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Abstract
This paper presents a simple general equilibrium model of precautionary saving and accidental bequests. This model is used to analyze the implications of individual lifetime uncertainty for aggregate consumption and capital accumulation. A precautionary demand for saving arises because an individual consumer does not know in advance the date at which he will die, and he wants to avoid low levels of consumption in the event that he lives longer than expected. An implication of this precautionary saving is that when death does occur, the consumer is generally holding some wealth, which is then passed on to his heirs in the form of an accidental bequest. Even if all consumers have the same ex ante mortality probabilities, there will be some intracohort variation in the date of death; consequently there will be a nondegenerate distribution of bequests left by consumers in a cohort. This nondegenerate distribution of bequests left by one generation induces variation in the distributions of wealth, consumption, and bequests of subsequent generations.

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Precautionary Saving and Accidental Bequests

By ANDREW B. ABEL*

This paper presents a simple general equilibrium model of precautionary saving and accidental bequests. This model is used to analyze the implications of individual lifetime uncertainty for aggregate consumption and capital accumulation. A precautionary demand for saving arises because an individual consumer does not know in advance the date at which he will die, and he wants to avoid low levels of consumption in the event that he lives longer than expected. An implication of this precautionary saving is that when death does occur, the consumer is generally holding some wealth, which is then passed on to his heirs in the form of an accidental bequest. Even if all consumers have the same ex ante mortality probabilities, there will be some intracohort variation in the date of death; consequently there will be a nondegenerate distribution of bequests left by consumers in a cohort. This nondegenerate distribution of bequests left by one generation induces variation in the distributions of wealth, consumption, and bequests of subsequent generations.

The importance of bequests in aggregate saving has been established by Laurence Kotlikoff and Lawrence Summers (1981) who reported that 80 percent of U.S. household wealth is inherited wealth. One interpretation of this finding is that the simple life cycle model without bequest motives is an inadequate description of saving behavior in the United States, but the model I present demonstrates that accidental bequests by selfish consumers can account for a potentially sizeable fraction of aggregate wealth. Although some part of bequests, especially by the wealthy, undoubtedly results from an explicit bequest motive, accidental bequests also play a role in the intergenerational transfer of wealth as well as in the intragenerational variation in wealth. In order to focus on the role of accidental bequests, I purposely exclude a bequest motive from the specification of the utility function.

The effects of lifetime uncertainty on individual consumption behavior were first examined formally in a seminal paper by Menachem Yaari (1965). Yaari's model provided the basic framework for virtually all subsequent work on uncertain lifetimes including well-known papers by Nils Häkansson (1969), Stanley Fischer (1973), Robert Barro and James Friedman (1977), David Levhari and Leonard Mirman (1977), and Kotlikoff and Avia Spivak (1981). However, all of these papers focused on the consumption decision of an individual and ignored the effect of accidental bequests on the behavior of the recipients of these accidental bequests. As will be shown at various points in this paper, changes in the economic environment can have effects on aggregate behavior which differ sharply from the effects on individual behavior because of the endogenous adjustment of bequests.

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1 In the presence of perfect annuity and life insurance markets, there may or may not be bequests (depending on the presence or absence of a bequest motive), but there would be no accidental bequests.

2 Kotlikoff and Spivak focus on the role of the family in providing an (incomplete) annuities market, but stop short of a full-scale overlapping generations model in which the distribution of bequests is determined endogenously.
the endogenous adjustment of bequests are modeled by embedding consumers with uncertain lifetimes into an overlapping-generations model à la Franco Modigliani and Richard Brumberg (1979), Paul Samuelson (1958), and Peter Diamond (1965). The model is dramatically different from the overlapping generations model with uncertain lifetimes proposed by Eytan Sheshinski and Yoram Weiss (1981), because Sheshinski-Weiss assume that all consumers who are born at the same date also die on the same date. Thus, unlike the model presented below, the Sheshinski-Weiss model does not generate intracohort variation in bequests, consumption, and wealth.

Zvi Eckstein, Martin Eichenbaum, and Dan Peled (1985a) have developed an overlapping generations model in which consumers have identical ex ante mortality probabilities but die at different ages. Since the Eckstein et al. model, which was developed independently of my model presented below, is similar to that model, it is worth commenting on the differences between the two models. First, and most importantly, the Eckstein et al. model has no capital although one could interpret that model as applying to an economy in which the net rate of return on capital is zero (i.e., a costless storage technology). However, as shown below, the effects of Social Security policy differ depending on whether or not the rate of return on capital is zero. Second, in my model, the instantaneous utility function is assumed to display hyperbolic absolute risk aversion (HARA), whereas Eckstein et al. use a more general concave utility function. However, their formulation is not as general as it might first appear because Eckstein et al. must at some point assume that the concavity of the derived saving function is “not too large” without presenting the implied restrictions on the utility function. An advantage of the HARA utility function used here is that it leads to linear decision rules, thereby making the analysis easily tractable. Third, my model presented below allows for nonzero rates of time preference and population growth, whereas each of these rates is assumed to be zero by Eckstein et al.

In Section I, I present a simple model of individual consumption behavior in the presence of an uncertain lifetime. In Section II, I trace the effects of accidental bequests on the saving and consumption of subsequent generations and calculate the steady-state intracohort distributions of consumption, wealth, and bequests. The next three sections analyze the aggregate and distributional consequences of introducing different types of annuities into the economy. Section III demonstrates that, in the absence of an annuity market, the introduction of a fully funded actuarially fair Social Security system leads to a reduction in the steady-state national capital stock. In addition, the introduction of actuarially fair Social Security reduces all central moments of the distributions of consumption, wealth, and bequests. Section IV is devoted to an analysis of the transition path to the new steady state. In Section V, I show that the introduction of a competitive annuity market can cause the steady-state capital stock either to rise or fall depending on whether the coefficient of relative risk aversion is less than or greater than a certain critical value. Concluding remarks and directions for further research are presented in Section VI.

I. Individual Consumption Behavior under Uncertain Lifetime

Consider an economy with many consumers and a single commodity. This commodity can be either consumed or invested. If one unit of the commodity is invested, it yields $R$ units of the commodity in the following period. Each consumer lives either one or two periods. A consumer works during the first period of his (or her) life earning a fixed labor income $Y$. Also in the first period of his life, a consumer consumes an amount $c_1$ and pays a tax $T$. At the end of the first period of his life, the consumer has $G - 1$ children. There is a probability $p$ that the consumer dies at the end of his first

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3It is assumed that the production function is linear in capital and employment. Let $N_t$ be the number of consumers born at the beginning of period $t$ and let $K_{t-1}$ be the average capital stock held at the end of period $t - 1$ by consumers born at the beginning of period $t - 1$. Then aggregate output in period $t$ is $N_t Y + R N_{t-1} K_{t-1}$. 

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period of life\(^4\) (after having the children). If the consumer survives to the second period of life, he does not work but receives a Social Security payment \(S\). He then consumes an amount \(c_2\). When a consumer dies (either at the end of period one or period two), any unconsumed wealth is divided equally among his children.

Each consumer chooses \(c_1\) and \(c_2\) to maximize the following utility function

\[
U(c_1) + (1 - p)\delta U(c_2),
\]

where \(0 < \delta \leq 1\). This utility function is based on the uncertain lifetime literature in which the discounted utility index for period \(j\) is multiplied by the probability of being alive in period \(j\). This formulation is simply the expected value of a state-contingent utility function in which \(U(c_j)\) is the utility index contingent on being alive at age \(j\), and the utility index is identically zero contingent on not being alive at age \(j\).\(^5\) According to the utility function in (1), consumers do not care about their children; they derive no utility from leaving bequests.\(^6\)

Up to this point it may appear that all consumers are identical: they have identical utility functions, labor income \(Y\), taxes \(T\), childbearing characteristics, probabilities of survival, and, if they survive, identical Social Security benefits \(S\). However, different consumers receive bequests of different sizes depending on the mortality history of the earlier generations of their families. Let \(B\) be the bequest a consumer receives from his parent when he is born.\(^7\) For the moment, take \(B\) as given; the determination of \(B\) will be discussed in Section II.

Finally, \(W\) is defined to be the wealth held by a consumer at the end of the first period of his life:

\[
W = B + Y - T - c_1.
\]

If a consumer dies at the end of his first period of life, each of his children receives \(RW/G\) as a bequest at the beginning of the following period. If the consumer survives into the second period, he consumes \(c_2 = RW + S\), because he derive no utility from leaving a bequest. Using equation (2), we have

\[
c_2 = R\left[B + Y - T - c_1\right] + S.
\]

The consumer's first-period consumption is determined by maximizing (1) with respect to \(c_1\), subject to the constraint in (3), to obtain

\[
U'(c_1) = (1 - p) R\delta U'(c_2).
\]

The first-order condition in (4) equates the marginal utility of a unit of first-period consumption with the expected present value of the utility from the \(R\) units of second-period consumption which could be obtained by reducing current consumption by one unit. I specify the utility index \(U(c)\) to be a member of the HARA (hyperbolic absolute risk aversion) family\(^8\)

\[
U(c) = \frac{1 - \gamma}{\gamma} \left( \frac{\beta c}{1 - \gamma} + \eta \right)^{-\gamma}.
\]

With HARA utility, the optimal value of

\(^4\)Although individual consumers face uncertainty about their date of death, there is no aggregate uncertainty: a fraction \(p\) of consumers in each generation dies at the end of the first period of life.

\(^5\)It is not necessary that the utility index is equal to zero in the case of death. All that is required is that utility in the state of death does not depend on the level of wealth.

\(^6\)Fischer and Sheshinski-Weiss model consumers as deriving utility from leaving a bequest. This utility is a function of the size of the bequest. Barro (1974) and Allan Drazen (1978), in models without lifetime uncertainty, assume that consumers derive utility from the utility of their children. Douglas Bernheim, Andrei Shleifer, and Summers (1984) argue that parents use the prospect of bequests as a way to induce their children to behave in ways that the parent wants. Thus, although parents care only about their own utility, they find it optimal to leave bequests.

\(^7\)If a parent dies after the first period of his life, his child receives a bequest \(B\) at the beginning of the first period of the child's life. If a parent lives two periods, then as shown below, the child receives no bequest in either period; in this case, of course, the bequest received at birth by the child is zero.

\(^8\)The utility function in (5) is subject to the following restrictions: \(\gamma \neq 1; \beta > 0; (\beta c/(1 - \gamma)) + \eta > 0; \eta = 1\) if \(\gamma = -\infty\). The HARA family of utility functions includes the following special cases: (i) constant relative risk aversion (\(\eta = 0\)), which includes logarithmic utility if \(\gamma = 0\); (ii) constant absolute risk aversion (\(\gamma = +\infty\)); and (iii) quadratic utility (\(\gamma = 2\)).
first-period consumption can be written as a linear function of the present value of lifetime income \( B + Y - T + R^{-1}S \),

\[
(6a) \quad c_1 = a(B + Y - T + R^{-1}S) + b,
\]

where

\[
(6b) \quad 0 < a = \left[ 1 + R^{-1}((1-p)RS)^{1/(1-\gamma)} \right]^{-1} < 1
\]

\[
(6c) \quad b = ((1-\gamma)\eta/\beta) aR^{-1}
\]

\[
\times \left[ 1 - ((1-p)R\delta)^{1/(1-\gamma)} \right].
\]

Note that \( a \), the marginal propensity to consume, is a positive constant less than one.

If \( U(c) \) has constant relative risk aversion \( (\eta = 0) \), then \( b = 0 \) and first-period consumption is proportional to the present value of disposable lifetime resources \( B + Y - T + R^{-1}S \). Let \( \sigma = 1 - \gamma \) be the (constant) coefficient of relative risk aversion and note that if \( R = \delta = 1 \) (i.e., zero time preference and zero net rate of return on capital), then the fraction of total disposable resources \( B + Y - T + S \) consumed in the first period of life is

\[
(9) \quad c_1 = a(Y - T + R^{-1}S) + b.
\]

II. Intergenerational Transfers

I have solved the consumer’s saving-consumption decision conditional on the bequest \( B \) received at birth. In this section, I calculate the bequests received by each consumer. The bequest received by a consumer depends on the mortality history of the earlier generations of his family. Specifically, let \( j \) be the number of consecutive previous generations in a consumer’s family that died at age 1 (i.e., did not live to the second period of their life). For example, \( j = 0 \) indicates that the consumer’s parent lived two periods and therefore left no bequest to the consumer. If \( j = 1 \), then the consumer’s parent died at age 1 leaving a bequest but the consumer’s grandparent lived two periods leaving no bequest. All consumers are indexed according to \( j \) and I use the superscript \( j \) written in parentheses to indicate that a variable pertains to a consumer of type \( j \). Observe that \( p'(1-p) \) is the fraction of consumers who are of type \( j \).

First consider type-0 consumers. As indicated above, the parents of these consumers lived two periods, leaving no bequest so that \( B^{(0)} = 0 \). The first-period consumption and end-of-first-period wealth of these consumers follow immediately from (6) and (7), respectively:

\[
(9) \quad c_1^{(0)} = a(Y - T + R^{-1}S) + b,
\]

\[
(10) \quad W^{(0)} = (1 - a)(Y - T) - aR^{-1}S - b.
\]

9If the consumer cannot borrow against his (uncertain) future Social Security benefit \( S \), then \( c_1 \rightarrow B + Y - T \) as \( \sigma \rightarrow 0 \). Of course, if \( S = 0 \), then the consumer will indeed consume all of his disposable lifetime resources in the first period.

10I assume that \( S \) and \( T \) are small enough and that the utility function and labor income are such that \( W^{(0)} > 0 \). Note that if \( b = 0 \) (as it would be with constant relative risk aversion), then \( W^{(0)} > 0 \) provided that \( S \) and \( T \) are small enough.
Now consider consumers who receive positive bequests at birth, that is, consumers of type \( j, j \geq 1 \). Because all consumers have the same constant marginal propensity to consume, the difference in first-period consumption between any two consumers is proportional to the difference in the bequests they received at birth. In particular, the first-period consumption of a type-\( j \) consumer exceeds the first-period consumption of a type-0 consumer by \( aB^{(j)} \):

\[
(11) \quad c^{(j)} = aB^{(j)} + c^{(0)}.
\]

Similarly, intracohort differences in end-of-first-period wealth are proportional to intracohort differences in the bequest received at birth, so that from (7) and (10) we obtain:

\[
(12) \quad W^{(j)} = (1 - a)B^{(j)} + W^{(0)}.
\]

Having related first-period consumption and end-of-first-period wealth to \( B^{(j)} \), the bequest received at birth, the next step is to calculate \( B^{(j)} \). If a type-\( j-1 \) consumer dies after one period, he leaves a bequest of \( G^{-1}W^{(j-1)} \) to each of his children (who are type-\( j \) consumers). The bequest earns a gross rate of return \( R \) so that:

\[
(13) \quad B^{(j)} = (\frac{R}{G})W^{(j-1)} \quad j = 1, 2, 3, \ldots
\]

Substituting (13) into (12) yields the first-order linear constant coefficient difference equation:

\[
(14) \quad W^{(j)} = W^{(0)} \sum_{i=0}^{j} (1 - a)^i (\frac{R}{G})^i \quad j = 0, 1, 2, \ldots
\]

According to (14), as we increase the number of previous generations that died early leaving bequests, we increase \( W^{(j)} \). We will assume that \( (1 - a)R < G \); hence, as \( j \) increases, \( W^{(j)} \) approaches \( W^{(0)}/[1 - (1 - a)(R/G)] \) asymptotically.\(^{11}\)

I have now obtained a complete formal solution of the model. Given any nonnegative integer \( j \), we know that a fraction \( (1 - p)p^j \) of the population is of type \( j \). Then, using equations (9)-(11), (13), and (14), it is a simple matter to calculate the consumption, wealth, and bequests received at birth by each type-\( j \) consumer. The next step is to summarize the distributions of consumption, wealth and bequests by calculating the values of aggregate first-period consumption \( C^*_1 \), aggregate second-period consumption \( C^*_2 \), aggregate private wealth \( W^* \), and aggregate bequests \( B^* \). Each of these aggregates is expressed on a per capita basis (more precisely, per person in the young generation).\(^{12}\)

Calculating the aggregate per capita values of both sides of (12), we obtain:

\[
(15) \quad W^* = (1 - a)B^* + W^{(0)}.
\]

Because a fraction \( p \) of each type of consumer dies early leaving a bequest, aggregate wealth held by consumers who die young is \( pW^* \). Including the accrued interest on this wealth and adjusting for the fact that each generation has \( G \) times as many consumers as the previous generation, we obtain:

\[
(16) \quad B^* = p(\frac{R}{G})W^*.
\]

Substituting (16) into (15) yields:

\[
(17) \quad W^* = W^{(0)}/(1 - (1 - a)pR/G).
\]

Therefore, per capita wealth is proportional to \( W^{(0)} \), the wealth of type-0 consumers, and the constant of proportionality is independent of the tax parameters \( T \) and \( S \).

\(^{11}\)Since \( 0 < a < 1 \), it follows immediately that if \( R \leq G \), then the convergence condition \( (1 - a)R < G \) holds. To examine the case where \( R > G \), observe from (6b) that \( (1 - a)R = \phi R^{1/(1-\gamma)}/[1 + \phi R^{\gamma/(1-\gamma)}] \), where \( \phi = [(1 - p)R^{1/(1-\gamma)}/(1 - \gamma)]. Therefore, \( (1 - a)R < G \) if and only if \( \phi R^{1/(1-\gamma)}/[1 - G/R] < G \). If \( R > G \), then the convergence condition holds if and only if \( \phi < [G/(R - G)]^{\gamma/(1-\gamma)} \).

\(^{12}\)For example, aggregate private wealth per capita is defined as \( W^* = \sum_{j=0}^{\infty} (1 - p)p^jW^{(j)} \). Since only a fraction \( (1 - p) \) of young consumers survives to the second period of life, and since each generation is only \( G^{-1} \) times as large as the succeeding generation, aggregate second-period consumption is:

\[
C^*_2 = (1 - p)G^{-1} \sum_{j=0}^{\infty} (1 - p)p^jW^{(j)}.
\]
Aggregate economywide private consumption per capita, \( C_1^* + C_2^* \), is equal to the sum of after-tax labor income, \( Y - T \), plus Social Security payments to the surviving fraction \((1 - p)\) of the old cohort, plus the net return on wealth, adjusted for population growth,\(^{13}\)

\[
C_1^* + C_2^* = Y - T + (1 - p) G^{-1} S + \left( \frac{R}{G} - 1 \right) W^*.
\]

A final useful relation between \( C_1^* \) and \( C_2^* \) is obtained by calculating the per capita values of both sides of the income expansion path in (8) and recalling that the old cohort has \((1 - p) G^{-1}\) times as many consumers as the young cohort,

\[
C_2^* = (1 - p) G^{-1} \left[ (1 - a) C_1^* - b \right] R/a.
\]

Thus, \( C_1^* \) and \( C_2^* \) move in the same direction in response to changes in labor income \( Y \), or in the Social Security parameters \( S \) and \( T \).

III. The Effects of Actuarially Fair Social Security

In this section I consider the effects on savings and consumption of the introduction of a fully funded actuarially fair Social Security system. Let us suppose that the only role of the government is to collect Social Security taxes from the young and distribute Social Security benefits to the old. Thus the taxes \( T \) levied on the young are Social Security taxes. An actuarially fair Social Security system would levy a tax of \((1 - p) R^{-1} S\) dollars for each dollar of expected benefits, that is, \( RT = (1 - p) S \). Under this system, a young consumer contributes \((1 - p) R^{-1} S\) to the Social Security system. He receives \( S \) if he survives to the second period of life, but receives zero if he dies after one period. Thus the expected present value of the Social Security benefit is \((1 - p) R^{-1} S\) which is equal to the consumer’s contribution. Put differently, the Social Security system runs a balanced account vis-à-vis each generation. The Social Security system collects taxes from the members of each generation when they are young, invests the tax revenue at a gross rate of return \( R \), and then returns all of the tax revenue with accrued interest to the surviving old members of the generation.

A. Aggregate Consumption and Capital Accumulation

In order to study the effects of actuarially fair Social Security on aggregate consumption, I proceed in three steps. First, I analyze the effects of Social Security on the saving and consumption behavior of type-0 consumers. Then, I use the results about the effects on \( W^{(0)} \) to analyze the effects on the private capital stock and on the total national capital stock. Finally, the relations between the national capital stock and aggregate consumption are used to determine the effects on \( C_1^* \) and \( C_2^* \).

To calculate the effects of actuarially fair social security on consumption and saving of young type-0 consumers, we substitute \( T = (1 - p) R^{-1} S \) into (9) and (10) to obtain

\[
c_1^{(0)} = a Y + b + a p R^{-1} S,
\]

\[
W^{(0)} = (1 - a) Y - b - T - a p R^{-1} S.
\]

The introduction of actuarially fair Social Security increases the present value of lifetime resources, \( B + Y - T + R^{-1} S \), by \(- T + R^{-1} S = p R^{-1} S \). A consumer who survives to the second period receives a Social Security payment \( S \) that exceeds the value of his contribution with accrued interest, \( RT \), because the surviving members of each generation receive (on a pro rata basis) the tax-cum-interest contributed by members of their generation who died after one period. The effect of this increase in lifetime resources is to increase \( c_1^{(0)} \) by \( a p R^{-1} S \). The wealth

\(^{14}\)An alternative explanation for the increase in consumption is that a consumer’s claim to Social Security

\(^{13}\)Observe from (2) and (16) that \( C^* = B^* + Y - T - W^* = Y - T - (1 - p R/G) W^* \). Since \( c_1^{(1)} = R W^{(1)} + S \), and since the old cohort is \((1 - p) G^{-1}\) times as large as the young cohort, we obtain \( C_2^* = (1 - p) G^{-1} (R W^* + S) \). Adding together the expressions for \( C_1^* \) and \( C_2^* \) yields (18).
held at the end of the first period by type-0 consumers is reduced for two reasons: first, disposable resources available in the first period fall by the amount of the tax $T$; second, the increase in first-period consumption further reduces wealth held at the end of the first period.\footnote{In a balanced budget pay-as-you-go system, $GT = (1 - p)S$. In this case, equation (9) implies $c^{(0)} = aY + b + a((G/(1 - p)R) - 1)T$ so that the introduction of Social Security causes $c^{(0)}$ to increase, decrease, or remain unchanged according to whether $G$ is greater than, less than, or equal to $(1 - p)R$. It follows from equation (10) that the introduction of Social Security causes $W^{(0)}$ to fall by $(1 - a)T + aR^{-1}S$.}

In a fully funded Social Security system, the total national capital stock per capita (measured at the end of a period) is equal to the sum of the aggregate private capital stock per capita, $W^*$, and the per capita capital stock held by the Social Security system $T$. Recall from (17), that the private capital stock $W^*$ is proportional to $W^*$, and that the constant of proportionality does not depend on the parameters of the Social Security system. Since, from (21), the introduction of Social Security reduces $W^{(0)}$, it also reduces the aggregate private capital stock. Since $B^* = (pR/G)W^*$, the reduction in the aggregate private capital stock implies an equiproportionate reduction in aggregate bequests per capita.

The effect of actuarially fair Social Security on the aggregate national capital stock per capita, $W^* + T$, is easily determined by first observing from the definition of end-of-first-period wealth in (2) that

\begin{equation}
W^* + T = Y + B^* - C_1^*.
\end{equation}

Then calculating the aggregate per capita values of both sides of (11) we obtain

\begin{equation}
C_1^* = aB^* + c^{(0)}.
\end{equation}

Substituting (23) into (22) yields

\begin{equation}
W^* + T = Y + (1 - a)B^* - c^{(0)}.
\end{equation}

Since the introduction of Social Security causes $B^*$ to fall and $c^{(0)}$ to increase, it is clear from (24) that the aggregate national capital stock $W^* + T$ is reduced by the introduction of Social Security.

Next I examine the effects on aggregate consumption of the introduction of actuarially fair Social Security. Substituting $RT$ for $(1 - p)S$ in (18) gives

\begin{equation}
C_1^* + C_2^* = Y + ((R/G) - 1)(W^* + T).
\end{equation}

Thus, aggregate private consumption is equal to the sum of labor income $Y$ and the net return (adjusted for population growth) on national wealth. Observe that if $R = G$, so that the net rate of return on capital is equal to the rate of population growth, then the coefficient on national wealth in (25) is zero. In this case, $C_1^* + C_2^*$ is independent of the level of actuarially fair Social Security taxes and benefits. Furthermore, in view of the aggregate income expansion path in (19), both $C_1^*$ and $C_2^*$ are independent of the level of actuarially fair Social Security taxes and benefits. Finally, if $R < G$, then $C_1^*$ and $C_2^*$ are each increased by the introduction of actuarially fair Social Security. If $R > G$, then $C_1^*$ and $C_2^*$ are each reduced by the introduction of actuarially fair Social Security.\footnote{Under a balanced budget pay-as-you-go system, $GT = (1 - p)S$, so that from (18) aggregate consumption is $C_1^* + C_2^* = Y + (R/G - 1)W^*$. As shown in fn.}
B. The Intracohort Distributions of Consumption and Wealth

Having analyzed the effects of Social Security on the aggregate consumption of the young cohort and the aggregate consumption of the old cohort, I now examine the intracohort distributions of consumption and wealth. As already shown (equation (20)), the first-period consumption of type-0 consumers increases by \( apR^{-1}S \) in response to the introduction of Social Security. Also I have shown that \( W^{(0)} \) falls by \( T + apR^{-1}S \) when Social Security is introduced. As a consequence of the fall in \( W^{(0)} \), there is a reduction in bequests, \( B^{(j)} \), received at birth by type-1 consumers. Indeed, the introduction of Social Security reduces \( B^{(j)} \) for all type-\( j \) consumers for \( j=1,2,3,\ldots \). This result follows from the facts that \( B^{(j)} \) is proportional to \( W^{(0)} \) (see equations (13) and (14)) and that \( W^{(0)} \) is reduced by the introduction of Social Security. Below I analyze the effects of the induced reduction in bequests in the intracohort distribution of consumption.

The deviation of a type-\( j \) consumer’s first-period consumption from the average level of first-period consumption is proportional to \( B^{(j)} - B^* \) (see equations (11) and (23)):

\[
(26) \quad c^{(j)}_1 - C^*_1 = a( B^{(j)} - B^*).
\]

Since \( B^{(j)} \) and \( B^* \) are each proportional to \( W^{(0)} \), it follows that \( c^{(j)}_1 - C^*_1 \) is also proportional to \( W^{(0)} \).17 Because the introduction of actuarially fair Social Security reduces \( W^{(0)} \), it also reduces the (magnitude of the) deviation of type-\( j \) consumer’s first-period consumption from the average first-period consumption.18 Thus, the distribution of consumption is narrowed by the introduction of Social Security. More precisely, all central moments of the intracohort distribution of \( c^{(j)} \) are reduced by the introduction of Social Security.

The effects of the introduction of Social Security on second-period consumption are easily calculated by observing from (8) that \( c^{(j)}_2 \) can be expressed as an increasing linear function of \( c^{(j)}_1 \). Therefore, the narrowing of the distribution of \( c^{(j)}_1 \) implies that the distribution of \( c^{(j)}_2 \) is also narrowed by the introduction of Social Security.

For the case in which \( R = G \), it is straightforward to analyze the (steady-state) welfare implications of the introduction of Social Security. In this case, the introduction of actuarially fair Social Security does not affect the average levels of consumption of the young or of the old as explained in Section III, Part A. However, it narrows the distribution of consumption of each cohort. Therefore, if each consumer has an identical utility function and receives equal weight in the social welfare function, the introduction of Social Security is welfare improving. If \( R < G \), then the introduction of Social Security raises the average level of consumption and reduces the variance of consumption. Each of these effects increases social welfare. However, if \( R > G \), then the introduction of Social Security reduces average consumption, which tends to reduce welfare, but also reduces the intracohort variance of consumption, which tends to raise welfare.

IV. The Transition Path to the New Steady State

The analysis in Section III of the effects of the introduction of actuarially fair Social Security was a comparative steady-state anal-

\[15, W^{(0)} \] is reduced by the introduction of pay-as-you-go Social Security. Therefore, since \( W^* \) is proportional to \( W^{(0)} \), it follows that \( W^* \) is also reduced. As in the text, aggregate consumption is reduced, increased, or left unchanged according to whether \( R \) is greater than, less than, or equal to \( G \).17 More formally, using equations (13), (14), (16), and (17), equation (26) can be rewritten as

\[
c^{(j)}_1 - C^*_1 = a \left( \frac{R}{G} \right) W^{(0)}
\]

\[
\times \left( \sum_{i=0}^{j-1} (1-a)^i \left( \frac{R}{G} \right)^i - \frac{p}{1-p} \left( \frac{R}{G} \right)(1-a) \right)
\]

where \( \sum_{i=0}^{j-1} (1-a)^i \left( \frac{R}{G} \right)^i \) is equal to zero for \( j = 0 \).

\[18\text{Recall from fn. 15 that } W^{(0)} \text{ is also reduced by pay-as-you-go Social Security. Hence, pay-as-you-go Social Security also narrows the distribution of consumption.}\]
ysis; it was assumed that the Social Security system had been in effect long enough so that essentially no one received a bequest that included part of the savings of an ancestor who lived in the initial regime without Social Security. Equivalently, it was assumed that each person had at least one ancestor who lived for two periods under the new regime, leaving no bequests and thus severing links to the old regime.

In this section, I examine the transition path to the new steady state, which accompanies the introduction of an actuarially fair Social Security system. I show that the introduction of Social Security reduces the intracohort variances of first-period and second-period consumption for every generation (except the first) born under the new Social Security regime. If \( R < G \), then the average levels of first-period and second-period consumption of each generation are at least as high under the Social Security regime as in the absence of Social Security. In this case, the introduction of Social Security increases the welfare of every generation born under the Social Security regime.

Suppose that actuarially fair fully funded Social Security is introduced at the beginning of period \( t^* + 1 \). It will be assumed that since the older cohort (born at time \( t^* \)) did not contribute to the Social Security system, they receive no benefits. The young generation (born at time \( t^* + 1 \)) pays a tax \( T = (1 - p)R^{-1}S \) and the survivors will each receive a Social Security payment of \( S \) as discussed in Section III. The bequests received by each individual in the young generation are invariant to the introduction of Social Security, and for a given level of bequests received at birth, the introduction of Social Security increases the present value of lifetime income by \( pR^{-1}S \). Thus, every consumer in this generation increases first-period consumption by \( apR^{-1}S \), and every survivor increases second-period consumption by \( (1 - a)pS \). This generation unanimously favors the introduction of actuarially fair Social Security.

Next I consider the effect of the introduction of Social Security on subsequent generations. Let the subscript \( m \) denote that a variable pertains to a consumer born at the beginning of period \( t^* + m \). Let \( \Delta \) denote the change in a variable induced by the introduction of Social Security (relative to the regime without Social Security). Thus, the effect of Social Security on the first-period consumption of type-\( j \) consumers born at time \( t^* + m \) is obtained from (11) and (13) as

\[
\Delta c_{1,m}^{(j)} = \Delta c_{1,m}^{(0)} + a\left(\frac{R}{G}\right)\Delta W_{m-1}^{(j-1)},
\]

where \( j = 1, 2, 3, \ldots \) and \( m = 2, 3, 4, \ldots \).

Equation (27) displays the two countervailing effects on the consumption of subsequent generations. It follows immediately from (20) that

\[
\Delta c_{1,m}^{(0)} = apR^{-1}S > 0.
\]

Thus, as explained earlier, the first-period consumption of type-0 consumers increases.

For consumers who receive positive bequests at birth, there is a second effect on consumption and lifetime income because these consumers receive smaller bequests as a result of the introduction of Social Security. A straightforward generalization of (14) yields

\[
\Delta W_{m-1}^{(j)} = \Delta W_{m-1}^{(0)} \sum_{i=0}^{j^*-1} (1 - a)^i \left(\frac{R}{G}\right)^i,
\]

where \( j^* = \min(j, m - 1) \). Observe from (21) that

\[
\Delta W_{m}^{(0)} = -(T + apR^{-1}S) < 0.
\]

Since \( \Delta W_{m}^{(0)} < 0 \), it is clear from (29) that \( \Delta W_{m-1}^{(j)} < 0 \) so that type-\( j \) consumers born at the beginning of period \( t^* + m \) receive smaller bequests at birth. The magnitude of the reduction in bequests is strictly increasing in \( j^* \). Thus, for the generation born at

\[\text{19}\]Recall from fn. 15 that \( \Delta W_{m}^{(0)} < 0 \) for pay-as-you-go Social Security also.
the beginning of period \( t^* + m \), the reduction in bequests received by type-\( j \) consumers is strictly increasing in \( j \) for \( j = 0, 1, \ldots, m - 1 \), and is constant for \( j = m - 1, m, m + 1, \ldots \). This finding combined with the fact that the level of bequests received by type-\( j \) consumers is strictly increasing in \( j \) for the Social Security regime as well as the regime without Social Security implies that the introduction of Social Security reduces the intracohort variance of bequests received by all generations born after period \( t^* + 1 \). Since first-period (second-period) consumption is a linear function of the bequest received at birth, the introduction of Social Security also reduces the intracohort variance of first-period (second-period) consumption for these generations.

I have derived unambiguous results about the intracohort variance of consumption along the transition path to the new steady state. The effects on the average level of consumption are less clear-cut. As already shown, for the generation born at the beginning of period \( t^* + 1 \), the average levels of first-period and second-period consumption are increased by the introduction of Social Security. It has also been shown that, in the new steady state, the average levels of \( c^{(1)}_{1,m} \) and \( c^{(2)}_{1,m} \) decrease, increase, or remain unchanged, depending on whether \( R \) is greater than, less than, or equal to \( G \). I show in the Appendix that \( \Delta C^{(1)}_{1,m} \), the change in the average level of first-period consumption of the generation born at time \( t^* + m \), is

\[
\Delta C^{(1)}_{1,m} = \frac{apR^{-1}S}{1-p(R/G)(1-a)} \times \left\{1 - \frac{(R/G)}{1-p(1-a)} \right\} \times \left\{(R/G)^{m} p^{m-1}(1-a)^{m-1}\right\},
\]

Since it has been assumed that \((1-a)pR < G\), it is clear from (31) that \( \Delta C^{(1)}_{1,m} \) decreases as \( m \) increases. The reason is that as \( m \) increases (i.e., as we increase the length of time for which the Social Security regime has been in effect), there is a decrease in the amount of bequests which represent accumulated saving from generations born before the introduction of Social Security, when private saving was higher.

In the case in which \( R = G \), equation (31) implies that \( \Delta C^{(1)}_{1,m} \) is equal to \( aR^{-1}Sp^{m} (1-a)^{m-1} \), which is positive for all finite \( m \). Thus, since the introduction of Social Security increases the average and reduces the variance of \( c^{(1)}_{1,m} \) for all finite \( m \), it also (see equation (8)) increases the average value and reduces the variance of \( c^{(2)}_{1,m} \) for all finite \( m \). Therefore, if \( R = G \), the introduction of Social Security is welfare improving for every generation born under the new Social Security regime. More generally, if \( R \leq G \), the welfare of every generation (except the current old generation which is unaffected) is improved by the introduction of Social Security.

The welfare effects of the introduction of Social Security are less clear-cut in the case in which \( R > G \). Clearly, the welfare of the generation born at time \( t^* + 1 \) is improved because, as explained earlier, the first-period consumption of every consumer in this generation increases by \( apR^{-1}S \) (and from equation (8), second-period consumption increases by \( (1-a)pS \)). For all generations born after time \( t^* + 1 \), the introduction of Social Security reduces the intracohort variance of consumption. For sufficiently small \( m \), it follows from (31) that the average level of first-period (and second-period) consumption is increased by the introduction of Social Security. Thus, for these generations, welfare is increased. The difficulty in my welfare analysis arises for generations born long after the introduction of Social Security. If \( R > G \), then it follows from (31) that for sufficiently large \( m \), the average first-period (and a fortiori average second-period) consumption of the generation born at time \( t^* + m \) is reduced by the introduction of Social Security. The effect on the welfare of this generation thus depends on whether the welfare-improving effects of reduced variance dominate the welfare-worsening effects of reduced average consumption.

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20 This statement is simply an application of the fact that if \( x(j) \) and \( y(j) \) are strictly increasing in \( j \), and if \( y(j) - x(j) \) is nonincreasing in \( j \) and strictly decreasing for some \( j \), then the variance \( y(j) \) is less than the variance of \( x(j) \).
V. Private Annuities

In previous sections in this paper, it was assumed that there is no private market for annuities, and I showed that the introduction of actuarially fair Social Security reduces the national capital stock. However, if there were a competitive market for annuities, then the introduction of Social Security would have no effect because the competitively supplied actuarially fair annuities would be perfect substitutes for actuarially fair Social Security; hence, consumers could completely offset the effects of Social Security by conducting transactions in the private annuity market. Since the introduction of actuarially fair Social Security reduces the steady-state capital stock, it is natural to ask whether the introduction of a competitive annuity market also reduces the steady-state capital stock. 21

With the introduction of private annuities, there are now two alternative forms in which a consumer can hold his wealth. As before, he can hold capital directly, earning a gross rate of return $R$. Alternatively, he can deposit his savings at an annuity company. The annuity company operates by accepting deposits from young consumers and using these deposits to buy capital which earns a gross rate of return $R$. At the beginning of the following period, the annuity company distributes its holdings (with accumulated interest) to its surviving depositors in proportion to their initial deposits. Thus, each surviving depositor at the annuity company receives $A = R/(1 - p)$ dollars for each dollar initially deposited. As shown by Yaari, consumers who do not have explicit bequest motives will choose to hold all of their wealth in the form of these annuities. Thus, there will be no bequests.

Consumers can, by holding annuities, earn a gross rate of return $A$ on their savings so that $c_2 = A[Y - c_1]$. The maximization problem of the representative consumer 22 is

$$\text{(32)} \quad \text{Max } U(c_1) + (1 - p) \delta U(A(Y - c_1)).$$

The first-order condition for this problem is

$$\text{(33)} \quad U'(c_1) = (1 - p) A \delta U'(c_2).$$

With actuarially fair annuities $(1 - p)A = R$, so that the first-order condition (33) can be written as

$$\text{(34)} \quad U'(\hat{c}_1) = R \delta U'(\hat{c}_2),$$

where a circumflex denotes the value of a variable in the presence of a private annuity market.

For the remainder of this section we assume that $R = \delta = 1$, that is, that the net rate of return on capital and the rate of time preference are each equal to zero. With $R\delta = 1$, (34) implies that $\hat{c}_1 = \hat{c}_2$ for any strictly concave utility function $U(\cdot)$. Since $\hat{c}_2 = A(Y - \hat{c}_1)$ and $A = (1 - p)^{-1}$, we obtain $\hat{c}_1 = \hat{c}_2 = (1/(2 - p))Y$. Therefore, since $W = Y - \hat{c}_1$, we obtain

$$\text{(35)} \quad W = ((1 - p)/(2 - p)) Y.$$

Now consider the economy without an annuity market. For the remainder of this section, it will be assumed that $U(c)$ exhibits constant relative risk aversion. Recall from Section I that with $R = \delta = 1$, and a constant coefficient of relative risk aversion equal to $\sigma$, the marginal (and average) propensity to consume is $a = [1 + (1 - p)^{1/\sigma}]^{-1}$. Thus, the first-period consumption of type-0 consumers is

$$\text{(36)} \quad c_1^{(0)} = Y = (1 + (1 - p)^{1/\sigma}).$$

It is straightforward to show that

$$\text{(37)} \quad \hat{c}_1 \geq c_1^{(0)} \quad \text{as } \sigma \geq 1.$$

21 See Kotlikoff, John Shoven, and Spivak (1983) for an analysis of the effects of various annuity arrangements on capital accumulation.

22 Since there are no bequests, there is no need to distinguish consumers according to the mortality history of their families. Also, since actuarially fair Social Security has no effect in the presence of an annuity market, I simply set $S = T = 0$. 
The intuition for this result is that the introduction of a private annuity market raises the rate of return on private savings from $R$ to $R/(1 - p)$. The income effect of this change is to raise first-period consumption whereas the substitution effect is to reduce first-period consumption. With $\sigma > 1$, the income effect dominates and with $\sigma < 1$, the substitution effect dominates. For logarithmic utility, $(\sigma = 1)$, the income and substitution effects exactly offset each other. By contrast, notice that although Social Security has the payoff characteristics of an annuity, the introduction of actuarially fair Social Security has a positive income effect but has no substitution effect because individual consumers cannot choose the level of savings to be held in the form of Social Security.

The analysis in the above paragraph examines the effect on type-0 consumers of the introduction of an annuity market. For type-j consumers, there is an additional effect, because these consumers receive bequests in the absence of private annuities but do not receive bequests in the presence of annuities. To calculate the effect of an annuity market on the long-run capital stock, observe from (10) and (17) that in the absence of annuities, and with $R = \delta = 1$, the steady-state capital stock is

$$W^* = \frac{Y}{1 + (1 - p)^{-1/\sigma} - p/G}.$$  

Comparing (35) and (38) it can be shown that

$$\hat{W} \leq W^* \quad \text{as} \quad \sigma \geq \hat{\sigma}$$

where

$$\hat{\sigma} = \left[1 - \frac{\ln(1 + ((1 - p)p/G))}{\ln(1 - p)}\right]^{-1} < 1.$$  

According to (39), there is a critical value of the coefficient of relative risk aversion that determines whether the long-run capital stock increases or decreases when a private annuity market is introduced. When $\sigma > 1$, the introduction of an annuity market raises first-period consumption of type-0 consumers and thus reduces their saving. In addition, the elimination of bequests received by type-j consumers for $j \geq 1$ also tends to reduce private wealth. On the other hand, when $\sigma < 1$, the introduction of private annuities reduce the first-period consumption and increases the saving of type-0 consumers. Whether this wealth-increasing effect dominates the wealth-reducing effect of eliminating bequests depends on whether $\sigma$ is less than $\hat{\sigma}$.

VI. Concluding Remarks

I have developed a general equilibrium model of precautionary saving and accidental bequests that is sufficiently rich to produce endogenous distributions of consumption, wealth, and bequests. The model is based on individual utility-maximizing behavior and yields decision rules for consumers that are linear and easily aggregated. After developing the model in Sections I and II, it was shown in Section III that, in the absence of a private annuity market, the introduction of actuarially fair Social Security crowds out private wealth by more than one for one, thereby reducing national wealth; in addition, it reduces all central moments of the distributions of consumption, wealth, and bequests. Section IV analyzes the transition path to the new steady state when Social Security is introduced. The immediate effect is for the average level of consumption by young consumers to increase and for the variance of their consumption to remain unchanged. However, both the mean and the variance of consumption by young consumers decreases continually as

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Kotlikoff, Shoven, and Spivak examine a more complex overlapping-generations model with uncertain lifetimes. They solve their model numerically, and in each of their numerical simulations, they assume that the constant coefficient of relative risk aversion is greater than one. Although they find that the introduction of a perfect annuity market reduces long-run aggregate wealth in each of their simulations, the results in (39) suggest caution in applying this result to consumers with a coefficient of relative risk aversion sufficiently below one. Of course, since their model differs somewhat from the model presented herein, the critical value of the coefficient of relative risk aversion, if one exists, would probably differ from that in (39).
each subsequent generation is born. In Section V, I switched attention from publicly provided annuities to privately traded annuities and showed that the introduction of a private annuity market would cause the steady-state capital stock to increase or decrease depending on the risk aversion of consumers.

The model presented in this paper was purposely designed to allow a simple examination of precautionary saving and accidental bequests in a general equilibrium framework. Toward this end, the following simplifying assumptions were made: 1) consumers live for either one or two periods; 2) the rate of return on capital, $R$, is constant; 3) there is no private market for annuities (except in Section V); and 4) consumers are selfish, that is, they have no bequest motive.

An implication of assumption 1 and the assumption that consumers give birth to their children at the end of the first period of life is that each consumer knows at birth exactly what bequest he will receive from his parent. If consumers lived potentially for many periods so that a parent's lifetime uncertainty were not resolved when the child is born, then we would have the additional problem of calculating optimal consumption behavior when there is the prospect of receiving a bequest of uncertain size at an unknown date in the future. Edi Karni and Itzhak Zilcha (1984) have examined the case in which consumers live for three periods. They prove the existence of the steady-state equilibrium in the absence of annuity and insurance markets, and demonstrate that the introduction of competitive life insurance and annuity markets leads to a Pareto optimal steady-state equilibrium. However, they do not provide closed-form solutions for consumption. Also, their model cannot be used to examine long-run capital accumulation since capital is absent from their model. Kotlikoff, Shoven, and Spivak also relax assumption 1, but provide numerical rather analytic solutions of their model.

Assumptions 1 and 2 are both relaxed by Glenn Hubbard (1984). In place of the linear technology assumed above, Hubbard introduces a neoclassical production function into a model with uncertain lifetimes. However, he assumes that the government confiscates the assets held by consumers when they die and then redistributes the assets in lump sum fashion. This assumption circumvents the technical difficulty mentioned above but this simplification also eliminates the intracohort variations in consumption and wealth. Hubbard does not solve his model analytically and resorts to numerical simulation to study the effects of Social Security.

Assumption 3 is crucial in order for Social Security to have an effect in this model. If there were a competitive annuity market, the rate of return on competitively supplied annuities would be equal to the implicit rate of return offered by Social Security. In this case consumers could undo the effects of Social Security by conducting offsetting transactions in the private annuity market. However, if the probability of dying after one period of life differed across consumers, then Social Security would have an effect on behavior. In another paper (1984a), I assume that an individual's probability of an early death is private information known only by the individual, so that the private annuity market is subject to adverse selection. However, a compulsory Social Security system is immune to adverse selection and can offer a higher rate of return than the equilibrium rate of return in the private annuity market. Thus, consumers cannot effectively undo the effects of Social Security by transacting in the private annuity market because private and social annuities are no longer perfect substitutes.

Eckstein, Eichenbaum, and Peled (1985b) also use the insight that the Social Security system is immune to adverse selection. They assume that consumers have no bequest motive so that with a private annuity market, there are no bequests, accidental or otherwise. However, in my 1984a paper, I relax assumption 4 and specify the utility function to have a bequest motive. I then show that the introduction of Social Security can either increase or decrease the steady-state capital stock depending on the strength of the bequest motive. In my 1984b paper, I assume that the probability of an early death differs across consumers, but these probabilities are public information. In this case, the intro-
duction of actuarially fair Social Security will have an effect if the government chooses not to discriminate on the basis of the probability of dying.

Although I have made some progress in incorporating a bequest motive into an overlapping generations model with uncertain lifetimes, further research is needed. My other papers (1984a,b) used the Hakansson, Fischer, and Scott Richard (1975) utility function which specifies a consumer's utility as a function of his own consumption and of the size of the bequest he leaves. An alternative formulation is based on Barro's intergenerational altruism in which a consumer derives utility from his own consumption and from the utility of his heirs. This formulation effectively converts the individual consumer's decision problem into an infinite horizon problem. In future research I plan to study the role for fiscal policy in an overlapping generations economy populated by consumers with uncertain lifetimes and altruistic bequest motives.

APPENDIX

Here I derive equation (31) in the text which shows the effect on $C_{1,m}^*$ of the introduction in period $t^*+1$ of actuarially fair Social Security.

It will be useful to define $x$ as

(A1) \[ x = (1-a)(R/G). \]

Under actuarially fair Social Security, $RT = (1-p)S$ so that

(A2) \[ \frac{R}{G}(T+apR^{-1}S) = \frac{R}{G}(1-p+ap)R^{-1}S = \left(\frac{R}{G} - px\right)R^{-1}S. \]

Substituting (28)-(30) into (27) and using (A1) and (A2), we obtain

(A3) \[ \Delta c_{1,m}^{(j)} = aR^{-1}S\left\{ p - \left(\frac{R}{G} - px\right) \sum_{i=0}^{j^*-1} x^i \right\}. \]

As a step toward calculating the average value of each side of (A3), I first calculate

(A4) \[ \sum_{j=0}^{\infty} (1-p)p^j \sum_{i=0}^{j^*-1} x^i = (1-p) \sum_{j=0}^{\infty} \frac{p}{1-x} \frac{1-x^{j^*}}{1-x}. \]

Recalling that $j^* = \min(j, m-1)$, (A4) can be rearranged to yield

(A5) \[ \sum_{j=0}^{\infty} (1-p)p^j \sum_{i=0}^{j^*-1} x^i = \frac{1-p}{1-x} \left[ \frac{1}{1-p} \sum_{j=0}^{m-1} p^j x^j - \sum_{j=m}^{\infty} p^j x^{j-1} \right]. \]

Calculating the sums on the right-hand side of (A5) yields

(A6) \[ \sum_{j=0}^{\infty} (1-p)p^j \sum_{i=0}^{j^*-1} x^i = \frac{1-p}{1-x} \left[ \frac{1-p^m x^{m-1}}{1-p} - \frac{1-p^m x^m}{1-p} \right], \]

which can be simplified to yield

(A7) \[ \sum_{j=0}^{\infty} (1-p)p^j \sum_{i=0}^{j^*-1} x^i = \frac{1-p}{1-x} \left[ \frac{1-p^m x^{m-1}}{1-p} - \frac{1-p^m x^m}{1-p} \right], \]

Now calculate the average value of each side of (A3) and use (A7) to obtain

(A8) \[ \Delta C_{1,m}^* = apR^{-1}S \left\{ 1 - \frac{(R/G) - px}{1-px} \right\} \times \left(1 - p^{m-1} x^{m-1}\right). \]

Rearranging (A8) yields

(A9) \[ \Delta C_{1,m}^* = \frac{apR^{-1}S}{1-px} \left\{ 1 - \frac{(R/G) - px}{1-px} \right\} + \left(\frac{R}{G} - px\right) p^{m-1} x^{m-1}. \]
Recognizing that
\[(A10) \quad \left[ (R/G) - px \right] p^{m-1} x^{m-1} \]
\[= \left[ 1 - p(1 - a) \right] p^{m-1}(1 - a)^{m-1} (R/G)^m \]
then yields equation (31) in the text.

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