries (4.2) are precisely the isometries of the reference metric. These theories are all ghost free [29–31] meaning that they propagate, nonlinearly, precisely the number of degrees of freedom of a massive graviton and no more.

Away from the unitary gauge, we have

$$\bar{g}_{\mu\nu}(x) = \frac{\partial X^\rho}{\partial x^\mu}\frac{\partial X^\sigma}{\partial x^\nu}G_{\rho\sigma}(X(x)), \quad (4.3)$$

which is nothing but the Stückelberg replacement used to restore diffeomorphism invariance to massive gravity [16,36].

V. FLAT BULK CASE

We now return to general codimension N, but specialize to a flat bulk metric $G_{AB} = \eta_{AB}$. The isometries are the Poincaré transformations of the bulk,

$$\delta_P X^A = K^A(X) = \omega^A{}_B X^B + \epsilon^A, \quad (5.1)$$

where ϵ^A and the antisymmetric matrix $\omega^A{}_B$ are the infinitesimal parameters of the bulk translations and Lorentz transformations, respectively.

[4] Our antisymmetrization weight is $[\mu_1\ldots\mu_n] = \frac{1}{n!}\times(\mu_1\ldots\mu_n + \cdots)$. The appearance of the symmetric polynomials and their relation to the absence of ghosts can be naturally seen in the vielbein formulation of the theory [35]. See Appendix A of Ref. [35] for more details on the symmetric polynomials.

The unitary gauge (2.7) is not in general preserved by the Poincaré transformations, but the gauge is restored by making the compensating gauge transformation, $\delta_g X^\mu = \xi^\nu \partial_\nu x^\mu = \xi^\mu$, with the choice

$$\xi^\mu_{\rm comp} = -\omega^\mu{}_\nu x^\nu - \omega^\mu{}_I \pi^I - \epsilon^\mu. \tag{5.2}$$

The combined transformation $\delta_{P'} = \delta_P + \delta_g$ then leaves the gauge fixing intact and is a ymmetry of the gauge fixed action. This symmetry acts on the remaining fields as

$$\delta_{P'}\pi^I = -\omega^\mu{}_\nu x^\nu \partial_\mu \pi^I - \epsilon^\mu \partial_\mu \pi^I + \omega^I{}_\mu x^\mu + \omega_J{}^\mu \pi^J \partial_\mu \pi^I + \epsilon^I + \omega^I{}_J \pi^J, \tag{5.3}$$

where the first two terms in this expression are unbroken spacetime rotations and translations, respectively. The second two terms are a DBI symmetry corresponding to the broken boosts in the extra dimensional directions (which becomes the Galileon symmetry for small π^I). The fifth term is a shift symmetry corresponding to broken translations into the transverse directions. Finally, the last term is the unbroken $SO(N)$ symmetry in the transverse directions, which appears as an internal rotation among the π fields. The symmetry breaking pattern is $ISO(1, D-1) \rightarrow ISO(1, d-1) \times SO(N)$.

The induced metric in unitary gauge is

$$\bar g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \pi^I \partial_\nu \pi_I, \tag{5.4}$$

and using (2.15) we can determine how the global symmetries extend to the metric. The metric transforms linearly, as a tensor, under the unbroken d-dimensional Poincaré symmetry, and is invariant under the unbroken $SO(N)$ internal symmetry. The broken DBI shift symmetries, on the other hand, extend nontrivially to the dynamical metric,

$$\delta g_{\mu\nu} = \omega_I{}^\lambda \pi^I \partial_\lambda g_{\mu\nu} + \omega_I{}^\lambda \partial_\mu \pi^I g_{\lambda\nu} + \omega_I{}^\lambda \partial_\nu \pi^I g_{\mu\lambda}. \tag{5.5}$$

This is a diffeomorphism with parameter $\xi^\mu = \omega_I{}^\mu \pi^I$, since its origin was nothing but a compensating diffeomorphism to restore unitary gauge. The induced metric (5.4) transforms in the same way as

$$\delta \bar g_{\mu\nu} = \omega_I{}^\lambda \pi^I \partial_\lambda \bar g_{\mu\nu} + \omega_I{}^\lambda \partial_\mu \pi^I \bar g_{\lambda\nu} + \omega_I{}^\lambda \partial_\nu \pi^I \bar g_{\mu\lambda}. \tag{5.6}$$

In summary, in the unitary gauge the Galileons π^I and the intrinsic metric $g_{\mu\nu}$ transform as tensors under the unbroken d-dimensional Poincaré symmetry, as a vector and singlet, respectively, under $SO(N)$, and as follows under the broken Galileon symmetries:

$$\begin{aligned}\delta\pi^I &= \omega^I{}_\mu x^\mu + \omega_J{}^\mu \pi^J \partial_\mu \pi^I + \epsilon^I,\\ \delta g_{\mu\nu} &= \omega_I{}^\lambda \pi^I \partial_\lambda g_{\mu\nu} + \omega_I{}^\lambda \partial_\mu \pi^I g_{\lambda\nu} + \omega_I{}^\lambda \partial_\nu \pi^I g_{\mu\lambda}.\end{aligned} \tag{5.7}$$

This is how the DBI Galileon symmetry extends to the metric. It is a global symmetry. In the unitary gauge we are working in, there is no diffeomorphism invariance (assuming there are interaction terms $S_{\rm mix}$ in the action). As a consequence, the graviton described by $g_{\mu\nu}$ will be massive. In this sense, it is natural for the Galileons to couple to a massive graviton.

If we choose not to go to unitary gauge, diffeomorphism invariance on the brane then remains intact. In this case we have the Stückelberg fields X^μ, and the induced metric takes the form

$$\bar g_{\mu\nu} = \frac{\partial X^\rho}{\partial x^\mu}\frac{\partial X^\sigma}{\partial x^\nu}\eta_{\rho\sigma} + \partial_\mu \pi^I \partial_\nu \pi_I. \tag{5.8}$$

Because we now still have diffeomorphism invariance, the induced and intrinsic metrics are invariant under the global symmetries (5.1), and transform as tensors under diffeomorphisms. The fields X^μ and π^I are scalars under the diffeomorphisms, and transform together as (5.1) under the global symmetries.

VI. SMALL FIELD EXPANSIONS AND DECOUPLING LIMITS

If the action has a background solution $g_{\mu\nu} = \eta_{\mu\nu}$ for the intrinsic metric, we may expand about it in fluctuations $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{6.1}$$

The unitary gauge nonlinear transformation laws (5.7) expanded around this background are then

$$\begin{aligned}\delta\pi^I &= \omega^I{}_\mu x^\mu + \omega_J{}^\mu \pi^J \partial_\mu \pi^I + \epsilon^I,\\ \delta h_{\mu\nu} &= \omega_{I\mu}\partial_\nu \pi^I + \omega_{I\nu}\partial_\mu \pi^I + \omega_I{}^\lambda \pi^I \partial_\lambda h_{\mu\nu}\\ &\quad + \omega_I{}^\lambda \partial_\mu \pi^I h_{\lambda\nu} + \omega_I{}^\lambda \partial_\nu \pi^I h_{\mu\lambda},\end{aligned} \tag{6.2}$$

and we see that the metric fluctuations must transform along with the Galileon fields.

Defining the fluctuation around the induced metric via

$$H_{\mu\nu} = g_{\mu\nu} - \bar g_{\mu\nu}, \tag{6.3}$$

we have in unitary gauge

$$H_{\mu\nu} = h_{\mu\nu} - \partial_\mu \pi^I \partial_\nu \pi_I. \tag{6.4}$$

To perform the Stückelberg expansion, we simply leave the gauge unfixed, so that the induced metric takes the form (5.8). The fluctuation (6.3) can then be written as

$$H_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu} - \partial_\mu X^\rho \partial_\nu X^\sigma \eta_{\rho\sigma} - \partial_\mu \pi^I \partial_\nu \pi_I. \tag{6.5}$$

As in massive gravity, we can then introduce another Stückelberg field ϕ to deal with the longitudinal mode through the following replacement by expanding X^μ around its unitary gauge value,

$$X^\rho = x^\rho + A^\rho - \eta^{\rho\sigma}\partial_\sigma \phi. \tag{6.6}$$

The action then has the infinitesimal gauge transformations,

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu},\\ \delta A_\mu &= \partial_\mu \Lambda - \xi_\mu + \xi^\nu \partial_\nu A_\mu,\\ \delta\phi &= \Lambda,\\ \delta\pi^I &= \xi^\mu \partial_\mu \pi^I.\end{aligned} \tag{6.7}$$

As is usually done in massive gravity, we ignore the vector mode A^μ which carries the helicity one components of the massive graviton at high energy (a consistent truncation), since these do not generally couple to matter at the linearized level. Putting them to zero is consistent with the equations of motion. Moreover, there is an enhanced $U(1)$ symmetry for this vector in the decoupling limit that guaranties that it propagates two degrees of freedom. We then have the replacement

$$H_{\mu\nu} = h_{\mu\nu} + 2\partial_\mu\partial_\nu\phi - \partial_\mu\partial_\lambda\phi\partial_\nu\partial^\lambda\phi - \partial_\mu\pi^I\partial_\nu\pi_I. \tag{6.8}$$

For generic choices of the action, these theories describe a massive graviton coupled to the Galileon fields π^I, with coupling such that the unitary gauge action is invariant under the Galileon symmetries (5.7). Away from the unitary gauge on the other hand, the longitudinal mode of the massive graviton is described by the scalar ϕ, which appears in addition to the π^I when we restore the diffeomorphism invariance by not fixing unitary gauge.

For definiteness, we now focus on $d = 4$. The action is

$$\begin{aligned}&S_g + S_{\text{mix}} + S_{\bar g}\\ &= \int d^4x \bigg\{ \frac{M_{\text{Pl}}^2}{2}\sqrt{-g}\bigg[R[g] - \frac{m^2}{4}\sum_{n=0}^{4}\beta_n S_n(\sqrt{g^{-1}\bar g})\bigg]\\ &\quad - M_{\text{Pl}}^2 m^2 \sqrt{-\bar g}(\lambda_0 + \cdots)\bigg\}.\end{aligned} \tag{6.9}$$

Here λ_0 and β_n are order one dimensionless constants, independent of the mass scales m and M_{Pl}. (We have chosen the mass scalings of the various terms so that there will be an interesting decoupling limit, with new ingredients beyond those appearing in the corresponding limit of massive gravity.) The ellipses in the final term denote the possible higher-order Lovelock and boundary terms, each of which has its own independent order one coefficient and is suppressed by the mass scale m (which will ensure that nontrivial Galileon interactions survive the decoupling limit).

The coefficient β_4 is redundant with λ_0, and so we set $\beta_4 = 0$, and expand around flat space in the unitary gauge $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\bar g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\pi^I\partial_\nu\pi_I$. Tadpole cancellation ensuring that flat space is a solution requires

$$\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3 = 0. \tag{6.10}$$

At quadratic order, we find a Fierz-Pauli massive graviton. One of the β's can be absorbed into m^2, which we do by demanding

$$8 - \beta_1 - 2\beta_2 - \beta_3 = 0. \tag{6.11}$$

This ensures that the Fierz-Pauli massive graviton with a mass m propagates at quadratic order.

In addition we find a kinetic term for the π fields,

$$-\frac{1}{2}M_{\text{Pl}}^2 m^2\bigg[\lambda_0 + \frac{1}{8}(\beta_1 + 3\beta_2 + 3\beta_3)\bigg](\partial\pi^I)^2. \tag{6.12}$$

Thus, provided $\lambda_0 + \frac{1}{8}(\beta_1 + 3\beta_2 + 3\beta_3) > 0$, the theory propagates, in addition to the massive graviton, N healthy (having the correct sign kinetic term) scalars on this background.

In the end, we have four free parameters for the flat space theory (plus those corresponding to the higher Lovelock terms): the graviton mass, two independent β's corresponding to the two free parameters in the interactions of dRGT massive gravity, and a remaining independent parameter which corresponds to the strength of the π kinetic term. Note that if we had been looking for curved (A)dS solutions, there would have been an additional parameter corresponding to the curvature of the background.

We now make the replacement (6.8) and expand in powers of the fields. After canonical normalization of the various kinetic terms via

$$\hat h \sim M_{\text{Pl}} h, \quad \hat\phi \sim m^2 M_{\text{Pl}}\phi, \quad \hat\pi \sim m M_{\text{Pl}}\pi, \tag{6.13}$$

one can examine the interaction terms to determine their associated interaction scales.

First focus on those interaction terms arising from S_{mix} (or from the DBI term in $S_{\bar g}$). By virtue of the Stückelberg replacement (6.8), ϕ always appears with two derivatives, π appears with one derivative, and h appears with no derivatives. A generic term with n_h powers of $h_{\mu\nu}$, n_π powers of π^I, and n_ϕ powers of ϕ reads

$$\begin{aligned}&\sim m^2 M_{\text{Pl}}^2 h^{n_h}(\partial\pi)^{n_\pi}(\partial^2\phi)^{n_\phi}\\ &\sim \Lambda_\lambda^{4-n_h-2n_\pi-3n_\phi}\hat h^{n_h}(\partial\hat\pi)^{n_\pi}(\partial^2\hat\phi)^{n_\phi},\end{aligned} \tag{6.14}$$

where the scale suppressing the term is written as

$$\Lambda_\lambda = (M_{\text{Pl}} m^{\lambda-1})^{1/\lambda}, \quad \lambda = \frac{3n_\phi + 2n_\pi + n_h - 4}{n_\phi + n_\pi + n_h - 2}. \tag{6.15}$$

Since we always assume $m < M_{\text{Pl}}$, the larger λ, the smaller is this scale. Note that n_π must be even, by virtue of the way it enters in (6.8), and we have $n_\phi + n_A + n_h \geq 3$, since we are only considering interaction terms. The terms suppressed by the smallest scale are ϕ self-interaction terms, $n_\pi = n_h = 0$, which are suppressed by scales $\geq \Lambda_5$ and $< \Lambda_3$. These terms, however, all cancel up to a total derivative due to the special structure of the ghost-free massive gravity interactions [12,13].

The scale $\Lambda_3 = (M_{\rm Pl} m^2)^{1/3}$ becomes the lowest scale, and is carried by terms of schematic form ($n \geq 1$)

$$\sim \frac{1}{\Lambda_3^{n-1}} \hat{h} (\partial^2 \hat{\phi})^n, \tag{6.16}$$

and[5]

$$\sim \frac{1}{\Lambda_3^{n}} (\partial \hat{\pi})^2 (\partial^2 \hat{\phi})^n. \tag{6.17}$$

(Note that here we have included the term mixing h and ϕ for $n = 1$, even though it is a kinetic mixing term and not an interaction term, because it is from this mixing that ϕ acquires its kinetic term and its canonical normalization.) All other terms carry scales higher than Λ_3. There are a finite number of terms of the first type (6.16), and they take the same form as they do in massive gravity [12,13]. However, there are an infinite number of terms of the second type (6.17).

We now return to the possibility of higher Lovelock terms in $S_{\bar{g}}$. In the unitary gauge, before any decoupling limit, these are the DBI Galileons [5,37] for $N = 1$, and their multifield generalizations for higher N [20]. As we have mentioned, these terms are suppressed by the scale m. For example, for $N = 1$ the leading Lovelock term beyond the DBI kinetic term is the trace of the extrinsic curvature, which leads to a cubic DBI Galileon in the unitary gauge action,

$$S_{\bar{g}} = -M_{\rm Pl}^2 m^2 \int d^4x \bigg(\lambda_0 \sqrt{1 + (\partial \pi)^2} - \frac{\lambda_1}{m} \frac{1}{1 + (\partial\pi)^2} \partial_\mu \partial_\nu \pi \partial^\mu \pi \partial^\nu \pi + \cdots \bigg). \tag{6.18}$$

Away from the unitary gauge on the other hand, restoring the Stückelberg field and canonically normalizing via $\hat{\pi} \sim m M_{\rm Pl} \pi$, the various Galileon terms yield interactions of the form

$$\sim M_{\rm Pl}^2 m^2 \bigg(\frac{\partial^2 \hat{\phi}}{\Lambda_3^3}\bigg)^{n_\phi} \bigg(\frac{\partial \hat{\pi}}{M_{\rm Pl} m}\bigg)^{n_\pi - n} \bigg(\frac{\partial^2 \hat{\pi}}{\Lambda_3^3}\bigg)^{n}, \quad n_\pi - 1 > n = 0, 1, \ldots, 4. \tag{6.19}$$

The terms with multiple powers of the the factor $\frac{\partial \hat{\pi}}{M_{\rm Pl} m}$ arise from expanding out the square roots and denominators of (6.18). The interactions suppressed by Λ_3 are those with $n_\pi = n + 2$. Of these, the ones with $n_\phi = 0$ are precisely the terms which survive the limit which recovers the normal Galileons from the DBI Galileons [5]. Those with $n_\phi > 0$ of which there are an infinite number describe the coupling of the Galileons with the longitudinal mode of the graviton. We work out these couplings for the case of the cubic Galileon in the Appendix. All other terms are suppressed by scales larger than Λ_3.

If we take the decoupling limit,

$$M_{\rm Pl} \to \infty, \qquad m \to 0, \qquad \Lambda_3 = (M_p m^2)^{1/3} \text{ fixed}, \tag{6.20}$$

then all interactions with scales greater than Λ_3 are set to zero, and the DBI-Galileon terms become the normal Galileons. The only part of the Einstein-Hilbert action that survives the decoupling limit is the quadratic part,

$$S_g \supset \int d^4x \frac{M_{\rm Pl}^2}{8} h_{\mu\nu} \mathcal{E}^{\mu\nu,\alpha\beta} h_{\alpha\beta}, \tag{6.21}$$

where the kinetic operator for the graviton is that of linearized Einstein gravity.[6] The gauge symmetries (6.7) in the decoupling limit become the linearized versions of those considered above

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, & \delta A_\mu &= \partial_\mu \Lambda, \\ \delta\phi &= 0, & \delta \pi^I &= 0. \end{aligned} \tag{6.23}$$

As an explicit example, consider the choice of coefficients $\beta_0 = -24$, $\beta_1 = 8$, $\beta_2 = \beta_3 = \lambda_0 = 0$ so that the DBI term is gone, and set to zero all the higher Lovelock terms within $S_{\bar{g}}$.[7] This choice corresponds in pure massive gravity to the model which contains no nonlinear scalar-tensor interactions in the decoupling limit [12]. The corresponding dRGT action written in terms of the quadratic, cubic, and quartic terms in the matrix $\mathcal{K}^\mu_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\lambda}\bar{g}_{\lambda\nu}}$ and its traces [13] can be rewritten for these particular coefficients in the form given in Ref. [38]

$$S = \int d^4x \frac{M_{\rm Pl}^2}{2} \sqrt{-g} [R[g] + 2m^2 (3 - \mathrm{Tr}\sqrt{g^{-1}\bar{g}})], \tag{6.24}$$

[5]Had we not neglected the vector mode A^μ, we would have seen that terms of the form $\partial A \partial A (\partial\partial\pi)^n / \Lambda^{3n}$ also survive in the limit at hand.

[6]Explicitly,

$$\mathcal{E}^{\mu\nu}{}_{\alpha\beta} \equiv (\eta_\alpha^{(\mu} \eta^{\nu)}{}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta}) \Box - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^\mu\partial^\nu \eta_{\alpha\beta} + \partial_\alpha\partial_\beta \eta^{\mu\nu}. \tag{6.22}$$

[7]This corresponds to the choice $c_3 = \frac{1}{6}$, $d_5 = -\frac{1}{48}$ in the notations of Refs. [12,14].

which is often referred as the "minimal model." Unlike in pure massive gravity though, here $\bar{g}$ is going to be an induced metric on the brane. Thus the interactions we find will be entirely due to the extension of the theory we have developed here.

The decoupling limit Lagrangian is[8]

$$\mathcal{L}_{\rm dec} = \frac{M_{\rm Pl}^2}{8} h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta} + M_{\rm Pl}^2 m^2\bigg[\frac{1}{2}h_{\mu\nu}(\eta^{\mu\nu}\Box\phi - \partial^\mu\partial^\nu\phi) - \frac{1}{2}\frac{1}{\delta^\mu_\nu - \partial^\mu\partial_\nu\phi}\partial_\mu\pi^I\partial^\nu\pi_I\bigg]. \tag{6.26}$$

Diagonalizing via $h_{\mu\nu} = h'_{\mu\nu} + m^2\phi\eta_{\mu\nu}$, we find a free decoupled graviton, and coupled interacting scalars,

$$\mathcal{L}_{\rm dec} = \frac{M_{\rm Pl}^2}{8} h'_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h'_{\alpha\beta} + M_{\rm Pl}^2 m^2\bigg[-\frac{3}{4}m^2(\partial\phi)^2 - \frac{1}{2}\frac{1}{\delta^\mu_\nu - \partial^\mu\partial_\nu\phi}\partial_\mu\pi^I\partial^\nu\pi_I\bigg]. \tag{6.27}$$

The terms involving π^I are new to this model, and do not appear in pure massive gravity. Note that, unlike massive gravity, there are an infinite number of scalar interaction terms that survive the decoupling limit.

VII. EQUATIONS OF MOTION AND GHOSTS

The equations of motion obtained from (6.27) are not second order. To see this, we need only expand to cubic order in the fields, $\mathcal{L}_{\rm cubic} \sim \partial_\mu\partial_\nu\phi\partial^\mu\pi^I\partial^\nu\pi_I$. The ϕ equation of motion, for example, is third order, $\sim\partial_\mu\partial_\nu(\partial^\mu\pi^I\partial^\nu\pi_I)$.

Higher order equations are generally associated with extra ghostly degrees of freedom. In dRGT massive gravity, the decoupling limit is second order and contains no extra degrees of freedom, as it must since the entire theory has no such extra degrees of freedom. The higher order equations we are finding here are naively worrisome, because if the decoupling limit contains extra degrees of freedom, the entire model is not ghost free.

[8]For deriving the decoupling limit, it is convenient to write the Lagrangian in terms of the tensor $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\lambda}\bar{g}_{\lambda\nu}}$, $\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}(R[g] + 2m^2(-1+\mathcal{K}^\mu{}_\mu))$. Putting in the Stückelberg fields,

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - g^{\mu\lambda}H_{\lambda\nu}}, \qquad H_{\mu\nu} = h_{\mu\nu} + 2\Pi_{\mu\nu} - \Pi^2_{\mu\nu} - P_{\mu\nu}, \tag{6.25}$$

with $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\phi$ and $P_{\mu\nu} = \partial_\mu\pi^I\partial_\nu\pi_I$. We then use the relation (in matrix notation with $\langle\rangle$ the trace) $\langle\delta\mathcal{K}\rangle = -\frac{1}{2}\langle(1-\Pi)^{-1}P\rangle$ to find the terms involving π, and $\mathcal{K} = \Pi + \frac{1}{2}h - \frac{1}{4}h\Pi - \frac{1}{4}\Pi h$ for the terms involving h.

However, higher order equations do not necessarily imply the existence of extra degrees of freedom. As we will now show, the Lagrangian (6.27) in fact contains no additional ghostly degrees of freedom, despite the higher derivatives.

A. A toy example

As a warm-up to proving this, consider first the simpler $0+1$ dimensional version of the scalar part of the action (6.27),

$$S = \int dt\bigg(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{\dot{\pi}^2}{1+\ddot{\phi}}\bigg). \tag{7.1}$$

This is a higher-derivative Lagrangian, and the equations of motion are naively fourth order (third order if we expand to cubic order in the fields, as we did in the full case above),

$$\frac{\delta S}{\delta\pi} = \frac{d}{dt}\bigg[\frac{\dot{\pi}}{1+\ddot{\phi}}\bigg], \tag{7.2}$$

$$\frac{\delta S}{\delta\phi} = \ddot{\phi} + \frac{1}{2}\frac{d^2}{dt^2}\bigg[\frac{\dot{\pi}^2}{(1+\ddot{\phi})^2}\bigg]. \tag{7.3}$$

As with the full theory, this seems worrisome, since it raises the possibility of extra degrees of freedom which are ghosts.

However, the number of initial data needed to solve this system is only four, consistent with there being only two degrees of freedom. To see this, note that the π equation implies $\frac{\dot{\pi}}{1+\ddot{\phi}}$ is a constant, which when substituted into the ϕ equation implies $\ddot{\phi} = 0$. Substituting this back into the π equation then yields $\ddot{\pi} = 0$. Thus the equations above are, in fact, equivalent to the free field equations $\ddot{\phi} = \ddot{\pi} = 0$, and there are therefore no extra degrees of freedom.

Note that Ostragradskii's theorem [39] does not apply to (7.1) [or to the full model (6.27)], since one of the conditions of the theorem, that the Lagrangian be nondegenerate in the higher derivatives (i.e., the matrix obtained by variation of the action with respect to second derivatives be nondegenerate), is not satisfied.

The absence of extra degrees of freedom can also be seen directly at the level of the action. Starting with (7.1), we introduce an auxiliary field σ to render the action polynomial in the fields,

$$S = \int dt\bigg[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\sigma^2(1+\ddot{\phi}) + \dot{\pi}\sigma\bigg]. \tag{7.4}$$

The equation of motion for σ is then $\sigma = \frac{\dot{\pi}}{1+\ddot{\phi}}$, which when substituted back into (7.4) recovers (7.1). Integrating by parts to remove the second derivatives from ϕ, we find an equivalent first order action,

$$S = \int dt\bigg[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\sigma^2 + \sigma\dot{\sigma}\dot{\phi} - \pi\dot{\sigma}\bigg]. \tag{7.5}$$

We now Legendre transform to find an equivalent Hamiltonian action. The canonical momenta are

$$p_\phi = \dot\phi + \sigma\dot\sigma, \qquad p_\pi = 0, \qquad p_\sigma = \sigma\dot\phi - \pi. \tag{7.6}$$

There is a primary constraint

$$p_\pi = 0, \tag{7.7}$$

for which we introduce the multiplier λ, and the action then takes the form

$$S = \int dt\Big[p_\phi\dot\phi + p_\pi\dot\pi + p_\sigma\dot\sigma - \Big(-\frac{1}{2\sigma^2}(p_\sigma + \pi)^2 + \frac{p_\phi}{\sigma}(p_\sigma + \pi) + \frac{\sigma^2}{2}\Big) - \lambda p_\pi\Big]. \tag{7.8}$$

It is now straightforward to see that the primary constraint (7.7) generates a secondary constraint leaving four phase space degrees of freedom, or two Lagrangian degrees of freedom.

Alternatively, we may solve the primary constraint directly in the action,

$$S = \int dt\Big[p_\phi\dot\phi + p_\sigma\dot\sigma - \Big(-\frac{1}{2\sigma^2}(p_\sigma + \pi)^2 + \frac{p_\phi}{\sigma}(p_\sigma + \pi) + \frac{\sigma^2}{2}\Big)\Big]. \tag{7.9}$$

Now π is an auxiliary field and can be eliminated through its equation of motion $\pi = p_\phi\sigma - p_\sigma$ leaving

$$S = \int dt\Big[p_\phi\dot\phi + p_\sigma\dot\sigma - \Big(\frac{1}{2}p_\phi^2 + \frac{1}{2}\sigma^2\Big)\Big]. \tag{7.10}$$

Thus, we see explicitly that there are exactly two degrees of freedom, since renaming $p_\sigma \to q$, $\sigma \to -p_q$, the action is equivalent to that of two free particles with positive energy,

$$S = \int dt\Big[p_\phi\dot\phi + p_q\dot q - \Big(\frac{1}{2}p_\phi^2 + \frac{1}{2}p_q^2\Big)\Big]. \tag{7.11}$$

B. No extra degrees of freedom in the decoupling limit

We now apply a Hamiltonian analysis to the full scalar action in the decoupling limit of the theory (6.27). Setting $3m^2/4 \equiv 1/2$, and specializing to the case of a single π field for simplicity, the action (6.27) we are studying is proportional to

$$S = \int d^4x\Big[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\partial_\mu\pi\frac{1}{\delta^\mu_\nu - \partial^\mu\partial_\nu\phi}\partial^\nu\pi\Big]. \tag{7.12}$$

To eliminate the inverse powers of derivatives and work with a local action, we introduce an auxiliary vector field Ω^μ, and write the following equivalent action:

$$S = \int d^4x\Big[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\Omega_\mu(\delta^\mu_\nu - \partial^\mu\partial_\nu\phi)\Omega^\nu - \Omega^\mu\partial_\mu\pi\Big]. \tag{7.13}$$

Integrating out Ω_μ through its equations of motion recovers (7.12). Now we make a $(3+1)$ decomposition of the Lorentz indices, and do some integrations by parts to remove all the double time derivatives from ϕ,

$$S = \int d^4x\Big[\frac{1}{2}\dot\phi^2 - \frac{1}{2}(\vec\nabla\phi)^2 - \frac{1}{2}(\Omega^0)^2 + \frac{1}{2}(\Omega^i)^2 + \Omega^0\dot\Omega^0\dot\phi\Big] + \int d^4x\Big[\partial_i(\Omega^0\Omega^i)\dot\phi - \frac{1}{2}\Omega^i\Omega^j\partial_i\partial_j\phi + \dot\Omega^0\pi - \Omega^i\partial_i\pi\Big]. \tag{7.14}$$

As in the toy model, we now have a Lagrangian which has at most first time derivatives, so we may pass to a Hamiltonian form of the action in standard fashion. The conjugate momenta are

$$p_\phi = \dot\phi + \Omega^0\dot\Omega^0 + \partial_i(\Omega^0\Omega^i), \tag{7.15}$$

$$p_{\Omega^0} = \Omega^0\dot\phi + \pi, \tag{7.16}$$

$$p_\pi = 0, \tag{7.17}$$

$$p_{\Omega^i} = 0. \tag{7.18}$$

From this, we see that we have the primary constraints,

$$p_\pi = 0, \qquad p_{\Omega^i} = 0. \tag{7.19}$$

The Hamiltonian density on the constraint surface is

$$\begin{aligned}\mathcal{H} &= p_\phi\dot\phi + p_{\Omega^0}\dot\Omega^0 + p_\pi\dot\pi + p_{\Omega^i}\dot\Omega^i - \mathcal{L}\\ &= \frac{p_\phi}{\Omega^0}(p_{\Omega^0} - \pi) - \frac{1}{2(\Omega^0)^2}(p_{\Omega^0} - \pi)^2 + \frac{1}{2}(\nabla\phi)^2\\ &\quad + \frac{1}{2}(\Omega^0)^2 - \frac{1}{2}(\Omega^i)^2 - \frac{1}{\Omega^0}\partial_i(\Omega^0\Omega^i)(p_{\Omega^0} - \pi)\\ &\quad + \frac{1}{2}\Omega^i\Omega^j\partial_i\partial_j\phi + \Omega^i\partial_i\pi.\end{aligned} \tag{7.20}$$

Thus, solving the constraint inside the action gives the following equivalent Hamiltonian form of the action:

$$S = \int d^4x[p_\phi\dot\phi + p_{\Omega^0}\dot\Omega^0 - \mathcal{H}(p_\phi, \phi, p_{\Omega^0}, \Omega^0, \pi, \Omega^i)]. \tag{7.21}$$

We can now see that the system describes precisely two fields' worth of degrees of freedom: both π and Ω^i appear algebraically and can be integrated out, leaving a Hamiltonian action depending only on the phase space variables p_ϕ, ϕ, p_{Ω^0}, Ω^0, for which there are standard unconstrained first order Hamiltonian equations.

Doing this explicitly, we first integrate out π with its equation of motion

$$\pi = p_{\Omega^0} - \Omega^0 p_\phi + \Omega^0\Omega^i\partial_i\Omega^0, \tag{7.22}$$

which, substituted back into the Hamiltonian density yields

$$\mathcal{H} = \frac{p_\phi^2}{2} - (\partial_i\Omega^i)p_{\Omega^0} - (\Omega^i\partial_i\Omega^0)p_\phi + \frac{1}{2}(\Omega^i\partial_i\Omega^0)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\Omega^0)^2 - \frac{1}{2}(\Omega^i)^2 + \frac{1}{2}\Omega^i\Omega^j\partial_i\partial_j\phi. \tag{7.23}$$

Next we eliminate Ω^i through its equation of motion

$$\Omega^i = (A^{-1})^{ij}(\partial_j p_{\Omega^0} - p_\phi\partial_j\Omega^0), \tag{7.24}$$

where $A^i_j \equiv \delta^i_j - (\partial^i\Omega^0)\partial_j\Omega^0 - \partial^i\partial_j\phi$ giving the Hamiltonian

$$\mathcal{H} = \frac{p_\phi^2}{2} + \frac{1}{2}(\partial_i p_{\Omega^0} - p_\phi\partial_i\Omega^0)(A^{-1})^{ij} \times(\partial_j p_{\Omega^0} - p_\phi\partial_j\Omega^0) + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\Omega^0)^2. \tag{7.25}$$

This describes two fields with nonlinear, spatially nonlocal interactions.

At the linear level, we have

$$\mathcal{H} = \frac{p_\phi^2}{2} + \frac{1}{2}(\nabla p_{\Omega^0})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\Omega^0)^2, \tag{7.26}$$

so if we redefine $p_{\Omega^0} \equiv \chi$ and $\Omega^0 \equiv -p_\chi$, we can see that this describes two ghost-free, nontachyonic modes

$$\mathcal{H} = \frac{p_\phi^2}{2} + \frac{p_\chi^2}{2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\nabla\chi)^2. \tag{7.27}$$

As already noted above, we have assumed that the vector mode also present in the decoupling limit does not get excited,[9] so that we may set $A^\mu = 0$. This is a consistent truncation of the action, since A^μ only enters at the quadratic level in the action. (It is also consistent quantum mechanically, in the decoupling limit, due to the Z_2 symmetry $A^\mu \to -A^\mu$.) For arbitrary excitations of this mode, our Hamiltonian treatment has to be modified. We have no handle on the infinite number of $\partial A\partial A(\partial\partial\pi)^n$ interactions present in the decoupling limit (even in dRGT gravity their form is in general not known beyond the cubic order [17], see however Refs. [34,40]). However, it is plausible to expect that the presence of the vector mode does not spoil the ghost-free property of the decoupling limit because of the enhanced $U(1)$ symmetry of the limiting action; this is precisely what happens in ghost-free massive gravity.

If curvature invariants composed of the induced metric are added as implied by an ellipsis in (3.1), the proof of ghost freedom becomes increasingly difficult; the special structure of the resultant decoupling limit interactions however leads us to conjecture that the absence of the extra degrees of freedom carries through to this case as well. The simplest such interaction corresponding to a cubic Galileon for π in the decoupling limit is considered in the Appendix.

[9]Note that the presence of the infinite number of interactions of the form $\partial A\partial A(\partial\partial\pi)^n$ in the decoupling limit is also essential for the (nonlinearly realized) invariance under the broken bulk Lorentz generators in (5.1). While the inhomogeneous piece in the Galileon transformation is $\delta\pi = \omega_\mu x^\mu$, the vector Stückelberg mode shifts under this generator as $\delta A_\mu = -\omega_\mu\pi$. The infinite number of these terms then should relate by this symmetry to the infinite number of scalar interactions found in (6.27).

VIII. SUMMARY AND PROSPECTS

We have introduced a model which couples a scalar π^I to a dynamical metric in a manner which respects the Galileon symmetries. The metric to which the Galileons couple is a massive graviton. The model can be considered as an extension to higher codimension of ghost-free dRGT massive gravity, or as an extension of the brane construction of the Galileons where a dynamical metric is added to the brane.

We have derived explicitly the decoupling limit of the model around flat space, for a specific choice of parameters, and have shown that there are no ghosts. We have not proven that there are no ghosts beyond the decoupling limit, though we expect that there should not be, since the model is based on the ghost-free constructions of dRGT massive gravity and Galileon theories.

This model should provide a completely consistent framework within which to investigate the implications of Galileon invariance in, for example, cosmology. To study cosmology, we must decide how the standard model matter is to be coupled in. In pure massive gravity, it is generally assumed that matter should couple minimally to the dynamical metric, so we might also make that choice here. This is not the only choice consistent with the absence of the Boulware-Deser mode—for instance coupling minimally to the fiducial metric is also consistent—but it is the choice which can be expected not to drastically violate direct tests of the equivalence principle. The questions of what the most general consistent couplings are and of what experimental bounds exist on these couplings are still open problems.

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APPENDIX: ADDING A CUBIC GALILEON

In this appendix, specializing to the case of a single extra dimension for simplicity, we derive the decoupling limit interactions resulting from adding the extrinsic curvature term $M_{\rm Pl}^2 m \int d^4x\sqrt{-\bar{g}}K(\bar{g})$ to the rhs of (3.1).

As above, we we work in the gauge $X^\mu = X^\mu(x)$, $X^5 = \pi(x)$, so that the function $\Phi(x^A) \equiv \pi(x(X)) - X^5 = 0$ defines the embedding. The vector n_A, normal to the brane has the following components:

$$n_A = \frac{\partial_A\Phi}{|\eta^{AB}\partial_A\Phi\partial_B\Phi|^{1/2}} \Rightarrow n_\mu = \frac{\bar{\partial}_\mu\pi}{(1+\bar{\partial}^\alpha\pi\bar{\partial}_\alpha\pi)^{1/2}},$$
$$n_5 = -\frac{1}{(1+\bar{\partial}^\alpha\pi\bar{\partial}_\alpha\pi)^{1/2}}, \tag{A1}$$

where the operator $\bar{\partial}$ denotes differentiation with respect to the bulk coordinate,

$$\bar{\partial}_\mu = \frac{\partial x^\rho}{\partial X^\mu}\frac{\partial}{\partial x^\rho} = (\partial_\rho X^\mu)^{-1}\frac{\partial}{\partial x^\rho} \equiv A_\mu{}^\rho(x)\partial_\rho. \tag{A2}$$

The trace of the extrinsic curvature is then given by

$$K^\mu_\mu = n_{A,B}e^A_\mu e^B_\nu \bar{g}^{\mu\nu} = n_{A,B}\eta^{AB}$$
$$= \frac{1}{(1+\bar{\partial}^\alpha\pi\bar{\partial}_\alpha\pi)^{1/2}}\left(\bar{\Box}\pi - \frac{\bar{\partial}_\mu\pi\bar{\partial}_\nu\pi\bar{\partial}^\mu\bar{\partial}^\nu\pi}{1+\bar{\partial}^\alpha\pi\bar{\partial}_\alpha\pi}\right). \tag{A3}$$

The last step is to evaluate $\sqrt{-\bar{g}}$,

$$\bar{g}_{\mu\nu} = \partial_\mu X^\alpha\partial_\nu X^\beta\eta_{\alpha\beta} + \partial_\mu\pi\partial_\nu\pi, \tag{A4}$$

which, multiplied by two factors of the operator A on both sides gives

$$\det(A_\lambda{}^\mu A_\rho{}^\nu\bar{g}_{\mu\nu}) = \det(\eta_{\lambda\rho} + \bar{\partial}_\lambda\pi\bar{\partial}_\rho\pi) = -1 - (\bar{\partial}\pi)^2$$
$$\Rightarrow \bar{g} = -(1+(\bar{\partial}\pi)^2)\det(A^{-2}),$$

so that we have[10]

$$M_{\rm Pl}^2 m\int d^4x\sqrt{-\bar{g}}K$$
$$= M_{\rm Pl}^2 m\int d^4x \det(\partial X)\left(\bar{\Box}\pi - \frac{\bar{\partial}_\mu\pi\bar{\partial}_\nu\pi\bar{\partial}^\mu\bar{\partial}^\nu\pi}{1+\bar{\partial}^\alpha\pi\bar{\partial}_\alpha\pi}\right). \tag{A5}$$

In the weak field and the decoupling limits this leads to an extra cubic Galileon with the 5D derivatives $\partial \to \bar{\partial}$ instead of ordinary ones in (6.27),

$$\mathcal{L}_{\rm dec} \supset -M_{\rm Pl}^2 m\int d^4x\det(1-\partial\partial\phi)\bar{\partial}_\mu\pi\bar{\partial}_\nu\pi\bar{\partial}^\mu\bar{\partial}^\nu\pi, \tag{A6}$$

where (in the decoupling limit) $\bar{\partial}_\mu \equiv [(1-\partial\partial\phi)^{-1}\cdot\partial]_\mu$. In $(0+1)$ one dimension, the latter expression becomes a surface term, as can be seen by reparametrizing the time coordinate $t \to t' = \int dt(1-\ddot{\phi})$.

[10]Of course, noting that $d^4x\det(\partial X) = d^4X$, this could be directly obtained by transforming the corresponding unitary gauge expression under a diffeomorphism $x^\mu \to X^\mu$.

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