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Abstract
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D-brane charges in Gepner models

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I. INTRODUCTION

It is by now firmly established[1,2] that the K-theory groups of space-time are the *D*-brane charge groups. More precisely, the claim is that the isomorphism classes of *D*-brane boundary states modulo boundary renormalization group (RG) flow are in one to one correspondence[3] with suitable K-theory classes of the string theory background in question. For geometrical backgrounds such as Calabi-Yau manifolds, one can construct a variety of *D*-branes by applying methods from boundary CFT, matrix factorizations, and geometry.[4–11] However, determining the endpoint of the RG flow[12] is unfortunately not easy.

Most well understood in this context are purely geometrical backgrounds of string theory, such as tori, orbifolds, and Calabi-Yau manifolds. In these instances, the K-theories were either already available in the mathematics literature or are easily computed by standard techniques and the complementary string theory computation of D-brane charges is relatively straightforward.

Less trivial is the situation of string theory backgrounds with nontrivial NSNS three-form flux H, where it is believed that twisted K-theory is the correct structure to classify D-brane charges.[2,13,14] Explicit checks of this claim have so far been restricted to backgrounds with large symmetries, namely supersymmetric WZW and coset conformal field theories (CFT)s.[15–26] The computation of twisted K-theories for compact Lie groups and coset models thereof were greatly simplified by the theorem of Freed, Hopkins, and Teleman.[27–29]

Our objective in this paper is to test the twisted K-theory proposal beyond standard CFT backgrounds by extending it to Gepner models. These are essentially orbifolds of tensor products of $\mathcal{N}=2$ minimal models, realized for our purposes in terms of $SU(2)/U(1)$ supersymmetric coset models. They are known to describe certain tori and Calabi-Yau spaces at particular points in their moduli space. Because the K-groups are a topological quantity, the *D*-brane charge group should be independent of the moduli. Therefore the twisted equivariant K-theory of the Gepner models has to agree with the topological K-theory of the corresponding Calabi-Yau manifold. This provides a non-trivial check of the brane charge classification.

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Technically, we are going to make use of the twisted equivariant Chern character. Consequently, we are going to compute the complexified K-groups,

$$K^*(X;\mathbb{C}) := K^*(X)\otimes_{\mathbb{Z}}\mathbb{C} \tag{1}$$

only. The downside is that one looses interesting torsion[30–32] information, since

$$K^*(X) = \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \Rightarrow K^*(X;\mathbb{C}) = \mathbb{C}^r. \tag{2}$$

However, since none of the Calabi-Yau threefolds with Gepner points actually have torsion in their K group, we do not expect to find any in the Gepner models either.

During the final stage of this work, we received a preprint[11] that constructs a basis of D-branes for the D-brane charge group. We will discuss a few details of their approach in Sec. VI.

II. THE QUINTIC

As an hors d'œuvre to our work, let us discuss[4,33] the $(k=3)^5$ Gepner model. It is known to correspond to the Fermat quintic

$$Q = \left\{ [x_0:x_1:x_2:x_3:x_4] \;\middle|\; \sum_{i=0}^{4} x_i^5 = 0 \right\} \subset \mathbb{P}^4 \Bigg\}. \tag{3}$$

The Hodge diamond of the quintic is by now quite familiar to all string theorists, and reads

$$h^{pq}(Q) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}. \tag{4}$$

We also know that there is no torsion in its cohomology, which then determines its K-theory to be

$$K^0(Q) = H^{\text{even}}(Q;\mathbb{C}) = \mathbb{Z}^4 \Rightarrow K^0(Q;\mathbb{C}) = \mathbb{C}^4,$$

$$K^1(Q) = H^{\text{odd}}(Q;\mathbb{Z}) = \mathbb{Z}^{204} \Rightarrow K^1(Q;\mathbb{C}) = \mathbb{C}^{204}. \tag{5}$$

We are going to arrive at the same answer for the complexified K-groups directly from the Gepner model, without making any reference to the quintic hypersurface.

The Gepner model corresponding to the quintic is a $\mathbb{Z}_5$ orbifold of 5 copies of the level $k=3$ minimal model; see Secs. III B and III F for more details. Moreover, the minimal model can be realized as an $\mathfrak{su}(2)_k/\mathfrak{u}(1)$ coset CFT. The coset CFT has a nice sigma model interpretation; it is an $SU(2)$ WZW model with a gauged $U(1)$ action. More precisely, the $U(1)$ acts as

$$U(1)\times SU(2) \to SU(2), \quad \left[e^{i\theta}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \mapsto \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}^{-1}. \tag{6}$$

[The cognoscente of course realize that our choice of maximal torus $U(1)\subset SU(2)$ is random. Since all maximal tori are conjugate, we just picked this one for explicitness.] Also see Fig. 1 for a picture of the orbits. The fixed point set of the $U(1)$ action is a circle inside $SU(2)\simeq S^3$, which we denote by

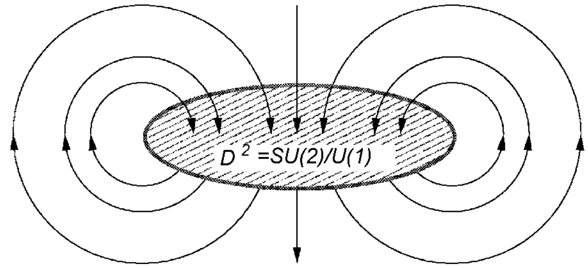

FIG. 1. $U(1)$ action on $SU(2)$.

$$S_A^1 \stackrel{\text{def}}{=} [SU(2)]^{U(1)} = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \middle| \varphi \in [0, \ldots, 2\pi) \right\}. \quad (7)$$

(For any space X with action of a group G, we write X^G for the G-fixed points. If $g \in G$, then we write X^g for the points fixed by the subgroup $\langle g \rangle \subset G$.) The space of orbits $SU(2)/U(1)$ is a disk, bounded by the fixed points S_A^1. Rotating this disk is another symmetry of the geometry, but arbitrary rotations are not a symmetry of the theory. The reason is that the H field is not symmetric under arbitrary rotations of the disk. Rather, the rotation group is broken to rotations by $2\pi/5$. This $\mathbb{Z}_5$ group action lifts to an action on the $SU(2)$ with fixed point set S_B^1; see Fig. 2. The fixed point sets S_A^1 and S_B^1 form a Hopf link inside $SU(2) \simeq S^3$.

By now it is firmly established that the charge group is given by the K-theory of space-time. More precisely, one has to pick the right "flavor" of K-theory depending on which $\mathcal{N}=1$ supersymmetric theory one formulates on the background.[19,22,26,34] For the coset model, the background is $SU(2)$ with an H flux. The latter implies that the correct K-theory is the so-called twisted K-theory, which we denote by ${}^tK$. Moreover, we want to gauge a $U(1)$ symmetry. As is familiar to all string theorists, this does *not* mean that we work on the set theoretic quotient $SU(2)/U(1)$. Instead, we have to correctly incorporate the twisted sectors, which on the level of cohomology means that we have to compute the $U(1)$ equivariant cohomology groups. Therefore, the correct K-theory for the minimal model is

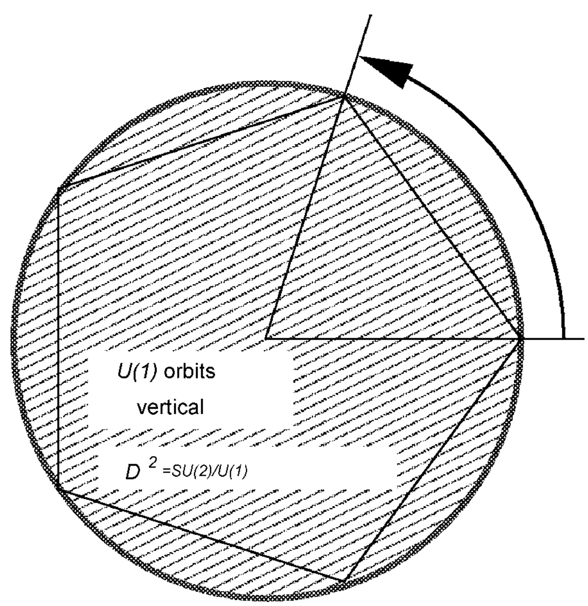

FIG. 2. $\mathbb{Z}_5$ action on $SU(2)$.

$$D\text{-brane charges in } \mathfrak{su}(2)_k/\mathfrak{u}(1) \text{ coset} = {}^tK_{U(1)}(SU(2)), \tag{8}$$

with the twist class

$$t = k+2 \in \mathbb{Z} = H^3_{U(1)}(SU(2);\mathbb{Z}). \tag{9}$$

Hence, the D-brane charges in the tensor product of five minimal models are

$${}^tK_{U(1)\times U(1)\times U(1)\times U(1)\times U(1)}(SU(2)\times SU(2)\times SU(2)\times SU(2)\times SU(2)), \tag{10}$$

where each $U(1)$ acts on just one of the $SU(2)$ factors. Finally, the Gepner model is the $\mathbb{Z}_5$ orbifold by the diagonal $\mathbb{Z}_5$ action. Therefore

$$D\text{-brane charges in the } (k=3)^5 \text{ Gepner model} = {}^tK_{U(1)^5\times\mathbb{Z}_5}(SU(2)^5) \tag{11}$$

To compute these K-groups, we are using a twisted version of the equivariant Chern isomorphism,

$$\mathrm{ch}: K_G^{0,1}(X;\mathbb{C}) \xrightarrow{\sim} \bigoplus_{g\in G} H_G^{\text{even,odd}}(X^g;\mathbb{C}). \tag{12}$$

(In this paper, we are only concerned with Abelian groups G. In general the sum is over conjugacy classes.) Adding an additional twist to the equivariant Chern character has two consequences. First, one is lead to twisted cohomology, which is roughly the cohomology of $d+[H]$ instead of d on differential forms. Second, the cohomology is with local coefficients, that is, with coefficients in a flat line bundle ${}^t\mathcal{L}$ instead of the trivial flat line bundle $\mathbb{C}$. The ensuing twisted equivariant Chern character (see Sec. III C)

$$\mathrm{ch}: {}^tK_G^*(X;\mathbb{C}) \to \bigoplus_{g\in G} {}^tH_G^*(X^g;{}^t\mathcal{L}(g)) \tag{13}$$

is an isomorphism, provided that only finitely many summands on the right are nonvanishing. This turns out to be the case here, and

$$\begin{aligned} {}^tK^*_{U(1)^5\times\mathbb{Z}_5}(SU(2)^5;\mathbb{C}) &\simeq \bigoplus_{g\in U(1)^5\times\mathbb{Z}_5} {}^tH^*_{U(1)^5\times\mathbb{Z}_5}([SU(2)^5]^g;{}^t\mathcal{L}(g)) \\ &= \bigoplus_{g\in U(1)^5\times\mathbb{Z}_5} [{}^tH^*_{U(1)^5}([SU(2)^5]^g;{}^t\mathcal{L}(g))]^{\mathbb{Z}_5} \end{aligned} \tag{14}$$

is indeed an isomorphism. More specifically, as we are going to show in Sec. III C the only contributions are from the 4^5+4 group elements

$$g = (\omega^{m_1},\omega^{m_2},\omega^{m_3},\omega^{m_4},\omega^{m_5},1), \quad m_i \in \{1,\ldots,4\}, \tag{15a}$$

$$g = (1,1,1,1,1,n), \quad n \in \{1,\ldots,4\}, \tag{15b}$$

in $U(1)^5\times\mathbb{Z}_5$, where we write $\omega \overset{\text{def}}{=} \exp(2\pi i/5)$. The corresponding fixed point sets are of the form

$$g = (\omega^{m_1},\omega^{m_2},\omega^{m_3},\omega^{m_4},\omega^{m_5},1) \Rightarrow [SU(2)^5]^g = (S^1_A)^5, \tag{16a}$$

$$g = (1,1,1,1,1,n) \Rightarrow [SU(2)^5]g = (S^1_B)^5. \tag{16b}$$

As we are going to discuss in more detail in the next section, the twisted equivariant cohomology for a single factor ${}^tH_{U(1)}(SU(2)^g;{}^t\mathcal{L}(g))$ for $g \in U(1)\times\mathbb{Z}_5$ is

$$g = (\omega^m,1) \Rightarrow {}^tH^0_{U(1)}(S^1_A;{}^t\mathcal{L}(g)) = 0, \quad {}^tH^1_{U(1)}(S^1_A;{}^t\mathcal{L}(g)) = \omega^m, \tag{17a}$$

$$g=(1,n)\Rightarrow {}^{t}H^{0}_{U(1)}(S^1_B;{}^{t}\mathcal{L}(g))=1, \quad {}^{t}H^{1}_{U(1)}(S^1_B;{}^{t}\mathcal{L}(g))=0, \tag{17b}$$

where we write the cohomology groups as $\mathbb{Z}_5$ characters. [By abuse of notation, we denote the generator for the character ring again ω. In other words, $m\in\mathbb{Z}_5=\{0,\ldots,4\}$ acts by multiplication with $\omega^m=\exp(2\pi i m/5)$.] The cohomology groups for the tensor product of 5 such factors is readily determined from the Künneth formula, and one obtains

$$\bigoplus_{g=(\omega^{m_1},\ldots,\omega^{m_5},1)} {}^{t}H^{*}_{U(1)^5}((S^1_A)^5;{}^{t}\mathcal{L}(g))=\begin{cases} 0, & *=0; \\ (\omega+\omega^2+\omega^3+\omega^4)^5, & *=5\equiv 1 \bmod 2; \end{cases} \tag{18a}$$

$$\bigoplus_{g=(1,\ldots,1,\omega^n)} {}^{t}H^{*}_{U(1)^5}((S^1_B)^5;{}^{t}\mathcal{L}(g))=\begin{cases} 4, & *=0; \\ 0, & *=1. \end{cases} \tag{18b}$$

It is now easy to determine the $\mathbb{Z}_5$-invariant part. Using the twisted equivariant Chern character, Eq. (14), we obtain

$${}^{t}K^{*}_{U(1)^5\times\mathbb{Z}_5}(SU(2)^5;\mathbb{C})=\begin{cases} \mathbb{C}^4, & *=0; \\ [(\omega+\omega^2+\omega^3+\omega^4)^5]^{\mathbb{Z}_5}=\mathbb{C}^{204}, & *=1 \end{cases} \tag{19}$$

which precisely equals the K-theory of the quintic hypersurface. (Perhaps not surprisingly, formally the same computation arises when one tries[35] to construct Gepner models using matrix factorizations. However, the authors of Ref. 35 fail to address the twisted sector branes that arise when the Gepner model contains minimal models of different levels.)

III. K-theory OF GEPNER MODELS

A. Group theory

As we saw in the quintic example discussed in Sec. II, one has to determine cohomology groups that form representations under a discrete group G_{GSO} ($=\mathbb{Z}_5$ for the quintic) that implements the GSO (After Gliozzi, Scherk, and Olive[36]) projection. Now we could always work with polynomials of characters as in Eq. (19), but this becomes cumbersome if one has to deal with tensor products of different minimal models.

For cyclic groups $\mathbb{Z}_\kappa$, the following representations will appear again and again. (In this paper, we are only going to consider complex representations.)

- The trivial representation $\mathbb{C}$.
- The regular representation $R\mathbb{Z}_\kappa$, which is defined as follows: Take the vector space $\mathbb{C}^\kappa$. The group acts by cyclically permuting the κ basis vectors. The regular representation can be diagonalized to the sum of all one-dimensional representations. Explicitly, if $\chi:\mathbb{Z}_\kappa\to\mathbb{C}$, $\chi(1)=\exp(2\pi i/\kappa)$ is the generating character, then

$$R\mathbb{Z}_\kappa=\bigoplus_{i=0}^{\kappa-1}\chi^i. \tag{20}$$

- The representation $\widetilde{R\mathbb{Z}}_\kappa$, which is the regular representation without its trivial subrepresentation,

$$\widetilde{R\mathbb{Z}}_\kappa=\bigoplus_{i=1}^{\kappa-1}\chi^i. \tag{21}$$

More formally, it is the cokernel

$$0 \to \mathbb{C} \to R\mathbb{Z}_\kappa \to \widetilde{R\mathbb{Z}}_\kappa \to 0. \tag{22}$$

Moreover, since we are actually computing cohomology groups, everything has a cohomological $\mathbb{Z}_2$ grade. By definition, we assign

$$\deg(\mathbb{C}) = 0, \tag{23a}$$

$$\deg(R\mathbb{Z}_\kappa) = \deg(\widetilde{R\mathbb{Z}}_\kappa) = 1. \tag{23b}$$

Of course, we have the usual operations of restriction and induction (transfer) to relate G_{GSO} representations and representations of subgroups of G_{GSO}. However, G_{GSO} is always a cyclic group and we have yet another operation that will occur frequently. This works as follows. Given any subgroup $\mathbb{Z}_\kappa \subset G_{\mathrm{GSO}}$, we have in addition to the inclusion i also a projection π,

$$\begin{tikzcd} \mathbb{Z}_\kappa \arrow[r, bend left, "i:\ n \mapsto \frac{|G_{\mathrm{GSO}}|}{\kappa} n"] & G_{\mathrm{GSO}} \simeq \mathbb{Z}_{|G_{\mathrm{GSO}}} \arrow[l, bend left, "\pi:\ n \mapsto n \bmod \kappa"] \end{tikzcd} \tag{24}$$

by modding out by κ. Given a representation $\rho:\mathbb{Z}_\kappa \to \mathbb{C}^n$, we can then define a representation $p_{\mathbb{Z}_\kappa}^{G_{\mathrm{GSO}}}(\rho)$ of G_{GSO} on the same vector space $\mathbb{C}^n$ by composing

$$p_{\mathbb{Z}_\kappa}^{G_{\mathrm{GSO}}}(\rho) \overset{\mathrm{def}}{=} \rho \circ \pi : G_{\mathrm{GSO}} \to \mathbb{C}^n, (n,v) \mapsto \rho(n \bmod \kappa, v). \tag{25}$$

Now, in general, the projection π depends on which generators you chose for G_{GSO}, a random choice. However, for the identity, the regular, and the reduced regular representation of $\mathbb{Z}_\kappa$ the resulting G_{GSO} representation does *not* depend on that choice. We are only going to use the $p_{\mathbb{Z}_\kappa}^{G_{\mathrm{GSO}}}$ operation in these cases.

For example, consider the group $\mathbb{Z}_{12}=\{0,1,\ldots,11\}$ with the character $\chi(1)=e^{2\pi i/12}$. Then the representation

$$p_{\mathbb{Z}_3}^{\mathbb{Z}_{12}}(\widetilde{R\mathbb{Z}}_3) \otimes_{\mathbb{C}} p_{\mathbb{Z}_4}^{\mathbb{Z}_{12}}(\widetilde{R\mathbb{Z}}_4) = (\chi^4+\chi^8)(\chi^3+\chi^6+\chi^9) = \chi+\chi^2+\chi^5+\chi^7+\chi^{10}+\chi^{11}, \tag{26}$$

is the six-dimensional representation of $\mathbb{Z}_{12}$ of cohomology degree $2 \equiv 0 \bmod 2$ generated by

$$\mathrm{diag}(e^{2\pi i/12}, e^{4\pi i/12}, e^{10\pi i/12}, e^{14\pi i/12}, e^{20\pi i/12}, e^{22\pi i/12}). \tag{27}$$

In the future, we are just going to write $\otimes$, and it will be understood that we are tensoring over $\mathbb{C}$.

B. Minimal model as coset

The minimal models for the $\mathcal{N}=2$ superconformal algebra have equivalent realizations in terms of super-GKO coset models,

$$\frac{\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2}{\mathfrak{u}(1)_{k+2}}, \tag{28}$$

as well as Landau-Ginzburg models. The modular invariant partition functions fall into an ADE classification.[37–39] From the coset CFT point of view these are obtained from the ADE modular invariants of the $\mathfrak{su}(2)_k$ WZW model. We shall focus on the A series minimal models. There are various subtleties concerning that modular invariant corresponds to the A-type superpotential, and it will turn out that there are essentially four distinct models that will be of interest. The fields of the coset CFT are labeled by (j,n,s), where $j=0,\ldots,k/2$ is the $\mathfrak{su}(2)_k$ highest weight, $n \in \mathbb{Z}_{2(k+2)}$ labels the representations of the denominator $\mathfrak{u}(1)_{k+2}$ and $s \in \mathbb{Z}_4$ labels the free fermion representations in $\mathfrak{u}(1)_2$. There is a $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ discrete group acting on the fields in the following fashion:

$$\alpha:\Phi_{(j,n,s)} \mapsto (-1)^{2n/k+2}\Phi_{(j,n,s)},$$

$$\beta:\Phi_{(j,n,s)} \mapsto (-1)^{s}\Phi_{(j,n,s)}. \tag{29}$$

The $\mathbb{Z}_{k+2}$ action is realized geometrically in the gauged WZW model by the rotation of the disk target space. Orbifolding the A-type theory with respect to these symmetries yields new modular invariants, as was first observed in Ref. 15. Note that a related issue arose in the context of WZW models for nonsimply laced groups in Ref. 22, where nontrivial automorphisms acting on the fermions gave rise to new modular invariants for the supersymmetric WZW models.

Since $s=1,3$ corresponds to the Ramond sector, the orbifold by $\mathbb{Z}_2=\langle\beta\rangle$ is from a space-time point of view the same as modding out $(-1)^F$. The state space of the (charge conjugate) diagonal modular invariant is

$$\mathcal{H}_{MM_k} = \bigoplus_{(j,n,s)} \mathcal{H}_{(j,n,s)} \otimes \overline{\mathcal{H}}_{(j,n,s)}, \tag{30}$$

where the direct sum is over the standard range of super-parafermion representations, including the selection and identification rules

$$(j,n,s) \equiv (k/2-j,n+k+2,s+2), \quad 2j+n+s \in 2\mathbb{Z}. \tag{31}$$

The state space of the $\mathbb{Z}_2$ orbifold is then obtained as

$$\mathcal{H}_{MM_k/\mathbb{Z}_2} = \bigoplus_{(j,n,s)} \mathcal{H}_{(j,n,s)} \otimes \overline{\mathcal{H}}_{(j,n,-s)}. \tag{32}$$

Orbifolding MM_k by $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$, it was observed in Ref. 15 that the partition function is the same as in MM_k, and that this model is, in fact, dual to MM_k. Likewise, $MM_k/\mathbb{Z}_{k+2}$ is T-dual to $MM_k/\mathbb{Z}_2$.

Gepner models are orbifolds of tensor products of minimal models with not necessarily equal level, which give rise to consistent GSO-projected string theory backgrounds. Consider a tensor product of r minimal models, of level $(k_1,\ldots,k_r)$, and define

$$\lambda = (j_1,\ldots,j_r), \quad \mu = (n_1,\ldots,n_r;s_1\ldots,s_r), \tag{33}$$

and $\beta_j=(0,\ldots,0,2,0,\ldots,0)$, with the nonzero entry at slot s_j and $\beta_0=(1,\ldots,1)$. Further define $K=lcm(2,k_j+2)$. Then the partition function for the Gepner model is given by

$$Z_{(k_1,\ldots,k_r)} = \sum_{\lambda,\mu}\sum_{b_0=0}^{2K-1}\sum_{b_j=0,1} \delta_\beta(-1)^{b_0}\chi_{\lambda,\mu}\bar{\chi}_{\lambda,\mu+b_0\beta_0+\Sigma b_j\beta_j}. \tag{34}$$

The characters of the tensor product of the minimal models are denoted by χ. In principle, one can define the conserved *D*-brane charges using RG flow,[3,40] but in practice this is not feasible.

C. Chern character of the minimal model

Now that we have defined all the ingredients, we can start to compute the relevant *K*-groups. Our main tool is going to be the twisted equivariant Chern character.[28,41] For explicitness, let us consider a single minimal model whose complexified *D*-brane charge group is

$${}^{\kappa}K_{U(1)}(SU(2))\otimes_{\mathbb{Z}}\mathbb{C} \overset{\text{def}}{=} {}^{\kappa}K_{U(1)}(SU(2);\mathbb{C}), \tag{35}$$

where $\kappa=k+2$ is going to be the twist class for the remainder of this section. [That is, the twist class is κ times the generator of $H^3_{U(1)}(SU(2);\mathbb{Z})$.] Now, given a twisted equivariant vector bundle, we can tensor it with any group representation, and get another equivariant vector bundle with the same twist. In other words, there is an action of $K_{U(1)}(\{\text{pt.}\};\mathbb{C})=RU(1)=\mathbb{C}[z,z^{-1}]$ on the twisted equivariant K-theory.

In geometrical terms, $\mathbb{C}[z,z^{-1}]$ is the ring of functions on $\mathbb{C}^{\times} \overset{\text{def}}{=} \mathbb{C}\backslash\{0\}$. And the twisted equivariant K-theory ${}^{\kappa}K_{U(1)}(SU(2);\mathbb{C})$ is a module over this function algebra, that is a sheaf over the base space $\mathbb{C}^{\times}$. The twisted equivariant Chern character identifies the stalks (fibers) of this sheaf over a point in $\mathbb{C}^{\times}$ with a certain cohomology group. More precisely, Freed-Hopkins-Teleman[28] identify the stalk over $\zeta \in \mathbb{C}^{\times}$ with

$$ {}^{\kappa}K^{*}_{U(1)}(SU(2);\mathbb{C})\Big|_{\zeta} \simeq {}^{\kappa}H^{*}_{U(1)}(SU(2)^{\zeta};{}^{\kappa}\mathcal{L}(\zeta)), \tag{36} $$

where ${}^{\kappa}\mathcal{L}(\zeta)$ is a certain flat line bundle. Note that when we say that ζ acts on $SU(2)$, we really mean that $\zeta/|\zeta| \in U(1)$ acts on $SU(2)$.

In general, the knowledge of the stalks is not enough to reconstruct the sheaf, for example, every fiber of a line bundle is just isomorphic to $\mathbb{C}$. However, in the case of a single minimal model, the sheaf turns out to be a skyscraper sheaf, and can indeed be reconstructed.

D. Twisted equivariant cohomology of the minimal model

In this section, we are going to determine the twisted equivariant cohomology groups that appear in the Chern character formula, Eq. (36). We advise the reader who is not interested in all the details to note the result, Eqs. (45a) and (45b), and then proceed with the next section.

In fact, the problem is very similar to ${}^{\kappa}K_{SU(2)}(SU(2);\mathbb{C})$ that is explicitly worked out as an example in Ref. 28. Depending on ζ, there are two different fixed point sets. One possibility is $\zeta \in \mathbb{R}_{>0}$, which acts trivially on the whole $SU(2)$. It turns out[28] that the line bundle ${}^{\kappa}\mathcal{L}(\zeta)$ is trivial in that case. Therefore, the *untwisted* equivariant cohomology is

$$ H^{*}_{U(1)}(SU(2)^{\zeta};{}^{\kappa}\mathcal{L}(\zeta)) = H^{*}_{U(1)}(SU(2);\mathbb{C}) = \mathbb{C}[u,t]/u^2, \tag{37} $$

where we used the Leray spectral sequence

$$ H^{p}(BU(1);H^{q}(SU(2);\mathbb{C})) \Rightarrow H^{p+q}_{U(1)}(SU(2);\mathbb{C}) \tag{38} $$

with $t \in H^2(BU(1);\mathbb{C})$ of degree 2 and $u \in H^3(SU(2);\mathbb{C})$ of degree 3. To determine the twisted equivariant cohomology ${}^{\kappa}H^{*}_{U(1)}(SU(2);\mathbb{C})$ from the untwisted one, we have to mode out by the additional differential $d_3 = \kappa u$. (More formally, we are using the untwisted to twisted cohomology spectral sequence. Note that $(d_3)^2 \sim u^2 = 0$ in $\mathbb{C}[u,t]/u^2$.) An easy computation shows that

$$ {}^{\kappa}H^{*}_{U(1)}(SU(2)^{\zeta};{}^{\kappa}\mathcal{L}(\zeta)) = {}^{\kappa}H^{*}_{U(1)}(SU(2);\mathbb{C}) = \frac{\ker(d_3)}{\mathrm{img}(d_3)} = 0. \tag{39} $$

This settles the case where the whole $SU(2)$ is fixed under the ζ action. The other possibility is the generic case where $SU(2)^{\zeta} = S^1_A$. In that case, the flat line bundle ${}^{\kappa}\mathcal{L}(\zeta)$ over S^1_A has[28] holonomy ζ^{κ}, so all cohomology groups vanish unless $\zeta^{\kappa} = 1$. In that case, that is over the $\kappa - 1$ points,

$$ \zeta_m \overset{\text{def}}{=} e^{2\pi i m/\kappa}, \quad m = 1, \ldots, \kappa - 1, \tag{40} $$

the untwisted cohomology is

$$ H^{*}_{U(1)}(S^1_A;{}^{\kappa}\mathcal{L}(\zeta_m)) = H^{*}(BU(1);\mathbb{C}) \otimes H^{*}(S^1_A;{}^{\kappa}\mathcal{L}(\zeta_m)) = \mathbb{C}[t] \otimes \mathbb{C}[v]/v^2 = \mathbb{C}[v,t]/v^2, \tag{41} $$

where $\deg(v) = 1$ and $\deg(t) = 2$. The twist class is in $H^3_{U(1)}(S^1_A;{}^{\kappa}\mathcal{L}(\zeta_m)) = \mathbb{C} \cdot tv$. Hence, if one normalizes the tv properly, then $d_3 = \kappa tv$. The d_3 cohomology is

$$ {}^{\kappa}H^{*}_{U(1)}(S^1_A;{}^{\kappa}\mathcal{L}(\zeta_m)) = \frac{\ker(d_3)}{\operatorname{img}(d_3)} = \mathbb{C}v = \begin{cases} \mathbb{C}, & * = 1; \\ 0, & \text{otherwise}. \end{cases} \tag{42} $$

In addition to the $U(1)$ action on $SU(2)$, we can also act with $\mathbb{Z}_\kappa$. We find two more cases: the fixed point set can be either S^1_B or empty. The cohomology of the empty set, of course, vanishes. In the former case, note that $U(1)$ acts simply transitive on S^1_B, so the equivariant cohomology is just the cohomology of a point. To summarize, there are four different cases corresponding to different $g \in U(1) \times \mathbb{Z}_\kappa$. The twisted cohomology groups are (ignoring the $\mathbb{Z}_\kappa$ action on the cohomology and the precise degrees for now)

$$ g = (1,0) \Rightarrow {}^{\kappa}H^{*}_{U(1)}(SU(2)^g;{}^{\kappa}\mathcal{L}(g)) = {}^{\kappa}H^{*}_{U(1)}(SU(2);\mathbb{C}) = 0, \tag{43a} $$

$$ g = (\zeta,0) \Rightarrow {}^{\kappa}H^{*}_{U(1)}(SU(2)^g;{}^{\kappa}\mathcal{L}(g)) = {}^{\kappa}H^{*}_{U(1)}(S^1_A;{}^{\kappa}\mathcal{L}(g)) = \delta_{\zeta^\kappa,1}\mathbb{C}, \tag{43b} $$

$$ g = (1,n) \Rightarrow {}^{\kappa}H^{*}_{U(1)}(SU(2)^g;{}^{\kappa}\mathcal{L}(g)) = H^{*}_{U(1)}(S^1_B;\mathbb{C}) = H^{*}(\{\mathrm{pt}.\}) = \mathbb{C}, \tag{43c} $$

$$ g = (\zeta,n) \Rightarrow {}^{\kappa}H^{*}_{U(1)}(SU(2)^g;{}^{\kappa}\mathcal{L}(g)) = H^{*}_{U(1)}(\varnothing;\mathbb{C}) = 0, \tag{43d} $$

where we took $n \in \mathbb{Z}_\kappa \backslash \{0\}$ and $\zeta \in \mathbb{C}^\times \backslash \{1\}$.

All that remains is to determine the precise action of $\mathbb{Z}_\kappa$ on the cohomology group, Eq. (43b). For that, note that even though the line bundle ${}^{\kappa}\mathcal{L}(\zeta_m)$ in Eq. (41) is trivial, the trivializing section winds m times around the S^1_A relative to *the* trivial line bundle. Therefore rotating S^1_A by $2\pi/\kappa$ multiplies v with the phase $\exp(2\pi i m/\kappa)$. In terms of the character $\chi:\mathbb{Z}_\kappa \to U(1)$, $m \mapsto \exp(2\pi i m/\kappa)$, this means that

$$ \bigoplus_{\zeta \in \mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)}(SU(2)^\zeta;{}^{\kappa}\mathcal{L}(\zeta)) = \left\{ \begin{array}{cc} 0, & * = 0 \\ \chi + \chi^2 + \cdots + \chi^{\kappa-1}, & * = 1 \end{array} \right\} = \widetilde{R\mathbb{Z}_\kappa}, \tag{44} $$

as the $\mathbb{Z}_\kappa$ representation. In other words, we can write the twisted equivariant cohomology groups as

$$ n = 0 \in \mathbb{Z}_\kappa \Rightarrow \bigoplus_{\zeta \in \mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)}(SU(2)^{(\zeta,n)};{}^{\kappa}\mathcal{L}(\zeta,n)) = \widetilde{R\mathbb{Z}_\kappa}, \tag{45a} $$

$$ n \neq 0 \in \mathbb{Z}_\kappa \Rightarrow \bigoplus_{\zeta \in \mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)}(SU(2)^{(\zeta,n)};{}^{\kappa}\mathcal{L}(\zeta,n)) = \mathbb{C}, \tag{45b} $$

using the conventions for cohomology degrees in Eqs. (23a), (23b).

E. Mirror symmetry for minimal models

As a quick application, let us compute the K-groups of the minimal model and its $\mathbb{Z}_\kappa$ orbifold. According to the twisted equivariant Chern character, the K-groups of the minimal model are

$$ {}^{\kappa}K^{*}_{U(1)}(SU(2);\mathbb{C}) = \bigoplus_{\zeta \in \mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)}(SU(2)^\zeta;{}^{\kappa}\mathcal{L}(\zeta)) = \widetilde{R\mathbb{Z}_\kappa} \simeq \begin{cases} 0, & * = 0; \\ \mathbb{C}^{\kappa-1}, & * = 1, \end{cases} \tag{46} $$

using the cohomology groups computed in Eq. (45a). We recover the known[19] D-brane charge groups for the coset minimal model.

Similarly, we can compute the D-brane charge group in the $\mathbb{Z}_\kappa$ orbifold, which is known to be the mirror of the minimal model. One obtains

$$\begin{aligned}
{}^{\kappa}K^{*}_{U(1)\times\mathbb{Z}_\kappa}\Big(SU(2);\mathbb{C}\Big) &= \bigoplus_{(\zeta,n)\in\mathbb{C}^\times\times\mathbb{Z}_\kappa} {}^{\kappa}H^{*}_{U(1)\times\mathbb{Z}_\kappa}\Big(SU(2)^{(\zeta,n)};{}^{\kappa}\mathcal{L}(\zeta,n)\Big)\\
&= \bigoplus_{n\in\mathbb{Z}_\kappa}\bigoplus_{\zeta\in\mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)\times\mathbb{Z}_\kappa}\Big(SU(2)^{(\zeta,n)};{}^{\kappa}\mathcal{L}(\zeta,n)\Big)\\
&= \bigoplus_{n\in\mathbb{Z}_\kappa}\left[\bigoplus_{\zeta\in\mathbb{C}^\times} {}^{\kappa}H^{*}_{U(1)}\Big(SU(2)^{(\zeta,n)};{}^{\kappa}\mathcal{L}(\zeta,n)\Big)\right]^{\mathbb{Z}_\kappa}\\
&= \Bigg[\underbrace{\widetilde{R\mathbb{Z}_\kappa}}_{n=0}\oplus\underbrace{\mathbb{C}}_{n=1}\ominus\cdots\oplus\underbrace{\mathbb{C}}_{n=\kappa-1}\Bigg]^{\mathbb{Z}_\kappa}\\
&= \mathbb{C}^{\kappa-1}\simeq\begin{cases}\mathbb{C}^{\kappa-1}, & *=0;\\ 0, & *=1.\end{cases}
\end{aligned} \tag{47}$$

Note that the $\mathbb{Z}_\kappa$ equivariant cohomology is simply the $\mathbb{Z}_\kappa$ invariant subspace of the cohomology group. For that, it is important to work with complex coefficients, because it would generate torsion contributions over the integers. Also note that the $\mathbb{Z}_\kappa$ equivariant K-theory is in general *not* the same as the $\mathbb{Z}_\kappa$ invariant K-groups.

To summarize, we observe that the $\mathbb{Z}_\kappa$ orbifold indeed exchanges $K^0\leftrightarrow K^1$, as we expect from the mirror involution. Furthermore, recall the distinction between A- and B-type branes.[15] The A-branes carry the charges in Eq. (45a), contributing to K^1 of the minimal model. On the other hand, the B-branes Eq. (45b), are only stable in the $\mathbb{Z}_\kappa$ orbifold of the minimal model where they contribute to K^0.

F. K-groups for Gepner models

Having tackled a single minimal model, we now proceed to Gepner models.[4,37–39] For that we take d copies of the $SU(2)$ with the action of d copies of $U(1)$ factor by factor. That is,

$$U(1)^d\times SU(2)^d\to SU(2)^d, \tag{48}$$

with a choice of twist

$$\bar{\kappa}=(\kappa_1,\kappa_2,\ldots,\kappa_d), \tag{49}$$

where $k_i=\kappa_i-2$ is the level in the CFT of the ith factor. The overall central charge is

$$c=\sum_{i=1}^{d}\frac{3k_i}{k_i+2}=\sum_{i=1}^{d}\frac{3(\kappa_i-2)}{\kappa_i}. \tag{50}$$

Whenever $c/3$ is integer, this could be the central charge of a geometric compactification of that dimension. However, a mere tensor product of minimal models is never geometric because of noninteger charges. In other words, it does not have space-times supersymmetry. The solution to this problem[37] is to orbifold by a certain discrete symmetry group G_{GSO}.

As we have seen, each of the minimal models has a discrete symmetry group $\mathbb{Z}_{\kappa_i}=\{0,1,\ldots,\kappa_i-1\}$. The GSO projection is the group generated by

$$(1,1,\ldots,1)\in\prod_{i=1}^{d}\mathbb{Z}_{\kappa_i}. \tag{51}$$

It follows that

$$G_{\mathrm{GSO}} = \mathbb{Z}_{\mathrm{lcm}(\kappa_1,\kappa_2,\ldots,\kappa_d)}. \tag{52}$$

According to the general dictionary between *D*-brane charge groups and K-theory group, the *D*-brane charges in the Gepner model are

$$^{\bar{\kappa}}K_{U(1)^d\times G_{\mathrm{GSO}}}(SU(2)^d). \tag{53}$$

We can again compute (the complexification) through the twisted equivariant Chern character. Once we translate the K-groups into cohomology, we can use the following:

- the G_{GSO} equivariant cohomology is the G_{GSO} invariant cohomology and
- the Künneth theorem for cohomology,

neither of which hold in general for twisted equivariant K-theory. Again, we have to complexify

$$U(1)^d \times G_{\mathrm{GSO}} \rightsquigarrow (\mathbb{C}^\times)^d \times G_{\mathrm{GSO}} \tag{54}$$

and think of the cohomology and K-groups as sheaves over this space. According to Sec. III D, the only potentially nonvanishing cohomology groups for the ith minimal model sit over the κ_ith roots of unity,

$$\mathcal{Z}_i \overset{\mathrm{def}}{=} \{e^{2\pi i m/\kappa_i} | m \in \mathbb{Z}_{\kappa_i} = \{0,\ldots,\kappa_i - 1\}\} \subset \mathbb{C}^\times, \tag{55}$$

therefore the only nonvanishing cohomology groups of the product are over the points

$$\mathcal{Z} \overset{\mathrm{def}}{=} \prod_{i=1}^{d} \mathcal{Z}_i = \{(e^{2\pi i m_1/\kappa_1},\ldots,e^{2\pi i m_d/\kappa_d} | m_i \in \mathbb{Z}_{\kappa_i}\} \subset (\mathbb{C}^\times)^d. \tag{56}$$

Using all that, we obtain

$$\begin{aligned}
^{\bar{\kappa}}K^*_{U(1)^d}(SU(2)^d;\mathbb{C}) &= \bigoplus_{g\in G_{\mathrm{GSO}}} \left[\bigoplus_{\bar{z}\in(\mathbb{C}^\times)^d} {}^{\bar{\kappa}}H^*_{U(1)^d}\left(\bigtimes_{i=1}^{d} SU(2)^{(z_i,g)}; \bigotimes_{i=1}^{d} {}^{\kappa_i}\mathcal{L}(z_i,g)\right)\right]^{G_{\mathrm{GSO}}} \\
&= \bigoplus_{g\in G_{\mathrm{GSO}}} \left[\bigoplus_{\bar{z}\in\mathcal{Z}} {}^{\bar{\kappa}}H^*_{U(1)^d}\left(\bigtimes_{i=1}^{d} SU(2)^{(z_i,g)}; \bigotimes_{i=1}^{d} {}^{\kappa_i}\mathcal{L}(z_i,g)\right)\right]^{G_{\mathrm{GSO}}} \\
&= \bigoplus_{g\in G_{\mathrm{GSO}}} \left[\bigotimes_{i=1}^{d} \left\{ \bigoplus_{z_i\in\mathcal{Z}_i} {}^{\kappa_i}H^*_{U(1)}(SU(2)^{(z_i,g)}; {}^{\kappa_i}\mathcal{L}(z_i,g))\right\}\right]^{G_{\mathrm{GSO}}}.
\end{aligned} \tag{57}$$

Note that according to Eqs. (45a) and (45b),

$$\bigoplus_{z_i\in\mathcal{Z}_i} {}^{\kappa_i}H^*_{U(1)}(SU(2)^{(z_i,g)}; {}^{\kappa_i}\mathcal{L}(z_i,g)) = \begin{cases} \widetilde{R\mathbb{Z}_{\kappa_i}}, & g \equiv 0 \bmod \kappa_i \Leftrightarrow \kappa_i | g; \\ \mathbb{C}, & \kappa_i \nmid g. \end{cases} \tag{58}$$

Moreover, G_{GSO} obviously acts on $\widetilde{R\mathbb{Z}_{\kappa_i}}$ as $p^{G_{\mathrm{GSO}}}_{\mathbb{Z}_{\kappa_i}}(\widetilde{R\mathbb{Z}_{\kappa_i}})$, see Eq. (25). Therefore, we can simplify Eq. (57) to

TABLE I. Elliptic curves with enhanced automorphism groups.

Complex structure	Symmetry	Gepner model	Hypersurface
$\tau=i$	$\mathbb{Z}_4$	$k=(0,2,2)$	$\{x_0^2+x_1^4+x_2^4=0\}\subset\mathbb{WP}_{2,1,1}$
$\tau=e^{2\pi i/3}$	$\mathbb{Z}_6$	$k=(1,1,1)$	$\{x_0^3+x_1^3+x_2^3=0\}\subset\mathbb{WP}_{1,1,1}$
$\tau=e^{2\pi i/3}$	$\mathbb{Z}_6$	$k=(0,1,4)$	$\{x_0^2+x_1^3+x_2^6=0\}\subset\mathbb{WP}_{3,2,1}$

$$ {}^{\kappa}K^{*}_{U(1)^d}(SU(2)^d;\mathbb{C}) = \bigoplus_{g\in G_{\mathrm{GSO}}}\left[\bigotimes_{\kappa_i|g} p^{G_{\mathrm{GSO}}}_{\mathbb{Z}_{\kappa_i}}(\widetilde{R\mathbb{Z}_{\kappa_i}})\right]^{G_{\mathrm{GSO}}}, \tag{59} $$

where we would like to remind the reader that $n|0$ for all n, that $\otimes_{i\in\varnothing}=\mathbb{C}$, and that we defined $\widetilde{R\mathbb{Z}_{\kappa_i}}$ to have cohomological degree 1.

IV. EXAMPLES

A. Toroidal theories

There are three Gepner models[42] that describe an elliptic curve. Two of them, $k=(1,1,1)$ and $k=(0,1,4)$, turn out to be the same CFT (for example, have identical partition functions). Hence, we obtain two different CFTs corresponding to the two orbifold singularities in the complex structure moduli space of the torus; see Table I. Recall that each elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ has a $\mathbb{Z}_2$ symmetry, but at $\tau=i$ and $\tau=\exp(2\pi i/3)$ the symmetry is enhanced to $\mathbb{Z}_4$ and $\mathbb{Z}_6$, respectively. We easily compute using Eq. (59) that in all three cases,

$$ {}^{\kappa}K^{*}_{U(1)^3\times G_{\mathrm{GSO}}}(SU(2)^3;\mathbb{C}) = \begin{Bmatrix} \mathbb{C}^2, & *=0 \\ \mathbb{C}^2, & *=1 \end{Bmatrix} = K^{*}(T^2;\mathbb{C}), \tag{60} $$

as expected, since we are dealing with a topological invariant of the torus. Note that the toroidal Gepner models always have three factors, even if that forces one of the levels to be zero. It is important to realize[43,44] that adding one factor with $c=0$ in the Gepner model does indeed have a physical effect. For example, we can easily compute the D-brane charges in the $k=(2,2)\Leftrightarrow\kappa=(4,4)$ model and obtain

$$ {}^{(4,4)}K^{*}_{U(1)^2\times G_{\mathrm{GSO}}}(SU(2)^2;\mathbb{C}) = \begin{cases} \mathbb{C}^6, & *=0; \\ 0, & *=1. \end{cases} \tag{61} $$

This is not the D-brane charge group of any geometric $c=3$ CFT. Note that the usual argument why $k=0$ factors do not matter is wrong: in the corresponding Landau-Ginzburg model, the $k=0$ factor corresponds to a field Φ that appears in the superpotential as

$$ W_{\mathrm{LG}} = \cdots + \Phi^2. \tag{62} $$

Folklore says that one can integrate out Φ at no cost. But that is only true if one restricts to the closed string sector, if one considers D-branes and open strings,[43] then one must include a boundary action that will contain Φ as well.

B. Twisted sectors

Let us have a closer look at the formula for the K-groups of a Gepner model, Eq. (59). First, let us rewrite it as

$$ {}^{\kappa}K^{*}_{U(1)^d}(SU(2)^d;\mathbb{C}) = \bigoplus_{g\in G_{\mathrm{GSO}}}(\mathcal{K}_g)^{G_{\mathrm{GSO}}}, \tag{63} $$

with

$$\mathcal{K}_g \overset{\text{def}}{=} \bigotimes_{\kappa_i | g} p^{G_{\text{GSO}}}_{\mathbb{Z}_{\kappa_i}}(\widetilde{R\mathbb{Z}}_{\kappa_i}). \tag{64}$$

Obviously, this has an interpretation of $\mathcal{K}_g^{G_{\text{GSO}}}$ being the contribution of the g-twisted sector in the G_{GSO} orbifold. Note that a single tensor factor $p^{G_{\text{GSO}}}_{\mathbb{Z}_{\kappa_i}}(\widetilde{R\mathbb{Z}}_{\kappa_i})$ does not have any G_{GSO}-invariant subspace, so the only way to obtain something invariant is to either have zero factors (which yields a B-type brane), or ≥ 2 factors. This is very familiar from the geometric interpretation as hypersurfaces in weighted projective spaces. If two or more weights $|G_{\text{GSO}}|/\kappa_i$ have a common factor, then the Calabi-Yau hypersurface inherits an orbifold singularity from the ambient space. The exceptional divisor from the resolution of the singularity increases the rank of the K-groups.

Specifically, in complex dimension ≥ 2 one can have genuine singularities that require resolutions and contribute twisted sector D-brane charges. To see that explicitly within the Gepner model context, let us consider the following two $K3$ Gepner models. First consider the $(k=2)^4$ Gepner model, corresponding to the Fermat quartic,

$$\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^3. \tag{65}$$

In this case, the ambient space and the hypersurface are nonsingular. The contribution of the untwisted and the three g-twisted sectors is

$g \in G_{\text{GSO}}$	0	1	2	3
$\mathcal{K}_g$	$(\widetilde{R\mathbb{Z}}_4)^4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
$\dim_{\mathbb{C}} \mathcal{K}_g^{G_{\text{GSO}}}$	21	1	1	1
Type: Even/Odd	O^4	E^4	E^4	E^4

$$\Rightarrow \; {}^{(4,4,4,4)}K^*_{U(1)^4}\big(SU(2)^4;\mathbb{C}\big) = \\ = \begin{cases} \mathbb{C}^{24}, & * = 0; \\ 0, & * = 1. \end{cases} \tag{66}$$

We can do the same for the $k=(1,2,2,4) \Leftrightarrow \kappa=(3,4,4,6)$ Gepner model. It corresponds to the singular $K3$ hypersurface,

$$X \overset{\text{def}}{=} \{x_0^3 + x_1^4 + x_2^4 + x_3^6 = 0\} \subset \mathbb{WP}_{4,3,3,2}. \tag{67}$$

The weighted projective space has a rational curve C_2 of $\mathbb{C}^2/\mathbb{Z}_2$ singularities and another rational curve C_3 of $\mathbb{C}^2/\mathbb{Z}_3$ singularities embedded as

$$C_2 \hookrightarrow \mathbb{WP}_{4,3,3,2}, \quad [s_0,s_1] \mapsto [s_0,0,0,s_1],$$

$$C_3 \hookrightarrow \mathbb{WP}_{4,3,3,2}, \quad [s_0,s_1] \mapsto [0,s_0,s_1,0]. \tag{68}$$

The surface inherits $4A_1$ and $6A_2$ orbifold singularities from

$$C_2 \cap X = 4, \quad C_3 \cap X = 6. \tag{69}$$

The resolution $\widetilde{X}$ is then a smooth $K3$ surface. This concludes the geometric point of view, now let us analyze the K-theory computation from the Gepner model side. Using Eq. (63), we find

$g \in G_{\text{GSO}}$	0	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{K}_g$	$\widetilde{R\mathbb{Z}}_{3,4,4,6}$	$\mathbb{C}$	$\mathbb{C}$	$\widetilde{R\mathbb{Z}}_3$	$\widetilde{R\mathbb{Z}}_{4,4}$	$\mathbb{C}$	$\widetilde{R\mathbb{Z}}_{3,6}$	$\mathbb{C}$	$\widetilde{R\mathbb{Z}}_{4,4}$	$\widetilde{R\mathbb{Z}}_3$	$\mathbb{C}$	$\mathbb{C}$
$\dim_{\mathbb{C}} \mathcal{K}_g^{G_{\text{GSO}}}$	10	1	1	0	3	1	2	1	3	0	1	1
Type: Even/Odd	O^4	E^4	E^4	–	EO^2E	E^4	OE^2O	E^4	EO^2E	–	E^4	E^4

(70)

where we abbreviated

$$\widetilde{R\mathbb{Z}}_{\kappa_1,\kappa_2,\ldots} \stackrel{\text{def}}{=} \bigotimes_{i=1,2,\ldots} p^{G_{\text{GSO}}}_{\mathbb{Z}_{\kappa_i}}(\widetilde{R\mathbb{Z}}_{\kappa_i}). \tag{71}$$

Of course, in the end we obtain again the K-groups of the $K3$ manifold. However, this times some of the charge groups involve mixtures of even- and odd-dimensional branes. In the same way one can analyze all $K3$ Gepner models; see Appendix A.

V. KNÖRRER PERIODICITY

If one adds two variables with a quadratic superpotential to the Landau-Ginzburg theory[45–47] with fields $\Phi=(\phi_1,\ldots)$,

$$W_{\text{LG}}(\Phi) \to \widehat{W}_{\text{LG}} = W_{\text{LG}}(\Phi) + x^2 + y^2, \tag{72}$$

then one obtains the same theory again. This is quite nontrivial, because adding a single variable with a quadratic superpotential certainly does yield an inequivalent theory as discussed in Sec. IV A.

The evidence for periodicity is that the topological B-branes, that is the category of matrix factorizations, are equivalent. This fact is known as Knörrer periodicity,[48]

$$\mathbf{MF}(\mathbb{C}[\Phi]/W_{\text{LG}}(\Phi)) \simeq \mathbf{MF}(\mathbb{C}[\Phi,x,y]/\widehat{W}_{\text{LG}}(\Phi,x,y)). \tag{73}$$

This periodicity manifests itself in our formula Eq. (59), as follows. Adding two factors with $k=0 \Leftrightarrow \kappa=2$ amounts to inserting

$$p^{G_{\text{GSO}}}_{\mathbb{Z}_2}(\widetilde{R\mathbb{Z}}_2) \otimes p^{G_{\text{GSO}}}_{\mathbb{Z}_2}(\widetilde{R\mathbb{Z}}_2) = \mathbb{C} \tag{74}$$

whenever $2|g$. But

$$(\cdots) \otimes \mathbb{C} = (\cdots) \tag{75}$$

is the identity, so we obtain again the same K-groups.

Note that the above argument is flawed since adding the $\kappa=2$ factors might change the G_{GSO} group Eq. (52). If the initial order $|G_{\text{GSO}}|$ was odd, that is,

$$\text{lcm}(\kappa_1, \ldots, \kappa_d) \in 2\mathbb{Z}+1, \tag{76}$$

then

$$\text{lcm}(\kappa_1, \ldots, \kappa_d, 2, 2) = 2\,\text{lcm}(\kappa_1, \ldots, \kappa_d). \tag{77}$$

Therefore, periodicity only holds if one had already an even κ_i. In general, Knörrer periodicity need not hold for the first time one adds two $k=0$ factors, but it always holds from the second time onward,

$$^{(\kappa_1,\ldots,\kappa_d,2)}K_{U(1)^{d+1}\times G_{\text{GSO}}}(SU(2)^{d+1};\mathbb{C}) = {}^{(\kappa_1,\ldots,\kappa_d,2,2,2)}K_{U(1)^{d+3}\times G_{\text{GSO}}}(SU(2)^{d+3};\mathbb{C}). \tag{78}$$

This is somewhat reminiscent of stabilization in K-theory.

VI. GENERALIZED PERMUTATION BRANES

In this section, we are going to focus on the Calabi-Yau ($c=9$) Gepner models. It is clear from Sec. III E that all D-brane charges can be found as suitable combinations of the D-branes in the coset or its mirror ($\mathbb{Z}_\kappa$ orbifold). In particular, the usual tensor product and permutation branes give

TABLE II. Gepner models associated to $K3$.

$\bar{k}=(1,1,1,1,1,1)$,	$\bar{k}=(0,1,1,1,1,4)$,	$\bar{k}=(2,2,2,2)$,	$\bar{k}=(1,2,2,4)$
$\bar{k}=(1,1,4,4)$,	$\bar{k}=(1,1,2,10)$,	$\bar{k}=(0,4,4,4)$,	$\bar{k}=(0,3,3,8)$
$\bar{k}=(0,2,6,6)$,	$\bar{k}=(0,2,4,10)$,	$\bar{k}=(0,2,3,18)$,	$\bar{k}=(0,1,10,10)$
$\bar{k}=(0,1,8,13)$,	$\bar{k}=(0,1,7,16)$,	$\bar{k}=(0,1,6,22)$,	$\bar{k}=(0,1,5,40)$

all the *D*-brane charges in the untwisted sector, corresponding to $g=0$ in Eq. (59). Similarly, one obtains zero or one brane in the twisted $(g=1,\ldots,|G_{\mathrm{GSO}}|)$ sectors. But the latter is not enough to fill out the *D*-brane charge lattice, in general, since sometimes there are two or more independent charges coming from a twisted sector. Of course, all that means is that the boundary state construction is incomplete. Using Landau-Ginzburg models and matrix factorizations one obtains[11,49] all brane charges.

Inspection of the formula for the K-groups, Eq. (59), shows that two or more brane charges can only come from a $g\in G_{\mathrm{GSO}}$ sector where some κ_i divides g. Moreover, if only a single κ_i divides g, then there is no contribution because

$$[p_{\mathbb{Z}_{\kappa_i}}^{G_{\mathrm{GSO}}}(\widetilde{R\mathbb{Z}_{\kappa_i}})]^{G_{\mathrm{GSO}}}=0 \tag{79}$$

has no invariant subspace. Hence, the interesting case is if two or more κ_i have a common factor. Following Ref. 11, let us consider the case where r of the shifted levels $\bar{\kappa}=(\kappa_1,\ldots)$ have the same divisor $d>2$. (If the common divisor $d=2$, then there is again only a one-dimensional contribution to the K group in the $g\in d\mathbb{Z}$ twisted sectors, which is not so interesting. Of course, our arguments hold in that case as well).

First, note that r odd contributes to K^1 only, as is evident from our degree convention, Eqs. (23a) and (23b). Not so surprisingly, if one[11] restricts oneself to K^0 then there are no *D*-brane charges for $r=1,3,5$. This leaves the cases $r=2$ and $r=4$. Looking at the list of Gepner models, $r=4$ can only occur if the Gepner model has more than five minimal model factors. There is nothing wrong with that, and our formula, Eq. (59), gives the correct answer for the K-groups. However, if one[11] were to restrict oneself to 5 minimal model factors, then $r=4$ cannot occur either.

VII. CONCLUSIONS

There is a very simple formula, Eq. (59), for the rank of the K-groups of Gepner models. The summands in the formula have a natural interpretation as the contributions from twisted sectors. We checked the computation in $c=3,6,9$ Gepner models and find agreement with the topology of the associated Calabi-Yau manifolds.

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APPENDIX A: K3 GEPNER MODELS

There are 16 Gepner models that are associated to $K3$ surfaces[50–52] listed in Table II. We checked that we obtain

$$^{\bar{k}}K^{*}_{U(1)^d\times G_{\rm GSO}}(SU(2)^d;\mathbb{C}) = \begin{Bmatrix} \mathbb{C}^{24}, & * = 0 \\ 0, & * = 1 \end{Bmatrix} = K^*(K3;\mathbb{C}) \tag{A1}$$

in all 16 cases. It is important that the right number of $k=0$ factors appears so that there are four minimal models altogether (exceptionally, six in the first two Gepner models).

In addition to the 16 known $K3$ Gepner models, we found that

$$^{(2,3,3,3,3,3,3)}K^{*}_{U(1)^7\times G_{\rm GSO}}(SU(2)^7;\mathbb{C}) = K^*(K3;\mathbb{C}), \tag{A2}$$

as well. Although it has not the conventional number of factors, this $\bar{k}=(0,1,1,1,1,1,1)$ Gepner model seems to yield yet another $K3$ CFT.

There is yet another combination of levels such that the total central charge $c=6$, which is $\bar{k}=(0,1,1,1,2,2)$. One can easily compute that

$$^{(2,3,3,3,4,4)}K^{*}_{U(1)^6\times G_{\rm GSO}}(SU(2)^6;\mathbb{C}) = \begin{Bmatrix} \mathbb{C}^{8}, & * = 0 \\ \mathbb{C}^{8}, & * = 1 \end{Bmatrix} = K^*(T^4;\mathbb{C}).$$

Clearly, this Gepner model describes a T^4 compactification with (accidentally) enhanced $\mathcal{N}=8$ space-time supersymmetry.

APPENDIX B: CALABI-YAU THREEFOLD GEPNER MODELS

First, note that a proper Calabi-Yau threefold X, that is a compact Kähler manifold of holonomy $SU(3)$ satisfies

$$\mathrm{rank}\, K^0(X) = 2h^{11}(X) + 2, \quad \mathrm{rank}\, K^1(X) = 2h^{21}(X) + 2. \tag{B1}$$

We can check this formula against the known list[53] of 168 Gepner models with central charge $c=9$, which are associated to Calabi-Yau threefolds. The list of all Gepner models is reproduced in Table III. If one uses these $N=(2,2)$ SCFTs as the compactification of the $E_8\times E_8$ heterotic string, then their low-energy spectrum consists of a number $n_{\overline{\mathbf{27}}}=h^{11}(X)$ of matter fields transforming in the $\overline{\mathbf{27}}$ and $n_{\overline{\mathbf{27}}}=h^{21}(X)$ of field in the $\mathbf{27}$ representation of E_6.

One can check that the formula, Eq. (B1), is obeyed for each Gepner model except for the seven cases with $n_{\overline{\mathbf{27}}}=n_{\mathbf{27}}=21$. The obvious explanation is that this misfit is associated $K3\times T^2$, which has Hodge numbers

$$h^{pq}(K3\times T^2) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 21 & & 1 & \\ 1 & & 21 & & 21 & & 1 \\ & 1 & & 21 & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}. \tag{B2}$$

Since $K3\times T^2$ has only $SU(2)$ holonomy, that is, it is not a proper Calabi-Yau manifold, it does not have to obey Eq. (B1). Adding up the even and odd cohomology groups, we find that

$$K^0(K3\times T^2) = \mathbb{Z}^{48}, \quad K^1(K3\times T^2) = \mathbb{Z}^{48}. \tag{B3}$$

These topological K-groups are in precise agreement with what we computed using the coset, Eq. (59).

TABLE III. $c=9$ Gepner models.

$\bar{k}=(k_1,k_2,\ldots)$	$n_{\overline{27}}$	n_{27}	$\mathrm{rk}K^1$	$\mathrm{rk}K^0$
(1, 1, 1, 1, 1, 1, 1, 1, 1)	0	84	2	170
(1, 1, 1, 1, 1, 1, 1, 4, 0)	0	84	2	170
(1, 1, 1, 1, 1, 1, 2, 2, 0)	21	21	48	48
(1, 1, 1, 1, 1, 2, 10),	2	62	6	126
(1, 1, 1, 1, 1, 4, 4)	1	73	4	148
(1, 1, 1, 1, 2, 2, 4)	11	35	24	72
(1, 1, 1, 2, 2, 2, 2)	21	21	48	48
(1, 1, 1, 1, 5, 40, 0)	23	47	48	96
(1, 1, 1, 1, 6, 22, 0)	16	52	34	106
(1, 1, 1, 1, 7, 16, 0)	8	68	18	138
(1, 1, 1, 1, 8, 13, 0)	17	41	36	84
(1, 1, 1, 1, 10, 10, 0)	7	79	16	160
(1, 1, 1, 2, 3, 18, 0)	21	21	48	48
(1, 1, 1, 2, 4, 10, 0)	2	62	6	126
(1, 1, 1, 2, 6, 6, 0)	21	21	48	48
(1, 1, 1, 3, 3, 8, 0)	21	21	48	48
(1, 1, 1, 4, 4, 4, 0)	0	84	2	170
(1, 1, 2, 2, 2, 10, 0)	10	46	22	94
(1, 1, 2, 2, 4, 4, 0)	3	51	8	104
(1, 2, 2, 2, 2, 4, 0)	1	61	4	124
(2, 2, 2, 2, 2, 2, 0)	0	90	2	182
(1, 1, 2, 11, 154)	71	71	144	144
(1, 1, 2, 12, 82)	40	76	82	154
(1, 1, 2, 13, 58)	26	86	54	174
(1, 1, 2, 14, 46)	26	86	54	174
(1, 1, 2, 16, 34)	16	100	34	202
(1, 1, 2, 18, 28)	31	55	64	112
(1, 1, 2, 19, 26)	41	41	84	84
(1, 1, 2, 22, 22)	11	131	24	264
(1, 1, 3, 6, 118)	55	55	112	112
(1, 1, 3, 7, 43)	19	67	40	136
(1, 1, 3, 8, 28)	19	69	40	140
(1, 1, 3, 10, 18)	31	31	64	64
(1, 1, 3, 13, 13)	7	103	16	208
(1, 1, 4, 5, 40)	17	65	36	132
(1, 1, 4, 6, 22)	10	70	22	142
(1, 1, 4, 7, 16)	7	79	16	160
(1, 1, 4, 8, 13)	12	48	26	98
(1, 1, 4, 10, 10)	5	101	12	204
(1, 1, 5, 5, 19)	17	65	36	132
(1, 1, 6, 6, 10)	19	43	40	88
(1, 1, 7, 7, 7)	4	112	10	226
(1, 2, 2, 5, 40)	35	35	72	72
(1, 2, 2, 6, 22)	8	68	18	138
(1, 2, 2, 7, 16)	19	43	40	88
(1, 2, 2, 8, 13)	27	27	56	56
(1, 2, 2, 10, 10)	5	89	12	180
(1, 2, 3, 3, 58)	23	47	48	96
(1, 2, 3, 4, 18)	15	39	32	80
(1, 2, 4, 4, 10)	2	74	6	150
(1, 2, 4, 6, 6)	7	55	16	112
(1, 3, 3, 3, 13)	3	75	8	152
(1, 3, 3, 4, 8)	15	39	32	80
(1, 4, 4, 4, 4)	1	103	4	208

TABLE III. (*Continued.*)

$\bar{k}=(k_1,k_2,\ldots)$	$n_{\overline{27}}$	n_{27}	$\mathrm{rk}K^1$	$\mathrm{rk}K^0$
(2, 2, 2, 3, 18)	5	65	12	132
(2, 2, 2, 4, 10)	3	69	8	140
(2, 2, 2, 6, 6)	2	86	6	174
(2, 2, 3, 3, 8)	15	39	32	80
(2, 2, 4, 4, 4)	6	60	14	122
(3, 3, 3, 3, 3)	1	101	4	204
(0, 1, 5, 41, 1804)	251	251	504	504
(0, 1, 5, 42, 922)	137	257	276	516
(0, 1, 5, 43, 628)	95	263	192	528
(0, 1, 5, 44, 481)	143	143	288	288
(0, 1, 5, 46, 334)	47	287	96	576
(0, 1, 5, 47, 292)	47	287	96	576
(0, 1, 5, 49, 236)	107	107	216	216
(0, 1, 5, 52, 187)	53	173	108	348
(0, 1, 5, 54, 166)	23	335	48	672
(0, 1, 5, 58, 138)	59	131	120	264
(0, 1, 5, 61, 124)	17	377	36	756
(0, 1, 5, 68, 103)	29	221	60	444
(0, 1, 5, 76, 89)	83	83	168	168
(0, 1, 5, 82, 82)	11	491	24	984
(0, 1, 6, 23, 598)	119	167	240	336
(0, 1, 6, 24, 310)	66	174	134	350
(0, 1, 6, 25, 214)	48	180	98	362
(0, 1, 6, 26, 166)	34	190	70	382
(0, 1, 6, 28, 118)	24	204	50	410
(0, 1, 6, 30, 94)	18	222	38	446
(0, 1, 6, 31, 86)	57	81	116	164
(0, 1, 6, 34, 70)	14	242	30	486
(0, 1, 6, 38, 58)	23	143	48	288
(0, 1, 6, 40, 54)	33	105	68	212
(0, 1, 6, 46, 46)	9	321	20	644
(0, 1, 7, 17, 340)	71	143	144	288
(0, 1, 7, 18, 178)	42	150	86	302
(0, 1, 7, 19, 124)	28	160	58	322
(0, 1, 7, 20, 97)	45	93	92	188
(0, 1, 7, 22, 70)	15	183	32	368
(0, 1, 7, 25, 52)	10	214	22	430
(0, 1, 7, 28, 43)	18	126	38	254
(0, 1, 7, 34, 34)	7	271	16	544
(0, 1, 8, 14, 238)	50	134	102	270
(0, 1, 8, 16, 88)	17	155	36	312
(0, 1, 8, 18, 58)	10	178	22	358
(0, 1, 8, 22, 38)	22	82	46	166
(0, 1, 8, 28, 28)	5	251	12	504
(0, 1, 9, 12, 229)	79	79	160	160
(0, 1, 9, 13, 108)	59	59	120	120
(0, 1, 9, 20, 31)	9	129	20	260
(0, 1, 10, 11, 154)	23	143	48	288
(0, 1, 10, 12, 82)	15	147	32	296
(0, 1, 10, 13, 58)	11	155	24	312
(0, 1, 10, 14, 46)	8	164	18	330
(0, 1, 10, 16, 34)	5	185	12	372
(0, 1, 10, 18, 28)	10	106	22	214
(0, 1, 10, 19, 26)	16	76	34	154

TABLE III. (*Continued.*)

$\bar{k}=(k_1,k_2,\ldots)$	$n_{\overline{27}}$	n_{27}	rkK^1	rkK^0
(0, 1, 10, 22, 22)	3	243	8	488
(0, 1, 11, 11, 76)	23	143	48	288
(0, 1, 12, 12, 40)	6	180	14	362
(0, 1, 12, 13, 33)	43	43	88	88
(0, 1, 12, 19, 19)	7	151	16	304
(0, 1, 13, 13, 28)	4	208	10	418
(0, 1, 13, 18, 18)	11	107	24	216
(0, 1, 14, 14, 22)	7	127	16	256
(0, 1, 16, 16, 16)	2	272	6	546
(0, 2, 3, 19, 418)	119	119	240	240
(0, 2, 3, 20, 218)	65	125	132	252
(0, 2, 3, 22, 118)	33	141	68	284
(0, 2, 3, 23, 98)	33	141	68	284
(0, 2, 3, 26, 68)	39	87	80	176
(0, 2, 3, 28, 58)	17	173	36	348
(0, 2, 3, 34, 43)	55	55	112	112
(0, 2, 3, 38, 38)	11	227	24	456
(0, 2, 4, 11, 154)	53	89	108	180
(0, 2, 4, 12, 82)	30	96	62	194
(0, 2, 4, 13, 58)	20	104	42	210
(0, 2, 4, 14, 46)	16	112	34	226
(0, 2, 4, 16, 34)	12	126	26	254
(0, 2, 4, 18, 28)	20	74	42	150
(0, 2, 4, 19, 26)	28	52	58	106
(0, 2, 4, 22, 22)	8	164	18	330
(0, 2, 5, 8, 138)	44	80	90	162
(0, 2, 5, 10, 40)	23	59	48	120
(0, 2, 5, 12, 26)	8	116	18	234
(0, 2, 6, 7, 70)	19	91	40	184
(0, 2, 6, 8, 38)	12	96	26	194
(0, 2, 6, 10, 22)	6	114	14	230
(0, 2, 6, 14, 14)	4	148	10	298
(0, 2, 7, 7, 34)	19	91	40	184
(0, 2, 7, 10, 16)	10	70	22	142
(0, 2, 8, 8, 18)	6	120	14	242
(0, 2, 8, 10, 13)	18	42	38	86
(0, 2, 10, 10, 10)	3	165	8	332
(0, 3, 3, 9, 108)	39	79	80	160
(0, 3, 3, 10, 58)	25	85	52	172
(0, 3, 3, 12, 33)	27	59	56	120
(0, 3, 3, 13, 28)	9	117	20	236
(0, 3, 3, 18, 18)	7	143	16	288
(0, 3, 4, 6, 118)	33	69	68	140
(0, 3, 4, 7, 43)	19	67	40	136
(0, 3, 4, 8, 28)	7	91	16	184
(0, 3, 4, 10, 18)	13	49	28	100
(0, 3, 4, 13, 13)	7	103	16	208
(0, 3, 5, 5, 68)	23	71	48	144
(0, 3, 6, 6, 18)	7	63	16	128
(0, 3, 8, 8, 8)	1	145	4	292
(0, 4, 4, 5, 40)	8	86	18	174
(0, 4, 4, 6, 22)	6	90	14	182
(0, 4, 4, 7, 16)	3	99	8	200
(0, 4, 4, 8, 13)	7	61	16	124

TABLE III. (*Continued.*)

$\bar{k}=(k_1,k_2,\ldots)$	$n_{\overline{27}}$	n_{27}	$\mathrm{rk}K^1$	$\mathrm{rk}K^0$
(0, 4, 4, 10, 10)	2	128	6	258
(0, 4, 5, 5, 19)	17	65	36	132
(0, 4, 6, 6, 10)	6	66	14	134
(0, 4, 7, 7, 7)	4	112	10	226
(0, 5, 5, 5, 12)	2	122	6	246
(0, 6, 6, 6, 6)	1	149	4	300

[1] R. Minasian and G. W. Moore, "*K*-theory and Ramond-Ramond charge," J. High Energy Phys. 11, 002 (1997).
[2] E. Witten, "*D*-branes and K-theory," J. High Energy Phys. 12, 019 (1998).
[3] G. Moore, "K-theory from a physical perspective" http://arXiv.org/abs/ hep-th/0304018.
[4] A. Recknagel and V. Schomerus, "*D*-branes in Gepner models," Nucl. Phys. B **531**, 185 (1998).
[5] A. Recknagel, "Permutation branes," J. High Energy Phys. 04, 041 (2003).
[6] A. Recknagel, "On permutation branes," Fortschr. Phys. **51**, 824 (2003).
[7] I. Brunner and M. R. Gaberdiel, "Matrix factorisations and permutation branes," J. High Energy Phys. 07, 012 (2005).
[8] I. Brunner and M. R. Gaberdiel, "The matrix factorisations of the *D*-model," J. Phys. A **38**, 7901 (2005).
[9] H. Enger, A. Recknagel, and D. Roggenkamp, "Permutation branes and linear matrix factorisations," hep-th/0508053.
[10] S. Fredenhagen and T. Quella, "Generalised permutation branes," J. High Energy Phys. 11, 004 (2005).
[11] C. Caviezel, S. Fredenhagen, and M. R. Gaberdiel, "The RR charges of A-type Gepner models," hep-th/0511078.
[12] S. Fredenhagen and V. Schomerus, "Branes on group manifolds, gluon condensates, and twisted K-theory," J. High Energy Phys. 04, 007 (2001).
[13] P. Bouwknegt and V. Mathai, "D-branes, B-fields and twisted K-theory," J. High Energy Phys. 0003, 007 (2000).
[14] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, and D. Stevenson, "Twisted K-theory and K-theory of bundle gerbes," Commun. Math. Phys. **228**, 17 (2002).
[15] J. M. Maldacena, G. W. Moore, and N. Seiberg, "Geometrical interpretation of *D*-branes in gauged WZW models," J. High Energy Phys. 07, 046 (2001).
[16] J. M. Maldacena, G. W. Moore, and N. Seiberg, "*D*-brane instantons and K-theory charges," J. High Energy Phys. 11, 062 (2001).
[17] V. Braun, "Twisted K-theory of Lie groups," J. High Energy Phys. 03, 029 (2004).
[18] S. Fredenhagen, "Organizing boundary rg flows," Nucl. Phys. B **660**, 436 (2003).
[19] S. Schafer-Nameki, "*D*-branes in $N=2$ coset models and twisted equivariant K-theory," hep-th/0308058.
[20] M. R. Gaberdiel and T. Gannon, "The charges of a twisted brane," J. High Energy Phys. 01, 018 (2004).
[21] M. R. Gaberdiel and T. Gannon, "*D*-brane charges on non-simply connected groups," J. High Energy Phys. 04, 030 (2004).
[22] V. Braun and S. Schafer-Nameki, "Supersymmetric WZW models and twisted K-theory of $SO(3)$," hep-th/0403287.
[23] S. Fredenhagen, "*D*-brane charges on $SO(3)$," http://arXiv.org/abs/ hep-th/0404017.
[24] M. R. Gaberdiel, T. Gannon, and D. Roggenkamp, "The *D*-branes of $SU(n)$," J. High Energy Phys. 07, 015 (2004).
[25] M. R. Gaberdiel, T. Gannon, and D. Roggenkamp, "The coset *D*-branes of $SU(n)$," J. High Energy Phys. 10, 047 (2004).
[26] S. Schafer-Nameki, "K-theoretical boundary rings in $N=2$ coset models," Nucl. Phys. B **706**, 531 (2005).
[27] D. S. Freed, "The Verlinde algebra is twisted equivariant K-theory," Turkish J. Math. 25, 159 (2001).
[28] D. S. Freed, M. J. Hopkins, and C. Teleman, "Twisted equivariant K-theory with complex coefficients," http://arXiv.org/abs/ math.AT/0206257. Volume 3 contains relevant changes.
[29] D. S. Freed, M. J. Hopkins, and C. Teleman, "Twisted K-theory and loop group representations," http://arXiv.org/abs/ math.AT/0312155.
[30] V. Braun, "K-theory torsion," hep-th/0005103.
[31] I. Brunner and J. Distler, "Torsion D-branes in nongeometrical phases," Adv. Theor. Math. Phys. **5**, 265 (2002).
[32] I. Brunner, J. Distler, and R. Mahajan, "Return of the torsion *D*-branes," Adv. Theor. Math. Phys. **5**, 311 (2002).
[33] I. Brunner, M. R. Douglas, A. E. Lawrence, and C. Romelsberger, "*D*-branes on the quintic," J. High Energy Phys. 08, 015 (2000).
[34] M. R. Gaberdiel and S. Schafer-Nameki, "*D*-branes in an asymmetric orbifold," Nucl. Phys. B **B654**, 177 (2003).
[35] A. Kapustin and L. Rozansky, "On the relation between open and closed topological strings," Commun. Math. Phys. **252**, 393 (2004).
[36] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, supergravity theories and the dual spinor model," Nucl. Phys. B **122**, 253 (1977).
[37] D. Gepner, "Space-time supersymmetry in compactified string theory and superconformal models," Nucl. Phys. B **296**, 757 (1988).
[38] D. Gepner and Z.-a. Qiu, "Modular invariant partition functions for parafermionic field theories," Nucl. Phys. B **285** 423, (1987).
[39] Z.-a. Qiu, "Modular invariant partition functions for $N=2$ superconformal field theories," Phys. Lett. B **198**, 497 (1987).
[40] S. Fredenhagen and V. Schomerus, "On boundary RG-flows in coset conformal field theories," Phys. Rev. D **67**, 085001 (2003).

[41] C. Teleman, "K-theory and the moduli space of bundles on a surface and deformations of the Verlinde algebra," in *Topology, Geometry and Quantum Field Theory*, Vol. 308 of London Math. Soc. Lecture Note Ser. (Cambridge University Press, Cambridge, 2004), pp. 358–378.
[42] K. Wendland, "Moduli spaces of unitary conformal field theories." Ph.D. thesis, Friedrich-Wilhelms Universität Bonn, 2000.
[43] A. Kapustin and Y. Li, "*D*-branes in topological minimal models: The Landau-Ginzburg approach," J. High Energy Phys. 07, 045 (2004).
[44] I. Brunner, K. Hori, K. Hosomichi, and J. Walcher, "Orientifolds of Gepner models," hep-th/0401137.
[45] E. Witten, "Phases of $N=2$ theories in two dimensions," Nucl. Phys. B **403**, 159 (1993).
[46] D. Orlov, "Triangulated categories of singularities and *d*-branes in Landau-Ginzburg models," math.AG/0302304.
[47] K. Hori, "Boundary RG flows of $N=2$ minimal models," hep-th/0401139.
[48] H. Knörrer, "Cohen-Macaulay modules on hypersurface singularities. I," Invent. Math. **88**, 153 (1987).
[49] J. Walcher, "Stability of Landau-Ginzburg branes," J. Math. Phys. **46**, 082305 (2005).
[50] W. Nahm and K. Wendland, "A hiker's guide to $K3$. Aspects of $N=(4,4)$ superconformal field theory with central charge $c=6$," Commun. Math. Phys. **216**, 85 (2001).
[51] K. Wendland, "Orbifold constructions of K3: A link between conformal field theory and geometry," in Orbifolds in Mathematics and Physics, Madison, WI, 2001, Vol. 310 of Contemp. Math. (Amer. Math. Soc., Providence, RI, 2002), pp. 333–358.
[52] W. Nahm and K. Wendland, "Mirror symmetry on Kummer type K3 surfaces," Commun. Math. Phys. **243**, 557 (2003).
[53] J. Fuchs, A. Klemm, C. Scheich, and M. G. Schmidt, "Spectra and symmetries of Gepner models compared to Calabi-Yau compactifications," Ann. Phys. **204**, 1 (2004).