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Mingqiang Yi
University of Pennsylvania

Haim H. Bau
University of Pennsylvania, bau@seas.upenn.edu

Howard H. Hu
University of Pennsylvania, hhu@seas.upenn.edu

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Abstract

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Keywords

peristaltic flow, finite element analysis, vibrations, fluid oscillations, confined flow

Comments

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Peristaltically induced motion in a closed cavity with two vibrating walls

Mingqiang Yi, Haim H. Bau,^{a)} and Howard Hu

Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, Philadelphia, Pennsylvania 19104-6315

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We consider incompressible, viscous fluid confined in a rectangular cavity. Both the bottom and top walls are made of thin membranes. Traveling waves with the same amplitude and frequency but opposite phase are transmitted in these membranes and induce peristaltic motion in the fluid. The induced velocity profiles are determined analytically for small-amplitude waves (ϵ) and numerically for waves of any amplitude. The analysis consists of a perturbation expansion in terms of the wave amplitude (ϵ) up to $O(\epsilon^2)$. Solutions for large-amplitude oscillations are obtained through the use of a finite element code that employs a moving grid. The numerical code is verified by critically comparing the simulations' predictions with analytical results. © 2002 American Institute of Physics. [DOI: 10.1063/1.1425841]

I. INTRODUCTION

In recent years, there has been a growing interest in developing minute laboratories on a "wafer." Often, in order to facilitate chemical and biological reactions, one needs to mix various reagents and chemicals. Although the characteristic lengths associated with microdevices are small, typically on the order of 100 μm , in the case of large molecules, diffusion alone does not provide a sufficiently rapid means for mixing. For example, at room temperature, myosin's coefficient of diffusion in water is about $10^{-11} \text{ m}^2/\text{s}$, and the time constant for diffusion along a length of 100 μm is intolerably large: 10^3 s . Commonly, one encounters only low Reynolds number flows in microdevices, and turbulence is not available to enhance mixing. Moreover, often it is not feasible to incorporate moving components such as stirrers into microdevices. Thus, one is forced to look for alternatives in order to make the mixing process more efficient.

We have considered a number of means for mixing enhancement in microconduits. Yi and Bau¹ studied a passive mixer, in which the conduit was shaped to form in- and out-of-plane bends. Each bend induced a pair of vortices that stretched and deformed material lines, thereby stirring the fluid. Unfortunately, to achieve effective mixing at very low Reynolds numbers, a very large number of bends was needed. Here, as an attractive alternative, we are studying theoretically a "peristaltic mixer." This theoretical study will guide us in the construction of the actual device. Our proposed mixer consists of a fluid-filled cavity bounded from below and above by flexible membranes. By printing electrical conductors on these membranes and passing electrical currents through these conductors in the presence of a magnetic field, one can cause the membranes to vibrate with relatively large amplitudes and in a prescribed way. Since the conductors can be shaped using photolithography, one can induce fairly complicated motions. For example, by appro-

priate phasing of the current, one can induce traveling waves in the membrane, which, in turn, will cause peristaltic pumping in the fluid. Of course, the results of our study are applicable to devices in which surface waves are actuated by other means. The construction of membranes made out of Kapton and their use in electromagnetic pumps is described in Kim *et al.*² The idea of using electromagnetic forces to induce deformations in a silicon membrane for the purpose of measuring added mass (microbalance) was implemented by Martin *et al.*³

Most prior studies of peristaltic pumping have focused on open conduits. For a lucid review of early literature see Jarrin and Shapiro.⁴ The mathematical problem depends on four dimensionless parameters: the wave amplitude ϵ , the wave number k , the Reynolds or Strouhal number R , and either the dimensionless pressure rise per wavelength (in the case of an open cavity) or the cavity's aspect ratio (in the case of a closed cavity). Since no general, closed-form solution for arbitrary values of all the above parameters seems possible, various authors have carried out their investigations with the asymptotic limit of some of these parameters being small or zero. Most studies have focused on the limit of small Reynolds numbers, very low-frequency oscillations (quasistatic approximation), and long wavelengths. This is essentially the lubrication approximation that in planar flows predicts an instantaneous parabolic (Poiseuille) velocity profile. The main advantage of this approximation is that it allows one to deal with large-amplitude (ϵ) oscillations. In addition to the references cited in Jarrin and Shapiro,⁴ lubrication theory was used more recently by Hung and Brown,⁵ Pozrikidis,⁶ Li and Brasseur,⁷ and Eytan and Elad.⁸ The lubrication theory is perhaps appropriate for physiological applications, in which the wave frequency is relatively small, and the wall deformations are large. In microsystems, however, one is often interested in high-frequency waves. In contrast to the above-cited investigations, Fung and Yih⁹ and Yin and Fung¹⁰ studied peristaltic pumping at arbitrary Reynolds numbers and at small wall deformations. Our study is an

^{a)} Author to whom correspondence should be addressed; electronic mail: bau@seas.upenn.edu

extension of these works^{9,10} to the case of a closed cavity.

All the above-cited studies focused on open cavities. Recently, however, using a perturbation expansion in terms of the wave amplitude (ϵ), Selverov and Stone¹¹ studied peristaltic motion in long (slender) cavities and low-amplitude vibrations. Their study, like ours, was motivated by the need to provide means for stirring fluids in microdevices.

On the experimental side, Moroney *et al.*¹² fabricated a peristaltically driven micromixer in silicon. The peristaltic motion was induced by traveling flexural waves in a 4 μm thick membrane. The fluid (water) motion was qualitatively observed by seeding the confined fluid with 2.5 μm diam spheres and by dye injection. The traveling waves were observed to induce fluid circulation with the fluid next to the membrane traveling in the direction of the wave and the fluid next to the stationary boundary traveling in the opposite direction.

In this paper, similarly to Fung and Yih,⁹ Yin and Fung,¹⁰ and Selverov and Stone,¹¹ we use a regular perturbation expansion in the wave amplitude (ϵ) to analyze peristaltic flow in a closed cavity. It turns out that in the closed-cavity case, the first-order solution consists of two terms: one identical to the expression for the open cavity^{9,10} and the other resulting from the pressure oscillations induced by the presence of sidewalls. This second term is specific to closed cavities, and it was ignored by earlier workers, who assumed that the presence of the sidewalls is felt only in the second-order approximation. The analytical solution allows us to determine the trajectories of passive tracers and assess the effectiveness of the peristaltic stirrer. Although the first-order approximation predicts net displacement of passive tracers, the magnitude of the displacement over one period of oscillation is of the order ϵ^2 . Thus, in order to obtain the correct description of the tracers' orbits, we are forced to compute the second-order approximation. Since it is not possible to treat analytically both high-frequency and large-amplitude oscillations, we resort to numerical techniques. Our numerical scheme employs finite elements in which an arbitrary-Lagrangian-Eulerian scheme is used to track the specified motion of the boundaries.¹³ The computer code is verified by critically comparing the computational results with theoretical predictions.

II. THE MATHEMATICAL MODEL

Consider an incompressible, Newtonian fluid confined in a rectangular cavity of length L^* and height H^* . The cavity's bottom and top boundaries deform, respectively, in the form of traveling waves $\epsilon H^* f(x^*, t^*, \omega^*, k^*)$ and $H^* - \epsilon H^* f(x^*, t^*, \omega^*, k^*)$. ω^* , k^* , and $\lambda = 2\pi/k$ are, respectively, the wave frequency, the wave number, and the wavelength. Here $\epsilon < 0.5$ is the dimensionless wave amplitude. The superscript (*) denotes dimensional quantities that will be made nondimensional. For example, previous workers^{9,10} considered $f(x^*, t^*, \omega^*, k^*) = \cos(k^* x^* - \omega^* t^*)$. In what follows, we use, respectively, the height H^* and the inverse of the angular wave frequency $(\omega^*)^{-1}$ as length and time scales. From now on, all quantities will be given in a nondimensional form. We employ the Cartesian coordinates $0 \leq x$

$\leq L$ and $\epsilon f(x, t, k) \leq y \leq 1 - \epsilon f(x, t, k)$. Moreover, we assume that the flow is predominantly two dimensional, i.e., the cavity is much wider than its height. We note in passing, however, that our proposed method of inducing boundary deformations can be modified to produce more complex motions.

The fluid motion in the cavity is induced by the transverse oscillations of the bottom and top boundaries. Since each membrane must be anchored at its two ends,

$$f(0, t) = f(L, t) = 0, \quad \forall t \geq 0. \quad (2.1)$$

Here, we select a function $f(x, t)$ that satisfies conditions (2.1) and that has the form

$$f(x, t) = \text{Real}\{e^{i(kx-t)}(1 - e^{-x/m})(1 - e^{(x-L)/m})\}. \quad (2.2)$$

The expressions $(1 - e^{-x/m})$ and $(1 - e^{(x-L)/m})$ were introduced to enforce the condition (2.1) of zero membrane deformation at the locations where the membrane is anchored to the substrate. Here m represents the "stiffness" of the anchor. Here, (m) is chosen to be small so that along most of the cavity's length (away from the side boundaries) the motion has the form of the traveling wave, $f(x, t) \sim \text{Real}\{e^{i(kx-t)}\}$.

Since the cavity is closed, volume conservation requires that

$$\int_0^L f(x, t) dx = 0. \quad (2.3)$$

With MATHEMATICA's¹⁴ aid, we compute the integral

$$\begin{aligned} \int_0^L f(x, t) dx = & -\frac{2}{k(1+k^2 m^2)} \cos\left(\frac{kL}{2} - t\right) \left((e^{L/m} - 1) km \right. \\ & \left. \times \cos\frac{kL}{2} - (e^{L/m} + 1) \sin\frac{kL}{2} \right). \end{aligned} \quad (2.4)$$

In order to satisfy condition (2.3), we must choose (k) such that

$$\tan\frac{kL}{2} = km \tanh\frac{L}{2m}. \quad (2.5)$$

For example, when $L/m = 100$, $kL \sim 6.41, 12.82, \dots$, $n \cdot 6.41$, where n is an integer.

The Strouhal number (which plays here a similar role to that of a Reynolds number) is defined as $R = H^* \sqrt{\omega^* / \nu}$, where ν is the fluid's kinematic viscosity. For example, in microfluidic applications, H^* may vary from 100 μm to 1 mm, the frequency may range from 10 Hz (when electromagnetic actuation is used) to 10^5 Hz or more (when piezoelectric actuation is used). When the working fluid is water, R may range from ~ 0.7 to ~ 700 . This can be contrasted with applications from physiology.⁸ For example, in the uterus, the wave frequency is typically smaller than 0.1 Hz and $R \ll 1$.

We scale the velocity $\mathbf{u} = \{u, v\}$ with $\epsilon H^* \omega^*$, and the pressure (p^*) with $\epsilon R^2 \mu \omega^*$, where μ is the shear viscosity. The fluid is incompressible and the flow satisfies the continuity equation,

$$\nabla \cdot \mathbf{u} = 0, \quad (2.6)$$

the momentum equation,

$$\frac{\partial \mathbf{u}}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + R^{-2} \nabla^2 \mathbf{u}, \quad (2.7)$$

and the boundary conditions,

$$\begin{aligned} \mathbf{u}[x, \epsilon f(x, t), t] &= \left\{ 0, \frac{\partial}{\partial t} f(x, t) \right\}, \\ \mathbf{u}[x, 1 - \epsilon f(x, t), t] &= \left\{ 0, -\frac{\partial}{\partial t} f(x, t) \right\} \quad (0 \leq x \leq L), \end{aligned}$$

and

$$\mathbf{u}(0, y, t) = \mathbf{u}(L, y, t) = 0 \quad [\epsilon f(x, t) \leq y \leq 1 - \epsilon f(x, t)]. \quad (2.8)$$

The first boundary condition implies that the wall is extensible and moves vertically only. By integrating the continuity equation with respect to (y) , we obtain

$$\begin{aligned} \int_{\epsilon f(x, t)}^{1 - \epsilon f(x, t)} \frac{\partial u(x, y, t)}{\partial x} dy \\ = v[x, \epsilon f(x, t), t] - v[x, 1 - \epsilon f(x, t), t] = 2 \frac{\partial f(x, t)}{\partial t}. \end{aligned} \quad (2.9)$$

Using Leibniz's rule and the boundary condition at the lateral wall, $u(0, y, t) = 0$, we integrate (2.9) from 0 to x and obtain

$$\int_{\epsilon f(x, t)}^{1 - \epsilon f(x, t)} u(x, y, t) dy = 2 \frac{\partial}{\partial t} \int_0^x f(x, t) dx. \quad (2.10)$$

Since we are conserving volume [Eq. (2.3)], this condition is also consistent with $u(L, y, t) = 0$. Condition (2.10) is instrumental in identifying the structure of the velocity field when we construct an approximate analytic solution for the flow.

Our prime interest is to investigate the flow field at moderate Stroh numbers (R) numbers. Therefore, we proceed with a perturbation expansion in terms of the amplitude (ϵ) and obtain expressions that are valid for a wide range of R numbers. Since most of the peristaltic pumping literature focuses on the limit $R \rightarrow 0$ and utilizes lubrication theory, we provide in Appendix A a derivation of the flow field for the closed conduit using lubrication theory and point out the differences between the closed and open conduits. We also show that in the appropriate limits the lubrication theory and the perturbation solution provided here collapse to the same limit.

III. PERTURBATION EXPANSION

In this section, we construct a perturbation solution valid for $\epsilon \ll 1$, $m \rightarrow 0$, and away from the lateral boundaries ($0 < x < L$). We expect the analytic solution to be valid at distances larger than one wavelength from the sidewalls. This expectation will be confirmed later when we compare the analytical solutions with the numerical ones. This estimate is consistent with the results of the numerical computations described later in the paper. The cavity need not be very long. We expand the velocity and pressure fields into a Taylor series in terms of ϵ :

$$\begin{aligned} \mathbf{u}(x, y, t; \epsilon) &= \mathbf{u}_0(x, y, t) + \epsilon \mathbf{u}_1(x, y, t) + \epsilon^2 \mathbf{u}_2(x, y, t) \\ &\quad + O(\epsilon^3), \\ p(x, y, t; \epsilon) &= p_0(x, y, t) + \epsilon p_1(x, y, t) + \epsilon^2 p_2(x, y, t) \\ &\quad + O(\epsilon^3). \end{aligned} \quad (3.1)$$

Likewise, with the aid of a Taylor series, the boundary conditions at the moving walls are specified at $y = 0$ and $y = 1$:¹⁵

$$\mathbf{u}[x, \epsilon f(x, t), t] = \mathbf{u}(x, 0, t) + \epsilon \frac{\partial \mathbf{u}(x, 0, t)}{\partial y} f(x, t) + O(\epsilon^2). \quad (3.2)$$

A. First-order solution

$$\nabla \cdot \mathbf{u}_0 = 0, \quad (3.3)$$

$$\frac{\partial \mathbf{u}_0}{\partial t} = -\nabla p_0 + R^{-2} \nabla^2 \mathbf{u}_0, \quad (3.4)$$

$$\begin{aligned} \mathbf{u}_0(x, 0, t) &= \left\{ 0, \frac{\partial}{\partial t} f(x, t) \right\}, \\ \mathbf{u}_0(x, 1, t) &= \left\{ 0, -\frac{\partial}{\partial t} f(x, t) \right\} \quad (0 \leq x \leq L), \end{aligned}$$

$$\mathbf{u}_0(0, y, t) = \mathbf{u}_0(L, y, t) = \mathbf{0} \quad (0 \leq y \leq 1), \quad (3.5)$$

and the condition (2.10) assumes the form

$$\int_0^1 u_0(x, y, t) dy = 2 \frac{\partial}{\partial t} \int_0^x f(x, t) dx = \text{Real}[I(x)]. \quad (3.6)$$

$I(x)$ represents the instantaneous flow rate at location x . The integral $I(x)$ is evaluated with MATHEMATICA:¹⁴

$$I(x, t) = 2(I_1 e^{i(kx-t)} + I_2 e^{-it}). \quad (3.7)$$

In the limit of $m \rightarrow 0$,

$$I_1 = -\frac{1}{k} \quad \text{and} \quad I_2 = \frac{1}{k}. \quad (3.8)$$

Equation (3.7) suggests that the velocity component in the x direction, u_0 , can be decomposed into two parts:

$$\begin{aligned} u_0(x, y, t) &= u_{0,1}(x, y, t) + u_{0,2}(y, t) \\ &= \text{Real}[U_{0,1}(y) e^{i(kx-t)} + U_{0,2}(y) e^{-it}]. \end{aligned} \quad (3.9)$$

In the above, far from the lateral walls, the first component is periodic in (x) and the second component is independent of (x) . Equation (3.9) is not valid next to the sidewalls ($x = 0, L$).

The second (x -independent) component results from the pressure gradient induced by the presence of the lateral boundaries. This pressure gradient is necessary to maintain a zero (u) velocity at the lateral solid walls, and the velocity component, $U_{0,2}(y)$, exists even in very long cavities. The velocity component $U_{0,1}(y)$ is identical to the flow field in an open cavity in the absence of lateral walls.^{9,10}

Equation (3.9) and the continuity equation suggest that the velocity component in the y direction has the form

$$v_0(x, y, t) = \text{Real}[V_{0,1}(y) e^{i(kx-t)}]. \quad (3.10)$$

Furthermore, upon substituting (3.9) into the x -direction momentum equation, we conclude that

$$p_0(x, y, t) = \text{Real}(A_0 x e^{-it} + P_{0,1}(y) e^{i(kx-t)}). \quad (3.11)$$

Since (3.4) and (3.5) are linear, we can solve independently for each of the velocity components (periodic in or independent of x) and then superimpose the various solutions.

The equation for $U_{0,2}(y)$,

$$-iU_{0,2} = -A_0 + R^{-2} \frac{d^2 U_{0,2}}{dy^2}, \quad (3.12)$$

with the boundary conditions

$$U_{0,2}(0) = U_{0,2}(1) = 0 \quad \text{and} \quad \int_0^1 U_{0,2} dy = 2k^{-1},$$

admits the solution

$$U_{0,2}(y) = \frac{2R \left[1 - \cos(\sqrt{i}Ry) - \tan\left(\frac{1}{2}\sqrt{i}R\right) \sin(\sqrt{i}Ry) \right]}{k \left[R + 2(-1)^{3/4} \tan\left(\frac{1}{2}\sqrt{i}R\right) \right]}. \quad (3.13)$$

Although valid for all values of R , when R is large, the numerical evaluation of (3.13) is susceptible to spurious numerical oscillations. For $R \gg 1$, using boundary layer theory and matched asymptotic expansions, we derive the approximate expression

$$U_{0,2}(y) \sim \frac{i\sqrt{i}R}{k} \frac{(1 - e^{i\sqrt{i}Ry} - e^{i\sqrt{i}R(1-y)})}{[\sqrt{i}R + 2(1 - e^{i\sqrt{i}R})]}. \quad (3.14)$$

When R is large, two thin boundary layers of thickness $O(R^{-1})$ form next to the boundaries $y=0$ and $y=1$.

To compute $U_{0,1}(y)$ and $V_{0,1}(y)$, it is convenient to introduce the streamfunction, $\Psi_{0,1}(x, y, t) = \text{Real}\{k^{-1} e^{i(kx-t)} \Phi_{0,1}(y)\}$, such that $u_{0,1} = \partial \Psi_{0,1} / \partial y$ and $v_{0,1} = -\partial \Psi_{0,1} / \partial x$. The streamfunction satisfies the momentum equation

$$\nabla^2 \left(R^{-2} \nabla^2 - \frac{\partial}{\partial t} \right) \Psi_{0,1} = 0, \quad (3.15)$$

and $\Phi_{0,1}(y)$ is the solution to the fourth-order ordinary differential equation,

$$\left(\frac{d^2}{dy^2} - k^2 \right) \left(\frac{d^2}{dy^2} - M^2 \right) \Phi_{0,1} = 0, \quad (3.16)$$

where $M^2 = k^2 - iR^2$. The boundary conditions are

$$\begin{aligned} \Phi_{0,1}(0) &= 1, \quad \Phi_{0,1}(1) = -1, \\ \frac{d\Phi_{0,1}}{dy}(0) &= 0, \quad \text{and} \quad \frac{d\Phi_{0,1}}{dy}(1) = 0. \end{aligned} \quad (3.17)$$

Equation (3.14) and the corresponding boundary conditions are identical to the open cavity problem, and the solution is given in Fung and Yih,⁹ i.e.,

$$\Phi_{0,1}(y) = A \sinh[k(y-1/2)] + B \sinh[M(y-1/2)], \quad (3.18)$$

where $A = -M \cosh(M/2)/F$, $B = k \cosh(k/2)/F$, and $F = M \sinh(k/2) \cosh(M/2) - k \cosh(k/2) \sinh(M/2)$.

Although (3.18) is valid for all R , when R is large, this equation, like Eq. (3.13), exhibits spurious numerical oscillations. For large values of R , it is convenient to use approximate expressions derived with the use of boundary layer theory. Such expressions were developed¹¹ for the case of a single moving wall. The extension for the two moving walls is straightforward and results in

$$\begin{aligned} u_{0,1} &= \text{Real}(e^{i(kx-t)} \{ \cosh(k/2)(e^{\xi y} + e^{\xi(1-y)}) \\ &\quad - \cosh[k(y-1/2)] / \sinh(k/2) \}), \end{aligned} \quad (3.19)$$

$$v_{0,1} = -\sin(kx-t) \sinh[k(y-1/2)] / \sinh(k/2), \quad (3.20)$$

where $\xi = R(\sqrt{2}/2)(i-1)$.

In the other extreme, when $R \rightarrow 0$ and $k \rightarrow 0$ (long wavelength), Eq. (3.18) predicts a parabolic velocity profile in the x direction. See Appendix A for additional details.

Not surprisingly, u_0 is $O(k^{-1})$. Since the fluid is incompressible, liquid displaced at the location of the wave's crest must travel to the location of the wave's trough that is located half a wavelength apart (π/k). As k decreases, the distance between the crest and the trough increases as well and so does the horizontal velocity.

The first-order streamfunction is

$$\Psi_0(x, y, t) = \text{Real}[k^{-1} e^{i(kx-t)} \Phi_{0,1}(y) + \Phi_{0,2}(y) e^{-it}]. \quad (3.21)$$

Although the velocity field is periodic in time, the first-order solution predicts a net displacement of fluid particles. It turns out (see the discussion section), however, that over one period of oscillations, the net tracer's displacement is $O(\epsilon^2)$. In other words, the displacement predicted by the first-order approximation is of the same order of magnitude as the second-order approximation. Hence, in order to track the tracer's trajectory, it is necessary to compute the second-order solution.

B. Second-order solution

To compute the second-order flow field, we solve the equation

$$\begin{aligned} \nabla^2 \left(R^{-2} \nabla^2 - \frac{\partial}{\partial t} \right) \Psi_1 &= \frac{\partial \Psi_0}{\partial y} \nabla^2 \left(\frac{\partial \Psi_0}{\partial x} \right) \\ &\quad - \frac{\partial \Psi_0}{\partial x} \nabla^2 \left(\frac{\partial \Psi_0}{\partial y} \right), \end{aligned} \quad (3.22)$$

with the boundary conditions

$$\begin{aligned} \mathbf{u}_1(x, 0, t) &= \left\{ \frac{\partial}{\partial y} \Psi_1(x, 0, t), -\frac{\partial}{\partial x} \Psi_1(x, 0, t) \right\} \\ &= \left\{ 0, -f(x, t) \frac{\partial \mathbf{u}_0}{\partial y}(x, 0, t) \right\} \end{aligned}$$

and

$$\left\{ \frac{\partial u_1}{\partial y} \left(x, \frac{1}{2}, t \right), v_1 \left(x, \frac{1}{2}, t \right) \right\} = \left\{ \frac{\partial^2 \Psi_1}{\partial y^2} \left(x, \frac{1}{2}, t \right), -\frac{\partial \Psi_1}{\partial x} \left(x, \frac{1}{2}, t \right) \right\} = \{0, 0\}.$$

In accordance with (2.10), the second-order velocity, $u_1(x, y, t)$, must satisfy $\int_0^1 u_1(x, y, t) dy = 0$.

With the aid of MATHEMATICA,¹⁴ we calculate the rhs of Eq. (3.22):

$$\begin{aligned} \frac{\partial \Psi_0}{\partial y} \nabla^2 \left(\frac{\partial \Psi_0}{\partial x} \right) - \frac{\partial \Psi_0}{\partial x} \nabla^2 \left(\frac{\partial \Psi_0}{\partial y} \right) &= \frac{i}{4k_1} [[\Phi_{0,1}(y)] * \Phi_{0,1}''(y) - [\Phi_{0,1}''(y)] * \Phi_{0,1}(y)]' \\ &+ \text{Real} \left(-\frac{i}{4} e^{ikx} \{ [\Phi_{0,2}'(y)] * [[M^2] * \Phi_{0,1}(y) - \Phi_{0,1}''(y)] + R^2 [A_0] * \Phi_{0,1}(y) \} \right. \\ &+ \frac{i}{4} e^{i(kx-2t)} [\Phi_{0,2}'(y) [\Phi_{0,1}''(y) - M^2 \Phi_{0,1}(y)] \\ &\left. - A_0 R^2 \Phi_{0,1}(y) \right] + \frac{i}{4k} e^{i(2kx-2t)} [\Phi_{0,1}'(y) \Phi_{0,1}''(y) - \Phi_{0,1}(y) \Phi_{0,1}'''] \Big). \end{aligned} \quad (3.23)$$

In the above, $()'$ and superscript $(*)$ denote, respectively, differentiation with respect to (y) and a complex conjugate. We also took advantage of the fact that $-i\Phi_{0,2}'(y) = -A_0 + R^{-2}\Phi_{0,2}'''(y)$ [Eq. (3.12)].

Equation (3.23) suggests that the second-order flow field has the form

$$\begin{aligned} \Psi_1(x, y, t) &= \text{Real}[\Phi_{1,1}(y) + \Phi_{1,2}(y)e^{ikx} \\ &+ \Phi_{1,3}(y)e^{i(kx-2t)} + \Phi_{1,4}(y)e^{i(2kx-2t)} \\ &+ \Phi_{1,h}(y)e^{i(kx-t)}]. \end{aligned} \quad (3.24)$$

Witness that Ψ_1 consists of time-independent and x -independent components, a term periodic in x but not in time, two terms with double periods in time, and a term of period one [the homogeneous solution, $\Phi_{1,h}(y)$]. The boundary conditions are

$$\begin{aligned} &\text{Real}[\Phi_{1,1}'(0) + \Phi_{1,2}'(0)e^{ikx} + \Phi_{1,3}'(0)e^{i(kx-2t)} \\ &+ \Phi_{1,4}'(0)e^{i(2kx-2t)}] \\ &= \text{Real} \left(-\frac{1}{k} \Phi_{0,1}''(0) - [\Phi_{0,2}''(0)] * e^{ikx} \right. \\ &\left. - \Phi_{0,2}''(0)e^{i(kx-2t)} - \frac{1}{k} e^{i(2kx-2t)} \Phi_{0,1}''(0) \right), \end{aligned} \quad (3.25)$$

$$\begin{aligned} &\text{Real}[-ik\Phi_{1,2}(0)e^{ikx} - ik\Phi_{1,3}(0)e^{i(kx-2t)} \\ &- 2ik\Phi_{1,4}(0)e^{i(2kx-2t)}] = 0, \end{aligned}$$

and symmetry boundary conditions at $y = \frac{1}{2}$.

Due to the absence of the $e^{i(kx-t)}$ term in the boundary conditions, $\Phi_{1,h} = 0$. By equating the coefficients of e^0 , e^{ikx} , $e^{i(kx-2t)}$, and $e^{i(2kx-2t)}$, we obtain equations for $\Phi_{1,1}(y)$, $\Phi_{1,2}(y)$, $\Phi_{1,3}(y)$, and $\Phi_{1,4}(y)$. These equations are solved individually. The computations are carried out with MATHEMATICA.¹⁴

By integrating the equation for $\Phi_{1,1}'(y)$ once, we obtain the equation for $u_{1,1}(y)$,

$$\begin{aligned} 4kR^{-2} \frac{d^2 u_{1,1}}{dy^2} &= i \{ [\Phi_{0,1}(y)] * \Phi_{0,1}''(y) - \Phi_{0,1}(y) \\ &\times [\Phi_{0,1}''(y)] * \} + \text{const}. \end{aligned} \quad (3.26)$$

The boundary conditions are $u_{1,1}(0) = (1/2k)\text{Re}\{\Phi_{0,1}''(0)\}$ and $u_{1,1}'(\frac{1}{2}) = 0$. Two subsequent integrations yield

$$\begin{aligned} u_{1,1}(y) &= C_0 + C_1 y + C_2 y^2 + \frac{1}{4k} \text{Real}(BA * \{ (k+M)^2 \\ &\times \cosh[(k-M)(y-1/2)] - (k-M)^2 \\ &\times \cosh[(k+M)(y-1/2)] \}) \\ &+ \frac{1}{16k} BB * \{ (M+M^*)^2 \\ &\times \cosh[(M+M^*)(y-1/2)] - (M-M^*)^2 \\ &\times \cosh[(M-M^*)(y-1/2)] \}. \end{aligned} \quad (3.27)$$

To determine the constants of integration C_0 , C_1 , and C_2 , we use two boundary conditions and the condition

$$\int_0^1 u_{1,1} dy = 0. \quad (3.28)$$

The expressions for C_0 , C_1 , and C_2 are lengthy, and they are not given here.

The equation for $\Phi_{1,2}(y)$ is

$$\begin{aligned} R^{-2} \left(\frac{d^2}{dy^2} - k^2 \right)^2 \Phi_{1,2} &= -\frac{1}{4} i \{ [\Phi_{0,2}'(y)] * [[M^2] * \Phi_{0,1}(y) \\ &- \Phi_{0,1}''(y)] + R^2 [A_0] * \Phi_{0,1}(y) \}, \end{aligned} \quad (3.29)$$

FIG. 1. The finite element mesh and the deformation of the cavity's boundary at different times, $t=0$ (a), $t=\pi/2$ (b), and $t=\pi$ (c) during one period. Here $\epsilon=0.1$, $L=5$, $k=3.8$, and $m/L=0.004$.

with the boundary conditions $\Phi'_{1,2}(0) = -\frac{1}{2}[\Phi''_{0,2}(0)]^*$, $\Phi_{1,2}(0) = \Phi_{1,2}(\frac{1}{2}) = \Phi'_{1,2}(\frac{1}{2}) = 0$.

The equation for $\Phi_{1,3}(y)$ is

$$\begin{aligned} R^{-2} \left(\frac{d^2}{dy^2} - k^2 \right) \left(\frac{d^2}{dy^2} - k^2 + 2iR^2 \right) \Phi_{1,3}(y) \\ = \frac{i}{4} \{ \Phi'_{0,2}(y) [\Phi''_{0,1}(y) - M^2 \Phi_{0,1}(y)] \\ - A_0 R^2 \Phi_{0,1}(y) \}, \end{aligned} \quad (3.30)$$

with the boundary conditions $\Phi'_{1,3}(0) = -\frac{1}{2}\Phi''_{0,2}(0)$ and $\Phi_{1,3}(0) = \Phi_{1,3}(\frac{1}{2}) = \Phi'_{1,3}(\frac{1}{2}) = 0$.

The equation for $\Phi_{1,4}(y)$ is

$$\begin{aligned} R^{-2} \left(\frac{d^2}{dy^2} - 4k^2 \right) \left(\frac{d^2}{dy^2} - 4k^2 + 2iR^2 \right) \Phi_{1,4}(y) \\ = \frac{i}{4k} [\Phi'_{0,1}(y) \Phi''_{0,1}(y) - \Phi_{0,1}(y) \Phi'''_{0,1}(y)], \end{aligned}$$

with the boundary conditions

$$\Phi'_{1,4}(0) = -\frac{1}{2k} \Phi''_{0,1}(0)$$

and

$$\Phi_{1,4}(0) = \Phi_{1,4}\left(\frac{1}{2}\right) = \Phi'_{1,4}\left(\frac{1}{2}\right) = 0. \quad (3.31)$$

Since the expressions for $\Phi_{1,2}(y)$, $\Phi_{1,3}(y)$, and $\Phi_{1,4}(y)$ are quite lengthy, they are not reproduced here. Copies of the MATHEMATICA sessions that describe the solution process are available upon request.

IV. NUMERICAL PROCEDURE

The analytical solution presented in the previous section is valid only for small-amplitude oscillations. In order to determine the flow field at arbitrary amplitudes, we use a Galerkin-based finite element discretization of the full Navier–Stokes equations. A second-order finite difference scheme is used for the temporal discretization of the governing equations. An arbitrary Lagrangian–Eulerian (ALE) technique¹³ is employed to track the specified motion of the boundaries. The ALE technique introduces a mesh velocity that conforms to the conditions specified by (2.8) on the domain boundary. The mesh velocity in the interior of the computational domain is determined by solving a Laplace equation to ensure smooth distribution of the grid points. At each time step, the mesh velocity is solved together with the governing equations, and the locations of the grid points in the mesh are then updated with this mesh velocity using a second-order scheme. The method uses mixed finite elements with quadratic shape functions for the velocities and linear shape functions for the pressure. Overall the method is third-order accurate in space and second-order accurate in time.

In order to be able to detect symmetry breaking bifurcations, we did not employ the symmetry conditions at $y = \frac{1}{2}$. Figure 1 depicts a coarse version of the finite element mesh at times $t=0$ (a), $t=\pi/2$ (b), and $t=\pi$ (c) during a single period of vibrations. A coarse grid is used so that the elements are visible at the reduction level of the figure. In the actual calculations, a much finer grid was employed. In Fig. 1, the cavity's aspect ratio $L=5$, $\epsilon=0.1$, $k=3.8$. As a result of the induction of traveling waves in the membranes, global fluid motion was observed. The code was verified for self-consistency, and the numerical results were in excellent agreement with the theoretical predictions. Detailed compar-

FIG. 2. When $t=0$ (solid diamonds), $\pi/2$ (upright, solid triangle), and π (solid square), $\epsilon=0.01$, $L=5$, $m/L=0.004$, $k=3.8$, and $R=7.93$, the figure depicts $u(x, 0.5, t)$ as a function of x (a), $v(x, 0.25, t)$ as a function of x (b), $u(L/2, y, t)$ as a function of y (c), and $v(L/2, y, t)$ as a function of y (d). The solid lines and symbols correspond, respectively, to the analytical and numerical solutions.

sons of the theoretical predictions and the numerical results are provided in the next section.

One of our objectives in this paper is to evaluate the potential of peristaltic motion to enhance stirring. To this end, we need to track particle trajectories. This can be done quite accurately when analytical expressions are available for the velocity field. The determination of particle trajectories based on numerically computed velocity fields is less straightforward. Below, we describe the procedure utilized to compute particles' trajectories.

The motion of passive tracers in a known flow field is given by the advection equations,

$$\frac{dx}{dt} = \epsilon u(x, y, z, t) \quad \text{and} \quad \frac{dy}{dt} = \epsilon v(x, y, z, t). \quad (4.1)$$

The particles' paths can be obtained by integrating these equations. When the flow field is periodic in time with period T , Poincaré sections are constructed by stroboscopic sampling of the tracers' positions, i.e.,

$$\{\mathbf{x}_n, \mathbf{y}_n\} = \{x(nT), y(nT)\},$$

$$\text{where } \{\mathbf{x}_0, \mathbf{y}_0\} = \{x(0), y(0)\}. \quad (4.2)$$

When the analytical expressions for the flow field are available, the above integrations are carried out using Matlab's fourth-order Runge–Kutta method. The integration is repeated with various time step sizes, and we confirm that the results are independent of the step size. When the analytical expressions for the flow field are not available, we write our own code to integrate the above equations. Typically, 40 time steps are applied within each period with a constant time interval $\Delta t = T/40$, i.e., we compute and store the flow fields $\{u_i(x, y, z, i\Delta t)\}$ and $\{v_i(x, y, z, i\Delta t)\}$, $i = 0, 1, 2, \dots, 39$. Passive tracers are advanced from their locations at time step k to time step $k+2$, using the fourth-order Runge–Kutta algorithm:¹⁶

$$\mathbf{q}_1 = \mathbf{u}(\mathbf{x}^k, k\Delta t),$$

FIG. 3. When $t=0$ (solid diamonds), $\pi/2$ (upright, solid triangle), and π (solid square), $\epsilon=0.01$, $L=5$, $m/L=0.004$, $k=3.8$, and $R=79.3$, the figure depicts $u(x, 0.5, t)$ as a function of x (a), $v(x, 0.25, t)$ as a function of x (b), $u(L/2, y, t)$ as a function of y (c), and $v(L/2, y, t)$ as a function of y (d). The solid lines and symbols correspond, respectively, to the analytical and numerical solutions.

FIG. 4. The second-order velocity component, u_{11} (a), $\{u_{12}, v_{12}\}$ (b), $\{u_{13}, v_{13}\}$ (c), and $\{u_{14}, v_{14}\}$ (d) are depicted as functions of y . Here $R=7.9$, $k=3.8$, $t=0$, and $x=1$.

FIG. 5. Stroboscopically probed tracers' trajectories (Poincaré sections) as predicted by the first-order analytical flow field (a), the first-order and the time-independent parts of the second-order analytical flow field [(b), solid line], and the analytical flow field up to the second order [(b), symbols]. Here $\epsilon=0.01$, $R=7.9$, and $k=3.8$.

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{u}[\mathbf{x}^k + \mathbf{q}_1 \Delta t, (k+1) \Delta t], \\ \mathbf{q}_3 &= \mathbf{u}[\mathbf{x}^k + \mathbf{q}_2 \Delta t, (k+1) \Delta t], \\ \mathbf{q}_4 &= \mathbf{u}[\mathbf{x}^k + \mathbf{q}_3 2 \Delta t, (k+2) \Delta t], \\ \mathbf{x}^{k+2} &= \mathbf{x}^k + (\mathbf{q}_1 + 2\mathbf{q}_2 + 2\mathbf{q}_3 + \mathbf{q}_4) \Delta t/3. \end{aligned} \quad (4.3)$$

In the above, $\mathbf{x}=(x,y)$ and $\mathbf{q}=(q_x, q_y)$. The flow field within each finite element is interpolated using a linear shape function.

To test the accuracy of our integration scheme, we compare (in Sec. V) the results obtained using the analytical solution for the flow field with the results obtained using the above described procedure.

V. RESULTS AND DISCUSSION

In all the numerical results presented in this section, we utilize 100×20 mesh points. The numerical results are given after initial transients have decayed. In all the simulations presented here, the flow eventually became periodic. It usually takes about nine periods for initial transients to die out and a periodic flow to be established. The time is synchronized so that $t=0$ when the velocity at the cavity's center, $u(L/2, 0.5, t)$, attains its maximum value. Since in the numerical simulations the boundaries are often subjected to significant deformations, to present the data for velocity profiles, it is convenient to rescale the y coordinate as

$$\xi = \frac{y - \epsilon f(x, t)}{1 - 2\epsilon f(x, t)}, \quad (5.1)$$

so that $0 \leq \xi \leq 1$.

All the results are presented in a laboratory reference frame. Whereas in the classical analysis^{4,6,8} of peristaltic waves in an infinitely long conduit, the model is steady when viewed from a reference frame that moves with the wave, in the closed-conduit case that we consider here, the flow field is inherently unsteady in any reference frame.

A. Code verification and velocity profiles

In order to verify the numerical code and gain some insights about the effects of lateral boundaries, we first simu-

late peristaltic pumping induced by small-amplitude wall oscillations. When $\epsilon=0.01$, $L=5$, $m/L=0.004$, $k=3.8$, and $R=7.9$, Fig. 2 depicts the velocity profiles at various times during a period. When $t=0$ (solid diamonds), $\pi/2$ (upright, solid triangles), and π (solid squares), Figs. 2(a), 2(b), 2(c), and 2(d) depict, respectively, $u(x, 0.5, t)$ as a function of x , $v(x, 0.25, t)$ as a function of x , $u(L/2, y, t)$ as a function of y , and $v(L/2, y, t)$ as a function of y . The solid lines and the symbols correspond, respectively, to the first-order analytical solution (Sec. III) and numerical predictions. Both velocity components, u and v , are periodic in time and in x (with the exception of the lateral walls' vicinities, $x \rightarrow 0$ and $x \rightarrow L$). In the absence of lateral walls (open cavity), the (u) velocity, when depicted as a function of x , exhibits oscillations about a zero mean. In contrast, in the case of the closed cavity [Fig. 2(a)], these oscillations are about a nonzero mean. Only the numerical solutions satisfy the boundary conditions at $x=0$ and $x=L$.

When R is large, spurious numerical oscillations preclude us from using Eqs. (3.13) and (3.18). Instead, we use the corresponding boundary layer approximations to produce the analytical solutions depicted in Fig. 3. When $\epsilon=0.01$, $L=5$, $m/L=0.004$, $k=3.8$, and $R=79$, Fig. 3 depicts the

FIG. 6. The leading-order prediction of the displacement of a passive tracer, $d = \|\mathbf{x}(t+T) - \mathbf{x}(t)\|$, during one period of oscillation as a function of ϵ . The tracer's initial position $(x, y) = (2.5, 0.25)$, $k=3.8$, and $R=7.93$. The symbols correspond to numerical calculations, and the solid line is the best-fit line, $d = 3\epsilon^2$.

FIG. 7. Stroboscopically probed passive tracers' trajectories (Poincaré sections) as predicted by the second-order analytical flow field. Here $\epsilon=0.01$, $R=1$, $k=3.8$, $m/L=0.004$, and $L=5$.

velocity profiles at various times during the period. When $t=0$ (solid diamonds), $\pi/2$ (upright, solid triangles), and π (solid squares), Figs. 3(a), 3(b), 3(c), and 3(d) depict, respectively, $u(x, 0.5, t)$ as a function of x , $v(x, 0.25, t)$ as a function of x , $u(L/2, y, t)$ as a function of y , and $v(L/2, y, t)$ as a function of y . The solid lines and the symbols correspond, respectively, to the first-order analytical solution (Sec. III) and numerical predictions. Only the numerical solution (symbols) satisfies nonslip and impermeable wall conditions at the side-walls. Witness the two thin horizontal boundary layers for the horizontal component of the velocity next to the vibrating walls and the two thin vertical boundary layers for the vertical component of the velocity next to the sidewalls. These boundary layers are of thicknesses on the order of R^{-1} . Even in the presence of the boundary layers, the numerical solution provides a reasonable description of the flow field.

Figures 2 and 3 illustrate that the analytical solution is valid at distances exceeding one wavelength away from the solid walls. In other words, when $2\pi/k \leq x \leq (L - 2\pi/k)$, the numerical results are in excellent agreement with theoretical predictions.

When $R=7.9$, $k=3.8$, $t=0$, and $x=1$, Figs. 4(a), 4(b), 4(c), and 4(d), depict, respectively, the second-order velocity components as functions of y : u_{11} , $\{u_{12}, v_{12}\}$, $\{u_{13}, v_{13}\}$, and $\{u_{14}, v_{14}\}$. Here $u_{1,i} = \partial \Psi_{1,i} / \partial y$ and $v_{1,i} = -\partial \Psi_{1,i} / \partial x$. $u_{1,1}$, $\Psi_{1,2}(y)$, $\Psi_{1,3}(y)$, and $\Psi_{1,4}(y)$ are specified, respectively, in Eqs. (3.27), (3.29), (3.30), and (3.31). The coefficient ϵ was not included in the magnitude of the various velocity components. In other words, to obtain the correct magnitude of the second-order velocity components, the values displayed in Fig. 4 must be multiplied by ϵ . Over the wave period, all the velocity components (Figs. 2–4), with the exception of $u_{1,1}$, average to zero. The Eulerian net flow [Fig. 4(a)] next to the vibrating walls is in the direction of the wave while the flow in the conduit's center is in the opposite direction. This, of course, does not imply that fluid particles will exhibit similar behavior. Although the second-order solution does not modify significantly the velocity profiles, it does play an important role in determining the trajectories of a passive tracer, as we demonstrate in the next section.

B. Advection of a passive tracer

In the previous section, we described the instantaneous, Eulerian velocity profiles. The Eulerian velocity does not, however, indicate the displacement history of material particles. In this section, we study the advection of a passive tracer. Using the fourth-order Runge–Kutta procedure, we integrate the advection equation (4.1) to obtain particle trajectories. All the figures presented in this section are stroboscopic images. In other words, the tracers' locations are documented once every period at the period's end. For better visibility, however, we connect the discrete stroboscopically probed points with solid lines. These lines do not represent the actual particles' trajectories. In dynamic system's theory, such images are often referred to as Poincaré cross sections.

First, we examine the relative role of the various terms in the perturbation solution on the tracers' trajectories. When $\epsilon=0.01$, $R=7.9$, and $k=3.8$, Fig. 5 depicts the tracers' trajectories predicted by the first-order (a) and the second-order (b) solutions. The arrows in the figure indicate the direction of the flow. In Fig. 5(a), we use only the first-order perturbation solution, $O(\epsilon)$. In Fig. 5(b), we depict the predictions obtained using the first-order solution and the time-independent part of the second-order solution (solid line) and the full first- and second-order solutions (symbols). Witness that there are significant qualitative differences between Figs. 5(a) and 5(b).

In order to better understand the reasons for the differences between Figs. 5(a) and 5(b), we examined the net displacement of a fluid particle during one period, as predicted by the first-order theory. Figure 6 depicts the net displacement $d = \|\mathbf{x}(t+T) - \mathbf{x}(t)\|$ as a function of ϵ when $\mathbf{x}(0) = (L/2, 0.25)$, $R=7.9$, and $k=3.8$. In the above, $\mathbf{x}$ is a position vector. Witness the nearly linear relationship between (d) and ϵ with the slope ~ 2 on a log–log scale, i.e., $d \sim \epsilon^2$. Although the first-order solution predicts a net displacement ($d \neq 0$) of tracer particles during one period of oscillation, this displacement is $O(\epsilon^2)$ and of the same magnitude as the displacement predicted by the second-order approximation. In Appendix B, we prove that this is, indeed, the case. In

FIG. 8. Stroboscopically probed particles' trajectories (Poincaré sections) when $\epsilon=0.01$, $R=7.9$, $k=3.8$, $m/L=0.004$, and $L=5$. The second-order analytical predictions and the results of the numerical simulations are shown, respectively, as solid lines and symbols.

FIG. 9. Stroboscopically probed particles' trajectories (Poincaré sections) when $R=7.9$, $L=5$, $k=3.8$, $\epsilon=0.1$, and $m/L=0.004$. Numerical simulation results.

summary, attempts to predict the Lagrangian motion of particles based on first-order theory may lead to qualitatively wrong conclusions. One must include both the first- and second-order terms in Eq. (4.1) in order to obtain qualitatively correct trajectories.

Figure 5(b) illustrates that the trajectories predicted when second-order, time-dependent terms are excluded (solid line) and included (symbols) in Eq. (4.1) are nearly identical. In order to obtain the correct trajectories, it would suffice to include in (4.1) the first-order velocity and the time-independent terms of the second-order velocity. The time-dependent terms of the second-order velocity make an $O(\epsilon^3)$ contribution to the net displacement. This observation is of some significance, as the algebra involved in obtaining the second-order solution is quite tedious and the realization that only the time-independent velocity terms in the second-order solution are needed for the calculation of particle trajectories may save considerable labor when studying peristaltic motion. Witness that when second-order terms are included [Fig. 5(b)], the net motion next to the solid boundaries is in the direction of the traveling wave while the motion at the center of the cavity is in the direction opposite to the wave's direction. The peristaltic waves also trap circulation bubbles or bolus of fluid particles. These bubbles form by closed particle paths, and they are stationary in the laboratory frame of reference; they do not move with the wave. The bubbles next to the vibrating walls rotate in a clockwise direction while the bubbles away from the vibrating walls rotate in the counterclockwise direction.

The topology of the flow becomes clearer when we depict the particle traces for even smaller R numbers. When $\epsilon=0.01$, $R=1$, and $k=3.8$, Fig. 7 depicts the tracers' trajectories predicted by second-order theory. The circulation bubbles next to the vibrating wall are clearly visible. Because of these clockwise rotating "bubbles," the Lagrangian velocity along certain segments of the wall is in the direction of the wave while along other segments, it is in a direction opposite to that of the wave. The size of the clockwise rotating circulation bubbles decreases as the R number increases, so the bubble may eventually disappear.

In Fig. 8, we compare the numerical (symbols) and analytical (solid lines) results of the particles' trajectories under the conditions in Fig. 5. Witness the good agreement between the numerical and analytical solutions.

Next, in order to investigate the flow patterns next to the lateral walls and with finite-amplitude oscillations, we repeat the computation of the particles' trajectories for cases when ϵ is not small. When $R=7.9$, $L=5$, $k=3.8$, $\epsilon=0.1$, and $m/L=0.004$, Fig. 9 depicts the numerically computed particle

FIG. 10. Stroboscopically probed particles' trajectories (Poincaré sections) when $R=79$, $L=5$, $k=3.8$, $\epsilon=0.05$, and $m/L=0.004$. Numerical simulation results.

trajectories. The arrows indicate the direction of the flow. Witness the global circulation in the entire cavity with the fluid moving in the direction of the traveling waves next to the membranes, turning around next to the sidewalls, and moving in the opposite direction in the center's region. In addition, six cells form in the cavity's interior. The topology of the flow field is similar to the one depicted in Fig. 5 when ϵ is small.

As the R number increases, so does the complexity of the flow. When $R=79$, $L=5$, $k=3.8$, $\epsilon=0.05$, and $m/L=0.004$, Fig. 10 depicts the numerically computed particles' trajectories. The flow direction next to the membranes and in the cavity's center is in the direction of the traveling wave. The fluid moves in the opposite direction between the cells next to the vibrating walls and the cells in the center's region. Although we did not invoke symmetry conditions in all our computations, the flow field maintained symmetry with respect to $y=0.5$. Our attempts to compute the flow field for even larger ϵ values were frustrated by the numerical difficulties resulting from the severe deformations of the finite elements.

Finally, in order to evaluate how well the peristaltic mixer stretches and deforms material lines, we depicted in Fig. 11 the deformation and stretching of the initially ($t=0$) vertical material blob ($1.5 < x < 2$) when $k=3.8$, $\epsilon=0.01$, and $R=25$. The material blob was advected using the second-order analytic solution. Figures 11(a), 11(b), and 11(c) depict, respectively, the blob at times $t=0$, $600T$, and $1200T$ where T is the period of the wave. One can think of two materials, one dark and one light, being stirred. As time goes by, the interface between the two materials stretches and deforms. The thickness of the dark blob reduces dramatically, enhancing the effects of molecular diffusion (not accounted for in the model). Right on the membranes, due to the nonslip condition, the material does not move. The material next to the membranes moves in the direction of the wave while (to conserve mass) the material in the center moves in the opposite direction. The stretching rate is nearly linear in time. This is much better than the rate of diffusion alone that scales like the square root of time but less effective than the exponential stretching associated with chaotic advection.

VI. CONCLUSIONS

In the paper we present a theoretical study of peristaltic motion in a closed cavity with traveling waves of the same amplitude and frequency but opposite phase along the bottom and top walls. A second-order analytic solution was ob-

FIG. 11. The deformation of the material blob that initially ($t=0$) occupied the region, $1.5 < x < 2$ (a) at $t=600T$ (b) and at $t=1200T$ (c), where T is the wave period. Here $k=3.8$, $\epsilon=0.01$, and $R=25$.

tained for the flow field in the cavity when the wall oscillations are small. It was demonstrated that the presence of lateral (side-) walls affects the first-order solution at $O(\epsilon)$, where ϵ is the amplitude of the membrane oscillations, and that it is sufficient to include only the time-independent terms in the second-order solution in order to obtain an accurate description of the passive tracers' trajectories.

In order to analyze the fluid motion at large amplitudes, we employed a numerical solution utilizing finite elements with a moving grid. The numerical code was verified by comparing its predictions with analytical results. The numerical computations indicate that for a range of Strohal numbers ($0 < R < 80$) and wave amplitudes ($\epsilon < 0.05$), the Eulerian flow field is periodic at the wave period. An interesting question that is left for future work is the stability of the periodic flow and the possibility of a period doubling sequence as the wave amplitude increases.

We also computed particle trajectories in the cavity. The flow topology consisted of "trapped" stationary circulation bubbles (or boluses) and gross circulation in the cavity. We also tracked the deformation of material blobs, and it appears that the flow stretches material lines at a nearly linear rate with time. The analysis was carried out for a case when both walls oscillated in a symmetrical fashion. When mixing enhancement is desired, it may be desirable to destroy the symmetry and to facilitate material transport through the cavity's

midplane ($y=0.5$). This can be accomplished either by oscillating only one wall or by oscillating both walls at different frequencies and different wave numbers.

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APPENDIX A: THE LUBRICATION THEORY APPROXIMATION

Since most of the works on peristaltic pumping in open conduits utilize lubrication theory, it would be useful to develop a theory for the case of a closed conduit under conditions when lubrication theory is applicable, i.e., $R \rightarrow 0$ (small Strohal number), $k = 2\pi/\lambda \rightarrow 0$ (long wavelengths) and $\omega \rightarrow 0$ (low frequency). Since the procedure is fairly standard, we omit details. Although not appropriate for this limiting case, to the extent possible and in order to avoid confusion, we use the same scaling that we utilized in the main body of the paper. Let the conduit height be $2h$. We introduce a new coordinate $\eta = y - h$ so that the conduit's midplane is fixed at $\eta = 0$ and rescale only the pressure, $p(x, \eta, t) = R^{-2} \hat{p}(x, \eta, t)$. We also note that $\partial/\partial x = O(\lambda^{-1})$, $\partial^2 u/\partial x^2$

$\ll \partial^2 u / \partial \eta^2$, and $\partial \hat{p} / \partial \eta \sim 0$. Consequently, to leading order, the momentum equation [Eq. (2.6)] in the x direction reduces to the equation

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{d\hat{p}}{dx}, \quad (\text{A1})$$

with the boundary conditions $\partial u(x, 0, t) / \partial \eta = u(x, h, t) = 0$, where $\eta = h(x, t) = \frac{1}{2} - \epsilon f(x, t) = \frac{1}{2} - \epsilon \cos(kx - t)$ describes the location of the upper membrane. Equation (A1) can be readily integrated to yield the parabolic (Poiseuille) profile:

$$u = -\frac{1}{2} \frac{d\hat{p}}{dx} (h^2 - \eta^2). \quad (\text{A2})$$

Condition (2.9) allows us to evaluate the pressure gradient and obtain

$$\frac{d}{dx} \left(\frac{1}{3} \frac{d\hat{p}}{dx} h^3 \right) = -\frac{\partial f}{\partial t}. \quad (\text{A3})$$

The term inside the derivative represents the instantaneous flow rate at location x . Here ϵ was eliminated from the equation because in Sec. II, we scaled the velocity (u) with ϵ . The pressure gradient can be readily obtained from (A3), i.e.,

$$\frac{d\hat{p}}{dx} = -\frac{3}{h^3} \int_0^x \left(\frac{\partial f}{\partial t} \right) dx = \frac{3}{h^3} \frac{\cos(kx - t) - \cos(t)}{k}. \quad (\text{A4})$$

The first term on the rhs is the same as in the open conduit case.⁴ The second term comes about because of the need to satisfy the impermeability condition on the sidewall at $x = 0$. The second term indicates that the phenomenon is unsteady also in a reference frame that is moving at the wave speed.

The vertical velocity can be readily obtained with the use of the continuity equation (2.6):

$$v = \frac{1}{2} \frac{d^2 \hat{p}}{dx^2} \left(h^2 \eta - \frac{\eta^3}{3} \right) + \frac{d\hat{p}}{dx} h \frac{dh}{dx} \eta. \quad (\text{A5})$$

The expressions computed in Sec. III A and the ones computed in this appendix share the common limit

$$u = \frac{12}{k} [\cos(kx - t) - \cos(t)] y(1 - y). \quad (\text{A6})$$

(A6) is obtained when one computes the leading-order term in the Taylor series of (A2) in terms of ϵ and the leading-order term in (3.9) when expanded into Taylor series in terms of R and k .

APPENDIX B: THE LAGRANGIAN DISPLACEMENT OF A PASSIVE TRACER DURING ONE PERIOD OF THE WAVE

In this appendix, we compute the net displacement of a passive tracer during one period of the wave oscillation and demonstrate that the net displacement predicted by the leading-order perturbation term is on the order of $O(\epsilon^2)$. For conciseness, we rewrite Eq. (4.1) in a vector form:

$$\frac{d\mathbf{x}}{dt} = \epsilon \mathbf{u}, \quad (\text{B1})$$

and assume, without loss of generality, that $\mathbf{x}(0) = \mathbf{x}_0$. As before, bold letters denote vectorial quantities. It is reasonable to expect (and numerical experiments support this notion) that during one period, $\|\mathbf{x}(t) - \mathbf{x}_0\| \sim O(\epsilon)$. Hence, we can approximate (B1) as

$$\begin{aligned} \frac{d(\mathbf{x} - \mathbf{x}_0)}{dt} &= \epsilon [\mathbf{u}_0(\mathbf{x}_0, t) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}_0(\mathbf{x}_0, t) + \cdots] \\ &\quad + \epsilon^2 [\mathbf{u}_1(\mathbf{x}_0, t) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}_1(\mathbf{x}_0, t) + \cdots]. \end{aligned} \quad (\text{B2})$$

Next, we expand $\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ into a Taylor series in terms of ϵ , i.e., $\mathbf{x} = \epsilon \chi_0 + \epsilon^2 \chi_1 + \cdots$, and collect terms of like powers in ϵ . To the leading order,

$$\begin{aligned} \frac{d\chi_0}{dt} &= \mathbf{u}_0(\mathbf{x}_0, t) \\ &= \mathbf{U}_0(x_0, y_0) e^{-it} + \text{c.c.} \\ &= \begin{pmatrix} U_{0,1}(y_0) e^{ikx_0} + U_{0,2}(y_0) \\ V_{0,1}(y_0) e^{ikx_0} \end{pmatrix} e^{-it} + \text{c.c.}, \end{aligned} \quad (\text{B3})$$

where the significance of $\mathbf{U}_0$ is clear from the context; “c.c.” stands for the complex conjugate and $U_{0,i}$ and $V_{0,1}$ are specified in Sec. III A. Equation (B3) can be readily solved to render

$$\chi_0(t) = -\frac{1}{i} \mathbf{U}_0(x_0, y_0) (e^{-it} - 1) + \text{c.c.} \quad (\text{B4})$$

Clearly, $\chi_0(t)$ is periodic in time and, to the leading order, the net displacement during one period equals zero. To $O(\epsilon^2)$, we have

$$\frac{d\chi_1}{dt} = \mathbf{u}_1(\mathbf{x}_0, t) + \chi_0 \cdot \nabla \mathbf{u}_0(\mathbf{x}_0, t). \quad (\text{B5})$$

In the first term above, only the component $u_{1,1}(y)$ [Eq. (3.27)] of $\mathbf{u}_1$ is nonperiodic in time. The second term in (B5) contains the nonperiodic component, $(1/i)(\mathbf{U}_0^* \cdot \nabla \mathbf{U}_0 - \mathbf{U}_0 \cdot \nabla \mathbf{U}_0^*)$, where the superscript (*) denotes a complex conjugate and all the variables are computed at $(\mathbf{x}_0)$. In other words, at the end of a period, the net displacement is

$$\chi_1(2\pi) \sim 2\pi \begin{pmatrix} u_{1,1}(y_0) \\ 0 \end{pmatrix} + \frac{1}{i} (\mathbf{U}_0^* \cdot \nabla \mathbf{U}_0 - \mathbf{U}_0 \cdot \nabla \mathbf{U}_0^*). \quad (\text{B6})$$

In conclusion, $\mathbf{x}(2\pi) - \mathbf{x}_0 = \epsilon^2 \chi_1 + O(\epsilon^3)$. The periodic components of $\mathbf{u}_1$ contribute only to higher-order terms in this expansion.

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