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Abstract
This paper extends the theory of investment under uncertainty to incorporate fixed costs of investment, a wedge between the purchase price and sale price of capital, and potential irreversibility of investment. In this extended framework, investment is a non-decreasing function of q, the shadow price of installed capital. There are potentially three investment regimes, which depend on the value of q relative to two critical values. For values of q above the upper critical value, investment is positive and is an increasing function of q, as is standard in the theory branch of the adjustment cost literature. For intermediate values of q, between two critical values, investment is zero. Although this regime features prominently in the irreversibility literature, it is largely ignored in the adjustment cost literature. Finally, if q is below the lower critical value, gross investment is negative, a possibility that is ruled out by assumption in the irreversibility of literature. In general, however, the shadow price q is not directly observable, so we present two examples relating q to observable varieties.

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A Unified Model of Investment Under Uncertainty

By Andrew B. Abel and Janice C. Eberly *

This paper extends the theory of investment under uncertainty to incorporate fixed costs of investment, a wedge between the purchase price and sale price of capital, and potential irreversibility of investment. In this extended framework, investment is a nondecreasing function of q, the shadow price of installed capital. The optimal rate of investment is in one of three regimes (positive, zero, or negative gross investment), depending on the value of q relative to two critical values. In general however, the shadow price q is not directly observable, so we present two examples relating q to observable variables. (*JEL* E22)

If a firm can instantaneously and costlessly adjust its capital stock, then, as shown by Dale W. Jorgenson (1963), its decision about how much capital to use is essentially a static decision in which the marginal product of capital is equated to the user cost of capital. The firm's investment decision becomes an interesting dynamic problem, in which anticipations about the future economic environment affect current investment, when frictions prevent instantaneous and costless adjustment of the capital stock. The investment literature of the last three decades has focused on two types of frictions: adjustment costs and irreversibility.

In this paper, we present a simple, more general framework that encompasses irreversibility as well as adjustment costs that may include a fixed component. Within this more general framework, the optimal investment behavior of the firm potentially comprises three regimes: (i) a regime of positive gross investment; (ii) a regime of zero gross investment; and (iii) a regime of negative gross investment. Most of the adjustment-cost literature tends to focus, either implicitly or explicitly, on the first of these regimes. The irreversibility literature is more explicit in its recognition of regimes of positive gross investment and zero gross investment, and it rules out the regime of negative gross investment by assumption. The more general model presented here allows a simple characterization of the conditions giving rise to each of these regimes.

In the adjustment-cost literature, based on the seminal work of Robert Eisner and Robert H. Strotz (1963), the adjustment-cost function is typically assumed to be strictly convex and to have a value of zero at zero investment. Although a few studies mention the possibility of fixed costs (see Michael Rothschild, 1971; Stephen J. Nickell, 1978), there is virtually no formal analysis of these fixed costs. The model presented in this paper incorporates fixed costs.

During the 1970's and 1980's, the adjustment-cost literature began to merge with the literature on Tobin's q. James Tobin (1969) argued that the optimal rate of investment is an increasing function of the ratio of the market value of the firm to the replacement cost of the firm's capital—a ratio that he called q and that has come to be known as "average q." Michael Mussa (1977) showed in a deterministic model, and

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Abel (1983) showed in a stochastic model, that the optimal rate of investment is the rate that equates the marginal adjustment cost with the marginal value of installed capital, a concept known as "marginal q." While average q is a potentially observable number, it is marginal q that is relevant for investment decisions. Fumio Hayashi (1982) presented conditions under which average q and marginal q are equal.

As indicated earlier, the assumption that investment is irreversible is another type of friction that makes the investment decision an interesting dynamic problem. In a seminal paper on irreversibility, Kenneth J. Arrow (1968 pp. 8–9) argued that "there will be many situations in which the sale of capital goods cannot be accomplished at the same price as their purchase.... For simplicity, we will make the extreme assumption that resale of capital goods is impossible, so that gross investment is constrained to be non-negative." Arrow showed that, in a deterministic model, optimal investment behavior under irreversibility will be characterized by alternating intervals of time corresponding to regimes of positive gross investment and regimes of zero gross investment. When the shadow price of capital is smaller than the cost of new capital, the firm will have zero investment; when the firm undertakes positive gross investment, the shadow price of capital equals the cost of new capital.[1]

We incorporate both adjustment costs and irreversibility in an extended model of adjustment costs. We note that adjustment costs and irreversibility are examined together in a deterministic model by Robert E. Lucas, Jr. (1981) and in a stochastic model by Lucas and Edward C. Prescott (1971). Curiously, both of these papers introduce the constraint that gross investment is nonnegative in the formal optimization problem, yet neither paper comments on this assumption, nor does either paper use the term "irreversibility." In effect, these papers take as a postulate that gross investment cannot be negative. In contrast, our model incorporates Arrow's observation that the resale price of capital may be below the price of new capital, and the model includes the special case in which the resale price is zero.

If we were simply to postulate that gross investment cannot be negative, then it would be easy to impose irreversibility in an adjustment-cost framework by simply assuming that infinite adjustment costs are incurred at any negative rate of investment, as in Caballero (1991 p. 281). Our approach avoids treating irreversibility as a postulate but rather allows for (and characterizes) cases in which the optimal rate of investment by the firm is never negative. We introduce an augmented adjustment-cost function that includes traditional convex adjustment costs, as well as the possibility of fixed costs and the possibility that the resale price of capital goods is below their purchase price and may even be zero. In this augmented adjustment-cost framework, investment is a nondecreasing function of the shadow price q, which is always positive. There are three regimes of optimal investment behavior characterized by two critical values of q, $q_1 \leq q_2$. Optimal gross investment is positive for $q > q_2$, zero for values of q between q_1 and q_2, and negative for $q < q_1$. If the lower critical value, q_1, is negative, then negative gross investment is never optimal, and investment would appear to be irreversible to an outside observer. It is worth noting that irreversibility does *not* require infinite adjustment costs at negative rates of gross investment, as assumed by Caballero (1991); indeed, as long as the augmented adjustment cost is strictly positive for all negative rates of gross investment, optimal investment behavior will appear to be irreversible.

[1]The same relationship among gross investment, the shadow price of capital, and the price of new capital was derived in a stochastic general-equilibrium model by Thomas J. Sargent (1980). Similarly, Giuseppe Bertola and Ricardo Caballero (1994) examine the behavior of an individual firm under uncertainty and find that the firm equates the marginal product of capital and the user cost of capital whenever it is undertaking gross investment; when the firm is not investing, the marginal product of capital is below the user cost.

In Section I we introduce the augmented adjustment-cost function and relate optimal investment to the shadow price q. Section II relates the shadow price q to observable variables in the context of two examples that restrict attention to the investment behavior of a competitive firm. Section III discusses the implications of competitive equilibrium for our analysis of these examples. Section IV summarizes and outlines future work.

I. The Model of the Firm

A. *The Operating-Profit and Augmented Adjustment-Cost Functions*

Consider a firm that uses capital and a vector of costlessly adjustable inputs, such as labor, to produce a nonstorable output. At each point of time, the firm chooses the amounts of costlessly adjustable inputs to maximize the value of its revenue minus expenditures on these inputs. Let $\pi(K_t, \varepsilon_t)$ denote the maximized value of this instantaneous operating profit at time t, where K_t is the capital stock at time t and ε_t is a random variable that could represent randomness in technology, in the prices of costlessly adjustable inputs, or in the demand facing the firm. Assume that $\pi_K(K_t, \varepsilon_t) > 0$, $\pi_{K,K}(K_t, \varepsilon_t) \leq 0$,[2] and that ε_t evolves according to a diffusion process:

$$d\varepsilon_t = \mu(\varepsilon_t)\,dt + \sigma(\varepsilon_t)\,dz \tag{1}$$

where z is a standard Wiener process.

Capital is acquired by undertaking gross investment at rate I, and the capital stock depreciates at a fixed proportional rate δ, so the capital stock evolves according to

$$dK_t = (I_t - \delta K_t)\,dt. \tag{2}$$

When the firm undertakes gross investment, it incurs costs that we can describe in terms of three components: (i) purchase or sale costs, (ii) costs of adjustment, and (iii) fixed costs per unit time.

(i) Purchase/sale costs are the costs of buying or selling uninstalled capital. Let p_K^+ be the price per unit at which the firm can buy any amount of uninstalled capital, and let p_K^- be the price per unit at which the firm can sell any amount of uninstalled capital. We assume that $p_K^+ \geq p_K^- \geq 0$. The sale price of capital may be strictly less than the purchase price of capital if, for example, capital is firm-specific.[3,4]

The purchase/sale cost function is $p_K^+ I$ for $I > 0$ and $p_K^- I$ for $I < 0$. It is a (weakly) convex and nondecreasing function that takes the value zero when gross investment is zero. Note that the purchase/sale cost function is twice differentiable everywhere except possibly at $I = 0$.

(ii) Adjustment costs are nonnegative costs that attain their minimum value of zero when $I = 0$. As is typical in the adjustment-cost literature, we assume that adjustment costs are continuous and strictly convex in I.[5,6]

In some formulations, adjustment costs also depend on the capital stock K, with the

[2] This formulation of the profit function allows the firm to be either a price-taker or a price-setter. In the examples in Section II, the firm is assumed to be a price taker.

[3] Alternatively, the sale of capital may be less than its purchase price if there is adverse selection in the market for used capital goods. The adverse-selection framework, however, implies heterogeneity in acquisition and sales prices across firms.

[4] In addition to Arrow (1968) cited in the Introduction, Nickell (1978 p. 40), Bertola and Caballero (1991 p. 1), and Robert S. Pindyck (1991 p. 1111) recognize that p_K^- may be lower than p_K^+ and choose to make the extreme assumption that $p_K^- = 0$. In the literature on consumer durables, Pok-sang Lam (1989), Sanford J. Grossman and Guy Laroque (1990), and Eberly (1994) include a proportional transaction cost when consumers resell durables, which corresponds to p_K^- being smaller than p_K^+.

[5] Notable exceptions are Alan S. Manne (1961) and Rothschild (1971), who analyze investment behavior under concave adjustment costs.

[6] In addition, the partial derivative of the adjustment-cost function with respect to investment goes to infinity as investment goes to infinity, and this partial derivative goes to negative infinity as investment goes to negative infinity.

partial derivative of the adjustment-cost function with respect to K being negative.[7] To accommodate this case as well as the case in which adjustment costs do not depend on K, we assume that the partial derivative of the adjustment-cost function with respect to K is nonpositive.

We assume that the adjustment-cost function is twice differentiable with respect to I everywhere except possibly at $I = 0$. The assumptions made so far imply that the partial derivative of the adjustment-cost function with respect to investment is positive for $I > 0$ and is negative for $I < 0$. If the adjustment-cost function is differentiable at $I = 0$, the partial derivative of the adjustment-cost function is zero at $I = 0$; more generally, the left-hand partial derivative is nonpositive and the right-hand derivative is nonnegative at $I = 0$.

(iii) Fixed costs of investment are nonnegative costs that are independent of the level of investment and are incurred at each point in time when investment is nonzero. Thus, a firm can avoid the fixed cost of investment at a particular point of time by setting investment equal to zero at that point of time.

We take account of all three of these types of costs associated with capital investment. The total cost of investment equals the product of a dummy variable ν and an "augmented adjustment-cost function" $c(I, K)$. The dummy variable ν takes the value 0 when $I = 0$ so that the total investment cost is zero when $I = 0$. When $I \neq 0$, the dummy variable ν equals 1 so that the total investment cost equals the augmented adjustment cost $c(I, K)$.

The augmented adjustment-cost function $c(I, K)$ represents the sum of purchase/sale costs, adjustment costs, and fixed costs. We assume that $\lim_{I \downarrow 0} c(I, K) = \lim_{I \uparrow 0} c(I, K)$ and denote the common value of these limits as $c(0, K)$. Note that $c(0, K)$ is *not* the total investment cost when $I = 0$, because when $I = 0$ the dummy variable ν equals 0 and total investment cost equals zero. Instead, $c(0, K)$ is interpreted as the fixed cost of investment because both the purchase/sale cost function and the adjustment-cost function are continuous functions that take on the value 0 when $I = 0$. Because the fixed cost is nonnegative, we have $c(0, K) \geq 0$. The augmented adjustment-cost function is continuous, strictly convex, and twice differentiable with respect to I everywhere except possibly at $I = 0$.[8] Let $c_I(0, K)^-$ and $c_I(0, K)^+$ denote the left-hand and right-hand partial derivatives, respectively, of $c(I, K)$ with respect to I evaluated at $I = 0$. It follows from the assumptions made above that $c_I(0, K)^+ \geq 0$, but $c_I(0, K)^-$ may be positive, negative, or zero. In addition, $c_I(0, K)^+ \geq c_I(0, K)^-$.

B. *Maximization: The Optimal-Investment Function*

Assume that the firm is risk-neutral and chooses investment to maximize the expected present value of operating profit $\pi(K, \varepsilon)$ less total investment cost $\nu c(I, K)$. The value of the firm is thus

$$(3)\quad V(K_t, \varepsilon_t) = \max_{I_{t+s}, \nu_{t+s}} \int_0^\infty E_t\{\pi(K_{t+s}, \varepsilon_{t+s}) - \nu_{t+s} c(I_{t+s}, K_{t+s})\} e^{-rs}\, ds$$

where $r > 0$ is the discount rate, and the maximization in (3) is subject to the evolution of ε_t and K_t described in (1) and (2), respectively.[9]

We will solve the maximization problem in (3) using the Bellman equation[10] (where

[7]For instance, Lucas (1967, 1981), Arthur B. Treadway (1969), Lucas and Prescott (1971), Hayashi (1982), and Abel and Olivier J. Blanchard (1983) all model adjustment costs as a decreasing function of K for a given I.

[8]The properties noted in footnote 6 imply that $\lim_{I \to \infty} c_I(I, K) = \infty$ and $\lim_{I \to -\infty} c_I(I, K) = -\infty$.

[9]While standard, this expression rules out bubbles in the value of the firm.

[10]A formal derivation of this Bellman equation is presented in Appendix A of Abel and Eberly (1993).

we have suppressed the time subscript t):

$$(4) \quad rV(K,\varepsilon) = \max_{I,\nu}\left\{\pi(K,\varepsilon) - \nu c(I,K) + \left(\frac{1}{dt}\right)E(dV)\right\}.$$

The left-hand side of equation (4) is the required return on the firm, and the right-hand side of (4) is the maximized expected return, which consists of two components: operating profits net of augmented adjustment costs, $\pi(K,\varepsilon) - \nu c(I,K)$; and the expected "capital gain" represented by the change in the value of the firm, $(1/dt)E(dV)$. To calculate the expected capital gain, we observe that the value of the firm, V, depends on K and ε, which evolve continuously over time according to (2) and (1), respectively. Thus, we can calculate $E(dV)$ using Ito's lemma, equations (1) and (2), and the facts that $(dK)^2 = (dK)(d\varepsilon) = (dt)^2 = (dz)(dt) = 0 = E(dz)$ to obtain

$$(5) \quad E(dV) = \left[V_K(I - \delta K) + \mu(\varepsilon)V_\varepsilon + \tfrac{1}{2}\sigma(\varepsilon)^2 V_{\varepsilon,\varepsilon}\right] dt.$$

Now define $q \equiv V_K$, which is the marginal valuation of a unit of installed capital. Substituting this definition and the expected capital gain from equation (5) into equation (4) yields

$$(6) \quad rV = \max_{I,\nu}\left\{\pi(K,\varepsilon) - \nu c(I,K) + q(I - \delta K) + \mu(\varepsilon)V_\varepsilon + \tfrac{1}{2}\sigma(\varepsilon)^2 V_{\varepsilon,\varepsilon}\right\}.$$

To solve the maximization problem on the right-hand side of (6), notice that the only terms that involve the decision variables I and ν are $-\nu c(I,K)$ and qI. Therefore, the optimal values of I and ν solve

$$(7) \qquad \max_{I,\nu}\left[qI - \nu c(I,K)\right].$$

It is convenient to solve the maximization problem in (7) in two steps. First, assume that $\nu = 1$, and choose the value of I that maximizes the maximand in (7) conditional on $\nu = 1$. Then choose ν to be either 0 or 1.

For the moment, assume that $\nu = 1$ and let $\psi(q,K)$ denote the maximized value of the maximand in (7) given that $\nu = 1$. Specifically,

$$(8) \quad \psi(q,K) \equiv \max_{I}\left[qI - c(I,K)\right].$$

Let $I^*(q,K)$ denote the value of I that maximizes the maximand in equation (8). Given that $c(I,K)$ is strictly convex in I and is differentiable everywhere except possibly at $I = 0$, the first-order conditions determining $I^*(q,K)$ are:

$$(9a) \quad c_I(I^*(q,K),K) = q \quad \text{for } q < c_I(0,K)^- \text{ or } q > c_I(0,K)^+$$

$$(9b) \quad I^*(q,K) = 0 \quad \text{for } c_I(0,K)^- \le q \le c_I(0,K)^+.$$

According to equation (9a) the firm equates the marginal cost of investment and the marginal benefit of investment, measured by q. Notice that $c_{I,I} > 0$ implies that $I^*(q,K)$ is a strictly increasing function of q over the range of q in equation (9a).

If $c(I,K)$ is differentiable at $I = 0$, then $c_I(0,K)^- = c_I(0,K)^+$ and $c_I(I^*(q,K),K) = q$ for all q. However, if $c(I,K)$ is not differentiable at $I = 0$, then for values of q between $c_I(0,K)^-$ and $c_I(0,K)^+$ there is no corresponding value of the marginal cost of investment. As shown in equation (9b) for values of q in this range, $I^*(q,K) = 0$. Looking at equations (9a) and, (9b) together, we see that $I^*(q,K)$ is a nondecreasing function over the entire range of q, and that

$$(10) \quad I^*(q,K)\begin{cases} < 0 & \text{for } q < c_I(0,K)^- \\ = 0 & \text{for } c_I(0,K)^- \le q \le c_I(0,K)^+ \\ > 0 & \text{for } q > c_I(0,K)^+. \end{cases}$$

Having determined the optimal value of I given that $\nu = 1$, we now turn to the choice of the optimal value of ν. If $\nu = 0$, gross investment is also zero, and the value of the maximand in equation (7) is zero. If $\nu = 1$, the optimal rate of investment is $I^*(q,K)$ and the value of the maximand in (8) is

$$(11)\quad \psi(q,K) = qI^*(q,K) - c(I^*(q,K),K).$$

The firm will therefore choose $\nu = 1$ when, and only when, $\psi(q,K)$ is greater than zero.[11] To determine the sign of $\psi(q,K)$, we now characterize the behavior of this function. Recall from equation (9b) that for $c_I(0,K)^- \le q \le c_I(0,K)^+$, $I^*(q,K) = 0$. Substituting zero investment into the right-hand side of (11) yields

$$(12)\quad \psi(q,K) = -c(0,K) \quad \text{if } c_I(0,K)^- \le q \le c_I(0,K)^+.$$

For values of q outside the interval $[c_I(0,K)^-, c_I(0,K)^+]$, $\psi(q,K) \ge -c(0,K)$ because the firm could always choose to set $I = 0$ and thereby attain a value of $-c(0,K)$ for $qI - c(I,K)$. Thus, the minimum value of $\psi(q,K)$ is attained for q in the interval $[c_I(0,K)^-, c_I(0,K)^+]$. Outside this interval, $\psi(q,K)$ is twice differentiable with respect to q. Differentiating equation (11) with respect to q and using equations (9a) and (10) yields

$$(13)\quad \psi_q(q,K) = I^*(q,K)\begin{cases} <0 & \text{if } q < c_I(0,K)^- \\ =0 & \text{if } c_I(0,K)^- \le q \le c_I(0,K)^+ \\ >0 & \text{if } q > c_I(0,K)^+ \end{cases}$$

$$(14)\quad \psi_{q,q}(q,K) = I_q^*(q,K) > 0 \quad \text{if } q < c_I(0,K)^- \text{ or if } q > c_I(0,K)^+.$$

Thus, the function $\psi(q,K)$ is a convex function that attains its minimum value of $-c(0,K)$ when q is in the interval $[c_I(0,K)^-, c_I(0,K)^+]$. Let q_1 and q_2 denote the smallest and largest roots, respectively, of $\psi(q,K) = 0$. It follows from equation (13) that

$$(15)\quad \psi(q,K) > 0 \qquad \text{if } q < q_1 \text{ or } q > q_2.$$

The function $\psi(q,K)$ is depicted in Figure 1 for a given value of K. The flat segment of $\psi(q,K)$ for values of q between $c_I(0,K)^-$ and $c_I(0,K)^+$ corresponds to equation (12). Figure 1 is drawn under the assumption that the fixed cost, $c(0,K)$ is positive, so that the minimum value of $\psi(q,K)$ is negative, and the flat segment lies below the horizontal axis. According to equation (13), $\psi(q,K)$ is strictly decreasing to the left of the flat segment and strictly increasing to the right of the flat segment. Thus, in the case depicted in Figure 1, the

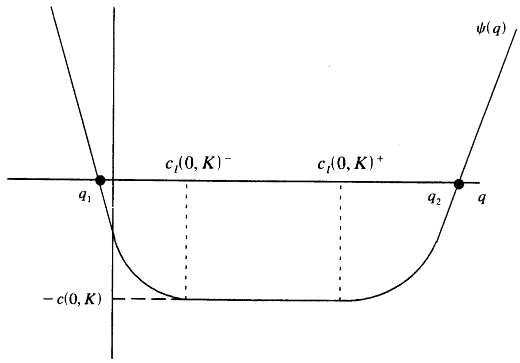

FIGURE 1. THE REWARD TO INVESTING

[11] When $\psi(q,K) = 0$, the firm is indifferent between $I = 0$ and $I = I^*(q,K)$. Of course, if $I^*(q,K) = 0$, the optimal rate of investment is zero. If $I^*(q,K) \ne 0$, we assume that the firm chooses to set investment equal to zero at these points of indifference. The time path of K is unaffected by this assumption because $q \equiv V_K(K,\varepsilon)$ follows a diffusion process, which implies that the set of times when $\psi(q,K) = 0$ and $I^*(q,K) \ne 0$ has zero measure.

equation $\psi(q,K)=0$ has two distinct roots, q_1 and q_2; $\psi(q,K)>0$ if $q<q_1$ or if $q>q_2$. Thus, optimal investment behavior $\hat{I}(q,K)$ is characterized by

$$\hat{I}(q,K)=\begin{cases} I^*(q,K)<0 & \text{if } q<q_1 \\ 0 & \text{if } q_1\le q\le q_2 \\ I^*(q,K)>0 & \text{if } q>q_2. \end{cases} \tag{16}$$

C. *Characteristics of Optimal Investment and q*

(i) As shown in equation (16), the optimal rate of investment is zero when q is in the interval $[q_1,q_2]$. We will show that $q_2>q_1$, so that this range of inaction for investment is nondegenerate, if there are fixed costs of investment $[c(0,K)>0]$ or if the augmented adjustment-cost function is nondifferentiable with respect to I at $I=0$.

In general $\psi(q,K)=0$ may have either a unique root or more than one root. If there is more than one root, there are either two roots or a continuum of roots. We describe each of these three cases below.

Case I: $\psi(q,K)=0$ has a unique root so that $q_1=q_2$ and the range of inaction is degenerate. The equation $\psi(q,K)=0$ has a unique root if (a) $c(I,K)$ is differentiable at $I=0$ so that $c_I(0,K)^-=c_I(0,K)^+$, and hence there is no flat segment at the bottom of the $\psi(q,K)$ function; and (b) the fixed cost $c(0,K)=0$ so that the minimum value of $\psi(q,K)$ is zero. These two assumptions are fairly standard in the adjustment-cost literature, and they account for the absence of a range of inaction in much of this literature (see e.g., John P. Gould, 1968; Richard Hartman, 1972; Abel and Blanchard, 1983).

Case II: $\psi(q,K)=0$ has exactly two roots so that there is a nondegenerate range of inaction. The equation $\psi(q,K)=0$ has exactly two distinct roots if the fixed cost $c(0,K)>0$ so that the minimum value of $\psi(q,K)$ is negative. With a positive fixed cost of investment, a nondegenerate range of inaction will arise regardless of the differentiability of $c(I,K)$ at $I=0$.

Case III: $\psi(q,K)=0$ has a continuum of roots so that there is a nondegenerate range of inaction. If (a) the augmented adjustment-cost function $c(I,K)$ is not differentiable at $I=0$ so that $c_I(0,K)^-<c_I(0,K)^+$ [implying that there is a flat segment at the bottom of $\psi(q,K)$] and (b) the fixed cost $c(0,K)=0$, so that the flat segment at the bottom of $\psi(q,K)$ lies along the horizontal axis, then $\psi(q,K)=0$ has a continuum of roots extending from q_1 to q_2. For any value of q in this range, the optimal rate of investment is zero.

(ii) The largest and smallest roots of the equation $\psi(q,K)=0$, q_1 and q_2, depend only on the specification of the augmented adjustment-cost function $c(I,K)$. They are independent of the specification of the operating-profit function $\pi(K,\varepsilon)$ and the specification of the diffusion process for ε_t.

(iii) If there are positive fixed costs or if $c(I,K)$ is not differentiable at $I=0$, there is a nondegenerate range of inaction. If there are positive fixed costs, the function $\hat{I}(q,K)$ is discontinuous; the optimal rate of investment jumps from a negative value to zero at $q=q_1$, and it jumps from zero to a positive value at $q=q_2$.

(iv) If $\min_I c(I,K)\ge 0$, it is never optimal for the firm to undertake negative gross investment; the firm's behavior is observationally equivalent to a situation of irreversible investment.[12] Note from the definition of $\psi(q,K)$ in equation (8) that $\psi(0,K) = \max_I -c(I,K) = -\min_I c(I,K)$ so that if $\min_I c(I,K)\ge 0$, then $\psi(0,K)\le 0$. But if $\psi(0,K)\le 0$, then q_1, the smallest root of $\psi(q,K)=0$, is nonpositive. Therefore it is impossible for q, which must be positive [see equation (20)], to be less than q_1, and

[12]Caballero (1991 p. 281) specifies the augmented adjustment-cost function $C(I)=I+[I>0]\gamma_1 I^\beta+[I<0]\gamma_2|I|^\beta$ where $\beta\ge 1$, $\gamma_1\ge 0, \gamma_2\ge 0$, and the brackets denote the indicator function. Caballero states that "the irreversible-investment case of Pindyck (1988) and Bertola (1988) corresponds to the case in which $\gamma_1=0$, $\gamma_2=\infty$, and $\beta=1$." In fact, however, if $\beta=1$, irreversibility will occur whenever $\gamma_2>1$. There is no need to make γ_2 infinite to prevent optimal investment from being negative.

the optimal rate of investment cannot be negative.[13] This result has a straightforward explanation. In order for a firm to find it optimal to give up some of its installed capital, which has a positive value, the augmented adjustment cost it incurs must be negative (i.e., the net sale price of the capital after taking account of the fixed cost and the adjustment cost must be positive). If there is no value of gross investment for which $c(I,K)$ is negative, then it will never be optimal for a firm to undertake negative gross investment.

(v) Note that the Bellman equation in equation (6) holds identically in K at a point in time so the partial derivative of the left-hand side with respect to K equals the partial derivative of the right-hand side with respect to K. Differentiating both sides of (6) with respect to K yields

$$(17)\quad rV_K = \pi_K(K,\varepsilon) - \hat{\nu} c_K(\hat{I},K) - \delta q + q_K(\hat{I} - \delta K) + \mu(\varepsilon)V_{\varepsilon,K} + \tfrac{1}{2}\sigma(\varepsilon)^2 V_{\varepsilon,\varepsilon,K}$$

where $\hat{I}$ is optimal investment from equation (16) and $\hat{\nu}$ is the optimal choice of ν. Recall that $q \equiv V_K$ so that $q_\varepsilon = V_{\varepsilon,K}$ and $q_{\varepsilon,\varepsilon} = V_{\varepsilon,\varepsilon,K}$. Now apply Ito's lemma and equations (1) and (2) to calculate $E\{dq\}$:

$$(18)\quad E\{dq\} = q_K(\hat{I} - \delta K)\,dt + \mu(\varepsilon)V_{\varepsilon,K}\,dt + \tfrac{1}{2}\sigma(\varepsilon)^2 V_{\varepsilon,\varepsilon,K}\,dt.$$

Substituting (18) into (17) and rearranging yields

$$(19)\quad (r+\delta)q = \pi_K(K,\varepsilon) - \hat{\nu} c_K(\hat{I},K) + \frac{E\{dq\}}{dt}.$$

Equation (19) is essentially an Euler equation from the calculus of variations. The left-hand side of equation (19) is the required return (gross return before subtracting depreciation) on the valuation of the marginal unit of capital, and the right-hand side is the expected return, which consists of three components: the marginal operating profit $\pi_K(K,\varepsilon)$, the marginal reduction in the augmented adjustment cost $-\hat{\nu} c_K(\hat{I},K)$, and the expected capital gain $E\{dq\}/dt$. In the special case in which there is no uncertainty, equation (19) becomes $(r+\delta)q = \pi_K(K,\varepsilon) - \hat{\nu} c_K(\hat{I},K) + dq/dt$, which is widely used in the deterministic literature on the q theory of investment (see e.g., Blanchard and Stanley Fishcher, 1989 p. 62).

(vi) The marginal valuation of installed capital, q, is the expected present value of the stream of marginal products of capital. This result can be shown formally using the following lemma, which is a special case of the Feynman-Kac formula (see Ioannis Karatzas and Steven E. Shreve, 1988 p. 267). A simple proof is given in appendix B of Abel and Eberly (1993).

LEMMA 1: *Suppose that χ_t is a diffusion and that $a > 0$ is constant. Then $\chi_t = E_t\{\int_0^\infty g_{t+s} e^{-as}\,ds\}$ is a solution to the differential equation $E_t(d\chi)/dt - a\chi_t + g_t = 0$.*

Using the fact that q_t is a diffusion and applying this lemma to equation (19) yields[14]

$$(20)\quad q_t = \int_0^\infty E_t\{\pi_K(K_{t+s},\varepsilon_{t+s}) - \nu_{t+s} c_K(I_{t+s},K_{t+s})\} e^{-(r+\delta)s}\,ds > 0.$$

[13]To show that q_2, the largest root of $\psi(q,K)=0$, is not negative, we suppose that $q_2 < 0$ and show that this assumption leads to a contradiction. If $q_2 < 0$, then according to equation (16), $\hat{I}(0,K) = I^*(0,K) > 0$. Recall that $c_I(0,K)^+ \geq 0$ so that equation (10) implies that $I^*(0,K) \leq 0$, which is a contradiction. Therefore $q_2 \geq 0$.

[14]We have chosen the solution to equation (19) that does not contain bubbles.

Thus, q_t is the present value of the stream of expected marginal profit of capital which consists of two components: $\pi_K(K,\varepsilon)$ is the marginal operating profit accruing to capital, and $-\nu c_K(I,K)$ is the reduction in the augmented adjustment cost accruing to the marginal unit of capital. The assumptions made above that $\pi_K(K,\varepsilon)>0$ and $c_K(I,K)\leq 0$ imply that q_t is always positive.

II. Relating the Shadow Price q to Observable Variables

We have shown that optimal investment is a nondecreasing function of the shadow price of capital, which is called q. In general, we cannot directly observe shadow prices. In this section we restrict our attention to perfectly competitive firms with linearly homogeneous production functions and derive expressions for q in terms of observable variables. In the first example, q equals the value of the firm divided by its capital stock (Tobin's q), and in the second example q is a function of the price of output, the real interest rate, and parameters describing the price of output and the production function.

Consider a competitive firm that uses capital, K, and a vector of costlessly adjustable inputs, $\mathbf{L}$, to produce output according to the production function $F(K,\mathbf{L},\varepsilon)$. Assume that the production function $F(K,\mathbf{L},\varepsilon)$ is linearly homogeneous in K and $\mathbf{L}$, and note that the production function may be subject to stochastic shocks. In addition, the competitive prices of output and inputs may be subject to stochastic shocks. It is well known that if the firm is a price-taker in output and factor markets, the operating profit can be written as

$$\pi(K,\varepsilon)=H(\varepsilon)K \tag{21}$$

where $H(\varepsilon)>0$.[15]

Case I: $c(I,K)$ is linearly homogenous in I and K.—We can show that if the operating profit function satisfies equation (21), and if $c(I,K)$ is linearly homogeneous in I and K, then

$$V(K,\varepsilon)=q(\varepsilon)K. \tag{22}$$

In this case, the shadow price of capital, $q(\varepsilon)$, equals the average value of capital, $V(K,\varepsilon)/K$, which is observable using security market prices and is known as Tobin's q. This result extends Hayashi's (1982) result, which was derived in a deterministic model, to a stochastic model that admits irreversibility.

The value function in equation (22) is implied by the following lemma, proved in Appendix A.

LEMMA 2: *Suppose that $\pi(K,\varepsilon)$ and $c(I,K)$ are homogeneous of degree ρ in I and K. Then the value function can be written as $V(K,\varepsilon)=\Lambda(\varepsilon)K^\rho$, and $q\equiv V_K(K,\varepsilon)=\rho\, V(K,\varepsilon)/K$.*

Thus, when the operating profit function and the augmented adjustment-cost function are of the same degree of homogeneity, marginal q and average q are proportional. In the special case where $\rho=1$ [so that equation (21) holds, and the augmented adjustment-cost function is linearly homogeneous], Lemma 2 indicates that average and marginal q are equal, as in equation (22).

We now discuss the content of the assumption that $c(I,K)$ is linearly homogeneous. Recall that $c(I,K)$ has three components: (i) a purchase/sale cost; (ii) an adjustment cost; and (iii) a fixed cost.

(i) As we discussed in Section I, the purchase/sale cost is $p_K^+ I$ for $I>0$ and $p_K^- I$ for $I<0$. Obviously, a doubling of

[15]The operating profit in this case can be written as $\pi(K,\varepsilon)=\max_{\mathbf{L}}[p(\varepsilon,Q)F(K,\mathbf{L},\varepsilon)-\mathbf{w}(\varepsilon)'\mathbf{L}]$, where $p(\varepsilon,Q)$ is the given price of the firm's output, $\mathbf{w}(\varepsilon)$ is the vector of given prices of the costlessly adjustable inputs, and Q is industry output. Note that all of the prices may be potentially random. Let $\boldsymbol{\lambda}\equiv\mathbf{L}/K$ be the vector of ratios of the costlessly adjustable inputs to the capital stock. It follows from the linear homogeneity of $F(K,\mathbf{L},\varepsilon)$ that $\pi(K,\varepsilon)=\max_{\boldsymbol{\lambda}}[p(\varepsilon,Q)F(1,\boldsymbol{\lambda},\varepsilon)-\mathbf{w}(\varepsilon)'\boldsymbol{\lambda}]K$. The maximand in square brackets is independent of the individual firm's capital stock, K, and thus the operating-profit function can be written as in equation (21), where $H(\varepsilon)=\max_{\boldsymbol{\lambda}}[p(\varepsilon,Q)F(1,\boldsymbol{\lambda},\varepsilon)-\mathbf{w}(\varepsilon)'\boldsymbol{\lambda}]$.

I and K doubles the purchase/sale cost, so the purchase/sale cost function is a linearly homogeneous function of I and K.

(ii) In the literature in which the adjustment-cost function depends on K as well as on I, it is commonly assumed that the adjustment-cost function is linearly homogeneous in I and K.[16]

(iii) The fixed cost of investment, $c(0,K)$, is independent of the amount of investment I. If this fixed cost reflects the cost of stopping production while new capital is installed,[17] it is proportional to the operating profit function $H(\varepsilon)K$ which is, of course, proportional to K. In this case, the fixed cost, $c(0,K)$, is a linearly homogeneous function of I and K (even though it is independent of I).

If the purchase/sale cost, the adjustment cost, and fixed cost are all linearly homogeneous functions of I and K, then $c(I,K)$ is linearly homogeneous in I and K and can be written as

$$(23)\quad c(I,K) \equiv Kc\left(\frac{I}{K},1\right) \equiv KG\left(\frac{I}{K}\right)$$

where $G(\cdot)$ is continuous and convex, and except possibly at zero, is twice differentiable. In this case, $c_I(I,K) = G'(I/K)$, so that equations (16) and (9a) yield

$$(24)\quad \frac{I}{K} = \begin{cases} G'^{-1}(q) < 0 & \text{if } q < q_1 \\ 0 & \text{if } q_1 \le q \le q_2 \\ G'^{-1}(q) > 0 & \text{if } q > q_2. \end{cases}$$

Notice that the optimal investment–capital ratio depends only on q, and since q is independent of the capital stock, the optimal investment–capital ratio is independent of the scale of the firm.[18] If $q_1 < 0$, then the negative investment regime is never operative, and as explained in Section I, investment would appear to be irreversible.

Case II: $c_K(I,K) \equiv 0$.—Now assume that the augmented adjustment-cost function does not depend on the capital stock (formally, $c_K(I,K) \equiv 0$).[19] We continue to assume that the firm is perfectly competitive and has a linearly homogeneous production function so that the operating profit function is proportional to the capital stock [equation (21)]. Under these assumptions, we show in Appendix B that the value function is a linear function of the capital stock regardless of the specification of the diffusion for ε. In particular,

$$(25)\qquad V(K,\varepsilon) = q(\varepsilon)K + J(\varepsilon)$$

where $J(\varepsilon) > 0$. To get an explicit expression for $q(\varepsilon)$ in terms of the underlying stochastic process, we will focus on particular parametric specifications of the operating profit function and the diffusion for ε. It is not necessary to restrict $c(I,K)$ further.

Consider a competitive firm that uses capital and labor to produce output according to the Cobb-Douglas production function $vL^{\alpha}K^{1-\alpha}$, where $0 < \alpha < 1$, and $v > 0$ is a productivity parameter that may be stochastic. The firm pays a constant wage rate w per unit of labor and sells its output at a price P that may be stochastic. Define $p \equiv Pv$ and observe that the instantaneous operating profit equals the revenue from selling output minus the cost of labor so that

$$(26)\quad \pi(K,p) \equiv \max_L\left[pL^{\alpha}K^{1-\alpha} - wL\right] = hp^{\theta}K$$

where $h \equiv (1-\alpha)\alpha^{\alpha/(1-\alpha)}w^{-\alpha/(1-\alpha)} > 0$ and $\theta \equiv 1/(1-\alpha) > 1$.

[16] Lucas (1967, 1981), Lucas and Prescott (1971), Hayashi (1982) and Abel and Blanchard (1983) all make this assumption.

[17] Rothschild (1971 p. 609) and Nickell (1978 p. 37) both suggest that the cost of stopping production would rise to a fixed cost of investment. In addition Rothschild suggests that breaking in new equipment or procedures is costly.

[18] Lucas (1967) highlights this feature in a deterministic model with a linearly homogeneous operating profit function and convex costs of adjustment.

[19] This assumption is adopted by Eisner and Strotz (1963), Gould (1968), Rothschild (1971), Richard Hartman (1972), Mussa (1977), Nickell (1978), Pindyck (1982), Abel (1983), and Cabellero (1991).

At time t, the present value of marginal profits accruing to the undepreciated portion of currently installed capital is[20]

$$(27) \qquad q_t = h\int_0^\infty E_t\{p_{t+s}^\theta\}e^{-(r+\delta)s}\,ds.$$

We calculate the expectations in equation (27), and the value of q_t, under the assumption that p evolves according to the geometric Brownian motion

$$(28) \qquad \frac{dp}{p} = \sigma\,dz$$

where z follows a standard Wiener process. In this case, the distribution of $\ln p_{t+s}$ conditional on p_t is $N(\ln p_t - \frac{1}{2}\sigma^2 s, \sigma^2 s)$ so that

$$(29) \qquad E_t\{p_{t+s}^\theta\} = p_t^\theta \exp\left[\tfrac{1}{2}\theta(\theta-1)\sigma^2 s\right].$$

Substituting equation (29) into equation (27) and simplifying yields[21]

$$(30) \qquad q_t = \frac{hp_t^\theta}{\left[r+\delta-\frac{1}{2}\theta(\theta-1)\sigma^2\right]}.$$

Now suppose that $0 < q_1 < q_2$ so that all three investment regimes are potentially operative,[22] and consider the effects of an increase in the instantaneous standard deviation σ.[23] It follows directly from equation (30) that an increase in σ increases q_t for a given p_t. If the initial value of q_t is less than q_1 or greater than or equal to q_2, the increase in q_t increases investment, which is consistent with Hartman (1972), Abel (1983), and Caballero (1991). But note that if the initial value of q_t is in the interval $[q_1, q_2)$, a small increase in σ will not move q_t out of this interval, and investment will remain unchanged and equal to zero. Thus, with the more general adjustment-cost function introduced in this paper, we have the result that investment is a nondecreasing function of σ for a given p_t.

III. Competitive Equilibrium[24]

In a recent paper Pindyck (1993) questions the relevance of adjustment costs in competitive equilibrium for firms with constant returns to scale. Pindyck also points out that considerations of industry equilibrium may reverse the findings of Hartman (1972), Abel (1983), and Caballero (1991) concerning the effects of uncertainty on investment by competitive firms.

We first address Pindyck's argument that adjustment costs are irrelevant in a perfectly competitive industry in which firms have constant returns to scale. His argument applies to the case in which the adjustment-cost function depends only on the rate of investment, and not on the capital stock (see Pindyck, 1993 p. 274). Observe from equation (25) that, under constant returns to scale and this form of the adjustment-cost function, the value of a competitive firm with capital stock K^* is $q(\varepsilon)K^* + J(\varepsilon)$. If this firm could costlessly divide itself into two firms with capital stock $K^*/2$, each of the two firms would be worth $q(\varepsilon)K^*/2 + J(\varepsilon)$; the total value of the two firms would be $q(\varepsilon)K^* + 2J(\varepsilon)$, which is greater than the value of the original firm. Thus, provided that new firms can be freely created, firms would have an incentive to

[20]As in equation (20), we assume that there is no bubble in the shadow price q.

[21]We assume that $r+\delta-\frac{1}{2}\theta(\theta-1)\sigma^2 > 0$ so that the integral in equation (27) converges.

[22]Recall that $\min_I c(I,K) < 0$ is necessary and sufficient for $q_1 > 0$. Either $c(0,K) > 0$ or $c_I(0,K)^- < c_I(0,K)^+$ is sufficient for $q_2 > q_1$.

[23]When we consider the effects of a change in a parameter such as σ, we are actually comparing the behavior of two otherwise identical firms with different constant values of the parameter in question. This analysis does not apply to the effect on a given firm of a change in the parameter because the firm's optimization problem assumes that the parameters are known with certainty to be constant over time.

[24]We thank an anonymous referee for raising the issues that motivated us to write this section.

divide into smaller parts. In addition, if there is free entry, potential entrants would enter the industry because even with no capital a firm has a positive value $J(\varepsilon)$. Pindyck argues that "in the limit, the industry would be composed of an infinite number of infinitesimally small firms, and so each firm would have no adjustment costs" (p. 274) because they would each have infinitesimally small rates of investment.

Pindyck's conclusion that each firm would have no adjustment cost is based on the assumption that $\lim_{I \to 0} c^*(I, K) = 0$ and $\lim_{I \to 0} c_I^*(I, K) = 0$, where $c^*(I, K)$ is the adjustment-cost function, rather than the augmented adjustment-cost function. While this assumption is fairly standard in formulations of the adjustment-cost function in which c_K is identically zero, adjustment costs will not become irrelevant in our Case I, in which the augmented adjustment-cost function is linearly homogeneous in I and K. As we have shown, the value of the firm is strictly proportional to its capital stock in this case. Therefore, a firm with zero capital has zero value, so that even with free entry there are no rents to be earned by potential entrants with zero capital. In this case, the size distribution of firms is indeterminate. It is possible that some firms will have infinitesimally small capital stocks and rates of investment, but even for these firms adjustment costs are not irrelevant. Arbitrarily small firms will have arbitrarily small values of I and K, but the value of I/K will still be given by equation (24), which depends on the augmented adjustment-cost function.[25] Thus, Pindyck's (1993) argument about the irrelevance of adjustment costs under constant returns to scale and perfect competition does not apply when the augmented adjustment-cost function is linearly homogeneous in I and K, as in our Case I.

Pindyck's second argument is that, even if for some reason firms cannot be arbitrarily small, the response of existing firms and free entry will cause the equilibrium price to respond endogenously to shocks. Most studies of investment behavior by competitive firms under uncertainty ignore this endogenous response of equilibrium price. Although a competitive firm is a price-taker, a competitive industry is not a price-taker. Specifically, a shock that hits all firms in an industry is likely to affect industry output and thus the equilibrium price. However, a shock that hits only one competitive firm in an industry will not affect industry output or the equilibrium price.

Pindyck (1993) analyzes endogenous price responses to industry-wide shocks to reexamine the results of Hartman (1972), Abel (1983), and Caballero (1991), who find that increased uncertainty increases the investment of competitive firms with constant returns to scale. Pindyck shows that if all firms in an industry face identical realizations of the random variable(s) impacting the industry, then taking account of the endogenous response of the equilibrium price tends to reverse the findings of Hartman, Abel, and Caballero. However, it should be noted that if competitive firms face only idiosyncratic shocks, then the results of Hartman, Abel, and Caballero continue to hold. Our analysis in Case II would be subject to Pindyck's criticism if we interpret the uncertainty about $p \equiv Pv$ as arising from demand shocks that affect the competitive price of output P, which is identical for all firms in a competitive industry; however, our analysis in Case II is immune to Pindyck's criticism if the uncertainty arises from a productivity shock v that is idiosyncratic to a particular firm.[26]

[25]Recall from equation (23) that in this case the augmented adjustment-cost function can be written as $KG(I/K)$, where $G(\cdot)$ is continuous and convex. Although the augmented adjustment cost $KG(I/K)$ goes to zero as K goes to zero, the marginal augmented adjustment cost $G'(I/K)$, evaluated at optimal I, does not go to zero as K goes to zero.

[26]The issue of the endogenous response of equilibrium price to shocks does not arise in our analysis of Case I, because we need not specify the relationship between price and the source of uncertainty. Indeed, our analysis of Case I did not use any specification for the evolution of the price of output. Whatever the behavior of the price of output, and however it responds to shocks, competitive firms take the price of output as given.

IV. Conclusion

In this paper we have extended the adjustment-cost framework under uncertainty to incorporate fixed costs of investment, a wedge between the purchase price and sale price of capital, and potential irreversibility of investment. In this extended framework, investment is a nondecreasing function of q, the shadow price of installed capital, and there are potentially three investment regimes which depend on the value of q relative to the critical values q_1 and q_2. Conveniently, these critical values depend only on the specification of the augmented adjustment-cost function. If q is greater than q_2, then, as is standard in the q-theory branch of the adjustment-cost literature, investment is positive and is an increasing function of q. If q is between q_1 and q_2, then the investment is zero. Although this regime features prominently in the irreversibility literature, it is largely ignored in the adjustment-cost literature. Finally, if q is less than q_1, gross investment is negative, a possibility that is simply ruled out by assumption in the irreversibility literature.

The shadow price q is in general not observable, so we presented two examples relating q to observable variables. In one example, restrictions on the production function and the augmented adjustment-cost function guarantee that q is identically equal to the average value of the capital stock, which is observable using security prices. In the other example, we tightly specify the production function and the diffusion process for the random variable p (the product of the output price and a productivity parameter) and derive an expression for q as a function of the contemporaneous value of p. In this example, p does not have a stationary distribution, and hence q does not have a stationary distribution. In ongoing research we are examining the behavior of q and investment in the presence of a mean-reverting process for p so that q will have a stationary distribution. The ultimate goal of this line of research is to derive an econometric specification to apply these models to aggregate and disaggregate data on investment.

Appendix A

PROOF OF LEMMA 2:

The operating-profit function and the augmented adjustment-cost function are homogeneous of degree ρ in I and K so that

$$\text{(A1)} \qquad \pi(K,\varepsilon) = H(\varepsilon)K^{\rho}$$

and

$$\text{(A2)} \qquad c(I,K) = G\left(\frac{I}{K}\right)K^{\rho}.$$

Then the value function in equation (3) can be written as

$$\text{(A3)} \quad V(K_t,\varepsilon_t) = \max_{I_{t+s},\nu_{t+s}} \int_0^{\infty} E_t\{[H(\varepsilon_{t+s}) - \nu_{t+s}G(i_{t+s})]K_{t+s}^{\rho}\}e^{-rs}\,ds$$

where $i_{t+s} \equiv I_{t+s}/K_{t+s}$ is the (gross) growth rate of the capital stock. Consider a firm with capital stock $K_t^{(1)}$ at time t, and let $\nu_{t+s}^{(1)}$ and $i_{t+s}^{(1)}$ denote the optimal values of the dummy variable ν and the investment–capital ratio chosen by this firm at time $t+s$. This optimal behavior leads to a capital stock of $K_{t+s}^{(1)}$ at time t. The value of the firm at time t is $V(K_t^{(1)},\varepsilon_t)$. Now consider a second firm with a capital stock at time t equal to $K_t^{(2)} = \alpha K_t^{(1)}$ with $\alpha > 0$. This firm has the option of choosing exactly the same values of the dummy variable ν and the investment–capital ratio I/K at every point of time as chosen by the firm with capital stock $K_t^{(1)}$. If the second firm were to set $\nu_{t+s}^{(2)} = \nu_{t+s}^{(1)}$ and $i_{t+s}^{(2)} = i_{t+s}^{(1)}$ for all $s > 0$, then $K_{t+s}^{(2)}$ would equal $\alpha K_{t+s}^{(1)}$ for all $s > 0$. Because the cash flow at time $t+s$ is proportional to K_{t+s}^{ρ} in equation (A3), the second firm has the option of obtaining an expected present value of cash flows equals to $\alpha^{\rho}V(K_t^{(1)},\varepsilon_t)$. Therefore,

$$\text{(A4)} \quad V(\alpha K_t,\varepsilon_t) \geq \alpha^{\rho}V(K_t,\varepsilon_t).$$

Equation (A4) holds for any K_t and for any positive factor α. In particular, consider a first firm that has a capital stock of αK_t at time t, and a second firm that has a capital stock of $K_t = (1/\alpha)\alpha K_t$ at time t. Therefore, the argument preceding equations (A4) implies that

$$\text{(A5)} \quad V(K_t, \varepsilon_t) \geq (1/\alpha)^{\rho} V(\alpha K_t, \varepsilon_t).$$

Putting together equations (A4) and (A5) we have $V(\alpha K_t, \varepsilon_t) \geq \alpha^{\rho} V(K_t, \varepsilon_t) \geq V(\alpha K_t, \varepsilon_t)$, which implies

$$\text{(A6)} \quad V(\alpha K_t, \varepsilon_t) = \alpha^{\rho} V(K_t, \varepsilon_t).$$

Because equation (A6) holds for any positive K_t and any positive α, the value of the firm is proportional to the capital stock to the power ρ, and hence the value function can be written as

$$\text{(A7)} \qquad V(K_t, \varepsilon_t) \equiv \Lambda(\varepsilon_t) K_t^{\rho}.$$

Partially differentiating (A7) with respect to K_t yields

$$q_t \equiv V_K(K_t, \varepsilon_t) = \rho \frac{V(K_t, \varepsilon_t)}{K_t}.$$

Appendix B

The Value Function When the Augmented Adjustment-Cost Function Does Not Depend on the Capital Stock

The optimal program of the firm is governed by the differential equation given in the text equation (6). Here we assume that $c_K(I, K) \equiv 0$, so we write the augmented adjustment-cost function, $c(I, K)$ as simply $c(I)$.

$$\text{(B1)} \quad rV(K, \varepsilon) = \max_{I,\nu} \{\pi(K, \varepsilon) - \nu c(I) + q(I - \delta K) + \mu(\varepsilon) V_{\varepsilon} + \tfrac{1}{2}\sigma(\varepsilon)^2 V_{\varepsilon\varepsilon}\}.$$

Now suppose that the firm is a price-taker in output and factor markets and has a production function that is linearly homogeneous in I and K so that $\pi(K, \varepsilon) = H(\varepsilon)K$ (see footnote 15). We will verify that $V(K, \varepsilon) = q(\varepsilon)K + J(\varepsilon)$ satisfies (B1). Substituting $V(K, \varepsilon) = q(\varepsilon)K + J(\varepsilon)$ and $\pi(K, \varepsilon) = H(\varepsilon)K$ into (B1) and recalling the definition of q yields

$$\text{(B2)} \quad rq(\varepsilon)K + rJ(\varepsilon) = \max_{I,\nu} \{H(\varepsilon)K - \nu c(I) + q(\varepsilon)(I - \delta K) + \mu(\varepsilon) q_{\varepsilon} K + \mu(\varepsilon) J_{\varepsilon} + \tfrac{1}{2}\sigma(\varepsilon)^2 q_{\varepsilon\varepsilon} K + \tfrac{1}{2}\sigma(\varepsilon)^2 J_{\varepsilon\varepsilon}\}.$$

Collecting terms in K yields

$$\text{(B3)} \quad [(r+\delta)q(\varepsilon) - \mu(\varepsilon)q_{\varepsilon} - \tfrac{1}{2}\sigma(\varepsilon)^2 q_{\varepsilon\varepsilon} - H(\varepsilon)]K = \max_{I,\nu}[q(\varepsilon)I - \nu c(I)] - rJ(\varepsilon) + \mu(\varepsilon)J_{\varepsilon} + \tfrac{1}{2}\sigma(\varepsilon)^2 J_{\varepsilon\varepsilon}.$$

In order for (B3) to hold for all K, the term in square brackets on the left-hand side must equal zero, and the right-hand side of (B3) must also equal zero. Note that from equation (8) we can write

$$\text{(B4)} \quad \max_{I,\nu}[q(\varepsilon)I - \nu c(I)] = \max[0, \psi(q)].$$

Setting the right-hand side and the left-hand side of (B3) equal to zero yields

$$\text{(B5)} \quad \max[0, \psi(q(\varepsilon))] - rJ(\varepsilon) + \mu(\varepsilon)J_{\varepsilon} + \tfrac{1}{2}\sigma(\varepsilon)^2 J_{\varepsilon\varepsilon} = 0$$

and

$$\text{(B6)} \quad H(\varepsilon) - (r+\delta)q(\varepsilon) + \mu(\varepsilon)q_{\varepsilon} + \tfrac{1}{2}\sigma(\varepsilon)^2 q_{\varepsilon\varepsilon} = 0.$$

Note that both differential equations are of the form

$$\text{(B7)} \quad g(\varepsilon) - a\chi(\varepsilon) + E(d\chi/dt) = 0.$$

According to Lemma 1, a solution to the differential equation in (B7) is

$$\text{(B8)} \quad \chi(\varepsilon_t) = E_t \int_0^\infty g(\varepsilon_{t+s}) e^{-as}\, ds.$$

Since equations (B5) and (B6) are both of the form in equation (B7), we substitute from these into equation (B8) to conclude:

$$\text{(B9)} \quad J(\varepsilon_t) = E_t \int_0^\infty \max[0, \psi(q_{t+s})] e^{-rs}\, ds$$

$$\text{(B10)} \quad q(\varepsilon_t) = E_t \int_0^\infty H(\varepsilon_{t+s}) e^{-(r+\delta)s}\, ds.$$

Therefore, $J(\varepsilon_t)$ can be interpreted as the present value of rents accruing to the firm from the augmented adjustment technology, and $q(\varepsilon_t)$ is the present value of marginal products of capital. Note that this solution was derived for any diffusion process governing ε_t.

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