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The Fair Price of Volatility During Financial Distress: A Look at the COVID-19 Pandemic and the Effect on the Volatility Paradigm

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THE FAIR PRICE OF VOLATILITY DURING FINANCIAL DISTRESS:
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VOLATILITY PARADIGM

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JWS Thesis Project

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## Contents

Introduction  

Literature Review  
- Black-Scholes Model  
- Stochastic Volatility Models  
- Jump Diffusion  
- Market Discussion  
  - Equity Market Volatility During COVID-19  
- Motivation  

Methodologies  
- Data  
- Model Selection  
  - Parameter Clarification  
- Model Calibration  

Analysis  
- In-Sample Performance and Structured Parameter Estimates  
  - Parameter Summary  
  - Implied Volatility Analysis  
  - In-Sample Pricing Fit  
- Out-of-Sample and Hedging Performance  
  - Out-of-Sample Pricing Error  
  - Hedging Performance
Introduction

Derivatives products allow investors to mitigate certain risks in their portfolios while also offering speculators instruments to express directional views on the underlying asset. At the center of derivative pricing is implied volatility, a forward-looking metric of future volatility embedded in options prices. One of the main focuses of option pricing theory is accounting for and accurately pricing implied volatility for an underlying asset. Although the technical and theoretical research behind options pricing and volatility modeling is getting much more sophisticated, many practitioners continue to use simpler models, namely Black-Scholes, since they are much more intuitive for traders to wrap their heads around.\(^1\) When looking specifically at periods of financial distress and large shocks – namely the COVID-19 Pandemic – with rapid fluctuations of volatility, the assumptions of Black-Scholes begin to break down. Under a constant volatility framework, periods of high volatility are drastically under-weighted,\(^2\) which opens the door for investors and institutions to lose lots of money due to mis-pricing and an incorrect evaluation of the risks associated with these products. In extreme scenarios (such as the Black Friday in 1987, the 2008 financial crisis, and most recently the onset of the COVID-19 pandemic), this can lead to large hedging errors for large institutions and entire asset management and hedge funds losing all of their capital in a matter of days and hours.

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In normal market conditions, being able to accurately price and assess volatility risk is important for investors that are either looking to hedge their portfolio to higher-order sensitivities, speculate on future levels of volatility in the market, or arbitrageurs looking to pick up any money that’s currently being left on the table in the derivatives market. In periods of financial distress, it becomes ever so crucial to be able to accurately price-in volatility risks.

My research will attempt to answer the question of how well stochastic volatility models are in pricing and assessing the volatility risks in the marketplace. To assess the effectiveness of these models, we will test how well the Heston Model\(^3\) and the Heston Model with embedded jump risk\(^4\) perform in-sample in terms of pricing and implied volatility fit, as well as out-of-sample in terms of pricing fit and hedging error. For the purposes of appealing to traders and practitioners who prefer the Black-Scholes equation for its interpretability, we have focused our efforts on the stochastic volatility models with analytical solutions, allowing for closed-form partials analogous to the Black-Scholes Greeks. This research will be most relevant to investors, private banks and hedge funds, investment banks, and retail investors. This research will provide a historical example of these models in action during the months of the onset of the COVID-19 pandemic and allow practitioners to see the performance of these

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models in an extreme period of financial distress. Being able to protect your investments to adverse shocks in the market is extremely relevant to all of the above mentioned.

There have been some similar experiments conducted in practical settings. Shanghai Luoshu Investment Company, a high-frequency, multi-strategy quantitative hedge fund has been exploiting a trading strategy that takes advantage of the price spreads between traditional volatility models (such as Black-Scholes and SABR) and more sophisticated rough stochastic-volatility models. Through this strategy, they were able to boost their revenue by roughly 20%, solely on an arbitrage strategy. Additionally, Natixis, a French investment bank, back-tested a similar arbitrage strategy for multiple assets to create a strategy that has a Sharpe Ratio above 2. Where this research differs, is to isolate the COVID-19 period and see if these models perform well under immense financial stress.

**Literature Review**

In this section, we will explore and examine some of the quantitative models that have been used in financial markets to either model/forecast volatility or assess volatility risk in financial markets. This will be a somewhat technical overview of the general model framework, important equations and other interesting notes about how volatility has behaved empirically in the market.

Throughout a majority of this literature review, interest rates will be assumed to be constant and will be denoted with $r$ in mathematical formulas that they arise in. There has been an extensive literature on implementing a stochastic interest rate component to the

---

5. Cesa and Mannix, “The Volatility Paradigm That’s Stirring up Options Pricing.”
following models, however, as Bakshi, Cao, and Chen\textsuperscript{6} discovered, incorporating stochastic interest rates did not improve model performance. Thus, for the purposes of pricing contingent claims accounting for non-constant equity volatility, the constant and deterministic $r$ assumption has been used in the literature and is robust.

Black-Scholes Model

In 1973, Black and Scholes\textsuperscript{7} first introduced their valuation formula for a European Call Option. Their initial valuation framework attempted to price the European Call Option in terms of the price of the underlying asset. In the derivation of their formula, they assumed “ideal” conditions in the market, most importantly:

1. The short-term interest rate is known and is constant;
2. The distribution of stock prices is log-normal;
3. The stock pays no dividend;
4. There are no frictions regarding transaction costs in both the options and equities markets;
5. Volatility is a constant

They used the hedging argument to construct a portfolio of stock and call option that does not depend on the price of the underlying asset, but rather on time and the parameters that are assumed to be constants.

Under the hedging argument, if you consider $w(x, t)$ to be the value of the option as a function of the stock price $x$ and time $t$, then they concluded that $\left(\frac{\partial w(x, t)}{\partial x}\right)^{-1}$ shares of the option must be sold short for every long position in the equity. When look-

\textsuperscript{6}Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”

ing solely at one share of equity and one option contract, the total value of the equity in this hypothetical portfolio would be \( x - w(x,t) \times \left( \frac{\partial w(x,t)}{\partial x} \right)^{-1} \). And subsequently, for any small change in the value of equity over some time \( \Delta t \), this formula would now shift to: \( \Delta x - \Delta w(x,t) \times \left( \frac{\partial w(x,t)}{\partial x} \right)^{-1} \). Using stochastic calculus to expand \( \Delta w \): \( \Delta w(x,t) = \frac{\partial w(x,t)}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 w(x,t)}{\partial x^2} \sigma^2 x^2 \Delta t + \frac{\partial w(x,t)}{\partial t} \Delta t \), where \( \sigma^2 \) represents the variance of the underlying asset. From these equations, they concluded that the change in the value of equity in their hedged portfolio must be: \( - \left( \frac{1}{2} \frac{\partial^2 w(x,t)}{\partial x^2} \sigma^2 + \frac{\partial^2 w(x,t)}{\partial t} \right) \times \Delta t \times \left( \frac{\partial w(x,t)}{\partial x} \right)^{-1} \). From the construction of their hedged portfolio, they concluded that the return on the equity in the hedged position is certain, it must be equivalent to \( r \Delta t \). By setting these two expressions equal to each other and dividing by \( \Delta t \) from both sides, they derived the following partial differential equation for the price of a European Call Option:

\[
\frac{\partial w(x,t)}{\partial t} = r w(x,t) - r x \frac{\partial w(x,t)}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 w(x,t)}{\partial x^2}
\]

Subject to the following boundary condition:

\[
w(x,T) = \begin{cases} 
  x - k & x \geq c \\
  0 & x < c 
\end{cases}
\]

Where \( T \) is the maturity date of the contract and \( k \) represents the strike price of the option.

The partial differential equation can be solved analytically and simplifies to (where \( \Phi(*) \) denotes the normal cumulative density function):

\[
w(x,t) = x \Phi(d_1) - k e^{r(t-T)} \Phi(d_2) \quad \text{where},
\]
The Black-Scholes methodologies of martingales and stochastic processes to price derivatives products served as the first building block for more complicated models in the future. Particularly, one of the first assumptions that needs to be relaxed when trying to model volatility is the constant volatility argument. However, the theoretical methodologies of using stochastic processes to model asset price movement would then begin to be incorporated to explain movements in volatility.

Stochastic Volatility Models

In 1993, Heston developed a new model to value the price of contingent claims while relaxing the assumption of constant variance in stock returns over time. Namely, the author assumed that the underlying asset at time $t$ follows a diffusion modeled by:

$$dS(t) = \mu Sdt + \sqrt{v(t)}Sdz_1(t)$$

Where $z_1(t)$ is a Weiner process. Heston assumed that volatility follows an Ornstein-Uhlenbeck process, $d\sqrt{v(t)} = -\beta \sqrt{v(t)}dt + \delta dz_2(t)$, thus allowing the variance, $v(t)$, to follow the process:

$$dv(t) = [\delta^2 - 2\beta v(t)] dt + 2\delta \sqrt{v(t)}dz_2(t)$$

Heston, “A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.”
The two processes $z_1(t)$ and $z_2(t)$ are assumed to have correlation $\rho$.

From Black-Scholes (1973) no-arbitrage arguments, Heston states that the price of an option, $U(S, v, t)$, must satisfy:

$$
\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 U}{\partial v^2} + r S \frac{\partial U}{\partial S} + \left[ \kappa \left( \theta - v(t) \right) - \lambda(S, v, t) \right] \frac{\partial U}{\partial v} - r U + \frac{\partial U}{\partial t} = 0
$$

Where $\lambda(S, v, t)$ is the price of volatility risk which has yet to be derived. Heston states that this volatility risk must be independent of the particular asset, and that for equity options this term is non-zero. Using consumption models derived from Breeden and Cox, Ingersoll and Ross, Heston concludes that, $\lambda(S, v, t) \propto v$, $\Rightarrow \lambda(S, v, t) = \lambda v$.

Heston states that the a European call Option, $U(S, v, t)$, with strike $K$ and maturity $T$ satisfies the previous partial differential equation with the following boundary conditions:

$$
U(S, v, T) = \max[0, S - K], \quad U(0, v, t) = 0, \quad \frac{\partial U}{\partial S}(\infty, v, t) = 1, \quad U(S, \infty, t) = S, \quad \text{and}
$$

$$
r S \frac{\partial U}{\partial S}(S, 0, t) + \kappa \theta \frac{\partial U}{\partial v}(S, 0, t) - r U(S, 0, t) + U_t(S, 0, t) = 0
$$

To solve this partial differential equation, Heston proposes a solution in the form $C(S, v, t) = SP_1 - KP(t, T)P_2$, where the first term represents the present value of the underlying asset upon optimal exercise of the option and the second term represents the present value of the strike-price payment.

For simplicity in calculations, Heston switches to expressing the spot price of the underlying asset $S$, in terms of $x$ where $x = \ln(S)$. This substitution cleans up some of the following expressions. After substituting his proposed solution, Heston states that $P_1$ and
$P_2$ both must satisfy:

\[
\frac{1}{2} v \frac{\partial^2 P_j}{\partial x^2} + \rho \sigma v \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 P_j}{\partial v^2} + (r + u_j v) \frac{\partial P_j}{\partial x} + (a - b_j) \frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0, \quad \text{for: } j \in (1, 2)
\]

s.t. \quad u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa \theta, \quad b_1 = \kappa + \lambda - \rho \sigma, \quad b_2 = \kappa + \lambda

Then, Heston states that in order for the option price to satisfy the terminal condition laid out previously, that for each $j \in (1, 2)$, $P_j(x, v, T; \ln(K)) = 1_{\{x \geq \ln(K)\}}$. Heston then explains that the way we defined the underlying asset price and variance processes, this probability becomes the conditional probability that the option expires “in the money,” $Pr[x(t) \geq \ln(K) | x(t) = x, v(t) = v]$. Heston states that these probabilities can be obtained from the characteristic function of this process.\(^9\)

Dunn et al.\(^10\) provide a nicer explanation in their technical note “Estimating Option Prices with Heston’s Stochastic Volatility Model” on the link between the characteristic function for the Heston’s stochastic volatility model. They provide some of the theoretical backgrounds into Ito’s Lemma, which is used in deriving the characteristic function for the Heston Model. Since this stochastic process contains three variables $x = \ln(S)$, $v$ and $t$, they explain that from Ito’s Lemma the characteristic function for the Heston Model will have the following form:

\[
f(x, v, t) = e^{A(T-t)+B(T-t)x+C(T-t)v+i\phi x}
\]

---


The simplified expressions that they provide in their note are in a slightly different form than in Heston’s paper, however, so we will continue with Heston’s simplified expressions. Solving the partial differential equation with the characteristic, Heston determines that the characteristic function for his model is as follows (where $\tau = T - t$):

$$f_j(x, v, t; \phi) = e^{C(\tau; \phi) + D(\tau; \phi)v + i\phi x}$$

where

$$C(\tau; \phi) = r\phi i\tau + \frac{a}{\sigma^2} \left[(b_j - \rho \sigma \phi i + d)\tau - 2 \ln \left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right],$$

$$D(\tau; \phi) = \frac{b_j - \rho \phi i + d}{\sigma^2} \left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right],$$

$$g = \frac{b_j - \rho \sigma \phi i + d}{b_j - \rho \sigma \phi - d}, \quad \text{and} \quad d = \sqrt{(\rho \sigma \phi i - b_j)^2 - \sigma^2(2u_j \phi i - \phi^2)}$$

From there Heston notes that:

$$Pr [x(t) \geq \ln(K)|x(t) = x, v(t) = v] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln(K)} f_j(x, v, T; \phi)}{i\phi}\right] d\phi$$

Thus the price for a European call option is as follows:

$$U(x, v, t) = S(t) \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln(K)} f_1(x, v, T; \phi)}{i\phi}\right] d\phi\right) -$$

$$Ke^{-r(T-t)} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln(K)} f_2(x, v, T; \phi)}{i\phi}\right] d\phi\right)$$
Jump Diffusion

In this section, we will begin to look at the existing stochastic volatility models that incorporate stochastic jumps. We will ignore models that incorporate jump diffusion processes but hold volatility constant.

In 1997, Bakshi, Cao, and Chen\textsuperscript{11} tested the performance of the Black-Scholes, Heston and other alternative options pricing models. For the purposes of this literature review, we will focus on the development of their stochastic volatility model with jumps in the stock price process. Their model formulation is as follows:

\[
\frac{dS(t)}{S(t)} = [r - \lambda \mu] dt + \sqrt{V(t)} d\omega_S(t) + J(t) dq(t),
\]

\[
dV(t) = [\theta - \kappa V(t)] dt + \sigma_v \sqrt{V(t)} d\omega_v(t)
\]

\[
\ln(1 + J(t)) \sim \mathcal{N} \left( \ln(1 + \mu_j) - \frac{1}{2} \sigma_j^2, \sigma_j^2 \right)
\]

Where \( r \) is the spot interest rate, \( \lambda \) is the frequency of jumps annually, \( V(t) \) is the volatility process, \( \omega_S \) and \( \omega_v \) are Brownian motion with \( Cov(d\omega_S, d\omega_v) = \rho_{SV} dt \), \( J(t) \) is the jump processes with log normal distribution, \( \kappa \) is the speed of adjustment, \( \theta/\kappa \) is the long-run variance mean, and \( \sigma_v \) is the volatility of volatility.

Using similar methodologies as Heston, they developed a closed-form solution for European call options under this risk-neutral measure in the form:

\[
C(t, \tau) = S(t) \Pi_1 - Ke^{-r\tau} \Pi_2,
\]

\textsuperscript{11.} Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”
\[
Π_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[ \frac{e^{-i\phi \ln(K)} f_j}{i\phi} \right] d\phi
\]

and:

\[
f_1 = \exp \left\{ i\phi r \tau - \frac{\theta}{\sigma_v^2} \left[ 2 \ln \left( 1 - \frac{[\zeta_1 - \kappa + (1 + i\phi)\rho_{SV}\sigma_v](1 - e^{-\zeta_1\tau})}{2\zeta_1} \right) \right] \right. \\
- \frac{\theta}{\sigma_v^2} \left[ \zeta_1 - \kappa + (1 + i\phi)\rho_{SV}\sigma_v \right] \tau + i\phi \ln(S) + \lambda\tau(1 + \mu_J) \left[ (1 + \mu_J)^{i\phi} e^{\left( \frac{i\phi}{2} \right)(1 + i\phi)\sigma_v^2} - 1 \right] - \lambda i\phi \mu_J \tau \\
+ \frac{i\phi(i\phi + 1)(1 - e^{\zeta_1\tau})}{2\zeta_1 - \left[ \zeta_1 - \kappa + (1 + i\phi)\rho_{SV}\sigma_v \right] (1 - e^{-\zeta_1\tau})} v_0 \right\},
\]

\[
f_2 = \exp \left\{ i\phi r \tau - \frac{\theta}{\sigma_v^2} \left[ 2 \ln \left( 1 - \frac{[\zeta_2 - \kappa + i\phi\rho_{SV}\sigma_v](1 - e^{-\zeta_2\tau})}{2\zeta_2} \right) \right] \right. \\
- \frac{\theta}{\sigma_v^2} \left[ \zeta_2 - \kappa + i\phi\rho_{SV}\sigma_v \right] \tau + i\phi \ln(S) + \lambda\tau \left[ (1 + \mu_J)^{i\phi} e^{\left( \frac{i\phi-1}{2} \right)(i\phi-1)\sigma_v^2} - 1 \right] - \lambda i\phi \mu_J \tau \\
+ \frac{i\phi(i\phi - 1)(1 - e^{-\zeta_2\tau})}{2\zeta_2 - \left[ \zeta_2 - \kappa + i\phi\rho_{SV}\sigma_v \right] (1 - e^{-\zeta_2\tau})} v_0 \right\},
\]

\[
\zeta_1 = \sqrt{[\kappa - (1 + i\phi)\rho_{SV}\sigma_v]^2 - i\phi(i\phi + 1)\sigma_v^2}, \quad \text{and} \quad \zeta_2 = \sqrt{[\kappa - i\phi\rho_{SV}\sigma_v]^2 - i\phi(i\phi - 1)\sigma_v^2}
\]

They also provide closed form solutions for the model sensitivities that will be utilized to calculate position Deltas later. As we begin more of our analysis, we will discuss more of the methodologies and results from their study.
In 2000, Duffie, Pan, and Singleton\(^{12}\) provided a substantial addition to the existing literature on securities pricing. In their paper, they derived a closed-form expression for an “extended transform” of an affine jump diffusion process, building upon the methodologies of solving for the Fourier inversion of the conditional characteristic function. They consider a generalized terminal payoff function: \((v_0 + v_1 \cdot X_T)e^{u \cdot X_T}\) of an affine jump process \(X_T\) where \(v_0\) and the \(n\) elements of \(v_1\) and \(u\) are scalars. They derive a closed-form expression for the transform:

\[
E_t \left[ \exp \left\{ - \int_t^T R(X_s, S) \, ds \right\} (v_0 + v_1 \cdot X_T)e^{u \cdot X_T} \right]
\]

Where \(E_t\) denotes the conditional expectation on the known information of the process \(X\) up until time \(t\) and \(R(X_t)\) is the stochastic discount factor.

They set the stage to their ordinary differential equation approach by introducing the price \(p\) of a derivative at time 0 with final payoff of \((e^{d \cdot X_T} - c)^+\), where \(c\) is a constant strike price, \(b \in \mathbb{R}^n\), and \(X\) is an \(n\)-dimensional affine jump diffusion process, with a short term interest rate process that is itself affine in \(X\). They let the function \(G_{a,b}(y)\) represent the price of a security that pays \(e^{a \cdot X_T}\) at time \(T\) in the event that \(b \cdot X_T \leq y\), as a call option is in the money when \(-d \cdot X_T \leq -\ln(c)\) and thus pays \(e^{d \cdot X_T} - ce^{0 \cdot X_T}\). They have the option priced at:

\[
p = G_{d,-d}(-\ln(c)) - cG_{0,-d}(-\ln(c))
\]

Since the function $G_{a,b}(y)$ is an increasing function, they compute the Fourier transform,

$$G_{a,b}(z) = \int_{-\infty}^{\infty} e^{izy} dG_{a,b}(y),$$

of $G_{a,b}(y)$. Duffie et al explain that this Fourier transform can be evaluated explicitly by noting that $G_{a,b}(z)$ is given by the conditional expectation above for the complex coefficient vector $u = a + izb$ and $v_0 = 1$ and $v_1 = 0$. Thus, due to the affine structure of the underlying processes, $G_{a,b}(z) = \exp\{\alpha(0) + \beta(0) \cdot X_0\}$. Where $\alpha$ and $\beta$ solve known, complex valued ordinary differential equations with boundary conditions at $T$ which are described by $z$.

In their proposed model, they fix a probability space $(\Omega, \mathcal{F}, P)$ with an information filtration $\mathcal{F}_t$ and assume $X$ is a Markov process in some state space $D \subset \mathbb{R}^n$ that solves the stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t$$

Where $W$ is a standard Brownian motion in $\mathbb{R}^n$; $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^{n \times n}$, and $Z$ is a pure jump process whose jumps have a fixed probability distribution $\nu$ on $\mathbb{R}^n$ which arrive with intensity $\{\lambda(X_t); t \geq 0\}$ for some $\lambda : D \to [0, \infty)$. They fix an affine discount rate function $R : D \to \mathcal{R}$ and the affine dependence of $\mu, \sigma\sigma^T, \lambda$ and $R$ to the coefficients $(K, H, l, \rho)$ such that:

$$\mu(x) = K_0 + K_1x, \quad \text{for} \quad K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$$

$$(\sigma(x)\sigma(x)^T) = (H_0)_{ij} + (H_1)_{ij} \cdot x, \quad \text{for} \quad H = (H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$$

$$\lambda(x) = l_0 + l_1 \cdot x, \quad \text{for} \quad l = (l_0, l_1) \in \mathcal{R} \times \mathbb{R}^n$$

$$R(x) = \rho_0 + \rho_1 \cdot x, \quad \text{for} \quad \rho = (\rho_0, \rho_1) \in \mathcal{R} \times \mathbb{R}^n$$
For \( c \in \mathcal{C}^n \), the set of \( n \)-tuples of complex numbers, they let \( \theta(c) = \int_{\mathbb{R}^n} \exp \{ d \cdot z \} \, dv(z) \), which determines the jump-size distribution.

The coefficient vector for \( X \) completely determines it’s distribution and give an initial condition for \( X(0) \). Using a “characteristic” \( \chi = (K, H, l, \theta, \rho) \), they define their transform as:

\[
\psi^\chi(u, X_t, t, T) = E^{\chi} \left[ \exp \left\{ - \int_t^T R(X_s) \, ds \right\} e^{u \cdot X_T} \mid F_t \right]
\]

From their proposition earlier regarding the form of their Fourier transform:

\[
\psi^\chi(u, x, t, T) = e^{\alpha(t)+\beta(t) \cdot x}
\]

Where \( \beta(t) \) and \( \alpha(t) \) solve the complex valued ordinary differential equations:

\[
\dot{\beta}(t) = \rho_1 - K^T_1 \beta(t) - \frac{1}{2} \beta(t)^T H_1 \beta(t) - l_1 [\theta(\beta(t)) - 1], \quad \text{and}
\]

\[
\dot{\alpha}(t) = \rho_0 - K_0 \cdot \beta(t) - \frac{1}{2} \beta(t)^T H_0 \beta(t) - l_0 [\theta(\beta(t)) - 1]
\]

With boundary conditions \( \beta(T) = u \) and \( \alpha(T) = 0 \). To allow for an analytically tractable solution, one should select a distribution for jump processes or an easily computable transform. They then provide a proof for obtaining \( G_{a,b}(y; X_0, T, \chi) \) utilizing the Lévy inversion formula:

\[
G_{a,b}(y; X_0, T, \chi) = \frac{\psi^\chi(a, X_0, 0, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im} \left[ \psi^\chi(a + ivb, X_0, 0, T) e^{-ivy} \right]}{v} \, dv
\]
They then extend this affine jump diffusion model to include jumps in both the volatility and underlying price process. They propose the following stochastic differential equation:

$$
    d \begin{pmatrix} Y_t \\ V_t \end{pmatrix} = \begin{pmatrix}
        r - \zeta - \bar{\mu} - \frac{1}{2} V_t \\
        \kappa_v (v - V_t)
    \end{pmatrix} dt + \sqrt{V_t} \begin{pmatrix}
        1 & 0 \\
        \rho \sigma_v & \sigma_v \sqrt{1 - \rho^2}
    \end{pmatrix} dW^Q_t + dZ_t
$$

Where $Y_t$ represents the log equity process $Y = \ln(S)$, $V_t$ is the volatility process $W^Q$ is a standard Brownian motion under the risk-neutral martingale measure $Q$, and $Z$ is a pure jump process in $\mathbb{R}^2$ with constant mean jump arrival rate $\bar{\lambda}$, whose bi-variate jump-size distribution $v$ has the transform $\theta$. To satisfy the risk-neutral coefficient restriction, $\bar{\mu} = \theta(1,0) - 1$. From their earlier results, they are able to derive the transform $\psi$ of the log equity price process as:

$$
    \psi(u, (y, v), t, T) = \exp \left\{ \bar{\alpha}(T - t, u) + uy + \bar{\beta}(T - t, u)v \right\}
$$

where if $b = \sigma \bar{\mu} u - \kappa_v$, $a = u(1 - u)$, and $\gamma = \sqrt{b^2 + a \sigma_v^2}$, then,

$$
    \bar{\beta}(\tau, u) = - \frac{a(1 - e^{-\gamma \tau})}{2\gamma - (\gamma + b)(1 - e^{-\gamma \tau})},
$$

$$
    \bar{\alpha}(\tau, u) = \alpha_0(\tau, u) - \bar{\lambda}\tau(1 + \bar{\mu} u) + \bar{\lambda} \int_0^\tau \theta(u, \bar{\beta}(s, u))\, ds
$$

where

$$
    \alpha_0(\tau, u) = -r\tau + (r - \zeta)u\tau - \kappa_v \left[ \frac{\gamma + b}{\sigma_v^2} \tau + \frac{2}{\sigma_v^2} \ln \left( 1 - \frac{\gamma + b}{2\gamma}(1 - e^{-\gamma \tau}) \right) \right]
$$
and the term \( \int_0^\tau \theta(u, \bar{\beta}(s, u)) \, ds \) depends on the specific formulation of the bi-variate jump diffusion. They provide a concrete example for “double-jump” processes which integrates cleanly:

\[
\theta(c_1, c_2) = \bar{\lambda}^{-1}(\lambda_y \theta^y(c_1) + \lambda^v \theta^v(c_2) + \lambda^c \theta^c(c_1, c_2))
\]

Where, \( \bar{\lambda} = \lambda_y + \lambda^v + \lambda^c \) and:

\[
\theta^y(c_1) = \exp \left\{ \mu_y c_1 + \frac{1}{2} \sigma^2_y c_1^2 \right\}, \quad \theta^v(c_2) = \frac{1}{1 - \mu_v c_2}, \quad \text{and} \quad \theta^c(c_1, c_2) = \exp \left\{ \mu_c y c_1 + \frac{1}{2} \sigma^2_c y^2 \right\}
\]

In this formulation, jumps in the log-equity processes with arrival intensity \( \lambda_y \) are \( \mathcal{N}(\mu_y, \sigma^2_y) \), jumps in \( V \) arrive with intensity \( \lambda^v \) and are exponentially distributed with mean \( \mu_v \), and correlated jumps in \( Y \) and \( V \) arrive with intensity \( \lambda^c \). The marginal distribution of the jump size in \( V \) for correlated jumps is exponential with mean \( \mu_v \) and, conditional on a realization \( z_v \) of the jump size in \( V \), the jump size in \( Y \) is normally distributed with mean \( \mu_c y + \rho_J z_v \) and variance \( \sigma^2_c y \).

This choice of jump diffusion allows for an explicit solution for option pricing through their transform framework as:

\[
\int_0^\tau \theta(u, \bar{\beta}(s, u)) \, ds = \bar{\lambda}^{-1}(\lambda_y f^y(u, \tau) + \lambda^v f^v(u, \tau) + \lambda^c f^c(u, \tau))
\]

Where,

\[
f^y(u, \tau) = \tau \exp \left\{ \mu_y u + \frac{1}{2} \sigma^2_y u^2 \right\}, \quad f^v(u, \tau) = \exp \left\{ \mu_c y + \sigma^2 c \left( \frac{y u^2}{2} \right) \right\}
\]
$f^v(u, \tau) = \frac{\gamma - b}{\gamma - b + \mu_v a} \tau - \frac{2\mu_v a}{\gamma^2 - (b - \mu_v a)^2} \ln \left( 1 - \frac{(\gamma + b) - \mu_v a}{2\gamma} (1 - e^{-\gamma \tau}) \right)$

where $a = u(1 - u)$, $b = \sigma_v \bar{p} u - \kappa_v$, $c = 1 - \rho \mu_{c,v} u$, and

$d = \frac{\gamma - b}{(\gamma - b)c + \mu_{c,v} a} \tau - \frac{2\mu_{c,v} a}{(\gamma c)^2 - (bc - \mu_{c,v} a)^2} \ln \left( 1 - \frac{(\gamma + b) - \mu_{c,v} a}{2\gamma c} (1 - e^{-\gamma \tau}) \right)$

Thus, the Fourier transform $\psi$ is now available in functional form as the integral in the expression earlier evaluates cleanly. With $\psi$ obtained in functional form, the solution to option pricing can be obtained by applying the Lévy inversion formula as before to obtain $G_{a,b}(y)$.

Market Discussion

In a recent article from Mauro Cesa and Rob Mannix published to *Risk Magazine*, they discussed the adoption of newer “rough volatility models.” By the time of their writing of this article, they mentioned that at least 9 firms were utilizing these models for pricing or strategies. In this section, I want to focus on the strategic element behind utilizing these models.

One example they provided was Shanghai Luoshu Investment Company, a high-frequency multi-strategy quantitative fund, used pricing discrepancies between rough volatility and traditional models in an exploitative arbitrage strategy. They note that this fund was able to boost its profits by roughly 20%. Additionally, a fund Natixis back-tested a theoretical rough volatility trade that arbitraged pricing discrepancies across 15 options markets, with daily delta hedging. This strategy reported a Sharpe ratio above 2.
However, these theoretical and practical results have been met with some push back from practitioners. The traditional Black-Scholes formulas are much easier for traders and practitioners to understand and the corresponding risk metrics, the Greeks, are simple for traders to grasp. Since, as we observed with the quadratic rough Heston model, these models factor in the price path that the underlying asset took. Thus, the corresponding Greeks will often suggest different hedging strategies than the Black-Scholes Greeks.

The discussion of the rich literature surrounding volatility modeling and structured volatility derivatives trading contrasted with slow adoption of these models is to motivate our exploration of these stochastic volatility models during times of financial distress. While the assumptions of Black-Scholes methodologies may hold well for longer dated vanilla options, these assumptions could be detrimental for investors in the short run in periods of financial duress.

**Equity Market Volatility During COVID-19**

As COVID-19 has been a hot topic in recent year in political and social discourse, it has received equal attention in financial literature for its effects on equity markets. The existing literature regarding the affects of COVID-19 on financial markets is still on-going, but there have been a number of empirical studies performed. Liu et al. performed an event study on the short-term effects of COVID-19 on stock markets of countries that were most

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affected, which concluded that there was a starch negative response in markets. Zaremba et al.\textsuperscript{15} showed that governmental interventions raised equity market volatility with large informational events and cancelling public occasions being some of the major driving forces behind rises in volatility. Duttilo, Gattone, and Di Battista\textsuperscript{16} performed a Gaussian autoregressive conditional heteroskedatistity (GARCH) analysis of various equity indexes in the Euro-zone throughout the onset of the pandemic, finding that different regions responded differently. However, there has not been an extensive consideration on the pricing of options contract under stochastic volatility frameworks.

Motivation

This research seeks to answer the question: “is volatility being fairly priced in equity markets during periods of financial distress?” Through this research, we hope to test whether or not the assumptions of the these stochastic volatility models are able to accurately capture the behavior and hence fairly price volatility risk during periods of financial uncertainty. This is an extremely important question to answer for investors, private funds and hedge funds as well as banks and retail investors. Not only is it important to be properly compensated for taking volatility risk but it is also important to be able to assess these risks properly through model sensitivities.


Through an exploration of the literature, we see that there have been many advancements in volatility modeling that have come extremely close to matching observed historical market trends in volatility and are extremely robust in pricing vanilla options as observed in the market discussion section. We want to specifically isolate the onset of the COVID-19 Pandemic in the United States, as it provides us an interesting case study of drastic fluctuations of volatility, and assess model performance strictly during this period. Additionally, we see that there exists a slow adoption of these models in practice, despite tests and studies that confirm the findings in the literature. Thus, we want to be able to determine if there should be much more of a harder push for more advanced pricing models to protect and hedge against possible periods of financial crisis and heightened volatility, or, if there needs to be additional features added to these existing volatility models to account for periods of financial crisis.

**Methodologies**

This project seeks to provide an empirical evaluation of the performance of existing stochastic volatility models throughout the onset of the COVID-19 Pandemic. Volatility is a crucial market component used in pricing many equity derivatives with optionality involved. Therefore, to test the market performance of some of these models, it makes the most sense to test its performance in the vanilla derivatives market. Throughout the period of March 2020 to May 2020, we will fit a series of stochastic volatility models across US stock indexes daily, analyze the implied parameter estimates, assess in and out of sample pricing errors, and compute model sensitivities to calculate hedging error. Similar to the methodologies
used by Bakshi, Cao, and Chen,\textsuperscript{17} we will determine if our model parameters produce results that are consistent with the observable implied volatility time series.

Data

For our analysis, we will look at vanilla European style options data throughout the time period of March 2\textsuperscript{nd}, 2020 to May 31\textsuperscript{st}, 2020, during the height of the COVID-19 Pandemic in the United States. We will focus our data on the S&P 500, the stock index of the 500 largest market cap firms in the United States. The options data was nested within the Wharton Research and Data Services (WRDS) OptionMetrics data set and provides us with the best bid, best offer, strike price, expiration date, and unique identifiers for each option contract as well as closing price, and dividend yields for our underlying assets.

We will implement some of the exclusionary filters that Bakshi, Cao, and Chen\textsuperscript{18} incorporated in their analysis. First, we will eliminate options that have less than 6 days until maturity as there could be liquidity-related biases in the price data, which may present more issues when looking at a period of heightened volatility and financial uncertainty. Additionally, we will eliminate price quotes that are less than \$\frac{3}{8}. Then, we will eliminate options from our calibration efforts that do not meet the following arbitrage restriction:

\[
C_{\text{MP}}(K_i, \tau_j) \geq \max \left[ 0, S(t) - K, S(t) - D(\tau) - Ke^{-r\tau} \right]
\]

\textsuperscript{17} Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”

\textsuperscript{18} Bakshi, Cao, and Chen.
for calls and

\[ P_{MP}(K_i, \tau_j) \geq \max \left[ 0, K - S(t), Ke^{-r\tau} + \overline{D(\tau)} - S(t) \right] \]

where \( \overline{D(\tau)} \) represents the present value of future expected dividend payments throughout the residual maturity of the option. Additionally, we will implement daily filters based on daily trading volume. Some contracts with little trading activity may produce stale prices and throw off some of our model results. To make these thresholds, we will evaluate the distribution of trading volume for each of the contracts we are examining. The histogram plots are as follows:

Figure 1: Daily Observations of Intra-day Option Contract Trading Volume

Note: right tail is truncated

Here, we see that many of the daily observations for our option contracts in our data set have little trading activity around them. The right tail of these distributions are extremely skewed right. For S&P 500, we will set a threshold of Trading Volume > 1250. Filtering
down our data to focus on options that are more regularly traded allows us to avoid pricing errors as well as analyze the empirical performance of these models over a wide range of strikes and maturities to be fit each day without being too computationally intensive. Table 1 summarizes the mean midpoint of the bid/ask spread as well as number of observations (in parenthesis) at each moneyness level:

To obtain the risk free interest rates, we will gather the bid/ask quotes from the zero coupon treasury bonds (ZCBs) during the time period of our option data. The data were obtained from WRDS under the Center for Research in Security Prices. Since the price of a ZCB with $100 of face value can be written as: \( P(t, T) = 100 \times e^{-rT} \), we can invert this formula to be in terms of the continuously compounded risk free rate \( r \):

\[
P(t, T) = 100 \times e^{-rT} \quad \Rightarrow \quad r = -\frac{\ln \left( \frac{P(t, T)}{100} \right)}{T}
\]

Since the ZCB maturities do not align exactly with the option contract maturities, we will utilize linear interpolation with the two ZCB maturities that straddle the option maturity date to estimate the price of a hypothetical ZCB that expires on the maturity dates of our options contracts. In other words, for an option that matures on some date \( T \in (T_1, T_2) \) s.t. \( T_1 < T < T_2 \), where \( T_1 \) and \( T_2 \) are observable market ZCB maturity dates, we will estimate the price of the hypothetical ZCB corresponding to that option maturity \( T \) as:

\[
P_h(t, T) = P(t, T_1) + [T - T_1] \times \frac{P_h(t, T_2) - P_h(t, T_1)}{T_2 - T_1}
\]
Table 1: Data Summary

<table>
<thead>
<tr>
<th>%Moneyness</th>
<th>Days-to-Expiration</th>
<th>Subtotal</th>
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<tr>
<td></td>
<td>&lt; 60</td>
<td>60 – 180</td>
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<tr>
<td>Calls</td>
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<tr>
<td>&lt;85%</td>
<td>$751.19</td>
<td>$738.87</td>
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<td></td>
<td>(28)</td>
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<tr>
<td>85% – 90%</td>
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<td>$415.73</td>
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<td></td>
<td>(8)</td>
<td>(6)</td>
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<tr>
<td>90% – 95%</td>
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<tr>
<td>95% – 100%</td>
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<td>$219.13</td>
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<tr>
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<td>(805)</td>
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<td>Subtotal</td>
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<tr>
<td></td>
<td>(2369)</td>
<td>(1105)</td>
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<td>Puts</td>
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<td>(716)</td>
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<tr>
<td>100% – 105%</td>
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<td>$248.03</td>
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<td></td>
<td>(379)</td>
<td>(196)</td>
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<tr>
<td>105% – 110%</td>
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<td>$322.98</td>
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<td>(43)</td>
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<tr>
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<tr>
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<td></td>
</tr>
<tr>
<td></td>
<td>(6087)</td>
<td>(2930)</td>
</tr>
</tbody>
</table>

Subtotal (2369) (1105) (411) (3885)
Then, we can use the previous equation to get the continuously compounded rate $r$ for our hypothetical ZCB price with maturity $T$.

Model Selection

For our analysis, we will focus our efforts on Heston’s stochastic volatility model\(^{(19)}\) (SV) and the stochastic volatility model with jumps in the stock price (SVJ) analyzed by Bakshi, Cao, and Chen.\(^{(20)}\) The main motivation behind these two models is that they provide a computable closed form solution for European style call options in a form analogous to the Black-Scholes model. As discussed in our literature review, many traders stray away from more complicated pricing methodologies as the results become less interpretable. Additionally, as vanilla options are not as complicated as some of the structured volatility and exotic derivative products on the market, practitioners avoid consideration of more complicated models. It is our goal to track the performance of the SV and SVJ models throughout the onset of the COVID-19 Pandemic, utilizing the closed form solution and hedging suggestions similarly to how one would utilize the outputs from the Black-Scholes model. If these alternative pricing models are not able to accurately capture the underlying of vanilla options, then it should be notable that these models will fail in capturing the risk for other more structured products.

However, with a closed form pricing solution in the form $S \Pi_1 - K e^{-r \tau} \Pi_2$ can be interpreted by trader’s similarly to the intuition behind Black-Scholes. Although the resulting

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probability distributions that provide us $\Pi_1$ and $\Pi_2$ are not as well-known as the standard bell curve, when plotted along side the normal distribution the resulting distribution looks remarkably close to a standard bell curve to the naked eye. Relaxing the assumption of constant volatility adds additional skewness and kurtosis to the log return distribution, in other words making the tails of the bell curve fatter or thinner than the standard normal density.

Parameter Clarification

This section is dedicated to clarifying our terminology and parameters. First, we will standardize our observable market notation:

- $S(t)$: The underlying price at the current time $t$;
- $K$: The strike price of the underlying option contract;
- $\tau$: $T - t$, or the residual maturity left on the option contract;
- $r$: The corresponding continuously compounded risk free interest rate, derived from the ZCB quotes as described previously;
- $\zeta$: The corresponding dividend yield.

Now, we will now standardize our parameters and clarify them for the reader.

- $v_0$: The “observed” current spot volatility. Since this value is not directly observable in the market, this will be estimated as a separate parameter;
- $\kappa$: The mean reversion rate for the volatility process;
- $\theta$: Where $\theta/\kappa$ represents the long-run mean for the volatility process;
- $\sigma_v$: The volatility of the volatility process;
- $\rho_{SV}$: The correlation between the log-return and volatility processes;
- $\lambda$: The mean jump arrival rate annually in the stock price process;
- $\mu$: The mean jump size in the stock process;
- $\sigma_J$: The volatility in the mean jump size for the stock price jump process.

Additionally, we will utilize the pricing formulas noted in Bakshi, Cao, and Chen as their SVJ model can be collapsed down to the traditional SV model by setting $\lambda = 0$. 
Model Calibration

As discussed in our model selection section, we will focus on implementing the closed-form solutions for the SV and SVJ models as provided by Bakshi, Cao, and Chen. Although they provide us with closed-form solutions, the integrals aren’t readily computable as the normal density is. As such, we will need to clearly define an integration scheme to compute the value of the Fourier integrals present in our equations. Mikhailov and Nögel\textsuperscript{21} propose an adaptive Gauss-Lobatto quadrature for numerical evaluation while Gilli and Schumann\textsuperscript{22} utilize a similar Gauss-Legendre to quadrature evaluate these Fourier integrals. In R, the base quadrature package, \texttt{integrate()},\textsuperscript{23} utilizes an adaptive Gauss-Kronrod quadrature and is able to accurately compute these integrals quickly and with ease.

Additionally, for certain parameter specifications there are some computational pitfalls that arise when evaluating complex-valued logarithms, as Kahl and Jäckel\textsuperscript{24} noted in their technical article focusing on the implementation of the closed-form solution to Heston’s model. They point out that this is a generic problem that arises when evaluating the Fourier transformation of a log-characteristic function due to the fact that a complex natural logarithm is multi-valued. Namely, for \( z = a + i b = re^{i(t+2\pi n)} \) with \( t \in [-\pi, \pi) \), \( n \in \mathbb{Z} \) and

\begin{itemize}
\item \textsuperscript{21} Sergei Mikhailov and Ulrich Nögel, “Heston’s Stochastic Volatility Model Implementation, Calibration and Some Extensions,” \textit{Wilmott Magazine}.
\item \textsuperscript{22} Manfred Gilli and Enrico Schumann, “Calibrating the Heston Model with Differential Evolution,” in \textit{Applications of Evolutionary Computation}, ed. Cecilia Di Chio et al. (Berlin, Heidelberg: Springer Berlin Heidelberg, 2010), 242–250.
\item \textsuperscript{23} \texttt{integrate: Integration of One-Dimensional Functions ()}, https://www.rdocumentation.org/packages/stats/versions/3.6.2/topics/integrate.
\item \textsuperscript{24} Christian Kahl and Peter Jäckel, “Not-so-complex Logarithms in the Heston Model,” \textit{Wilmott Magazine}.
\end{itemize}
$r \in \mathbb{R}$ ($r = |z|$); $\ln(z) = \ln(r) + i(t + 2\pi n)$. They note that keeping track of the branch path along the integral can lead to inaccurate results along as the integral will still become discontinuous at certain points. These issues tend to arise when focusing on options with longer maturity dates, however, since we are restricting our options to shorter-dated options these computational pitfalls should not present any immediate problems.

To calibrate our model to market data, we will use a similar methodologies as discussed by Bakshi, Cao, and Chen,\textsuperscript{25} Mikhailov and Nögel,\textsuperscript{26} and Gilli and Schumann.\textsuperscript{27} We will use least squared error to fit our models to the given volatility surface for a given stock index on a given day. If we let $\tau_1, \ldots, \tau_J$ correspond to the given residual maturities, with their corresponding interest rates $r_1, \ldots, r_J$, and $K_1, \ldots, K_I$ corresponding to a set of strikes. Thus for any given day:

$$\text{SqrErr}(\Theta) = \sum_{i=1}^{I} \sum_{j=1}^{J} w_{i,j} [O_{\text{MP}}(K_i, \tau_j) - O_{\text{Model}}(S(t), K_i, r_j, \tau_j, \Theta)]^2$$

Where $O_{\text{MP}}(K, \tau)$ corresponds to the observed market price for an option with strike $K$ and a residual maturity $\tau$. $O_{\text{Model}}$ corresponds to our model’s closed form price for an option with a given strike and residual maturity. This price will depend on a vector of parameters, $\Theta$. For the SV model $\Theta_{SV} = \{v_0, \kappa, \theta, \sigma_v, \rho_{SV}\}$; and for the SVJ model $\Theta_{SVJ} = \{v_0, \kappa, \theta, \sigma_v, \rho_{SV}, \lambda, \bar{\mu}, \sigma_J\}$. The term $w_{i,j}$ corresponds to a weighting scheme for the sum.

\textsuperscript{25} Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”

\textsuperscript{26} Mikhailov and Nögel, “Heston’s Stochastic Volatility Model Implementation, Calibration and Some Extensions.”

\textsuperscript{27} Gilli and Schumann, “Calibrating the Heston Model with Differential Evolution.”
of square dollar pricing error. As Bakshi, Cao, and Chen\textsuperscript{28} note, when \( w_{i,j} = 1 - \) and the objective function simply becomes a sum of square pricing errors – this naturally places more weight on the options that have a higher price. In other words, the in-the-money and longer term options. Setting weights inversely proportional to the market price would change the objective function to optimize the sum of squared percent errors and place more weight on out-of-the-money and shorter dated options. Since we are focusing our data set on options with maturities less than a year and are filtering out contracts with low prices and minimal trading volume, we will set \( w_{i,j} = 1 \) and minimize the squared error of price differences.

Our optimization problem \( \min \left[ \text{SqrErr}(\Theta) \right] \) is a non-linear optimization problem. Additionally, as Mikhailov and Nögel\textsuperscript{29} note, the objective function is not convex and thus there tend to be many local extrema. Thus, there becomes a trade off when using local vs. global optimization. When given an initial “good guess” for our parameter vector \( \Theta \) a local optimization scheme can be beneficial to use for computational time. However, this runs the risk of missing the “true” parameters. Thus, Gilli and Schumann\textsuperscript{30} propose the implementation of a differential evolution algorithm to optimize parameter selection as opposed to a more traditional local-optimizer that uses gradient decent. There are existing software packages that can implement these calibration methods such as \texttt{DEoptim} in R.\textsuperscript{31} However, computa-

\begin{itemize}
  \item \textsuperscript{28}Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”
  \item \textsuperscript{29}Mikhailov and Nögel, “Heston’s Stochastic Volatility Model Implementation, Calibration and Some Extensions.”
  \item \textsuperscript{30}Gilli and Schumann, “Calibrating the Heston Model with Differential Evolution.”
  \item \textsuperscript{31}David Ardia et al., \textit{Package ‘DEoptim’} (2021 [Online].), https://cran.r-project.org/web/packages/DEoptim/DEoptim.pdf.
\end{itemize}
tional time became an issue when calibrating multiple models throughout this time period and a Nelder-Mead algorithm was chosen instead.

Analysis

In this section, we will analyze our model results. We will begin by analyzing our models in-sample performance, assessing the calibrated parameter summaries, the in-sample pricing fit, as well as the implied volatilities suggested from our models. Then, we will test the out-of-sample performance of our models again summarizing pricing fit, and computing the hedging error for our models, utilizing a single-instrument hedging strategy re-balanced daily throughout our calibration period.

In-Sample Performance and Structured Parameter Estimates

In this section, we will analyze the in-sample performance of the models. To do so, we will first provide a detailed interpretation of our structured parameter estimates recovered from our model calibration, discuss the in-sample pricing error and implied volatility calculations.

Parameter Summary

The table below provides the mean and standard deviation (in parenthesis) for the implied parameter fits over the course of our model calibrations.

For all three models, the recovered spot volatility was extremely high on average, roughly 45.26% for the SV model and 40.57% for the SVJ model. Intuitively, these results make sense as we are focusing our efforts on an extremely volatile period and would expect high estimated spot volatilities. On average, both the SV and SVJ models recover similar
estimates for the spot volatility with similar degrees of accuracy. However, the SVJ model requires a lower spot volatility as we have jumps incorporated into the stock price process, which mitigates the effect of the recovered spot volatility on the diffusion process.

Interestingly, despite the heightened volatility on average, the SV and SVJ models recover very small estimates for the long-run volatility, with somewhat minimal deviation. For the SV and SVJ models the estimated long run variance is $\sqrt{\frac{\theta}{\kappa}} = 12.62\%$ and $\sqrt{\frac{\theta}{\kappa}} = 9.46\%$ respectively. It is quite remarkable that our parameter estimates recovered such low estimates for the long run volatility despite isolating our analysis on a period of heightened volatility. The SVJ model estimated both a lower $\theta$ and higher $\kappa$ than the SV, resulting in the average long run volatility being less than 10%.
Additionally, both models recover higher (absolute) estimates for the $\rho_{SV}$ parameters than that which were recovered by Bakshi, Cao, and Chen. During the initial onset of the COVID-19 Pandemic in the United States, the market crashed while volatility spiked and observable market data would show that correlations reached all time (absolute) highs. This observation is again directly supported from our recovered parameter estimates. Additionally, we would expect to see the SV model recover a more higher (absolute) estimate for the $\rho_{SV}$ parameter than the SVJ model, as the SV model solely relies on the correlation and $\sigma_v$ parameter to drive the extra skew and kurtosis in the log-returns distribution, whereas the SVJ model is able to spread the burden of capturing the residual skew and kurtosis with the jump related parameters.

For the jump process, the SVJ model estimates that a jump in the stock price will occur roughly $\lambda = .76$ times per year with an average jump intensity of $\bar{\mu} = -14.44\%$ and uncertainty of $\sigma_J = 18.72\%$. This jump mean and jump volatility are astoundingly large, but when considering the period we are looking at, these results are in-line with out intuition. We would expect to recover large, negative estimates for the jump mean as the S&P rapidly tumbled downward during the beginning and middle of March.

**Implied Volatility Analysis**

The implied volatility of an asset is defined as the volatility, $\sigma$, that would return the corresponding market price under the Black-Scholes model framework. In other words, we must solve for $\sigma$ s.t. $BS(\sigma) = \text{Market Price}$, where $BS(\sigma)$ is the Black Scholes price.

---

assuming $\sigma$ as the volatility parameter and \textbf{Market Price} represents the market price of the option. Unlike the other values that go into the Black-Scholes framework, $\sigma$ is not directly observable in the market. Additionally, the implied volatility can not be directly computed, as there is no way to invert the Black-Scholes formula to obtain volatility implied by the current market price, and thus it can only be estimated. Manaster and Koehler\textsuperscript{33} propose utilizing a Newton-Raphson procedure to back-out the implied volatility. The Newton-Raphson procedure is defined as follows:

$$\sigma_i = \sigma_{i-1} - \frac{BS(\sigma_i) - \text{Market Price}}{\text{Vega}_i}$$

Where $\sigma_i$ is the $i^{th}$ “guess” of the implied volatility, and Vega\textsubscript{i} is the Vega for the Black-Scholes price with $\sigma = \sigma_i$, where Vega = $\frac{\partial BS}{\partial \sigma}$, or the sensitivity of the current Black-Scholes price to changes in the volatility parameter. They state that a great advantage to using this technique is that for any $\sigma_i \in [a, b]$, then this search algorithm is guaranteed to converge to the true market implied volatility. They provide an initial guess $\sigma_1$ that falls within the range that guarantees a convergence to the true implied volatility: $\sigma_1 = \sqrt{\ln \left(\frac{S}{K}\right)} \frac{2}{r}$. However, the Newton-Raphson procedure will only properly converge given the option is reasonably priced. In other words, the option does not exceed the maximum arbitrage bounds: $\max[0, S - Ke^{-rT}] \leq \text{Market Price} \leq S$. In this case, there is no implied volatility that can be plugged into the Black-Scholes equation to retrieve the market price. Using this

methodology, we will estimate the implied volatilities for the options priced under our model as well as produce figures for the implied volatility surface.

In our analysis of the implied volatility smiles, we will perform similar calculations as Bakshi, Cao, and Chen\textsuperscript{34} did to assess model mispecification. To focus on the most tumultuous portion of our data set, we will center our focus our implied volatility analysis around March 13\textsuperscript{th}, 2020. On March 13\textsuperscript{th}, 2020 the White House officially declared the novel coronavirus outbreak a national emergency,\textsuperscript{35} and the market response occurred on the next trading day with the VIX Index reaching its highest closing level throughout this time period. There have been many assessments of the SV and SVJ models’ ability to capture the implied volatility smiles, so an analysis during days of lower volatility and more regular market conditions would be a mute point. Thus, we will focus our market observations between March 2\textsuperscript{nd}, 2020 and April 3\textsuperscript{rd}, 2020, back out the Black-Scholes implied volatilities as detailed in the procedure above, and compute an average in bucketed bins of $\% MNY = \frac{K}{S}$ and maturities, $\tau$, for the out of the call options in our sample. We will calculate $\% MNY$ in increments of 2.5%, and we will divide $\tau$ into short term (less than 60 days of residual maturity), medium-term (between 60 – 180 days of residual maturity) and longer term (between 180 – 365 days of residual maturity) to summarize our results. We understand that using a longer period to compute these average implied volatilities would be better, but we want to isolate our calculations on the period of our data set when the markets were under

\textsuperscript{34} Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”

the most uncertainty and distress. The following page contains the graphs which summarize the results of our calculations:
Figure 1: Implied Volatility Estimates Summary
From these graphs, we see that for the most part the SV and SVJ do a good job on average replicating the market implied volatility for shorter term options. Although the plot is much more jagged due to the rapidly changing prices in the market, it is clear to see that the for lower strike options, the estimated implied volatilities tend to drastically spike upwards. During the initial crash during the onset of COVID-19, the downside (lower strike price and moneyness) option implied volatility drastically rose. Our model is able to capture these trends well on average when comparing our results to the implied volatility in the market. It is notable that across maturities, our models tend to estimate higher levels of implied volatility on average, when compared to the market mean.

Additionally, to exemplify these observations, we will provide linearly interpolated volatility surfaces for our model, as well as what is implied in the market, using the entire period and averaging over duplicate points. These implied volatility surfaces are provided on the following page.
Figure 2: Average Implied Volatility Surfaces

(a) SV

(b) SVJ

(c) Market
We see that for the shorter-term options, our models are flexible enough to capture the wild behavior amongst the lower strike options. However, the implied volatility estimates of our models begin to break down as we lengthen $\tau$. Namely, it is also evident to see that the implied volatility for the in-the-money calls seems to decay much faster with the SV and SVJ models than what we observe within the market.

**In-Sample Pricing Fit**

Now, we will use our daily parameter estimates to recover our model’s options prices and compare them with the observable prices in the market. We will compute the % Pricing Error $\frac{O(t, \tau, \Theta) - O_{MP}(t, \tau)}{O_{MP}(t, \tau)}$ across different moneyness levels, and compute the mean, median, and standard deviation of these estimates. These results are summarized below in Table 3:

<table>
<thead>
<tr>
<th>Model</th>
<th>%Moneyness</th>
<th>Mean</th>
<th>Median</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>&lt;85%</td>
<td>-33.231030%</td>
<td>-27.755945%</td>
<td>74.70158%</td>
</tr>
<tr>
<td></td>
<td>85% – 90%</td>
<td>7.782268%</td>
<td>-10.25517%</td>
<td>156.4352%</td>
</tr>
<tr>
<td></td>
<td>90% – 95%</td>
<td>25.837929%</td>
<td>-3.516815%</td>
<td>154.75182%</td>
</tr>
<tr>
<td></td>
<td>95% – 100%</td>
<td>9.760375%</td>
<td>3.121976%</td>
<td>32.87848%</td>
</tr>
<tr>
<td></td>
<td>100% – 105%</td>
<td>35.474726%</td>
<td>7.400810%</td>
<td>139.92644%</td>
</tr>
<tr>
<td></td>
<td>105% – 110%</td>
<td>95.312461%</td>
<td>28.861210%</td>
<td>180.30718%</td>
</tr>
<tr>
<td></td>
<td>&gt;110%</td>
<td>48.978075%</td>
<td>5.035205%</td>
<td>180.30200%</td>
</tr>
<tr>
<td>SVJ</td>
<td>&lt;85%</td>
<td>-14.9922886%</td>
<td>-8.551542%</td>
<td>57.36029%</td>
</tr>
<tr>
<td></td>
<td>85% – 90%</td>
<td>3.2020967%</td>
<td>-3.347064%</td>
<td>100.10355%</td>
</tr>
<tr>
<td></td>
<td>90% – 95%</td>
<td>5.0576317%</td>
<td>-2.878923%</td>
<td>88.92186%</td>
</tr>
<tr>
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<td>95% – 100%</td>
<td>0.4549297%</td>
<td>-0.568810%</td>
<td>18.00355%</td>
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<td>13.4297056%</td>
<td>1.686383%</td>
<td>66.73369%</td>
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<tr>
<td></td>
<td>105% – 110%</td>
<td>53.2539635%</td>
<td>5.438436%</td>
<td>157.48721%</td>
</tr>
<tr>
<td></td>
<td>&gt;110%</td>
<td>61.4428208%</td>
<td>9.044185%</td>
<td>216.17081%</td>
</tr>
</tbody>
</table>
Immediately looking at this table, we see that the SVJ model provides a better and more consistent in-sample pricing fit across moneyness levels when compared to the SV. Although the SVJ begins to over-price the higher-strike price options with large errors (namely at the 105% − 110% and ≥ 110% moneyness levels), it provides a much tighter fit around the near-money options, and drastically improves the fit around the < 85% moneyness level. The SVJ model is able to more consistently price the lower strike options when compared to the SV model, as out of the money options tend to decay faster under the SV model. However, due to numerical methods utilized to calibrate these models and the data used, it is impractical to draw any solid conclusions while assessing these models on the basis of in-sample fit. We will need to test the out-of-sample performance of the model performance.

Out-of-Sample and Hedging Performance

Since the three models were calibrated using the closing price of the index and option prices, we will discuss an analysis of the calibrated parameters lagged one day. We will assess the out-of-sample pricing error and the hedging error of the SV and SVJ models.

Out-of-Sample Pricing Error

To assess model bias, we will conduct an analysis of our model’s pricing performance, when the previous day’s recovered parameters are used to price the options in the market. The table below summarizes the out-of-sample % Pricing Error of our models:
### Table 4: Out-of-Sample % Pricing Error

<table>
<thead>
<tr>
<th>Model</th>
<th>%Moneyness</th>
<th>Mean</th>
<th>Median</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV</td>
<td>&lt;85%</td>
<td>-28.43983%</td>
<td>-29.5697853%</td>
<td>76.33863%</td>
</tr>
<tr>
<td></td>
<td>85% – 90%</td>
<td>11.69027%</td>
<td>-7.5690447%</td>
<td>122.12568%</td>
</tr>
<tr>
<td></td>
<td>90% – 95%</td>
<td>35.44307%</td>
<td>4.4262800%</td>
<td>170.38370%</td>
</tr>
<tr>
<td></td>
<td>95% – 100%</td>
<td>13.13510%</td>
<td>9.1952640%</td>
<td>38.60551%</td>
</tr>
<tr>
<td></td>
<td>100% – 105%</td>
<td>33.85683%</td>
<td>6.8516076%</td>
<td>110.46221%</td>
</tr>
<tr>
<td></td>
<td>105% – 110%</td>
<td>106.30640%</td>
<td>25.0729152%</td>
<td>251.15522%</td>
</tr>
<tr>
<td></td>
<td>&gt;110%</td>
<td>75.00280%</td>
<td>-0.1419401%</td>
<td>357.61531%</td>
</tr>
<tr>
<td>SVJ</td>
<td>&lt;85%</td>
<td>-30.808440%</td>
<td>-38.886860%</td>
<td>133.82598%</td>
</tr>
<tr>
<td></td>
<td>85% – 90%</td>
<td>-15.761898%</td>
<td>-12.054538%</td>
<td>71.92203%</td>
</tr>
<tr>
<td></td>
<td>90% – 95%</td>
<td>-6.718376%</td>
<td>-12.158700%</td>
<td>68.74643%</td>
</tr>
<tr>
<td></td>
<td>95% – 100%</td>
<td>-4.271119%</td>
<td>-3.802151%</td>
<td>17.49361%</td>
</tr>
<tr>
<td></td>
<td>100% – 105%</td>
<td>8.123494%</td>
<td>-1.491602%</td>
<td>59.51273%</td>
</tr>
<tr>
<td></td>
<td>105% – 110%</td>
<td>55.653501%</td>
<td>4.533627%</td>
<td>133.72345%</td>
</tr>
<tr>
<td></td>
<td>&gt;110%</td>
<td>62.229635%</td>
<td>8.171458%</td>
<td>197.97092%</td>
</tr>
</tbody>
</table>

From these results, we see that out-of-sample, the SVJ performs much worse on average when compared to its in-sample pricing fit, whereas the SV is able to produce roughly similar errors when used out-of-sample compared to its in-sample pricing performance. These pricing error discrepancies with the SVJ model most notably appear at the ≤ 85%, 85% – 90%, and 95% – 100% %MNY levels. This suggests that the SVJ model is more biased to in-sample performance and is over-fitting to the in-sample data, which causes some of these drastic errors when the parameters are used out-of-sample. These biases could present some issues when using these model sensitivities to assess risk and hedge off of.
Hedging Performance

To assess the hedging performance of our models, we will implement similar methodologies as Bakshi, Cao, and Chen. We will incorporate single instrument hedges and compute the mean dollar and absolute hedging error across our time frame.

Since we are incorporating stochastic volatility and jumps into our models, we will utilize the minimum variance hedge solution as follows:

\[ X_S(t) = \frac{v_0}{v_0 + V_J} \Delta_s(t, \tau) + \rho_{SV} \sigma_v \Delta_v(t, \tau) \frac{v_0}{v_0 + V_J} + \frac{\lambda}{S(t)(v_0 + V_J)} \left[ \Lambda_1(t) - \Lambda_2(t) - \mu O(t, \tau) \right] \]

Where, \( O(t, \tau) \) denotes the computed price of the option under our model formulation, \( \Delta_S \) corresponds to the first order partial derivative with respect to the stock price \( S(t) \), \( \Delta_V \) corresponds to the first order partial derivative with respect to the volatility process, and \( V_J \) is the instantaneous jump volatility. \( V_J = \lambda \left[ \mu^2 + (e^{\sigma_v^2} - 1)(1 + \mu)^2 \right] \). \( \Delta_S(t), \Delta_V(t), \Lambda_1(t) \) and \( \Lambda_2(t) \) are computed as follows:

\[ \Delta_S(t) = \Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi \ln(K)} f_1(t, \tau)}{i\phi} \right] d\phi \]

Note, for put options, \( \Delta_S(t) = \Pi_1 - 1 \)

\[ \Delta_V(t) = S(t) \frac{\partial \Pi_1}{\partial V} - K e^{-r\tau} \frac{\partial \Pi_2}{\partial V} \]

\[ \Lambda_1(t) = \frac{S(t)}{2} \left[ \mu + \mu^2 + (e^{\sigma_v^2} - 1)(1 + \mu)^2 \right] + \frac{S(t)}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi \ln(K)} f_1 m_1}{i\phi} \right] d\phi \quad \text{and} \]

\[ \Lambda_2(t) = \frac{K e^{-r \tau} \bar{\mu}}{2} + \frac{K e^{-r \tau}}{\pi} \int_0^\infty \text{Re} \left[ e^{-i \phi \ln(K)} \frac{f_2 m_2}{i \phi} \right] d\phi \]

where, \( f_1 \) and \( f_2 \) are the characteristic functions as defined under the model formulation from Bakshi, Cao, and Chen,\(^{37}\)

\[ \frac{\partial \Pi_i}{\partial V} = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i \phi \ln(K)}}{i \phi} \frac{\partial f_i}{\partial V} \right] d\phi, \]

\[ m_1 = \exp \left\{ (1 + i \phi) \left( \ln[1 + \bar{\mu}] - \frac{1}{2} \sigma_y^2 \right) + \frac{1}{2}(2 + i \phi)^2 \sigma_y^2 \right\} - \exp \left\{ (1 + i \phi) \left( \ln[1 + \bar{\mu}] - \frac{1}{2} \sigma_y^2 \right) \right. \]

\[ + \left. \frac{1}{2}(2 + i \phi)^2 \sigma_y^2 \right\}, \text{ and} \]

\[ m_2 = \exp \left\{ (1 + i \phi) \left( \ln[1 + \bar{\mu}] - \frac{1}{2} \sigma_y^2 \right) + \frac{1}{2}(1 + i \phi)^2 \sigma_y^2 \right\} - \exp \left\{ i \phi \left( \ln[1 + \bar{\mu}] - \frac{1}{2} \sigma_y^2 \right) \right. \]

\[ - \frac{1}{2} \phi^2 \sigma_y^2 \right\}. \]

Using these results, we can compute the equivalent position in the underlying in order to minimize the minimum variance hedge, \( X_s(t) \). With this, we can compute the cash value of this position at time \( t \), \( X_0(t) = O(t, \tau) - X_s(t) \times S(t) \). To compute the hedging error, we will take the position of the previous observation of the option in the market and then re-balance the position after \( \Delta t \) days: \( H(t + \Delta t) = X_s(t) \times S(t + \Delta t) + X_0(t) e^{r \Delta t} - O(t + \Delta t, \tau - \Delta t) \).

Then we will compute the average dollar and absolute hedging error across our observations as a function of \( \Delta t \). Since we are focusing our efforts on a shorter horizon with high levels of volatility, we will filter observations in which \( \Delta t = 1 \). Table 5 summarises the results of these calculations:

\( ^{37} \) Bakshi, Cao, and Chen, “Empirical Performance of Alternative Option Pricing Models.”
Table 5: Hedging Error (March 2020 – May 2020)

<table>
<thead>
<tr>
<th>Hedging Error</th>
<th>%Moneyness</th>
<th>Model</th>
<th>SV</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>SVJ</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>N</th>
</tr>
</thead>
<tbody>
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<td>Dollar</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;85%</td>
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<td></td>
<td></td>
<td></td>
<td>-0.16</td>
<td>10.41930</td>
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<td>-$4.52</td>
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<td>105% – 110%</td>
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<tr>
<td>Absolute</td>
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<td></td>
<td></td>
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<tr>
<td>&lt;85%</td>
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<td></td>
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<td>$10.32</td>
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<td>90% – 95%</td>
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<td></td>
<td></td>
<td>$11.55</td>
<td>13.310190</td>
<td>$14.71</td>
<td>10.825807</td>
<td>(336)</td>
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<tr>
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<td>23.43809</td>
<td>$21.63</td>
<td>27.69738</td>
<td>(7004)</td>
</tr>
</tbody>
</table>

The first thing that immediately captures one’s attention are the large hedge errors, as Bakshi, Cao, and Chen\(^{38}\) reported much smaller magnitudes for dollar and absolute hedging errors. However, the time period were are examining has been unprecedented in terms of large volatility swings, so our results must be put into better context. Thus, we perform the same calculations over an equal length period immediately following this one, from June 2020 – August 2020. The table below summarizes the results of the dollar and absolute hedging errors for this contextualizing time period:

---

Table 6: Hedging Error (June 2020 – August 2020)

<table>
<thead>
<tr>
<th>Hedging Error</th>
<th>%Moneyness</th>
<th>Model</th>
<th>SV</th>
<th>SVJ</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dollar</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mean</td>
<td>St. Dev.</td>
<td>Mean</td>
<td>St. Dev.</td>
</tr>
<tr>
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Throughout the June 2020 – August 2020 calculations, we see that the dollar and absolute hedge errors are significantly lower across the board, showing a decrease of roughly 200% from the dollar and absolute errors calculated above. Although the hedging errors are still high, our results confirm that the initial onset of the COVID-19 Pandemic resulted in drastically larger hedging errors for both models as there were unprecedented swings in the stock price and volatility.

Going back to the results from Table 5, we notice that the SV model seems to perform much better, both in terms of dollar and absolute hedging error, when compared to the SVJ model. As we assessed previously with the out-of-sample pricing error, this is most likely
due to the SVJ model over-fitting to the calibrated data, and thus the computed model sensitivities are heavily biased towards the previous tables and perform much worse when assessing risks out-of-sample. Although the in-sample pricing fit for the SVJ was remarkably better than the SV model, the out-of-sample % Price Error performance of the SVJ rapidly declines.

Results

Our results confirm much of what has been analyzed regarding COVID-19 and equity volatility – heightened volatility and increased correlation corroborated by our estimates for the $v_0$ and $\rho_{SV}$ parameters – but has also provided us with some important insights regarding model bias and performance, imperative when assessing the risk-management aspect of the implementation of these models. It is difficult to predict tumultuous events such as the COVID-19 pandemic, but being able to accurately and efficiently assess the risks throughout periods of financial stress. Although the SVJ model performed much better in-sample, when we used the previous day parameter estimates to compute prices and sensitivities for the following day’s option prices, we noticed that there was a significant decline in performance in terms of pricing fit and hedging error. The SV model provided us with a parsimonious

This is due to the SVJ model becoming increasingly more biased in a setting where there is heightened volatility and financial distress. The more parsimonious SV model is able to control its in-sample biases and provide us with a more “risk-neutral” and un-biased risk assessment and hedging errors out-of-sample compared to the SVJ model.

This suggests that under the current model formulation for the SVJ model, there is an implicit bias to the in-sample data present when we isolate periods of financial distress.
As the skewness and kurtosis of the log-normal return distributions for the SVJ model can be influenced heavily by the jump related parameters $\lambda, \mu, \nu,$ and $\sigma_J$, the more flexible SVJ model will end up stretching its parameter estimates too much to conform as close as possible to the in sample data, and hence over-fit these jump-related parameters to conform to the in-sample data. As this period was much more volatile than the period examined by Bakshi, Cao, and Chen, these observations would not have immediately presented themselves, but some of these issues are able to come to light now.

**Conclusion**

The COVID-19 Pandemic has drastically changed the landscape of many aspects of our lives and has presented us with many unprecedented challenges. When analyzing the performance of stochastic volatility models in the equity derivatives market during the onset of the COVID-19 Pandemic in the United States on the equity derivatives market, we begin to see how COVID-19 has affected the volatility paradigm. Our results suggest that when placed under stress, the SVJ model tends to behave in a more biased manner towards its in-sample data, resulting in a poorer out-of-sample performance during this more tumultuous period.

The COVID-19 pandemic provided an interesting case example to test the performance of these models. Through this empirical analysis we were able to see that the SVJ model was able to fit the option prices very well in sample, and both the SV and SVJ models were able to closely recover the market implied volatilities when assessing the average

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implied volatility throughout the most volatile month in our sample. Additionally, we found that the more parsimonious SV model was able to provide much better out-of-sample fit as the SVJ model ended up being highly biased towards the in-sample results, resulting in worse performance out-of-sample and hedging.

Perhaps, there could be similar studies performed on other options pricing models that do not have a readily analytically tractable solution to test the performance of a wider range of models.
Bibliography


Dunn, Robin, Paloma Hauser, Tom Seibold, and Hugh Gong. “Estimating Option Prices with Heston’s Stochastic Volatility Model.”


