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Larry A. Shepp

University of Pennsylvania

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NOTES

A LOCAL LIMIT THEOREM

By L. A. Shepp

University of California, Berkeley and Bell Telephone Laboratories

0. Introduction. In this note we shall obtain asymptotic estimates for $c_n = P\{a \leq S_n \leq b\}$, where $a$ and $b$ are fixed, $a < b$, and $S_n = X_1 + \cdots + X_n$ is a sum of independent random variables with a common distribution and finite variance. It is shown that for $X_1$, nonlattice, with mean zero and variance $\sigma^2$, $c_n \sim (b - a)(2\pi n\sigma^2)^{-1}$. This will appear as an application of the central limit theorem of Cramér and Esseen. On the other hand, in the special lattice case (integer-valued $X_1$), we have $c_n = \nu(a, b)(2\pi n\sigma^2)^{-1} + o(n^{-1})$, where $\nu[a, b]$ is the number of integers in $[a, b]$.

The theorems appear unified in the following formulation. If $F_n$ is the measure induced on the real line by $S_n$, then $(2\pi n\sigma^2)^{-1}F_n$ converges in the sense of distributions to Lebesgue measure $\lambda$ in the nonlattice case, and to the measure $\nu$ in the special lattice case. Each of these measures may be viewed as a Fourier transform of Dirac’s distribution. The remaining general lattice case will be treated similarly.

These results are simple but do not appear to have been published previously. They are related to some known results as follows: Under certain additional assumptions on distributions in the domain of attraction of a nonnormal stable law ($\sigma = \infty$), Gnedenko ([2], p. 236; [3]) has obtained strong local theorems. Our results complement those of Gnedenko in the lattice case ([2], p. 233).

1. Statement of results. If with probability one, $X_1$ has only the values $\alpha + k\beta$, $k = 0, \pm 1, \cdots$, with $\alpha$ and $\beta$ fixed, then $X_1$ is said to have a lattice distribution. We suppose $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$.

Theorem 1. If $X_1$ does not have a lattice distribution, then for all continuous functions, $g$, with compact support

$$
\int g(y)H_n\{dy\} \to \int g(y)\,dy,
$$

where $H_n = (2\pi n\sigma^2)^{-1}F_n$.

Theorem 2. If $X_1$ does not have a lattice distribution, then

$$
P\{a \leq S_n \leq b\} \sim (b - a)(2\pi n\sigma^2)^{-1}.
$$

In the lattice case, assuming first that $\beta$ is chosen maximally, we may set our

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units so that $\beta = 1$. We shall prove the following theorem, otherwise obtainable as a corollary of results of Gnedenko ([2], p. 233). Here $[x]$ denotes the fractional part of $x$ and $\nu$ is the measure assigning unit mass to each integer.

**Theorem 3.** If $X_1$ has a lattice distribution, then

$$
(1.3) \quad (2\pi n\sigma^2)^{\frac{1}{4}} P\{a \leq S_n \leq b\} = \nu[a - \{n\alpha\}, b - \{n\alpha\}] + o(1).
$$

If $\alpha$ is rational, we may take $\alpha = 0$, and then for all continuous functions, $g$, with compact support,

$$
(1.4) \quad \int g(y)H_n\{dy\} \to \int g(y)\nu\{dy\}.
$$

If $\alpha$ is irrational, we have convergence in the Césaro sense,

$$
(1.5) \quad \int g(y)H_n\{dy\} \to \int g(y)\, dy, \quad (C, 1).
$$

Here, (1.5) also holds if $g$ is the indicator of an interval.

2. **Nonlattice case with finite third moment.** If $X_1$ does not have a lattice distribution and $E|X_1|^3 < \infty$, then an application of a deep theorem (stated below) of Cramér and Esseen ([2], p. 210) proves Theorem 1 for smooth functions, $g$. This theorem has been previously applied to similar situations by Kallianpur and Robbins [7] and by Bahadur and Rao [1].

**Theorem (Cramér-Esseen).** If $X_1$ does not have a lattice distribution and has finite third moment, then uniformly in $x$,

$$
(2.1) \quad F_n(x\alpha n^{\frac{1}{4}}) = \Phi(x) + \alpha_3 (1 - x^2)(72\pi n\sigma^2)^{-\frac{1}{4}} \exp \{-x^2/2\} + o(n^{-\frac{1}{4}}),
$$

where $\alpha_3 = EX_1^3$.

If $b \in L$ is absolutely continuous and vanishes at infinity, we obtain by integration by parts, applying (2.1),

$$
(2.2) \quad \int b(y)H_n\{dy\} = -(2\pi n\sigma^2)^{\frac{1}{4}} \int F_n(y)b'(y)\, dy = \int b(y)\, dy + o(1).
$$

The absolute continuity assumption is easily removed, but to prove (1.1) for general $X_1$, we must adopt a different attack.

We shall assume first that $g \in C^2$ (i.e., has two continuous derivatives). The transform, $\gamma(z) = \int g(y)e^{-i\gamma z}\, dy = O(1 + z^2)^{-\frac{3}{4}}$ and so $\gamma \in L$. The characteristic function, $\varphi(z) = E(e^{i\gamma z})$, satisfies Parseval's identity,

$$
(2.3) \quad \int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{\frac{1}{4}} \int \gamma(z)\varphi''(z)\, dz.
$$

For any $K > 0$, by normal convergence ([2], p. 181),

$$
(2.4) \quad (n\sigma^2/2\pi)^{\frac{1}{4}} \int_{|z| \leq K} \gamma(z)\varphi''(z)\, dz \to \gamma(0) = \int g(y)\, dy,
$$
using the fact that sup \(|\varphi(z)|^n: 0 < \delta \leq |z| \leq K\) is exponentially small ([2], p. 59), as \(n \to \infty\).

The proof of Theorem 1 is completed by showing that

\[
(2.5) \quad n^\frac{1}{2} \int_{|z| > K} \gamma(z) \varphi^n(z) \, dz = o(1).
\]

We remark that (2.5) holds if \(X_1\) satisfies condition \(C\) of Cramér, \(\lim \sup |\varphi(z)| < 1\), as \(z \to \infty\).

3. Nonlattice case with infinite third moment. We now assume that \(X_1\) is nonlattice and \(E|X_1|^3 = \infty\). We shall prove (1.1) by a truncation method. Assume first that \(g\) has four continuous derivatives. By integrations by parts, \(|\gamma(z)| < c\beta(z)\), where \(\beta(z) = (1 + z^4)^{-1}\), and \(c > 0\). We put \(\psi(z) = |\varphi(z)|^2\).

**Lemma.** If \(\psi\) is a nonnegative, nonlattice characteristic function, then for every \(K > 0\), we have

\[
(3.1) \quad n^\frac{1}{2} \int_{|z| \geq K} \beta(z) \psi^n(z) \, dz = o(1).
\]

If \(X\) is a random variable with characteristic function \(\psi\), let

\[
Y = X \quad \text{if} \quad |X| < T,
\]

\[
= 0 \quad \text{else}
\]

where \(T\) is chosen so large that \(Y\) is nonlattice. Let \(\psi_1\) denote the characteristic function of \(Y\). For all \(z\),

\[
(3.3) \quad \psi(z) \leq \psi_1(z).
\]

If \(G_n\) denotes the distribution function corresponding to \(\psi_1^n\), then by (2.2) with \(b(x) = (2\pi)^{-\frac{1}{2}} e^{ixz}\beta(z) \, dz\), we obtain

\[
(3.4) \quad 2\pi n^\frac{1}{2} \int b(x)G_n\, dx \to (\sigma_1^2 / 2\pi)^{-1} \int b(x) \, dx,
\]

where \(\sigma_1 = EY^2\), since \(E|Y|^3 < \infty\). By (2.4), we obtain

\[
(3.5) \quad n^\frac{1}{2} \int_{|z| \leq K} \beta(z) \psi_1^n(z) \, dz \to (\sigma_1^2 / 2\pi)^{-1} \int b(x) \, dx.
\]

Subtracting (3.5) from (3.4) and using (2.3), we obtain

\[
(3.6) \quad n^\frac{1}{2} \int_{|z| \geq K} \beta(z) \psi_1^n(z) \, dz \to 0.
\]

The lemma follows from (3.6) and (3.3). Since \(\gamma\) is bounded by \(\beta\), (2.5) follows from the lemma, and so (1.1) holds whenever \(EX_1^4 < \infty\), and \(g \in C^4\). To remove the latter restriction, we employ the usual technique, proving somewhat more than necessary for Theorem 1. Let \(f_m, h_m, m = 1, 2, \cdots\), be functions of \(C^4\)
with compact support, for which,

\[(3.7) \quad f_m \leq g \leq h_m \quad \text{and} \quad \int (h_m - f_m) \to 0.\]

If \( g \) satisfies the hypothesis of Theorem 1, or if \( g \) is the indicator of an interval, then such sequences \( f_m, h_m \) exist. Hence, for \( m = 1, 2, \cdots, n = 1, 2, \cdots \), we have

\[(3.8) \quad \int f_m(y)H_n\{dy\} \leq \int g(y)H_n\{dy\} \leq \int h_m(y)H_n\{dy\}.
\]

Letting \( n \to \infty \), then \( m \to \infty \), we learn that \( \int g(y)H_n\{dy\} \to \int g(y)\,dy \). This proves Theorems 1 and 2.

4. Lattice case. We consider first the simple lattice case, when \( \beta \) may be chosen so that \( \alpha = 0 \). We assume that \( \beta \) is maximal, and then adjust units so that \( \beta = 1 \). With \( \gamma(z) = \sum g(n)e^{-inz} \), where \( g \) has compact support, we obtain

\[(4.1) \quad \int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{1/2}\int_{-\pi}^{\pi} \gamma(z)\varphi^n(z)\,dz.
\]

Arguing as in (2.4) gives immediately, since we are now dealing with a compact interval,

\[(4.2) \quad \int g(y)H_n\{dy\} \to \gamma(0) = \sum g(n) = \int g\,dv.
\]

This proves (1.4). In the general lattice case, \( \alpha \) is irrational, and (1.3) receives a direct proof based on the same argument.

If \( g \in C^2 \) has compact support, we have, with \( \gamma(z) = \int e^{-izx}g(x)\,dx \).

\[(4.3) \quad \int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{1/2}\int \gamma(z)e^{i\alpha x}(e^{-i\alpha x}\varphi(z))^n\,dz.
\]

Since \( e^{-i\alpha x}\varphi(z) \) is periodic with period \( 2\pi(\beta = 1) \), we may write

\[(4.4) \quad \int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{1/2}\int \gamma_n^*(z)\varphi^n(z)\,dz
\]

where \( \gamma_n^*(z) = \sum \gamma(z + 2\pi k)e^{i\alpha x k n} \). Applying the normal convergence and the equicontinuity of the sequence \( \gamma_n^* \), \( n = 1, 2, \cdots \),

\[(4.5) \quad \int g(y)H_n\{dy\} = \gamma_n^*(0) + o(1).
\]

Using the Poisson summation formula ([8], Section 2.85), we have

\[(4.6) \quad \gamma_n^*(0) = \sum \gamma(2\pi k)e^{i\alpha x k n} = \sum g(k + n\alpha).
\]

By Weyl's Lemma ([6], pp. 115–117), for any irrational \( \alpha \),

\[(4.7) \quad \frac{1}{n} \sum_{k=1}^{n} \gamma_k^*(0) \to \int g(y)\,dy.
\]
Thus, for $g \in C^2$, with compact support,

$$\int g(y)H_n(dy) \to \int g(y) \, dy, \quad (C, 1).$$

By imitating the steps in (3.7) and (3.8), we extend the class of functions, $g$, for which (4.8) holds. This proves (1.5) and Theorem 3.

REFERENCES


