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Abstract

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EXPECTED NUMBER OF REAL ZEROS OF A RANDOM POLYNOMIAL WITH INDEPENDENT IDENTICALLY DISTRIBUTED SYMMETRIC LONG-TAILED COEFFICIENTS*

L. SHEPP† AND K. FARAHMAND‡

Abstract. We show that the expected number of real zeros of the nth degree polynomial with real independent identically distributed coefficients with common characteristic function $\phi(z) = e^{-A(\ln|1/z|)^{-a}}$ for $0 < |z| < 1$ and $\phi(0) = 1$, $\phi(z) \equiv 0$ for $1 \leqq |z| < \infty$, with $1 < a$ and $A \geqq a^{(a-1)}$, is $(a-1)/(a-\frac{1}{2}) \log n$ asymptotically as $n \to \infty$.

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1. Introduction. Probability theory has long been used to study the relationship between the coefficients of a polynomial and its roots. For example, the expected number of real zeros of a polynomial of degree n, with independent identically distributed (i.i.d.) coefficients, whose common characteristic function is $\phi(z) = e^{-A|z|^{\alpha}}$, is known to be asymptotic to $c(\alpha)\log n$ as $n \to \infty$, where $c(2) = 2/\pi$, $c(0^+) = 1$, and $c(\alpha)$ *decreases* in $0 < \alpha \leqq 2$. If we allow the characteristic function $\phi(z)$ to be *a Pólya-type characteristic function*, i.e., ϕ is symmetric, real, decreasing, and convex in $z \geqq 0$, one might expect that the expected number of real zeros will be asymptotic to $c(\phi)\log n$, where $c(\phi)$ depends only on the behavior of $\phi(z)$ near $z = 0$. One would believe, because of the above results, that $c(\phi)$ would *increase* as the cusp of $\phi(z)$ at $z = 0$ becomes sharper, that is, as the common symmetric coefficient random variables have longer tails. But a surprising recent theorem of Zaporozhets [12], [15] shows that there are long-tailed symmetric ϕ's where the expected number of real zeros is *bounded*. Thus, as the tails get longer, or as the cusp gets sharper, there must eventually be a *decrease* in the expected number of real roots, despite the fact that within the stable class longer tails imply more real zeros. In light of this paradoxical situation, one would like to be able to calculate $c(\phi)$ for some Pólya-type characteristic function with a sharper cusp than $e^{-|z|^{\alpha}}$ for any α. Kac [4] and Rice [8] gave analytic formulas for the expected number of real zeros in terms of an arbitrary ϕ, but it has been difficult to obtain $c(\phi)$ from their formulas for any characteristic function except those with a cusp like that of $e^{-|z|^{\alpha}}$ because the method to obtain $c(\alpha)$ used the "Frullani methodology", which works *only* for the stable case, $e^{-A|z|^{\alpha}}$. Zaporozhets used other methods to obtain his result. Here we circumvent the use of Frullani's formula, and that enables us to move forward.

This paper calculates $c(\phi)$ for the two-parameter subclass of the Pólya characteristic functions, those of the form

$$\phi_{a,A}(z) = e^{-A(\log(1/|z|))^{-a}}, \qquad 0 < |z| < 1,$$

and $\phi_{a,A}(0) = 1$, $\phi_{a,A}(z) \equiv 0$, $|z| \geqq 1$, which is of Pólya type for $A \geqq a^{a-1}$, $a \geqq 1$, but is not infinitely divisible. We show that the expected number of real zeros of the n-th degree polynomial with i.i.d. coefficients with characteristic function $\phi_{a,A}$ is, asymptotically,

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$c(\phi_{a,A})\log n$ as $n \to \infty$, where $c(\phi_{a,A}) = C(a)$ depends only on a and is given (for $a > 1$) by

$$C(a) = \frac{2(a^2-1)}{\pi^2}\int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, \frac{\log^2|1-y/x|}{(1+y)^{a+2}} \tag{1}$$
$$- \frac{2a(a-1)^2}{\pi^2}\int_{-\infty}^{\infty} dx \left(\int_0^{\infty} dy\, \frac{\log|1-y/x|}{(1+y)^{a+1}}\right)^2.$$

At first, this looks formidable and suitable for numerics, as in [10], but this complicated expression collapses to a simple form, as we will show, namely, to

$$C(a) = \frac{a-1}{a-1/2}, \qquad a \geqq 1.$$

We can draw two interesting conclusions from this formula.

1. The case $a = 1^+$ "verifies" Zaporozhets' theorem, although his conclusion is much stronger than ours; our examples only give that the *limiting* expected number of real zeros is $< \varepsilon \log n$, while his example, obtained by a different, nonconstructive method, gives a *bounded* expected number of real zeros. It appears at first somewhat strange that the zero appears at $a = 1$, rather than at $a = 0^+$, but this was foreshadowed by Ibragimov and Zaporozhets [13], who showed that if $\mathbf{E}\,|\log X| = \infty$, then there are few complex zeros near the unit circle.

2. As $a \to \infty$, the limiting expected number of real zeros is *unity* times $\log n$, which meets the stable case at $\alpha = 0^+$; i.e., $C(\infty) = c(0^+) = 1$. As we run through the stables, from $\alpha = 2$ down to $\alpha = 0^+$, $c(\alpha)$ *increases* from $2/\pi$ to unity; then if we continue to run though the special Pólya class $\phi_{a,A}$ from $a = \infty_-$ down to $a = 1^+$ (for any $A(a) \geqq a^{(a-1)}$), $C(a)$ *decreases* from unity to zero; then, putting the two runs together, we get a running class of ever sharper characteristic functions, and hence ever longer tails. In the first part of the run the expected number of real zeros *increases*, but in the second part, the expected number of real zeros *decreases*. The *maximum* mean number of zeros along this path occurs only in a limiting sense at the transition, $\alpha = 0^+$, for $c(\alpha)$, and $a = \infty^-$, for $C(a)$, since $c(0_+) = C(\infty_-) = 1$. Of course it is possible that there is another surprise ahead, and that the number of real zeros is not just a function of the cuspiness of the characteristic function or of the fatness of the tails, but since we have now gotten the complete range of cusps and tails, it seems likely that further surprises will not appear. Descartes' rule of signs seems incapable of foretelling the constant $\times \log n$ behavior, which we propose to call the "Kac phenomenon" because he set out to shed light on Descartes' rule of signs by applying probability theory.

It would be interesting to know whether any other $c(\phi) \geqq 1$ is possible for some characteristic function. Indeed, from the above result, one is tempted to conjecture that the answer is negative; i.e., for any characteristic function the expected number of real zeros is always less than $\log n$, i.e., $c(\phi) \leqq 1$ for all ϕ. Further, one is tempted to conjecture that the maximum is not achieved; indeed the "pseudodensity" at the common limit, $a \to \infty$, $\alpha \to 0$, in some formal sense, is

$$f(x) \sim \frac{1}{|x|(\log|x|)^{\infty}} = \frac{1}{|x|}\left(\frac{1}{|x|}\right)^{(1/\infty)} \qquad \text{as} \quad |x| \to \infty,$$

in both limiting cases, though of course this f is *not* a density. On the other hand, one would think there is a density that achieves the limiting constant, unity. Namely, a density such as

$$f(x) \sim \frac{C}{|x|e^{b(\log|x|)^{\gamma}}}, \qquad 0 < \gamma < 1,$$

which tends to zero faster than any power $|x|^{-a+1}$ but more slowly than any of the form $C/(|x|(\log|x|)^{a+1})$. We can make a Pólya function whose density has this behavior, for example, if $0 < \gamma < 1$, $b > 0$, $a > 0$, and A is sufficiently large; then for $0 < |z| < 1$, set

$$\phi(z) = \exp\left\{ - A\left(\log\frac{1}{|z|}\right)^{-a} \exp\left\{ - b\left(\log\frac{1}{|z|}\right)^{\gamma}\right\}\right\}$$

and let $\phi(0) = 1$, and $\phi(z) = 0$, for $|z| \geqq 1$. This ought to have $c(\phi) = 1$, since it lies in the "intersection domain" of both results, but the calculations carried out below for $\phi_{a,A}$ seem to be difficult for this case; see, however, the last section for more details.

Another conjecture which is suggested by the results of this paper is that for Pólya characteristic functions with sharper cusps than any of $\phi_{a,A}$, the expected number of real zeros is $o(\log n)$. This last conjecture was again foreshadowed by Ibragimov and Zaporozhets [13], which showed, as mentioned above, that if $\mathbf{E}\log|X| = \infty$, where X is the random variable of each i.i.d. coefficient, then there are relatively few complex zeros of magnitude close to one. This result was very helpful to us in finding our way to the results here because it indicates that $a = 1$ is the value of the parameter in $\phi_{a,A}$, where one might expect $o(\log n)$ real zeros.

It is tempting to assume that if any smooth, symmetric density $f(x) \sim A/(x(\log x)^{a+1})$ as $x \to \infty$ with $a > 1$, then the expected number of real zeros will be asymptotically $C(a)\log n$ with $C(a) = (a-1)/(a-1/2)$; i.e., the density does not have to be exactly the density of $\phi_{a,A}(z)$. Ibragimov and Maslova [11] proved the analogous statement in the stable case, and we will use their result in a later section along with a big gap condensation argument to show that there are Pólya-type characteristic functions, ϕ, that do not have a limiting constant, $c(\phi)$.

There certainly is a mystery that still needs to be explored. Why does the number of real zeros increase and then decrease with the fatness of the tails? Of course one might say that something else besides the fatness of the tails might be involved, but the fatness does seem to tell the whole story since the examples we have seem to cover the entire set of possibilities, assuming that the above "domain of attraction" results will be proven to be valid. Perhaps it has to do with fact that in the case when

$$\mathbf{P}(X > t) \sim \frac{C}{(\log t)^a},$$

all but a few of the coefficients are negligible with respect to the largest ones (in absolute value), which perhaps is all that matters.

2. Real zeros. Kac [4] started the use of *exact* probabilistical methods to explore the complicated connection between the coefficients of a polynomial and the number of its real roots. He solved a problem posed by Littlewood and Offord by finding the *exact* asymptotics of the expected number of real zeros when the coefficients are i.i.d. zero mean normal and independent (which he argued [5] has some philosophical significance, although see footnote below[1]) then the expected number of real roots is asymptotically $(2/\pi)\log n$ for large n. This was generalized by Erdös and Offord to zero mean Bernoulli coefficients with the same result, and by others for any i.i.d. coefficients in the domain of attraction of the normal law, i.e., with zero mean and finite variance. Later it was shown [10] that if the finite variance condition is dropped, then other behavior is possible: the expected number of real zeros of a polynomial whose coefficients are i.i.d. and symmetric stable with parameter $0 < \alpha \leqq 2$ is known to be asymptotic to $c(e^{-|z|^\alpha})\log n$ as $n \to \infty$, where

$$c(e^{-|z|^\alpha}) = \frac{4}{\pi^2\alpha^2}\int_{-\infty}^{\infty} dx\, \log\int_0^\infty \frac{|x-y|^\alpha}{|x-1|^\alpha}\, e^{-y}\, dy.$$

Ibragimov and Maslova [11] showed that if the common symmetric density, $f(x)$, is in the domain of attraction of the stable law, essentially if $f(x) \sim \text{const}\cdot x^{-(a+1)}$ as $a \to \infty$, then

[1] Kac [6] gives a clue to his real interest in the problem of zeros of polynomials — as a teenager, he gave a proof of Cardano's formula for the roots of a cubic polynomial. This won him a scholarship and saved his life since it took him out of the path of the Holocaust.

$c(\phi)$ exists and is equal to $c(e^{-|z|^\alpha})$. For additional results and background on zeros of random polynomials, see [1], [14].

3. Pólya's class of characteristic functions. Pólya's criterion is that a function $\phi(z) \equiv \phi(-z)$ which for $z \geqq 0$ is continuous, nonnegative, decreasing, and convex, with $\phi(0) = 1$ is a characteristic function [2, Chap. 15. 2a, p. 509].

Pólya's theorem may be extended a bit as follows. Call a characteristic function *strongly Pólya* if $\phi(z) = e^{-\psi(z)}$, where $\psi(0) = 1$, and ψ is even, increasing, and *concave*. Such a ϕ is Pólya because $\phi'(z)/\phi(z) = -\psi'(z)$, and

$$\frac{\phi''(z)}{\phi(z)} = -\psi''(z) + \left(\frac{\phi'(z)}{\phi(z)}\right)^2 \geqq 0.$$

The converse is false, since the left side may be positive without ψ being concave, as the example $\phi_{a,A}(z)$ shows. It is easy to check that $\phi_{a,A}$ is of Pólya type if $a > 0$, and $A \geqq a^{a-1}$.

Remark 1. There is a number $A_0(a) \leqq a^{a-1}$, for which $A \geqq A_0$ implies that $\phi_{a,A}$ is Pólya, but we do not know what $A_0(a)$ is exactly. We can say that $A_0(a)$ is strictly greater than zero, because $\phi_{a,A}$ is not infinitely divisible. This is because for sufficiently small A, it is not a characteristic function since infinitely divisible characteristic functions have no zeros, and $\phi_{a,A}(z) = 0$ for $|z| \geqq 1$.

Remark 2. It is clear that a *strong* Pólya characteristic function is infinitely divisible, since ψ/n has the same properties as ψ. Conversely, if ϕ^ε is Pólya for every $\varepsilon > 0$, then ϕ is strongly Pólya because if

$$(\phi^\varepsilon)'' = \varepsilon(\varepsilon - 1)\phi^{\varepsilon-2}(\phi')^2 + \phi''\varepsilon\phi^{\varepsilon-1}$$

is nonnegative for every $z > 0$ and $\varepsilon > 0$, then it follows that $\phi''/\phi - (\phi'/\phi)^2 \geqq 0$ and so $-\log\phi$ is concave.

Remark 3. To find the behavior of the density of $\phi_{a,A}$, we write

$$f_{a,A}(x) = \frac{1}{\pi}\int_0^\infty \cos(xz)\phi_{a,A}(z)\,dz = \frac{1}{\pi x^2}\int_0^\infty (1 - \cos(xz))\phi''_{a,A}(z)\,dz.$$

Note that for large x, crudely replacing $1 - \cos(xz)$ with $\frac{1}{2}$, we get the very rough estimate,

$$\begin{aligned} f_{a,A}(x) &\approx \frac{1}{2\pi x^2}\int_0^{1/x} z^2\phi''_{a,A}(z)\,dz + \int_{1/x}^\infty \phi''_{a,A}(z)\,dz \approx \frac{1}{2\pi x^2}\left(-\phi'_{a,A}\left(\frac{1}{x}\right)\right) \\ &\approx \frac{aA}{2\pi x \log^{-(a+1)} x}. \end{aligned}$$

4. The Kac–Rice formula for the expected number of real zeros of a random polynomial with i.i.d. coefficients with an arbitrary symmetric characteristic function. Let $N_n(a, b, \phi)$ denote the number of real zeros in the interval $a < t < b$ of the polynomial $X_n(t) = \sum_0^n \xi_k t^k$, where the coefficients ξ_k are i.i.d. with common characteristic function on $\phi(z)$. Kac [5] introduced the use of probabilistic methods to gain understanding of the complicated relationship between the real coefficients of a polynomial and the number of its real zeros by studying the *expected number* of its real zeros in the case when the coefficients are i.i.d. random variables with common characteristic function $\phi(z) = \mathbf{E}\, e^{i\xi z}$.

At about the same time, Kac [5] and Rice [8] gave slightly different formulas for the expected number of zeros of a general stochastic process. If the stochastic process is a random polynomial, Kac realized that the formula can be obtained for the expected number of real zeros of the polynomial. Kac's formula leads [7] to the following neat general formula for the expectation of $N_{n-1}(a, b, \phi)$, in terms of the *expected density*, f_n, of zeros (Kac used $n - 1$; we will use n);

$$\mathbf{E}\, N_{n-1}(a, b, \phi) = \int_a^b f_{n-1}(t, \phi)\,dt, \tag{2}$$

where the expected density, $f_{n-1}(t,\phi)$, is given by

$$\frac{1}{2\pi^2}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\left[\prod_{k=0}^{n-1}\phi(\xi t^k)-\frac{1}{2}\left(\prod_{k=0}^{n-1}\phi(\xi t^k+k\eta t^{k-1})+\prod_{k=0}^{n-1}\phi(\xi t^k-k\eta t^{k-1})\right)\right]. \tag{3}$$

This formula for the expected number of real zeros is slightly different from Rice's formula used in [9], [10] and seems to be better suited for the general problem, although for the case of the stables treated in [9], [10] the formula of Rice appeared to be better suited.

The advantage of Kac's formula is that there is no need to take a limit on a small parameter, ε, as in [9], [10], though Rice's method, used in [9], [10], also gives a usable formula, and the convergence of Kac's formula is also similarly delicate and nonabsolute. In Rice's formula, one needs to supply a term which depends heavily on ϕ (the A, B in [9], [10]). This is avoided if one uses Kac's approach. Kac's approach could have been used in [9], [10], but the formula based on (3) seems less simple for that case. In the last analysis either method works as well; the difference in the Kac and Rice approaches should not be overly emphasized.

Returning to Kac's formula for $\mathbf{E}\,N_n(a,b)$ and $f_n(t)$, above, we want to compute for the given characteristic function, $\phi_{a,A}(z)$, the limit

$$c(\phi_{a,A}(z))=\lim_{n\to\infty}\frac{\mathbf{E}\,N_n(-\infty,\infty,\phi_{a,A}(z))}{\log n},$$

where $\mathbf{E}\,N_n$ is given as in (2), (3). If we make the substitution in (3), $\eta=\xi t/v$, and do some rearranging (though one has to be careful about convergence — Kac's integral is a "principal value"), we have for any real ϕ,

$$\mathbf{E}\,N_n(a,b,\phi)=\frac{1}{\pi^2}\int_a^b\frac{dt}{t}\int_0^{\infty}\frac{d\xi}{\xi}\int_{-\infty}^{\infty}dv\left[\prod_{j=0}^{n}\phi(\xi t^j)-\prod_{j=0}^{n}\phi\left(\xi t^j\left(1+\frac{j}{v}\right)\right)\right]. \tag{4}$$

The trick is to find where the contribution to the expected number of roots in this integral is coming from in (t,ξ,v) space. We first quickly review the stable case, because this case is similar but more difficult. For the stable case, both the Kac and Rice approaches give the same answer for $c=c(\alpha)$. We sketch the derivation based on (3); for the similar derivation based on (4), see [10]. One first shows [10] that the expected number of zeros in $[0,1-\delta]\cup[1-t/n,1]$ is bounded in n. Then it follows from (3) that $\mathbf{E}\,N_n(-\infty,\infty)$ is given by

$$\begin{aligned}\frac{4}{\pi^2}\int_{1-\delta}^{1-T/n}\frac{dt}{t}\int_0^{\infty}\int_0^{\infty}\frac{d\eta}{\eta}\Bigg[&2\exp\left\{-\sum_{k=0}^{n}\xi^{\alpha}t^{k\alpha}\eta^{\alpha}\right\}\\ &-\exp\left\{-\sum_{k=0}^{n}(\xi+k)^{\alpha}t^{k\alpha}\eta^{\alpha}\right\}-\exp\left\{-\sum_{k=0}^{n}|\xi-k|^{\alpha}t^{k\alpha}\eta^{\alpha}\right\}\Bigg].\end{aligned}$$

Now one proceeds, similarly to [10], by making the substitutions, $t=e^{-x/(n\alpha)}$, $\xi=nv/x$, and $\eta^{\alpha}=(ux^{\alpha}/n^{\alpha})x/n$, comparing Riemann sums with integrals, and taking a limit to obtain

$$\begin{aligned}c(\alpha)=\frac{4}{\pi^2\alpha^2}\int_0^{\infty}dv\int_0^{\infty}\frac{du}{u}\Bigg[&2e^{-v^{\alpha}u}-\exp\left\{-u\int_0^{\infty}(v+y)^{\alpha}e^{-y}\,dy\right\}\\ &-\exp\left\{-u\int_0^{\infty}|v-y|^{\alpha}e^{-y}\,dy\right\}\Bigg].\end{aligned}$$

This agrees with the formula for $c(\alpha)$ in [10], namely,

$$c(\alpha)=\frac{4}{\pi^2\alpha^2}\int_{-\infty}^{\infty}dx\,\log\int_0^{\infty}\frac{|x-y)|^{\alpha}}{|x-1|^{\alpha}}e^{-y}\,dy.$$

If one uses the Frullani integral,

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = f(0)\log\frac{b}{a},$$

which holds for any bounded continuous function, f, which falls off to zero fast enough for convergence at infinity, as is easy to see, we can evaluate the former expression as

$$c(\alpha) = \frac{4}{\pi^2\alpha^2}\int_0^\infty dv\,\log\left[\int_0^\infty \Big(1+\frac{y}{v}\Big)^{\alpha} e^{-y}\,dy \int_0^\infty \Big|1-\frac{y}{v}\Big|^{\alpha} e^{-y}\,dy\right].$$

To show that both formulas give the same value for $c(\alpha)$, use the simple formula that

$$0 = \int_0^\infty dv\,\log|1-v^{-2}|.$$

Unfortunately, unless we are in the stable case, neither (3) nor (4) is a Frullani integral, which is why the problem is more difficult in the case of $\phi_{a,A}(z)$.

We write from the general formula above for $\phi = \phi_{a,A}$, noting that the integral on ξ only goes up to $\xi = 1$ because $\phi_{a,A}$ vanishes outside this range and the $j = 0$ term makes the product zero,

$$\mathbf{E}\,N_n(\alpha,\beta,\phi) = \frac{1}{\pi^2}\int_\alpha^\beta \frac{dt}{t}\int_0^1 \frac{d\xi}{\xi}\int_{-\infty}^\infty dv\left[\prod_{j=0}^n \phi(\xi t^j) - \prod_{j=0}^n \phi\Big(\xi t^j\Big(1+\frac{j}{v}\Big)\Big)\right].$$

Choose $\alpha = \exp\{-\delta^{1-1/a}\}$, $\beta = \exp\{-(T/n)^{1-1/a}\}$, and make the substitutions

$$\log\frac{1}{t} = \Big(\frac{x}{n}\Big)^{1-1/a}, \quad \log\frac{1}{\xi} = z\,\Big(\frac{n}{x}\Big)^{1/a}, \quad v = w\frac{n}{x}.$$

Note that

$$\frac{dt}{t} = -\Big(1-\frac{1}{a}\Big)\Big(\frac{n}{x}\Big)^{1/a}\frac{dx}{n}, \quad \frac{d\xi}{\xi} = -dz\,\Big(\frac{n}{x}\Big)^{1/a}, \quad dv = dw\,\frac{n}{x},$$

and we get, after a calculation, that

$$\begin{aligned}&\mathbf{E}\,N_n\Big(\exp\{-\delta^{1-1/a}\}, \exp\Big\{-\Big(\frac{T}{n}\Big)^{1-1/a}\Big\}\Big)\\ &= \Big(1-\frac{1}{a}\Big)\frac{1}{\pi^2}\int_T^{n\delta}\frac{dx}{x}\int_0^\infty dz\int_{-\infty}^\infty dw\,\Big(\frac{n}{x}\Big)^{2/a}\exp\Big\{-A\,\frac{x}{n}\sum_{j=0}^n\Big(z+\frac{jx}{n}\Big)^{-a}\Big\}\\ &\quad\times\left[1-\chi\exp\Big\{A\,\frac{x}{n}\sum_{j=0}^n\Big(z+\frac{jx}{n}\Big)^{-a}\Big(1-\Big(1-\frac{x}{n}\,\frac{1}{a}\,\frac{\log|1+jx/(nw)|}{z+jx/n}\Big)^{-a}\Big)\Big\}\right],\end{aligned}$$

where $\chi = 1$ or $\chi = 0$ according as every argument that gets raised to the $-a$th power is positive or not, i.e., for every $0 \leqq j \leqq n$,

$$\Big(1-\frac{x}{n}\,\frac{1}{a}\frac{\log|1+jx/(nw)|}{z+jx/n}\Big) \geqq 0.$$

For all points, χ will be equal to one in the limit $n\to\infty$ because the term x/n is going to zero. If we set $x/n = \varepsilon$ to simplify notation and set $\theta = (z+jx/n)^{-1}\log|1+jx/(nw)|$, we can write

$$1-(1-\varepsilon\theta)^{-a} = -\varepsilon a\theta - \varepsilon^2\,\frac{a(a+1)\theta^2}{2!} + \cdots$$

by using the power series. If we now set

$$R = Aa\,\frac{x}{n}\sum_{j=0}^n \frac{\log|1+jx/(nw)|}{(z+jx/n)^{a+1}}, \quad S = A\,\frac{a(a+1)}{2}\,\frac{x}{n}\sum_{j=0}^n\frac{\log^2|1+jx/(nw)|}{(z+jx/n)^{a+2}},$$

then it is easy to see by comparison with integrals that

$$R \to Aa \int_0^x dy \, \frac{\log|1+y/w|}{(z+y)^{a+1}}, \quad S \to A\, \frac{a(a+1)}{2} \int_0^x dy \, \frac{\log^2|1+y/w|}{(z+y)^{a+2}}.$$

We now expand the term $[1-\chi\exp\{\dots\}]$ into a power series in ε and, together with the term $(n/x)^2 = 1/\varepsilon^2$ in the integrand of the expression for $\mathbf{E}\, N_n(\alpha,\beta) = \mathbf{E}\, N_n(\alpha,\beta,\phi)$, we get that

$$\mathbf{E}\, N_n(\alpha,\beta) = \Big(1-\frac{1}{a}\Big)\frac{1}{\pi^2}\int_T^{n\delta}\frac{dx}{x}\int_0^\infty dz\int_0^\infty dw\, \frac{1-\varepsilon R+\varepsilon^2(S-R^2/2)}{\varepsilon^2}.$$

If we now use the identity

$$0 = \int_0^\infty dw\, \log|1-w^{-2}|,$$

also mentioned above, then the R/ε term integrates to zero in w, and so drops out, and we can pass to the limit to get

$$\begin{aligned}\mathbf{E}\, N_n\Big(&\exp\{-\delta^{1-1/a}\}, \exp\Big\{-\Big(\frac{T}{n}\Big)^{1-1/a}\Big\}\Big)\\ &= \log n\Big(1-\frac{1}{a}\Big)\frac{1}{\pi^2}\int_0^\infty dz\int_0^\infty dw\,\frac{1-\varepsilon R+\varepsilon^2(S-R^2/2)}{\varepsilon^2}.\end{aligned}$$

Note that we have used the fact that

$$\int_T^{n\delta}\frac{dx}{x} = \log n + O(1)$$

to replace the integral on x by $\log n$. But this means that we must replace x by ∞ in the remaining integrals over z and w. This is exactly analogous to the calculation in [9], [10]. We next allow $\delta \to \infty$, so that we have

$$\mathbf{E}\, N_n(0,1) = \lim_{\delta\to\infty}\mathbf{E}\, N_n(e^{-\delta^{1-1/a}},1).$$

To get $\mathbf{E}\, N_n(-\infty,\infty)$ we have to multiply by 4 because $\mathbf{E}\, N_n(0,1) = \mathbf{E}\, N_n(1,\infty)$, as we see by letting t be replaced by $1/t$, and because $\mathbf{E}\, N_n(0,\infty) = \mathbf{E}\, N_n(-\infty,0)$, as we see by letting t be replaced by $-t$. We thus have

$$C(a) = 4\Big(1-\frac{1}{a}\Big)\frac{1}{\pi^2}\int_0^\infty dz\int_0^\infty dw\,\Big(S-\frac{R^2}{2}\Big),$$

which is easily seen to be the same as (1). In order to reduce $C(a)$ to $C(a) = c(\phi_{a,A}) = (a-1)/(a-1/2)$, we use the identity

$$\int_{-\infty}^{\infty}\log\Big|1-\frac{y_1}{x}\Big|\log\Big|1-\frac{y_2}{x}\Big|\,dx = \frac{\pi^2}{2}\,(|y_1|+|y_2|-|y_1-y_2|),$$

and now simple calculations finish the proof. The last identity can be proved in several ways and surely is well known although we could not find it in the standard tables. A proof can be based on the singular value Fourier transform of $f_y(x) = \log|1-y/x|$ as follows:

$$\begin{aligned}\hat{f}_y(\omega) = \int_{-\infty}^{\infty} f_y(x)e^{i\omega x}\,dx &= \int_{-\infty}^{\infty}\log\Big|1-\frac{y}{x}\Big|\,d_x\,\frac{e^{i\omega x}-e^{i\omega y}}{i\omega}\\ &= -\int_{-\infty}^{\infty}\frac{e^{i\omega x}-e^{i\omega y}}{i\omega}\,\frac{y}{x(x-y)}\,dx.\end{aligned}$$

Using the Parseval formula,

$$\int_{-\infty}^{\infty} f_{y_1}(x) f_{y_2}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_{y_1}(\omega)\hat{f}_{y_1}(\omega)^*\,d\omega,$$

and

$$\frac{d}{dy}\int_{-\infty}^{\infty}\frac{1-\cos(\omega y)}{\omega^2}\,d\omega = \int_{-\infty}^{\infty}\frac{\sin(\omega y)}{\omega}\,d\omega \equiv \pi$$

completes the proof.

5. Final remarks on the "intersection domain." If $\phi(z)$ has a sharper cusp than any of the stables, $e^{-|z|^{\alpha}}$, $\alpha > 0$, and a duller cusp than any of $\phi_{a,A}$ above, for example, if, as mentioned earlier, for $0 < \gamma < 1$, $b > 0$, $a > 0$, and A sufficiently large, we set $\phi(0) = 1$, set $\phi(z) = 0$ for $|z| \geqq 1$, and for $0 < |z| < 1$ set

$$\phi(z) = \exp\Big\{ - A\Big(\log\frac{1}{|z|}\Big)^{-a}\exp\Big\{ - b\Big(\log\frac{1}{|z|}\Big)^{\gamma}\Big\}\Big\},$$

then it is easy to check that ϕ is of (weak) Pólya type. This ϕ "ought" to have $c(\phi) = 1$, since it lies in the intersection domain of both results, assuming that fatness alone determines the expected number of real zeros (despite the lack of monotone dependence on the cusp or tails!). The calculations carried out above for $\phi_{a,A}$ seem to be more difficult for this ϕ. It perhaps can be done as follows, but we have not done it. Suppose, in general, we have

$$\phi(z) = e^{-F(\log(1/|z|))}.$$

The case above has $F(v) = Av^{-a}e^{-bv^{\gamma}}$, but suppose we have any F with similar asymptotics at $v = \infty$. This will give some ϕ_F in the intersection domain. Now, we let $g = F^{-1}$ be the inverse function to F, and we make the substitutions in (4),

$$\log\frac{1}{t} = g\Big(\frac{x}{n}\Big), \quad \log\frac{1}{\xi} = z\,\frac{x}{n}\,g\Big(\frac{n}{x}\Big), \quad v = w\,\frac{n}{x}.$$

This gives, with $f(u) = ug(u)$, the equation for $0 < \alpha < \beta < 1$,

$$\begin{aligned}\mathbf{E}\,N_n(\alpha,\beta,\phi) = {} & \frac{1}{\pi^2}\int_{nf^{-1}(\log\beta^{-1})}^{nf^{-1}(\log\alpha^{-1})}\frac{dx}{x}\int_0^{\infty}dz\int_{-\infty}^{\infty}dw\, f'\Big(\frac{x}{n}\Big)f\Big(\frac{x}{n}\Big)\\ & \times\Big[\exp\Big\{ -\sum_{j=0}^{n}F\Big(\Big(z+\frac{jx}{n}\Big)g\Big(\frac{x}{n}\Big)\Big)\Big\}\\ & - \exp\Big\{ -\sum_{j=0}^{n}F\Big(\Big(z+\frac{jx}{n}\Big)g\Big(\frac{x}{n}\Big) - \log\Big|1+\frac{jx}{nw}\Big|\Big)\Big\}\Big].\end{aligned}$$

It seems that one should be able to complete these asymptotics to show that this expression is asymptotic to $\frac{1}{4}\log n$ for any α and for β closer and closer to one. We leave this for others to carry out, since we have run out of energy, and it seems it "must work." But we will be very pleased to see it actually done.

Corrections. We take this opportunity to correct some misprints in [9], [10].
In formula (4) on page 29 of [9] replace dt by $t^{-1}\,dt$.
In formula (9) on page 309 of [10] replace $1 - t^2$ by $|1 - t^2|$.
In formula (12) on page 310 of [10] replace $A^{1/\alpha}$ by $A^{1-\alpha}$.
In formula (25) on page 312 of [10] replace $\log(v,\infty;\alpha)$ by $\log\lambda(v,\infty;\alpha)$.

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