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By ROBERT E. VERRECCHIA*

In an attempt to provide a precise analytical interpretation for the notion of "information efficiency" introduced by Eugene Fama (1970), Mark Rubinstein suggests that a necessary and sufficient condition for an individual to perceive all his information fully reflected in prices is that he have "consensus beliefs." That is, consensus beliefs are those beliefs which, if held by all individuals in an otherwise similar economy, would generate the same equilibrium prices as in the actual heterogeneous economy. However, the notion of information efficiency embodied in Rubinstein's discussion has been more broadly interpreted as follows.[1] A market is efficient with respect to an information set **A**, say, if the prices it generates are identical to those generated in an otherwise identical economy in which the set **A** accurately describes the information available to each and every market participant; the common, or homogeneous, belief induced by knowledge of **A** is the consensus belief.[2]

If the set of information **A** happens to be the union of the information sets privately available to each and every market participant, then efficiency with respect to **A** can be characterized as a situation in which the consensus belief is at least as correct an assessment (on the basis of **A**) as the assessment privately held by any single investor. Stated more informally, all investors perceive that no less information is fully reflected in prices than that which each individual investor privately knows. With regard to welfare implications, when the consensus belief is at least as correct, investors face a type of fair game in the market. That is, let "excess returns" be defined as the difference in the expected utility some well-informed investor achieves by trading at a price which reflects information inferior to his own vis-à-vis the expected utility he achieves when price reflects information at least as good. Then a price reflecting no less information than what is privately known prohibits excess returns. In effect, prices provide an effective barrier to the exploitation of less well-informed investors by better-informed investors, because no investor can exploit his superior information (in the classical price-taking setting usually attributed to a securities market) whenever a price exists which reflects information which is in no way inferior to his own.[3]

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[1] See, for example, Avraham Beja, William Beaver, and my 1979b paper.

[2] For example, suppose that a market consisted of two investors, the first of whom had as his sole source of information a set **O**, say, which contained only optimistic bits and pieces of data about a security, and which consequently induced him to regard the future return of the security as "abnormally high." Alternatively, the second investor had as his sole source of information a set **P**, say, which contained only pessimistic bits and pieces of data about the security, and which consequently induced him to regard the future return of the security as "abnormally low." Suppose further that trading against one another in the market resulted in a price that appeared to reflect the fact that the future return for the security would be "average"; this is a natural supposition if both traders have equal weight in the market with regard to the supply and demand of the security. Now if in an otherwise identical market the set of information **A**, where **A** is the union of **O** and **P**, is substituted for both traders' original private sources, **O** and **P**, respectively, and if this knowledge induces them to regard the future return of the security as average, then this will likely cause no price change since prices have already averaged out their diverse opinions. Thus the market for the security will be efficient with respect to **A**, where **A** is the union of **O** and **P**.

[3] To illustrate this, consider the situation described in fn. 2 in which one investor knows **O**, and another knows **P**. Suppose that a third investor enters the market who happens to know **A**, the union of **O** and **P**. Clearly, the third investor is "better informed" than either of the other two, where it will be assumed

In brief, the consensus belief concept is an intuitively plausible definition of market information efficiency which, unlike some of its competitors, lends itself to formal analyses. The purpose of this paper is to consider its robustness. For example, it is well known that prices are determined in part by a type of geometric averaging of each investor's assessment weighted by his tolerance for risk. Therefore, efficiency with respect to the union of investors' private information sets can be loosely described as a phenomenon in which the consensus belief implied by this geometric average is at least as correct as any assessment that contributes to the average. But there is no intrinsic reason why this should be the case: if two bettors have an opinion as to which horse will win a race, and one bettor's opinion is vastly superior to the other's (because of the time and expense invested in formulating that opinion), there may be no averaging of their opinions that will be superior to that of the more sagacious bettor.

This is the problem. If one investor is motivated to invest in the acquisition of information, it is within reason that he could achieve an assessment that is more accurate than that implied by the consensus belief, despite the fact that his assessment is among those that are impounded in price. To achieve a better understanding of this problem, it can be modeled along the following lines. Suppose that the mental process of determining an assessment of the return of a security is thought of as analogous to the statistical process of determining an estimate of some unknown parameter. That is, suppose that at the end of a future period, nature will draw from an urn a numbered ball that represents the return on the security. For the benefit of this discussion, it will be assumed that numbers on the balls are normally distributed in the urn with known variance and unknown mean. Before trading occurs, however, investors can draw sample observations from this urn; the observations are private information sets, and **A** can represent the union of these sets. With his private observations, each investor forms an estimate of the unknown mean which, in turn, determines his individual assessment. Then the consensus belief is at least as accurate as any single investor's assessment whenever the precision (where precision is defined to be the inverse of the variance) of the estimate of the return implicit in the consensus belief is greater than or equal to the precision in any single investor's estimate. This, in turn, implies that the market is efficient with respect to **A**, the union of investors' information sets.

The subsequent analysis considers the behavior of investors with different tolerances for risk in this situation. What is demonstrated is that whenever all investors acquire what each perceives to be an optimal quantum of information (in the form of sample observations), the degree of precision implicit in the consensus belief is indeed no less than the degree of precision in a single investor's estimate. This result implies that the Fama-Rubinstein insight as to what constitutes information efficiency is robust in that it evolves naturally from the behavior of investors operating in their own self-interests. The fact that it is robust suggests that it may provide a more viable and interesting characterization than competing theories.[4] The following analysis formally demonstrates this result.

without controversy that possession of more information signifies being better informed. The only way investors interact in a market is through prices; i.e., an equilibrium price is established for a security on the basis of investors' expectations, and at this price investors exchange holdings of the security. Therefore, when the price the third investor encounters is the same price that would prevail if each and every investor knew **A**, and in our previous example this was the case, there is no way he (the third investor) can earn an excess return (as I have narrowly defined the term) on the basis of **A**, despite the fact that it represents information superior to that available to the two other investors.

[4] The chief competing theory is one of "rational expectations." A recent survey article by Sanford Grossman provides a comprehensive guide to this work. The theory of rational expectations provides a particularly elegant argument for what constitutes market information efficiency. However it is also fragile in that it relies on a number of assumptions, such as the absence of "noise" (i.e., indiscernable randomness), which may not be met in a situation embodying the full complexity of a real world setting. Furthermore, it is a

I. A Description of the Market

To initiate the analysis, consider an investor i who trades in a market which offers the opportunity to purchase an unlimited amount of a risk-free security denoted by S_F, or a risky security denoted by S_R, whose total supply is S_R^T. At the end of some future period, investment in the risk-free security yields a fixed return of F units of wealth for each unit of S_F purchased, and investment in the risky security yields an uncertain return of R units of wealth for each unit of S_R purchased. At the beginning of a trading period, investor i will choose an amount S_F^i and S_R^i of the risk-free and risky security, respectively, which maximizes his expected utility subject to his initial wealth, which is denoted by W^i. That is, investor i chooses an amount S_F^i and S_R^i of the risk-free and risky security, respectively, to maximize his utility when S_F and S_R yield their respective returns.

Before trading, an investor also has the opportunity to acquire information. The information acquisition process will be characterized as a sampling process from a normal probability distribution.[5] That is, suppose that R is assumed to be some real number which is initially known to all investors to be normally distributed with mean $\bar{m}$ and precision $\bar{h}$ (where precision is defined to be the inverse of the variance $\bar{h}^{-1}$).[6] The acquisition of additional information is characterized as a sample of a predetermined size n consisting of the observation of random variables $X_1, \dots, X_n$, about which it is known initially that each is independently and identically distributed normally with mean R and precision h. Then it is well known (see Morris DeGroot, p. 167) that conditional on the observed sample values $x_1, \dots, x_n$, an investor regards the distribution of R to be normal with mean

$$\hat{m} = \frac{\bar{h}\bar{m} + nhm}{\hat{h}}$$

and precision $\hat{h} = \bar{h} + nh$

where $$m = \sum_{k=1}^{n} \frac{x_k}{n}$$

is the observed mean of the sample. It will also be assumed that each investor's observations are distributed independently of the observations of any other investor.

The sample mean itself, however, is not known until the information has been processed; prior to its revelation, it is a random variable. Its unconditional distribution can be determined as follows. Conditional upon knowledge of R, m is normally distributed with mean R and precision nh, because it is the mean of n sample observations, each of which is normally distributed about R with mean h. An investor's initial or prior belief about R is that it is normally distributed with mean $\bar{m}$ and precision $\bar{h}$. Thus, it can be shown that the unconditional distribution of m is a normal probability distribution with mean $\bar{m}$ and precision $nh\bar{h}/\hat{h}$.[7]

It will also be assumed that investor i has a (negative) exponential utility function of

notion of efficiency which is stronger than that suggested by Fama (1976), or implied by the empirical evidence concerning the existence of efficiency in the semistrong form.

[5] This characterization of information acquisition is suggested by Robert Wilson, p. 191.

[6] It should be emphasized that while prior assessments are identical, it is assumed that each investor arrives at the assessment $\bar{m}$ independent of other investors. This allows these prior assessments to be treated as though they were independent sample observations about R, each with precision $\bar{h}$.

[7] This result can be explained, intuitively, by noting that $\bar{m}$ is simply the mean of the prior distribution of R, and $nh\bar{h}/\hat{h}$ is equal to

$$\frac{nh\bar{h}}{\hat{h}} = \left[\frac{\bar{h}+nh}{nh\bar{h}}\right]^{-1} = \left[\frac{1}{nh} + \frac{1}{\bar{h}}\right]^{-1}$$

That is, the precision $nh\bar{h}/\hat{h}$ is simply the inverse of the sum of the variance $(nh)^{-1}$ of the conditional distribution of the sample mean for any given value of R, and the variance $\bar{h}^{-1}$ of the prior distribution of R. In effect, m is normally distributed with the mean investors attribute to R before they process the information, $\bar{m}$, and precision which, in a broad sense, represents the precision of their prior beliefs plus the precision of the information itself.

the form

$$U_i(w) = -r_i e^{-(1/r_i)w}$$

for wealth w, where $r_i > 0$. The coefficient $1/r_i$ is investor i's coefficient of risk aversion; its inverse r_i is referred to as his coefficient of risk tolerance. The purpose in assuming utility functions of this form is that it permits a consensus belief to be explicitly determined.[8] The extent to which the results of the analysis depend upon this assumption is not entirely clear. However, as the discussion unfolds, it becomes clear that the salient feature of the analysis is the relationship between degrees of risk tolerance and information acquisition activities; therefore, it is reasonable to conjecture that the form of the utility function assumes no more than a secondary role, provided that it evidences global risk aversion.[9]

When investor i enters the market at the beginning of the period, he encounters the following problem. If the cost associated with making n sample observations is represented by a function $C(n)$ (which is assumed to be a twice differentiable function of n), investor i must choose

1) an optimal number of sample observations, and

2) based upon the information in those observations, as well as his prior information, an optimal amount S_R^i and S_F^i of risky and risk-free securities, respectively, so as to maximize his future period expected utility; and, of course, his choices must be constrained by his initial wealth.

Suppose that a unit of the risk-free and risky security sell in the market for 1 and P units of wealth, respectively (i.e., the price of a risk-free security is a numeraire). Investor i's choice problem can be characterized mathematically as choosing those elements n, S_F^i, S_R^i that

$$(1) \qquad \max_{\{n, S_F, S_R\}} \int_m \int_R -r_i e^{-1/r_i\{FS_F + RS_R\}} \cdot dN(\hat{m}, \hat{h})\, dN(\bar{m}, nh\bar{h}/\hat{h})$$

subject to $W^i = S_F + PS_R + C(n)$, where $dN(\mu, \rho)$ denotes a normal density function with mean μ and precision ρ.

For example, consider optimal amounts of S_F^i and S_R^i conditional upon the choice of n sample observations. Conditional upon n, (1) reduces to solving the Lagrangian equation

$$L(S_F, S_R, \lambda) = -\int_R r_i e^{-1/r_i\{FS_F + RS_R\}} dN(\hat{m}, \hat{h}) + \lambda\{W^i - (S_F + PS_R + C(n))\}$$

Differentiating L with respect to S_R yields

$$(2) \qquad \frac{\partial L}{\partial S_R} = \int R e^{-1/r_i\{FS_F + RS_R\}} dN(\hat{m}, \hat{h}) - \lambda P$$

Differentiating L with respect to S_F yields

$$(3) \qquad \frac{\partial L}{\partial S_F} = \int F e^{-1/r_i\{FS_F + RS_R\}} dN(\hat{m}, \hat{h}) - \lambda$$

Necessary first-order conditions require $\partial L / \partial S_R = \partial L / \partial S_F = 0$ in (2) and (3). Thus the two equations can be combined to yield

$$(4) \qquad P = \frac{\int R e^{-1/r_i\{FS_F + RS_R\}} dN(\hat{m}, \hat{h})}{\int F e^{-1/r_i\{FS_F + RS_R\}} dN(\hat{m}, \hat{h})}$$

Recognizing that for any parameter α,

$$e^{-\alpha R} dN(\mu, v) = e^{-\alpha\mu + \alpha^2/2v} dN\left(\mu - \frac{\alpha}{v}, v\right)$$

and using a common property of the moment generating function of a normal dis-

[8] The *existence* of a consensus belief, however, can be established even in the absence of an explicit determination. See my 1979b paper.

[9] Note that the assumption of constant risk tolerance implies that an investor's tolerance for risk is independent of his wealth. However, if one postulates that risk tolerance increases as wealth increases, investors with large constant risk tolerance can be thought to assume the role of investors with large wealth.

tribution, it can be shown that (4) reduces to

(5)
$$P=\frac{\int R\,dN\left(\hat{m}-\frac{S_R}{r_i\hat{h}},\hat{h}\right)}{F\int dN\left(\hat{m}-\frac{S_R}{r_i\hat{h}},\hat{h}\right)}=\frac{\hat{m}-(S_R/r_i\hat{h})}{F}$$

Rearranging terms yields the amount of the risky security investor i demands,

$$S_R^i=r_i\hat{h}\{\hat{m}-FP\} \tag{6}$$

Suppose it is assumed, however, that investor i is representative of all other investors in the market; specifically, each investor j, say, has a utility for wealth characterized by a constant tolerance for risk r_j and an assessment of the distribution of $\tilde{R}$ which is normal with mean $\hat{m}_j$ and precision $\hat{h}_j=\bar{h}+n_jh$ (where n_j is investor j's optimal number of sample observations). Then summing (6) over the demand of all investors yields

$$S_R^T=\sum_{i=1}^{N} r_i\hat{h}_i\hat{m}_i-v_0FP \tag{7}$$

where N is the total number of investors,

$$v_0\equiv\sum_{i=1}^{N} r_i\hat{h}_i$$

is the sum of the precision in each investor's assessment of the unknown mean weighted by his risk tolerance, and

$$S_R^T\equiv\sum_{i=1}^{N} S_R^i$$

represents the total supply of risky securities. Finally (7) implies

$$P=\frac{1}{F}\left\{\sum_{i=1}^{N}\left(\frac{r_i\hat{h}_i}{v_0}\right)\hat{m}_i-\frac{S_R^T}{v_0}\right\} \tag{8}$$

The expression derived in (8) is the market-clearing or equilibrium price of the risky security S_R, assuming that the price of the risk-free security is set at one.

II. Consensus Beliefs and Market Information Efficiency

To provide some intuition, the expression for P in (8) can be interpreted as the ratio of

1) the sum of each investor's assessment or estimate of the unknown mean of the return for R, $\hat{m}_i$, weighted by the coefficient $r_i\hat{h}_i/v_0$, and adjusted by a factor of supply represented by $-S_R^T/v_0$, to

2) the known fixed return of the risk-free security F.

Using the definition of a consensus belief (see Rubinstein or my 1979b paper), it is a simple exercise to show that the expression

$$\sum_{i=1}^{N}\left(\frac{r_i\hat{h}_i}{v_0}\right)\hat{m}_i$$

is a consensus belief. This is because this expression is precisely that belief, or estimate, of the unknown mean of R, which, if held by all investors in some otherwise identical market, would effect a market-clearing price identical to the one derived in (8).[10]

Consider the variance of the consensus belief. Assuming that each sample observation made by an investor is independent of all other observations made by all other investors, as well as those other observations made by himself, the variance of the consensus belief is

(9)
$$\begin{aligned}\operatorname{Var}\left[\sum_{i=1}^{N}\left(\frac{r_i\hat{h}_i}{v_0}\right)\hat{m}_i\right]&=\sum_{i=1}^{N}\left(\frac{r_i\hat{h}_i}{v_0}\right)^2\operatorname{Var}[\hat{m}_i]\\&=\sum_{i=1}^{N}\left(\frac{r_i\hat{h}_i}{v_0}\right)^2\hat{h}_i^{-1}\\&=\sum_{i=1}^{N}\left(\frac{r_i}{v_0}\right)^2\hat{h}_i\end{aligned}$$

[10]Specifically, it can be shown that if each investor regards the distribution of R to be normal with mean $\Sigma(r_i\hat{h}_i/v_0)\hat{m}_i$ and precision $v_0/(\Sigma r_i)$, a price identical to the one in equation (8) obtains.

remembering that the $r_i\hat{h}_i$ are all constant.[11]

As discussed heuristically in the introduction, the notion of market information efficiency that underlies the concept of a consensus belief is that the variance in the consensus belief assessment of uncertainty (in this case the unknown mean of R) is less than or equal to the variance in each investor's personal assessment. (This, of course, implies that the precision in the consensus belief is at least as great as the precision in each investor's assessment.) Since the variance of the consensus belief assessment is given by (9), and the variance of each investor's assessment is given by $\hat{h}_j^{-1}$, this can be expressed formally as the requirement that for each investor j, $j=1,\ldots,N$,

$$\sum_{i=1}^{N}\left(\frac{r_i}{v_0}\right)^2\hat{h}_i \leq \hat{h}_j^{-1} \tag{10}$$

The interesting question is under what circumstances the expression in (10) would hold. Casual observation of (10) suggests that if investors with large risk tolerance have little precision in their assessments, and vice versa, then the variance in the consensus belief will be large, because in effect poor assessments are given greater weight in determining the consensus belief than good assessments. Therefore, for (10) to hold, it must be that investors with greater risk tolerance somehow contribute better assessments to determining the market-clearing price than investors with lower risk tolerance. The next section considers how investors choose an optimal degree of precision on the basis of their risk tolerance, and suggests a circumstance in which the appropriate behavior results.

III. A Sufficient Condition to Ensure Information Efficiency

The characterization of market efficiency implied by the consensus belief interpretation requires that equation (10) hold. The validity of equation (10) is demonstrated in a number of steps. First, a function $H(r)$ is introduced to represent an investor's optimal degree of precision as a function of his tolerance for risk through the relationship

$$H(r)=\hat{h}=\bar{h}+n_0(r)h$$

where $n_0(r)$ represents the optimal number of sample observations an investor requires as a function of his risk tolerance. Second, in this section, it is demonstrated that a sufficient condition that equation (10) hold is that $H(r)$ is a nondecreasing function of r; that is, each investor's optimal degree of precision is a nondecreasing function of his risk tolerance. Finally, the fact that $H(r)$ is indeed a nondecreasing function of r is demonstrated in the next section.

If it is assumed that $H(r)$ is nondecreasing, proving the validity of equation (10) is straightforward. To begin, consider the investor whose risk tolerance is no less than the risk tolerance of any investor in the market; this investor is referred to as investor k. That is, $r_k > r_i$ for all i. This implies that for all i,

$$r_iH(r_k) \leq r_kH(r_k) \leq \sum_{i=1}^{N} r_iH(r_i) \equiv v_0$$

which in turn implies that for all i,

$$H(r_k) \leq \frac{v_0}{r_i} \tag{11}$$

However, for equation (10) to be valid for investor k, it must be that

$$\sum_{i=1}^{N}\frac{r_i^2H(r_i)}{v_0^2} \leq \{H(r_k)\}^{-1}$$

or

$$\sum_{i=1}^{N}\frac{r_i^2H(r_i)H(r_k)}{v_0^2} \leq 1 \tag{12}$$

Substituting the inequality in (11) into the

[11]This also requires that each investor's prior assessment of the mean of R, $\bar{m}$ is arrived at independent of other investors. See fn. 6.

left-hand side of (12) yields

$$\sum_{i=1}^{N} \frac{r_i^2 H(r_i) H(r_k)}{v_0^2} \leqslant \sum_{i=1}^{N} \frac{r_i^2 H(r_i)}{v_0^2} \cdot \frac{v_0}{r_i}$$

$$= \sum_{i=1}^{N} \frac{r_i H(r_i)}{v_0} = 1$$

which implies (12). Thus, the degree of precision in investor k's assessment is no greater than the degree of precision in the consensus belief assessment.

But note that if $H(r)$ is a nondecreasing function of r, then it must be that for all i, $H(r_k) \geqslant H(r_i)$. That is, investor k attains a degree of precision that is no less than the degree attained by anyone else in the market. Consequently, no one else in the market has a degree of precision which is greater than the degree of precision in the consensus belief assessment, which ensures equation (10). In brief, market information efficiency holds in the sense of (10) whenever each investor attains as an optimal degree of precision a level which is a nondecreasing function of his risk tolerance.

It remains to show that $H(r)$ is a nondecreasing function. However, even in the absence of the analysis performed below, one would intuitively expect this to be the case. This is because investors with greater risk tolerance invest a greater proportion of their wealth in the risky security relative to investors with the same wealth and the same values for mean and precision, but with lower tolerances for risk. For example, this relationship can be seen from the expression for investor demand derived in (6):

$$S_R^i = r_i \hat{h}_i \{\hat{m}_i - FP\}$$

Therefore to safeguard their increased amount of investment in the risky security, one might reasonably expect risk tolerant investors to acquire more information. This is shown in the next section.

IV. Optimal Sampling Sizes

In this section it is formally demonstrated that $n_0(r)$ is a nondecreasing function of r; that is, an investor's optimal number of sample observations is a nondecreasing function of his risk tolerance. Since the optimal degree of precision is a linear function of the optimal sample size, this implies that the optimal degree of precision is a nondecreasing function as well.

To show that $n_0(r)$ is a nondecreasing function of r, an expression for equation (1) is needed. Recall that since it was shown that $S_R^i = r_i \hat{h}\{\hat{m} - FP\}$ and $S_F^i = W^i - C(n) - PS_R^i$ are optimal amounts of the risky and risk-free security, respectively, (1) can be expressed as

$$\max_{\{n\}} G(n, r)$$

$$\text{subject to} \quad W = S_F + PS_R + C(n)$$

where $G(n,r)$

$$= \int_m \int_R -re^{-F/r\{W - C(n)\} - \hat{h}\{R - FP\}\{\hat{m} - FP\}}$$

$$\times dN(\hat{m}, \hat{h})\, dN(\bar{m}, nh\bar{h}/\hat{h})$$

and the i sub- and superscripts used to identify investor i are also dropped to ease the notational burden. That is, $G(n,r)$ is the expected utility an investor with risk tolerance r achieves when he chooses to make n sample observations.

Using the properties of a normal moment generating function discussed above, integration with respect to R yields

(13)

$$G(n,r) = \int_m -re^{-F/r\{W - C(n)\} - \frac{1}{2}\hat{h}\{\hat{m} - FP\}^2}$$

$$\times dN(\bar{m}, nh\bar{h}/\hat{h})$$

As a means of integrating the expression in (13), let the variable y be introduced, where y is defined as

$$y = (nh\bar{h}/\hat{h})^{1/2}\{m - \bar{m}\}$$

which implies $dy = (nh\bar{h}/\hat{h})^{1/2}\, dm$

Also note that a priori an investor does not know what value P will assume. It will be assumed here that each investor will behave *as if* he believes that price will simply reflect the ratio of what all investors know initially about the expected return for R, i.e., $\bar{m}$, to the risk-free return F; that is, all investors will behave as if they believe that $P=\bar{m}/F$. Stated differently, investors act as if prices reflect only what is already well known, and not (among other things) their own additional information acquisition activities, or the information acquisition activities of others.[12] This assumption is consistent with the notion that as price takers in a traditional market setting, investors behave as if they perceive no relationship between their own activities and price. Analytically, this assumption implies (with some algebraic manipulation) that

$$\hat{h}\{\hat{m}-FP\}^2=\{(nh)^2/\hat{h}\}\{m-\bar{m}\}^2$$
$$=\{nh/\bar{h}\}y^2$$

where the variable y is used as a substitution.

It can be shown that a further substitution of y in place of the variable m transforms (13) into

$$G(n,r)$$
$$=\int_y -re^{-F/r\{W-C(n)\}-1/2\{1+(nh/\bar{h})\}y^2}dy$$

which, when integrated using properties of a normal probability distribution, yields

$$G(n,r)=-r(\bar{h}/\hat{h})^{1/2}e^{-F/r\{W-C(n)\}}$$

With an expression for $G(n,r)$ now available, define a function $n_0(r)$ which represents the value of n which maximizes $G(n,r)$ for each value of r. The function $n_0(r)$ represents the optimal number of sample observations an investor should acquire when his risk tolerance is r. It is necessary to show that $n_0(r)$ is a nondecreasing function of r. This is done using the above expression for $G(n,r)$.

Allowing the derivative with respect to n to be denoted by an asterisk, observe that

$$G^*(n,r)=\left\{\frac{F}{r}C^*(n)-\frac{1}{2}h(nh+\bar{h})^{-1}\right\}\times G(n,r)$$

since

$$\frac{\partial}{\partial n}(\bar{h}/\hat{h})^{1/2}=-\frac{1}{2}(\bar{h})^{1/2}h(nh+\bar{h})^{-3/2}$$
$$=-\frac{1}{2}h(nh+\bar{h})^{-1}(\bar{h}/\hat{h})^{1/2}$$

Assuming that $C(n)$ is a strictly increasing function of n, a necessary first-order condition for maximization requires that $n_0(r)$ satisfy

$$C^*(n_0(r))=\frac{rh}{2F\{n_0(r)h+\bar{h}\}} \tag{14}$$

since in this case $G^*=0$. Furthermore,

$$(15)\quad G^{**}(n,r)$$
$$=\left\{\frac{F}{r}C^{**}(n)+\frac{1}{2}h^2(nh+\bar{h})^{-2}\right\}G(n,r)$$
$$+\left\{\frac{F}{r}C^*(n)-\frac{1}{2}h(nh+\bar{h})^{-1}\right\}^2G(n,r)$$

Therefore, a sufficient second-order condition for maximization is that for all n,

$$\frac{F}{r}C^{**}(n)+\frac{1}{2}h^2(nh+\bar{h})^{-2}>0 \tag{16}$$

since in this case G is a concave function. (Concavity follows from the fact that $G(n,r)$ is negative for all n and r, and the expressions in (15) which precede those of $G(n,r)$ are both positive; thus G^{**} is negative

[12] This assumption is incompatible with theories, such as that of rational expectations, which rely substantially upon investors' ability to decode prices, or infer the information acquired by others through prices, over time (see Grossman). In effect, it suggests that investors evidence somewhat myopic behavior. Its advantage is that it may describe what occurs in an actual market setting.

whenever (16) holds.) Of course, maximization also requires that there is sufficient wealth for an investor to achieve this optimum, i.e.,

$$PS_R + C(n_0(r)) \leqslant W$$

But this will also be assumed since we are specifically concerned with the circumstance in which each investor achieves his optimal sample size.[13]

As an aside, note that while (16) does not require that the cost function is convex, for all practical purposes it does since otherwise a counterexample to the claim that G is concave can likely be constructed by an appropriate choice of parameters. This limitation notwithstanding, consider the objective of a cost function more generally. For equation (10) to hold, the cost function must prohibit a low-risk-tolerant investor from attaining a greater degree of precision than a high-risk-tolerant investor, or else the (geometric) average of their precisions may no longer exceed the precision of the investor with greatest precision. Increasing marginal cost of increased sampling is one circumstance which achieves this.

Finally, we demonstrate that the behavior of an optimal number of sample observations, as expressed in equation (14), is sufficient to imply that $n_0(r)$ is a nondecreasing function of r. Assuming that n_0 is differentiable with respect to r, differentiating both sides of (14) yields (where differentiation with respect to r is denoted by a prime):

$$C^{**}(n_0)\cdot n_0' = \frac{h}{2F}\left\{\frac{1}{(n_0 h+\bar{h})} - \frac{rhn_0'}{(n_0 h+\bar{h})^2}\right\}$$

or, rearranging terms,

$$(17)\quad n_0' = \frac{\dfrac{h}{2F(n_0 h+\bar{h})}}{C^{**}(n_0) + \dfrac{rh^2}{2F(n_0 h+\bar{h})^2}}$$

But the numerator of (17) is clearly nonnegative because all its terms are; the denominator is positive under the assumption ensuring concavity of G expressed in (16). Therefore, n_0' is a positive function of r, which implies that n_0 is (at least) a nondecreasing function of r.

The fact that $n_0(r)$ is nondecreasing immediately implies that $H(r)$ is also nondecreasing through the relationship

$$H'(r) = n_0'(r)h$$

In summary, if each investor can attain his optimal precision by acquiring his optimal number of sample observations, the degree of precision will be a nondecreasing function of his risk tolerance. As discussed previously, this behavior is sufficient to ensure that a market is informationally efficient in that equation (10) holds.

V. Conclusion

The interesting elements of any characterization of information efficiency are those conditions that are sufficient to induce it. This analysis attempts to point out that although there is no intrinsic reason why the geometric averaging performed by prices should yield an assessment (i.e., the consensus belief) which is no less precise than the individual assessments which contribute to the average, it holds whenever all investors acquire what each perceives to be an optimal quantum of information in the presence of some common cost function. For example, suppose that publicly available information is thought of as an urn from which investors make sample observations at some cost (in wealth, time, or effort). Then, within the context of the notion of market efficiency discussed here, a market efficient with respect to all publicly available information would naturally result; sufficient conditions would require no more than that investors act in their own self-interest. But, investors acting in their own self-interest is an assumption common to most discussions of competitive markets. Therefore, this interpretation of market information efficiency may imply a phenome-

[13] Note that given the nature of each investor's choice problem, the nonnegative amount $W-PS_R-C(n)$ which remains after n and S_R are selected is simply that amount allocated to investment in the risk-free security.

non that arises naturally from the market process.

Of course, there are a number of assumptions made in this analysis that limit generalization. Most significantly, it is assumed that investors face a common cost function which, although not necessarily convex, may approximate that requirement. A further restriction, but less obvious in its impact, is the assumption that all investors have constant risk tolerance. Thus, it is easy to imagine situations in which efficiency within the context of the consensus belief interpretation will not obtain. However, situations of this nature may be useful for explaining anomalies in the literature of efficient markets. The point of this paper is to demonstrate that the consensus belief interpretation of market information efficiency is both viable and robust.

APPENDIX

S_F = the risk-free security

S_R = the risky security

S_R^T = the total supply of the risky security

F = the risk-free rate of return

R = the (uncertain) risky rate of return

i = the ith investor

N = the total number of investors

W^i = the endowed wealth of investor i

S_F^i = the amount of investment in the risk-free security which is optimal for investor i

S_R^i = the amount of investment in the risky security which is optimal for investor i

$\bar{m}$ = the mean of the distribution of R prior to acquiring additional information

$\bar{h}$ = the precision of the distribution of R prior to acquiring additional information

m = the mean of a sample of observations from a process distributed normally with mean R and precision h

nh = the precision of a sample of n observations from a process distributed normally with mean R and precision h

$\hat{m}$ = the mean of the distribution of R posterior to acquiring additional information

$\hat{h}$ = the precision of the distribution of R posterior to acquiring additional information

r_i = the risk tolerance of investor i

$C(n)$ = the cost of sampling n observations

$dN(\mu, \rho)$ = a normal density function with mean μ and precision ρ

v_0 = the value of the sum over all investors of the product $r_i\hat{h}_i$

$n_0(r)$ = the optimal number of sample observations acquired by an investor with risk tolerance r

$H(r)$ = the optimal degree of precision attained by an investor with risk tolerance r, i.e., $H(r)=\bar{h}+n_0(r)h$

$G(n, r)$ = the expected utility an investor with risk tolerance r achieves when he chooses n sample observations.

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