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An analysis of (linear) exponentials based on extended sequents*

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Abstract: We apply the 2-sequents approach to the analysis of several logics derived from linear logic. In particular, we present a uniform formal system for Linear Logic, Elementary Linear Logic and Light Linear Logic.

1 Introduction

There is not a single reason for proposing an extension of Gentzen's format for sequents; often, it is rather a blend of distinct issues to inspire the design of a particular system [Kri63, Došš85, GdQ92a, Wan94]. The *2-sequent* approach [Mas92, Mas93, MM95a, MM95b] is not an exception to that rule. The original goal was notational: providing symmetric and local (i.e., context-free) rules for the minimal deontic logic KD. But soon, we discovered that 2-sequents could be used as a *uniform* tool for several logical systems. In particular, starting with a common core—the logical rules—we could shift from one system to another (e.g., from KD to S4) just changing the way in which syntactical objects are manipulated—say, the *structural rules*. In [MM95b], we applied this scalar approach to the modal logics in the K-S4 range. Here, we shall apply the same methodology to Girard's Linear Logic (LL).

This study started in [MM95a], where we gave natural deduction style presentations for certain LL fragments. As a result, we discovered some unsuspected connections with proof-nets and—because of the correspondence between λ-calculus and the multiplicative exponential fragment of LL (MELL)—with the so-called optimal or sharing implementation of lambda-terms (see [GMM96, GMM97]). Here, we focus on full LL (i.e., with additives, second order quantifiers, and constants) and on some subsystems with an intrinsic bound on the complexity of the representable functions. For instance, we shall see a 2-sequent presentation of the so-called Light Linear Logic (LLL), see [Gir95b].

The main issue of LLL is that, not only all the polynomial functions are representable by a (second order) LLL proof, but there is a suitable cut-elimination for LLL that can be performed in polynomial time. The main drawback of LLL is however its awkward syntax. Girard's idea for achieving the reduction bound of LLL is to have a tight control on the structure of the proof-nets associated to

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deductions, and on the way in which that nets growth along reduction. More precisely, the structural constraints on LLL proof-nets allow to reduce them by stages (i.e., pursuing an outermost-innermost strategy according to the nesting of exponential *boxes*), controlling at the same time the number of stages required to complete the task and the number of duplications performed at each stage. Unfortunately, mainly because of the presence of additives, the translation of that constraints on nets into logical rules leads to a system with an heavy ad hoc syntax, whose generalized sequents share very few structure with the standard ones of LL.

Because of the already mentioned tight relations between 2-sequents and proof-nets, it should not be particularly surprising that the same structural constraints of LLL find instead a very simple formulation in the 2-sequent framework: the 2-sequent formulation of LLL (or 2LLL) is just the restriction of the 2-sequent formulation of LL (or 2LL) to the case in which only two levels are used, plus the new modality of LLL and the other structural restrictions on the auxiliary doors of boxes. It is also remarkable that the same approach scales in a natural way to Elementary Linear Logic (ELL), an intermediate system between LLL and LL with (Kalmar) elementary cut-elimination.

Since the main focus of the paper is on the use of 2-sequent we will omit to the dynamics of the proposed logical systems, that is indeed the relevant issue of LLL. At the same time, we omit for lack of space the definition and study of the indexed proof-nets corresponding to LLL and ELL.

2 2-sequents for linear logic

We refer to [Gir87, Gir95a] for notation and preliminaries on Linear Logic. We will write many times (even in formal sequents) the linear implication $A \multimap B$ instead of $A^\perp \parr B$.

Definition 2.1 (2-sequents). A *2-sequence* is an expression

$$\begin{matrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{matrix}$$

in which each α_i is an ordinary (possibly empty) sequence of linear formulas. The formulas of α_i are *at level* i. A *2-sequent* is an expression $\vdash \Gamma$, where Γ is a 2-sequence in which at least one of the α_i is not empty.

Informally, levels express a form of modal (exponential) dependency.

Definition 2.2 (interpretation of 2-sequences). The *interpretation* $[\Gamma]^\sharp$ of the 2-sequence Γ is defined as

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}^\sharp = \parr\alpha_j \parr !(\parr\alpha_{j+1} \parr !(\dots!(\parr\alpha_n)\dots))$$

where j is the minimum level s.t. α_j is not empty; $\parr\alpha_k$ denotes the par of all the formulas in α_k, and in particular, the term $\parr\alpha_k$ is missing in the case that α_k is empty,

According to this interpretation, (modality related) structural rules naturally correspond to vertical rearrangement of formulas. Anyhow, since towers of sequences of formulas are not handy to write and manipulate, in the following we shall prefer an equivalent indexed representation. Namely, we will index by its level i any formula occurrence A in some α_i—say that A^i is the correspondent *indexed formula*. Any tower of α_i can then be merged into an ordinary linear *sequence of indexed formulas*. However, we will resort again to two-dimensional sequents for the relevant case of LLL (Section 5.1).

3 2LL: a 2-sequent calculus for Linear Logic

For the 2-sequence Γ, write $\max(\Gamma) = \max\{i : A^i \in \Gamma\}$.

3.1 The calculus

The axioms and rules in Figure 1 define 2LL, whose provability relation will be denoted by $\vdash_{2LL}$. In the next sections we will prove its equivalence (with respect to provability) to the standard presentation of LL. We write $\Gamma^{=i}$ to mean that all the formulas in Γ are at level i.
By the way, even if not explicitly stated above, the $\forall$ rule has the usual proviso on the second order variable it binds, that is, X must not occur free in any formula in Γ but A.

3.2 Correctness

In order to relate provability in 2LL to provability in standard LL ($\vdash_{LL}$), it is useful to isolate a class of formulas behaving as ?-modal formulas.

Definition 3.1 (Essentially exponential formulas). The class Exp is inductively defined as follows:

1. $\bot \in \mathsf{Exp}$;
2. for any formula A, $?A \in \mathsf{Exp}$;
3. if $A, B \in \mathsf{Exp}$, then $A \parr B, A \,\&\, B \in \mathsf{Exp}$.

The following is the main property of Exp (its proof is just an easy induction on the definition of Exp).

Lemma 3.2. *For any* $A \in \mathsf{Exp}$, $\vdash_{LL} A, !A^\perp$.

Therefore, weakening and contraction are admissible on formulas in Exp, and, moreover, the promotion rule with Exp contexts is derivable.

Lemma 3.3. *Let* $\Gamma = A_1, \ldots, A_n$ *be a (standard) sequent such that any* $A_i \in \mathsf{Exp}$. *Then:*

$$\vdash_{LL} \Gamma, A \quad \Rightarrow \quad \vdash_{LL} \Gamma, !A$$

Identity/Negation

$$\vdash A^i, A^{\perp i} \qquad\qquad \frac{\vdash \Gamma, A^i \quad \vdash \Delta, A^{\perp i}}{\vdash \Gamma, \Delta}\,\mathsf{cut}_{\,i \geq \max(\Gamma), \max(\Delta)}$$

Structure

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A^i}\,W_{\,i \leq \max(\Gamma)} \qquad\qquad \frac{\vdash \Gamma, ?A^i, ?A^i}{\vdash \Gamma, ?A^i}\,C$$

Logic

$$\frac{\vdash \Gamma, A^i}{\vdash \Gamma, ?A^{i-j}}\,?_{\,i \geq j \geq 0} \qquad\qquad \frac{\vdash \Gamma, A^i}{\vdash \Gamma, !A^{i-1}}\,!_{\,i > \max(\Gamma)}$$

$$\frac{\vdash \Gamma, C^i \quad \vdash \Gamma, D^i}{\vdash \Gamma, C \,\&\, D^i}\,\& \qquad \frac{\vdash \Gamma, C^i}{\vdash \Gamma, C \oplus D^i}\,\oplus_{L\; i \geq \max(\Gamma)} \qquad \frac{\vdash \Gamma, D^i}{\vdash \Gamma, C \oplus D^i}\,\oplus_{R\; i \geq \max(\Gamma)}$$

$$\frac{\vdash \Gamma, C^i \quad \vdash \Delta, D^i}{\vdash \Gamma, \Delta, C \otimes D^i}\,\otimes_{\,i \geq \max(\Gamma), \max(\Delta)} \qquad\qquad \frac{\vdash \Gamma, C^i, D^i}{\vdash \Gamma, C \parr D^i}\,\parr$$

$$\frac{\vdash \Gamma, A^i}{\vdash \Gamma, \forall X.A^i}\,\forall_{\,i \geq \max(\Gamma)} \qquad\qquad \frac{\vdash \Gamma, A^i[B/X]}{\vdash \Gamma, \exists X.A^i}\,\exists_{\,i \geq \max(\Gamma)}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp^i}\,\perp_{\,i \leq \max(\Gamma)} \qquad\qquad \vdash 1^i \qquad\qquad \vdash \Gamma^{=i}, \top^i$$

Figure 1: 2LL: 2-sequent presentation of LL

We may use 2LL to show that many formulas are in Exp.

Lemma 3.4. *Let* $\vdash_{2LL} \Gamma$. *For any* $A^i \in \Gamma$ *such that* $i < \max(\Gamma)$, $A \in \mathsf{Exp}$.

Proof. Induction on the derivation, exploiting the side-conditions. □

Indeed, we could constrain the rules $\parr$, $\&$ and $\perp$ to be applied only at maximal levels (like $\forall$, $\exists$, $\otimes$, $\oplus$ and cut), without losing in provability. In this more restrained system, all the non maximal formulas of a provable 2-sequent are of the shape ?A.

Correctness of 2LL with respect to LL is now an easy corollary, since any 2LL rule may be "flattened" to its corresponding LL rule. Define the following new interpretation of 2-sequences, simpler than the one presented in Definition 2.2:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^\flat = \alpha_0, \ldots, \alpha_n$$

Theorem 3.5 (correctness).

$$\vdash_{\mathsf{2LL}} \Gamma \quad \Rightarrow \quad \vdash_{\mathsf{LL}} [\Gamma]^\flat$$

Proof. Any 2LL rule but promotion (!) becomes the corresponding LL rule via the ♭-translation. Promotion is instead handled by Lemma 3.3, since by the side condition and Lemma 3.4, $[\Gamma]^\flat$ is an Exp-context. □

In view of the previous theorem, the reader might wonder why we gave Definition 2.2, if we rather used the ♭-translation for correctness. The reason is that ♭, though correct, hides the level "semantics." Indeed, if we expressed LL in terms of proof-nets, the levels would be a very intuitive notion: the box nesting depth of each formula. It is clear, however, that the two translations are related, as the following shows.

Proposition 3.6. *Let* Γ *be a 2-sequence such that, for any* $A^i \in \Gamma$ *with* $i < \max(\Gamma)$, $A \in \mathsf{Exp}$. *Then:*

$$\vdash_{\mathsf{LL}} [\Gamma]^\sharp \multimap \parr [\Gamma]^\flat \qquad \textit{and} \qquad \vdash_{\mathsf{LL}} \parr [\Gamma]^\flat \multimap [\Gamma]^\sharp$$

Proof. The implication $[\Gamma]^\sharp \multimap \parr [\Gamma]^\flat$ follows easily from the comonad law $\vdash_{\mathsf{LL}} !A \multimap A$. For the other direction, use repeatedly Lemmas 3.4 and 3.3. □

Observe that, in view of Lemma 3.4, the hypothesis of the previous proposition hold for any *provable* 2-sequence.

3.3 Completeness

Lemma 3.7. *The following rule is admissible in* 2LL*:*

$$\frac{\vdash ?\Gamma^{=0}, A^0}{\vdash ?\Gamma^{=0}, !A^0}$$

Proof. Observe first that, whenever $\vdash_{\mathsf{2LL}} ?\Gamma^{=0}, A^0$, we also have $\vdash_{\mathsf{2LL}} ?\Gamma^{=1}, A^1$ (simply add one to all the levels in the proof). Now, with several ? rules get $\vdash ??\Gamma^{=0}, A^1$, and hence $\vdash ??\Gamma^{=0}, !A^0$. Finally, cut this sequent against the provable sequent $\vdash !!\Gamma^{\perp=0}, ?\Gamma^{=0}$. □

Theorem 3.8 (completeness).

$$\vdash_{\mathsf{LL}} \alpha \quad \Rightarrow \quad \vdash_{\mathsf{2LL}} \alpha^{=0}$$

Proof. By induction on the length of the proof of $\vdash_{\mathsf{LL}} \alpha$ and by cases on the last rule. Most cases are trivial. For instance, the ? rule is handled directly, taking $j = 0$. For the ! rule, apply instead Lemma 3.7. □

A careful inspection of the proof above shows that all the non modal rules are applied with the principal formulas at maximal levels. Therefore, the system with this restriction would maintain completeness.

Corollary 3.9 (equivalence).

$$\vdash_{\mathsf{LL}} \alpha \quad \Leftrightarrow \quad \vdash_{\mathsf{2LL}} \alpha^{=0}$$

4 Taming the complexity of LL

Following the ideas in [MM95a], several interesting subsystems of LL can be obtained constraining the range of variation of the j parameter in the ? rule. Indeed:

1. setting $i \geq j > 0$, we avoid the principle $!A \multimap A$;
2. setting $j \in \{0, 1\}$, we avoid the principle $!A \multimap !!A$;
3. while, with $j = 1$, we get rid of both of them

Remark 4.1. Let 2ELL be the system in which $j = 1$, for any ? rule. That 2ELL does not prove $!A \multimap !!A$ and $!A \multimap A$ is established by a simple translation of linear formulas into classical modal ones: linear propositional connectives are replaced with classical conjunction and disjunction; ! with necessity and ? with possibility (see also section 4.1). The system 2ELL is translated in this way into a subcalculus of the 2-sequent calculus for KD (see [Mas92]). Obviously, if 2ELL proved one of $!A \multimap !!A$, $!A \multimap A$, also the 2-sequent calculus for KD would do the same for $\Box A \rightarrow \Box\Box A$, and $\Box A \rightarrow A$, which is instead impossible.

The interest of these subsystems stems from the fact that they avoid the rules that are the main culprits for the superexponential cost of cut-elimination for LL. In fact, questing for a logical system with an intrinsic polynomial complexity (i.e., with a polynomial cut-elimination, ensuring at the same time that all polynomial functions are representable) Girard proposed in [Gir95b] the system of Light Linear Logic (LLL), dropping, among others, the laws mentioned above. The purpose of this section is to reconstruct LLL in a 2-sequent notation, which, we believe, offers a better view of its key features.

In the same paper, Girard also briefly sketched ELL (Elementary Linear Logic), a supersystem of LLL with a (Kalmar) elementary cut-elimination. Also ELL is easily formulated in our notation: it is simply 2ELL, once Girard's syntax (especially what he calls "blocks") is expressed in the 2-sequent language. The equivalence between ELL and 2ELL will be stated at the end of the subsection devoted to LLL, with which ELL shares most syntax. Before getting through LLL, however, let us pause for a moment, discussing why (presumably) ELL was presented with a complicated syntax. A complication which, as we have seen, completely disappears in 2ELL.

4.1 ELL and the deontic logic KD

From the modal point of view, ELL is akin to KD (see Remark 4.1). Sequent rules for KD are well known; the standard ones are the following, where both modalities are introduced *at once*:

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A} \qquad \frac{\vdash \Gamma}{\vdash \Diamond\Gamma}$$

It is this "one-shot" formulation of modalities that gets rid of the two laws $\Box A \rightarrow \Box\Box A$ and $\Box A \rightarrow A$. ELL, however, wants to avoid these two rules, but without changing the outer setting; in particular, those laws regulating the interaction between exponential and propositional connectives. Namely, the sequent $\vdash_{\mathsf{ELL}} ?A, ?B, !(A^\perp \,\&\, B^\perp)$ *is* provable in ELL. But, look to its proof in LL:

$$\dfrac{\dfrac{\dfrac{\vdash A, A^\perp}{\vdash ?A, A^\perp}\,?}{\vdash ?A, ?B, A^\perp}\,W \qquad \dfrac{\dfrac{\vdash B, B^\perp}{\vdash ?B, B^\perp}\,?}{\vdash ?A, ?B, B^\perp}\,W}{\dfrac{\vdash ?A, ?B, A^\perp \& B^\perp}{\vdash ?A, ?B, !(A^\perp \& B^\perp)}\,!}\,\&$$

By inspection of this proof, we see that in this case it is *essential* to have separate rules for the two exponentials, for the introduction of ! is subordinated to a ? context built by two separate ? rules. Therefore, the two exponentials cannot be simultaneously introduced by a unique rule; we must rather have a system with two distinct rules for ? and ! in which the accumulation of unbalanced ? is forbidden. It is because of this problem that Girard, in its original formulation of ELL (and LLL), had to resort to a complicated notation.

The relevant point of 2LL is that we can ban the incriminated laws just "weakening" the exponential rules of LL (adding a constraint to the corresponding indexes), that is, maintaining two disjoint introduction rules for ? and !. For instance, it is immediate to see that the proof above is also a proof in 2ELL:

$$\dfrac{\dfrac{\dfrac{\vdash A^1, A^{\perp 1}}{\vdash ?A^0, A^{\perp 1}}\,?}{\vdash ?A^0, ?B^0, A^{\perp 1}}\,W \qquad \dfrac{\dfrac{\vdash B^1, B^{\perp 1}}{\vdash ?B^0, B^{\perp 1}}\,?}{\vdash ?A^0, ?B^0, B^{\perp 1}}\,W}{\dfrac{\vdash ?A^0, ?B^0, A^\perp \& B^{\perp 1}}{\vdash ?A^0, ?B^0, !(A^\perp \& B^\perp)^0}\,!}\,\&$$

5 Light Linear Logic

ELL is still too powerful. To exactly capture polynomial time, many of its rules must be restricted. From an axiomatic point of view, many laws valid in ELL have to be abandoned; among these, the generalization rule:

$$\frac{\vdash A}{\vdash !A}\,\mathsf{gen},$$

the law $!(A \multimap B) \multimap (!A \multimap !B)$ and the law $!A \otimes !B \multimap !(A \otimes B)$. To restore, in a harnessed way, some of the lost power, a new autodual modality § must be added to the system.

In the framework of 2-sequents, many constraints have to be added to 2ELL to get LLL. First, the rule for ! must be formulated as to avoid generalization as a special case:

$$\frac{\vdash \Gamma, A^i}{\vdash \Gamma, !A^{i-1}}\,!_{i>\max(\Gamma),\Gamma\neq\varnothing}.$$

Next we must avoid the two laws mentioned above, amounting to a drastic restriction of cut and $\otimes$: all the premises (and conclusions) must be at a same single level; for the same reason, also rule ? has to be strongly harnessed, asking the context to consist of exactly one formula:

$$\frac{\vdash B^i, A^i}{\vdash B^i, ?A^{i-1}}\,?.$$

Finally, also weakening must be restricted:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A^i}\,W._{i \leq \max(\Gamma), \Gamma^{\leq i} \neq \emptyset}$$

The new autodual ($(\S A)^\perp = \S A^\perp$) modality is regulated by the single rule:

$$\frac{\vdash \Gamma^{=i+1}, \Delta^{=i+1}}{\vdash ?\Gamma^{=i}, \S\Delta^{=i}}\,?\S,$$

which makes clear that § stays midway between ! and ?, for $\vdash_{\mathsf{LLL}} !A \multimap \S A$ and $\vdash_{\mathsf{LLL}} \S A \multimap ?A$.

All these restrictions make (essentially) useless the abundance of levels of the calculus. Indeed, in the next section we shall present an equivalent (and we believe more elegant) calculus with only two levels, to which we reserve the official name 2LLL. The equivalence of 2LLL to the sketched formulation with many levels is left to the interested reader and may be proved along the lines of the following Theorem 5.3.

5.1 2LLL

In Figure 2, we formulate the system 2LLL resorting to the two-dimensional notation for 2-sequents. All the sequences have at most two levels; recall that α, β range over ordinary sequences of formulas.

We prove next that 2LLL is equivalent to the standard formulation of Girard. In the following, $A, B, A_i, \dots$ range over formulas; $\mathbf{A}, \mathbf{B}, \mathbf{A}_i \dots$ range over blocks. Moreover, the block $A_1, \dots, A_n$ stands for (or to put it as in [Gir95b]"is hypocrisy for") $A_1 \oplus \dots \oplus A_n$; while, if $\mathbf{A}_1, \dots, \mathbf{A}_n$ stand for the formulas $A_1, \dots, A_n$, the sequence $\mathbf{A}_1; \dots; \mathbf{A}_n$ stands for $A_1 \parr \dots \parr A_n$.

5.2 Soundness

We slightly modify the interpretation of Definition 2.2 to take into account the block notion.

Definition 5.1 (interpretation of LLL 2-sequences).

1. $\overline{A_1, \dots, A_n} = A_1; \dots; A_n;$
2. $\parr(A_1, \dots, A_n) = A_1 \parr \dots \parr A_n;$

Identity/Negation

$$\vdash\ A, A^{\perp} \qquad\qquad \frac{\vdash\ \alpha, A \qquad \vdash\ \beta, A^{\perp}}{\vdash\ \alpha, \alpha}\,\mathsf{cut}$$

Structure

$$\frac{\vdash\ \begin{array}{c}\alpha\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A\end{array}}\,W1_{\ \beta\neq\varnothing} \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, ?A\\ \beta\end{array}}\,W2_{\ \alpha\neq\varnothing} \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A, ?A\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, ?A\end{array}}\,C1 \qquad \frac{\vdash\ \begin{array}{c}\alpha, ?A, ?A\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, ?A\\ \beta\end{array}}\,C2$$

Logic

$$\frac{\vdash\ \alpha, \beta}{\vdash\ ?\alpha, \S\beta}\,?\S \qquad \frac{\vdash\ A, B}{\vdash\ \begin{array}{c}?A\\ B\end{array}}\,? \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ A\end{array}}{\vdash\ \alpha, !A}\,!_{\ \alpha\neq\varnothing}$$

$$\frac{\vdash\ \begin{array}{c}\alpha\\ \beta, C\end{array} \qquad \vdash\ \begin{array}{c}\alpha\\ \beta, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \,\&\, D\end{array}}\,\& \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ \beta, C\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \oplus D\end{array}}\,\oplus_{\mathsf{L}} \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ \beta, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \oplus D\end{array}}\,\oplus_{\mathsf{R}}$$

$$\frac{\vdash\ \alpha, C \qquad \vdash\ \beta, D}{\vdash\ \alpha, \beta, C \otimes D}\,\otimes \qquad \frac{\vdash\ \begin{array}{c}\alpha\\ \beta, C, D\end{array}}{\vdash\ \begin{array}{c}\alpha\\ \beta, C \parr D\end{array}}\,\parr_1 \qquad \frac{\vdash\ \begin{array}{c}\alpha, C, D\\ \beta\end{array}}{\vdash\ \begin{array}{c}\alpha, C \parr D\\ \beta\end{array}}\,\parr_2$$

$$\frac{\vdash\ \alpha, A}{\vdash\ \alpha, \forall X.A}\,\forall \qquad\qquad \frac{\vdash\ \alpha, A}{\vdash\ \alpha, \exists X.A}\,\exists$$

$$\frac{\vdash\ \alpha}{\vdash\ \alpha, \bot}\,\bot \qquad\qquad \vdash\ 1 \qquad\qquad \vdash\ \alpha, \top$$

Figure 2: 2LLL: 2-sequent presentation of LLL

3. $$[\Gamma]^{\circledcirc} = \begin{cases} [\alpha]^{\circledcirc} = \overline{\alpha} & \text{when} \quad \Gamma = \alpha \\ \left[\begin{array}{c}\alpha\\ \beta\end{array}\right]^{\circledcirc} = \overline{\alpha}; ! \parr \beta & \text{when} \quad \Gamma = \begin{array}{c}\alpha\\ \beta\end{array} \end{cases}$$

In the following, the facts that $\vdash_{\mathsf{LLL}} A \parr B$ and $\vdash_{\mathsf{LLL}} B \multimap C$ implies $\vdash_{\mathsf{LLL}} A \parr C$, and that $\vdash_{\mathsf{LLL}} !B$ whenever $\vdash_{\mathsf{LLL}} A \multimap B$ and $\vdash_{\mathsf{LLL}} !A$, will be used over and over again without explicit reference (in particular, for the second case, recall that we cannot use $!(A \multimap B) \multimap (!A \multimap !B)$, for it *fails* in LLL).

Lemma 5.2. $\vdash_{\mathsf{LLL}} !(B \parr C) \otimes !(B \parr D) \multimap !(B \parr (C \,\&\, D))$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash B^{\perp}; B \quad \vdash C^{\perp}; C}{\vdash B^{\perp} \otimes C^{\perp}; B; C}}{\vdash B^{\perp} \otimes C^{\perp}, B^{\perp} \otimes D^{\perp}; B; D} \qquad \dfrac{\dfrac{\vdash B^{\perp}; B \quad \vdash D^{\perp}; D}{\vdash B^{\perp} \otimes D^{\perp}; B; D}}{\vdash B^{\perp} \otimes C^{\perp}, B^{\perp} \otimes D^{\perp}; B; D}}{\vdash B^{\perp} \otimes C^{\perp}, B^{\perp} \otimes D^{\perp}; B; C \,\&\, D}}{\vdash B^{\perp} \otimes C^{\perp}, B^{\perp} \otimes D^{\perp}; B \parr C \,\&\, D}}{\vdash [B^{\perp} \otimes C^{\perp}]; [B^{\perp} \otimes D^{\perp}]; !(B \parr (C \,\&\, D))}}{\vdash ?(B^{\perp} \otimes C^{\perp}); [B^{\perp} \otimes D^{\perp}]; !(B \parr (C \,\&\, D))}}{\vdash ?(B^{\perp} \otimes C^{\perp}); ?(B^{\perp} \otimes D^{\perp}); !(B \parr (C \,\&\, D))}}{\vdash ?(B^{\perp} \otimes C^{\perp}) \parr ?(B^{\perp} \otimes D^{\perp}); !(B \parr (C \,\&\, D))}}{\vdash !(B \parr C) \otimes !(B \parr D) \multimap !(B \parr (C \,\&\, D))}$$

□

Theorem 5.3 (soundness of 2LLL).

$$\vdash_{\mathsf{2LLL}} \Gamma \quad \Rightarrow \quad \vdash_{\mathsf{LLL}} [\Gamma]^{\circledcirc}$$

Proof. Induction on the proof of $\vdash_{\mathsf{2LLL}} \Gamma$ and by cases on the last rule. The case of axiom, cut, $\otimes$, $\forall$, $\exists$, $\perp$, 1, and $\top$ are obvious, since they are interpreted into the corresponding LLL rules. Also trivial are the cases of ! and $\parr$, since premise and conclusion have the same interpretation. For the other rules we have instead:

Rules $\oplus$: Easy, since $\vdash_{\mathsf{2LLL}} C \multimap C \oplus D$ and $\vdash_{\mathsf{2LLL}} C \multimap D \oplus C$.

Rule §: Apply rule neutral first and then rule why-not many times.

Rule ?: Apply rule of − course first and then rule why-not once.

Rule &: If α is empty, use rule with of LLL. Otherwise, the premises of the rule are interpreted as $\overline{\alpha}; !(\parr\overline{\beta}\parr C)$ and $\overline{\alpha}; !(\parr\overline{\beta}\parr D)$. Thus, by induction hypothesis, $\vdash_{\mathsf{LL}} \overline{\alpha}; \overline{\alpha}; !(\parr\overline{\beta} \parr C) \otimes !(\parr\overline{\beta} \parr D)$. By a cut, using Lemma 5.2, obtain $\vdash_{\mathsf{LL}} \overline{\alpha}; \overline{\alpha}; !(\parr\overline{\beta} \parr (C \,\&\, D))$. Observe now that, for $\overline{\alpha} = A_1; \ldots; A_n$, any A_i is (in the class Exp and in particular is) of the form $?B_1 \parr \ldots \parr ?B_{k_i}$. Therefore, using $\vdash_{\mathsf{LL}} ?B \parr ?B \multimap ?B$, by several successive cuts get the thesis.

Rule W1: By $\vdash_{\mathsf{LL}} B \multimap B \parr ?A$.

Rule W2: Use first mult-W, and then rule why-not of LLL.

Rules C1 and C2: Use again $\vdash_{\mathsf{LL}} ?B \parr ?B \multimap ?B$.

□

5.3 Completeness

Definition 5.4 (Translation of LLL sequents). Let $A, A_1, \ldots A_n$ be formulas and $\mathbf{A}, \mathbf{A}_1, \ldots \mathbf{A}_m$ be blocks:

1. $(A)^{\circledast} = A$ if A is a formula;

2. $([A])^{\circledast} = ?A$;

3. $(A1, \dots, A_n)^{\circledast} = A1 \oplus \dots \oplus A_n$;

4. $(\mathbf{A}_1; \dots ; \mathbf{A}_n)^{\circledast} = (\mathbf{A}_1)^{\circledast}, \dots , (\mathbf{A}_n)^{\circledast}$.

Lemma 5.5.

$$\vdash_{\mathsf{2LLL}} ?B_1, \dots , ?B_n, !(B_1^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp})$$

Proof.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\vdash B_1, B_1^{\perp}}{\vdash \begin{array}{c} ?B_1 \\ B_1^{\perp} \end{array}}\,?
}{\vdots}\,W2
}{\vdash \begin{array}{c} ?B_1, \dots , ?B_n \\ B_1^{\perp} \end{array}}\,W2
\qquad
\dfrac{
\begin{array}{c} \vdash B_2, B_2^{\perp} \\ \vdots \\ \vdash \begin{array}{c} ?B_1, \dots , ?B_n \\ B_2^{\perp} \end{array} \end{array}
\qquad
\begin{array}{c} \vdash B_3, B_3^{\perp} \ \cdots \vdash B_n, B_n^{\perp} \\ \vdots \\ \vdash \begin{array}{c} ?B_1, \dots , ?B_n \\ B_3^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp} \end{array} \end{array}
}{\vdash \begin{array}{c} ?B_1, \dots , ?B_n \\ B_2^{\perp} \,\&\, B_3^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp} \end{array}}\,\&
}{\vdash \begin{array}{c} ?B_1, \dots , ?B_n \\ B_1^{\perp} \,\&\, B_2^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp} \end{array}}\,\&
}{\vdash ?B_1, \dots , ?B_n, !(B_1^{\perp} \,\&\, B_2^{\perp} \,\&\, \cdots \,\&\, B_n^{\perp})}\,!
$$

□

Theorem 5.6 (completeness of 2LLL).

$$\vdash_{\mathsf{LLL}} \mathbf{A}_1; \dots ; \mathbf{A}_m \quad \Rightarrow \quad \vdash_{\mathsf{2LLL}} (\mathbf{A}_1)^{\circledast}, \dots , (\mathbf{A}_m)^{\circledast}$$

Proof. Induction on the proof of $\vdash_{\mathsf{LLL}} \mathbf{A}_1; \dots ; \mathbf{A}_m$ and by cases on the last rule. All the non modal rules of LLL (plus why-not) are handled trivially. The only interesting cases are those of of-course and neutral.
In the original formulation of LLL the of-course rule is:

$$\dfrac{\vdash B_1, \dots , B_n; A}{\vdash [B_1], \dots , [B_n]; !A}\ \textsf{of-course}$$

By definition, $(B_1, \dots , B_n; A)^{\circledast} = B_{\oplus}, A$, where $B_{\oplus} = B_1 \oplus \cdots \oplus B_n$. By induction hypothesis, $\vdash_{\mathsf{2LLL}} B_{\oplus}, A$. Therefore, since $([B_1], \dots , [B_n]; !A)^{\circledast} = ?B_1, \dots , ?B_n, !A$, the required 2LLL proof is:

$$
\dfrac{
\dfrac{\text{Lemma 5.5}}{\vdash ?B_1, \dots , ?B_n, !B_{\oplus}^{\perp}}
\qquad
\dfrac{\dfrac{\vdash B_{\oplus}, A}{\vdash \begin{array}{c} ?B_{\oplus} \\ A \end{array}}\,?}{\vdash ?B_{\oplus}, !A}\,!
}{\vdash ?B_1, \dots , ?B_n, !A}\ \textsf{cut}
$$

In the original formulation of LLL the neutral rule is:

$$\dfrac{\vdash C_1^1, \dots , C_{p_1}^1; \dots ; C_1^n, \dots , C_{p_n}^n; A_1; \dots ; A_m}{\vdash [C_1^1]; \dots ; [C_{p_1}^1]; \dots ; [C_1^n]; \dots ; [C_{p_n}^n]; \S A_1; \dots ; \S A_m}\ \textsf{neutral}$$

Let $C^k_\oplus = C^k_1 \oplus \cdots \oplus C^k_{p_k}$ and $\widetilde{C}^k_\& = (C^k_1)^\perp \& \cdots \& (C^k_{p_k})^\perp = (C^k_\oplus)^\perp$. The premise of the rule becomes $(C^1_1, \ldots, C^1_{p_1}; \ldots; C^n_1, \ldots, C^n_{p_n}; A_1; \ldots; A_m)^\circledast = C^1_\oplus, \ldots, C^n_\oplus, A_1, \ldots, A_m$.
By induction hypothesis, we have $\vdash_{2LLL} C^1_\oplus, \ldots, C^n_\oplus, A_1, \ldots, A_m$. Therefore, for $([C^1_1]; \ldots; [C^n_{p_n}]; \S A_1; \ldots; \S A_m)^\circledast = ?C^1_1, \ldots, ?C^n_{p_n}, \S A_1, \ldots, \S A_m$, the required 2LLL proof is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash C^1_\oplus, \ldots, C^n_\oplus, A_1 \ldots, A_m}{\vdash ?C^1_\oplus, \ldots, ?C^n_\oplus, \S A_1, \ldots, \S A_m}\S \qquad \dfrac{\text{Lemma 5.5}}{\vdash ?C^1_1, \ldots, ?C^1_{p_1}, !\widetilde{C}^1_\&}}{\vdash ?C^1_1, \ldots, ?C^1_{p_1}, ?C^2_\oplus \ldots, ?C^n_\oplus, \S A_1, \ldots, \S A_m}\mathsf{cut} \qquad \vdots}{\vdots}\mathsf{cut} \qquad \vdots}{\vdash ?C^1_1, \ldots, ?C^n_{p_n}, \S A_1, \ldots, \S A_m}\mathsf{cut}$$

□

Corollary 5.7 (equivalence).

$$\vdash_{2LLL} \mathbf{A}_1; \ldots; \mathbf{A}_m \iff \vdash_{2LLL} (\mathbf{A}_1; \ldots; \mathbf{A}_m)^\circledast.$$

6 Why 2-sequents

Although we believe that the clean and modular presentation of the previous systems, otherwise really contrived, is already a good justification for our approach; we want to give some more examples of the advantages of the 2-sequents formulation of Linear Logic. Firstly, we shall see that the clear syntactical formulation of the rules has also a counterpart in some properties of the system, as the elimination of a counter-example to cut-freeness of ELL; secondly, we shall see that the modular approach might apply to suitable extensions of Linear Logic and not only to sub-systems of it.

6.1 An ELL example

As already said, the system ELL was only sketched by Girard in its paper. We have already remarked that our 2-sequent presentation greatly simplifies the syntax. This is not only a matter of esthetic; the syntactical complications of the original formulation also hidden some poison in their tail.

Let us take the following proof in 2ELL:

$$\dfrac{\dfrac{\dfrac{\vdash A^{\perp 1}, A^1}{\vdash A^{\perp 1}, ?A^0}? \qquad \dfrac{\vdash 1^1}{\vdash 1^1, ?A^0}W}{\vdash A^\perp \& 1^1, ?A^0}\&}{\vdash !(A^\perp \& 1)^0, ?A^0}!$$

Its ending sequent is also derivable in ELL, nevertheless, it is not cut-free provable. By inspection of the previous proof, we see that to obtain a cut-free proof of $\vdash !(A^\perp \& 1), ?A$ we should be able to weaken $\vdash 1$ by the formula A, adding the ? in front of A only while closing the box of the ! (the firsts who discovered that ELL was not cut-free were Kanovich, Okada and Scedrov while extending to ELL their semantics for LLL [KOS97]). More formally, we can prove that not only this cut cannot be eliminated, but that in ELL there is no cut-free proof of $\vdash\ !(A^{\perp} \,\&\, 1), ?A$.

Beside the positive solution to this counterexample, we stress that we have not yet completed the study of the dynamics of the systems presented in the paper; therefore, we cannot yet say anything about their cut-elimination, neither if they are cut-free (indeed, note that in the proof of the correspondence between LL and 2LL we use cut).

6.2 Questing for more freedom

So far, the aim was at showing that the 2-sequents approach scales to subsystems of LL obtained constraining the nesting of the corresponding proof-net boxes. In particular, we have seen that these constraints find an immediate translation in 2LL as suitable restrictions of the provisos already present in its rules. Moreover, we have also seen that some subsystems of 2LL still enjoy the correspondence with LL (for instance, restricting all the non-exponential rules to formulas at maximal levels, Corollary 3.9 still holds).

On the other hand, rather to strength the restrictions already present in 2LL, another possible direction is relaxing or removing the side-conditions. Let us see in details the cases in which this make sense and with which consequences on provability. In particular, we will see that the proposed rules are the most liberal ones for representing LL.

In the rules of 2LL, we have several kind of constraints: (*i*) in the promotion rule for !; (*ii*) in the weakening rule; (*iii*) in the rules for $\forall$ and $\exists$; (*iv*) in the rules for cut, $\otimes$, and $\oplus$. Let us analyze the previous cases in order.

6.2.1 Promotion

In 2LL, dropping the constraint of the rule for the introduction of ! we would kill the exponential structure. In fact, since the sequent $\vdash ?A \multimap !A$ would become provable (note that this is not provable in LL), from the law $\vdash !A \multimap A$ (valid in LL too), we might conclude the uselessness of exponentials, i.e., $A \simeq ?A \simeq !A$.

The previous relaxation would however have a less drastic effect on the fragments 2LLL and 2ELL. For instance, let us take 2ELL. The law $\vdash !A \multimap A$ is not derivable in 2ELL and, moreover, we would not get it neither relaxing the constraint on the promotion rule of 2ELL. Therefore, in this case exponential formulas would not collapse to simple ones, but ! and ? would be self-dual, i.e., $!A \simeq ?A$. From a semantical point of view (and from a modal perspective), this would mean that ! is a sort of non-branching next operator; while, from a syntactical point of view, A^{i} would just become syntactic sugar for $!^{i}A$ (i.e., A preceded by i occurrences of !).

6.2.2 Weakening

Removing the constraint on weakening would lead to a system midway to a linear and affine system (i.e., with unrestricted weakening). In fact, this would correspond to extend the weakening rule of LL to the case of formulas like $!^k?A$, for any $k \geq 0$.

6.2.3 Quantifiers

This is maybe the least interesting case. Anyhow, let us note that at least for $\exists$, the proviso seems mandatory. The problem is that the constraint is in some sense forced by the usual proviso that introducing $\forall X$ in front of a formula A in the sequent Γ, then A is the only formula of Γ in which X may occur free. For instance, let us take the following proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash X^{\perp 1}, X^1}{\vdash ?X^{\perp 0}, X^1}\,?}{\vdash \exists X.?X^{\perp 0}, X^1}\,\exists}{\vdash \exists X.?X^{\perp 0}, \forall X.X^1}\,\forall}{\vdash \exists X.?X^{\perp 0}, !\forall X.X^0}\,!$$

in which the side-condition of $\exists$ is violated, and let us see why the ending sequent is not provable in LL. According to the interpretation of levels as boxes, the $\exists$ rule in the proof is outside the box of the !, while the $\forall$ is inside the box. Moreover, the axiom $\vdash X^{\perp 1}, X^1$ is the only rule inside that box. Therefore, applying $\forall$ we would violate the side-condition on the occurrences of X. In other words, the extension of $\mathsf{2LL}$ corresponding to a free application of $\exists$ is not a logically sound extension of LL.

6.2.4 Multiplicatives and additives

Let us use $\vdash_{\mathsf{2SLL}}$ for the provability relation obtained allowing an unrestricted use of cut, $\otimes$ and $\oplus$. We have:

$$\vdash ?(A^\perp) \oplus ?(B^\perp), !(A \mathbin{\&} B)$$

$$\vdash ?(A^\perp) \otimes ?(B^\perp), !(A \parr B)$$

These are essentially the only new sequents introduced by the relaxation. In fact, let S be the the set of axiom (schemas) derived from the pair of sequents above. We have that:

$$\vdash_{\mathsf{2SLL}} A^i \qquad \Longleftrightarrow \qquad \vdash_{\mathsf{LL}+S} A$$

and in particular, when A is exponential free, we have:

$$\vdash_{\mathsf{2SLL}} A^i \qquad \Longleftrightarrow \qquad \vdash_{\mathsf{LL}} A$$

The interesting point to remark is that the axioms in S fits our interpretation of exponentials as a sort of quantifiers, for it is well known that the corresponding sequents in which $\forall$ and $\exists$ replace ! and ? are provable in LL. Namely:

$$\vdash_{\mathsf{LL}} \exists X.(A^{\perp}) \oplus \exists X.(B^{\perp}), \forall X.(A \mathbin{\&} B)$$

$$\vdash_{\mathsf{LL}} \exists X.(A^{\perp}) \otimes \exists X.(B^{\perp}), \forall X.(A \parr B)$$

Unfortunately, the prevision extension of LL is not cut-free. In fact, from the second sequent in S and from $\vdash_{\mathsf{LL}} ?(A^{\perp} \parr B^{\perp})^0, (A \parr b) \otimes (A \parr B)^0$, we can derive:

$$\frac{\vdash ?A^{\perp} \otimes ?B^{\perp 0}, !(A \parr B)^0 \qquad \vdash ?(A^{\perp} \parr B^{\perp})^0, (A \parr B) \otimes (A \parr B)^0}{\vdash ?A^{\perp} \parr ?B^{\perp 0}, (A \parr B) \otimes (A \parr B)^0}\ \mathsf{cut}$$

whose ending sequent is not cut-free derivable.

7 Conclusions and further work

In this paper, our main aim was at showing how the 2-sequents approach can greatly simplify and improve the presentation of calculi whose rule must encode involved structural constraints of the corresponding proofs. Because, of this we omitted to study the dynamics of the systems that we proposed and we focused on examples and observations made possible by the use of our generalized notion of sequent.

However, the most relevant point of the 2-sequents approach to Linear Logic is the tight correspondence between the indexes it assigns to formulas and the box nesting of links in the corresponding proof-nets. Therefore, the natural next step is the analysis of the indexed proof-nets induced by our systems and, in particular, a detailed study of their dynamics. As we showed in [GMM96, GMM97], the dynamics of the indexed proof-nets corresponding to 2-sequents of LL can be implemented via a set of local an distributed rules (i.e., no more global rules for duplication of boxes). We believe that this approach might scale in a smooth way to the systems presented here. We hope that this not only would preserve the complexity bound that motivated Girard, but it would also give a clearer explanation of them.

References

[Doš85] Kosta Došen. Sequent-systems for modal logic. *Journal of Symbolic Logic*, 50:149–168, 1985.

[GdQ92a] Dov M. Gabbay and Ruy J.G.B. de Queiroz. Extending the Curry-Howard interpretation to linear, relevant and other resource logics. *Journal of Symbolic Logic*, 57:1319–1365, 1992.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[Gir95a] J.-Y. Girard. Linear logic: Its syntax and semantics. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, pages 1–42. Cambridge University Press, 1995.

[Gir95b] Jean-Yves Girard. Light Linear Logic. Manuscript. Available via anonymous ftp from `lmd.univ-mrs.fr` as `/pub/girard/LLL.ps.Z`, 1995.

[GMM96] S. Guerrini, S. Martini, and A. Masini. Coherence for sharing proof nets. In H. Ganzinger, editor, *Proc. of the 7th International Conference on Rewriting Techniques and Applications (RTA-96)*, volume 1103 of *LNCS*, pages 215–229, 1996. Springer-Verlag.

[GMM97] Stefano Guerrini, Simone Martini, and Andrea Masini. Proof nets, garbage, and computations. In editor?, editor, *TLCA 97*, volume vol. 1210 of *LNCS*, pages 181–195, 1997. Springer-Verlag.

[KOS97] Max I. Kanovich, Mitsuhiro Okada, and Andre Scedrov. Phase semantics for Light Linear Logic. In *Mathematical Foundations of Programming Semantics, Thirteenth Annual Conference*, Electronic Notes in Theoretical Computer Science, Pittsburgh, PA, 1997. ULR `http://www.elsevier.nl/locate/entcs`.

[Kri63] Saul Kripke. Semantical analysis of modal logic I. *Zeitschr. f. math. Logik und Grund. d. Mathematik*, 9:67–96, 1963.

[Mas92] Andrea Masini. 2-sequent calculus: A proof theory of modalities. *Annals of Pure and Applied Logic*, 58:229–246, 1992.

[Mas93] Andrea Masini. 2-sequent calculus: Intuitionism and natural deduction. *Journal of Logic and Computation*, 3:533–562, 1993.

[MM95a] S. Martini and A. Masini. On the fine structure of the exponential rule. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, pages 197–210. Cambridge University Press, 1995.

[MM95b] Simone Martini and Andrea Masini. A computational interpretation of modal proofs. In H. Wansing, editor, *Proof theory of Modal Logics*, volume 2 of *Applied Logic Series*, pages 213–241, Kluwer, 1996.

[Wan94] H. Wansing. Sequent calculi for normal modal propositional logic. *Journal of Logic and Computation*, 4(2),125–142, 1994.

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The rules of LLL are given in Figure 3. Remind that $\mathbf{A}$ stands for a block of formulas $A_1, \ldots, A_n$. Moreover, $A_1, \ldots, A_n$ “is hypocrisy for” $A_1 \oplus \ldots \oplus A_n$, while if $\mathbf{A}_1, \ldots, \mathbf{A}_n$ stand for the formulas $A_1, \ldots, A_n$, the sequence $\mathbf{A}_1; \ldots; \mathbf{A}_n$ “is hypocrisy for” for $A_1 \parr \ldots \parr A_n$.

For a complete treatment, including the relevant issues of the dynamics of LLL, see [Gir95b] (for a semantical approach, see instead [KOS97]).

ELL is obtained from LLL replacing the following of-course

$$\frac{\vdash C^1_1, \ldots, C^1_{p_1}; \ldots; C^n_1, \ldots, C^n_{p_n}; A}{\vdash [C^1_1]; \ldots; [C^1_{p_1}]; \ldots; [C^n_1]; \ldots; [C^n_{p_n}]; !A} \text{ of-course}$$

for the one of LLL. At the same time the modality § becomes superfluous and is removed from the calculus.

Identity/Negation

$$\frac{}{\vdash A; A^\perp} \text{ axiom} \qquad \frac{\vdash \Gamma; A \quad \vdash \Delta; A^\perp}{\vdash \Gamma; \Delta} \text{ cut}$$

Structure

$$\frac{\vdash \Gamma}{\vdash \Gamma; [A]} \text{ mult-W} \quad \frac{\vdash \Gamma; \mathbf{A}}{\vdash \Gamma; \mathbf{A}, B} \text{ add-W} \qquad \frac{\vdash \Gamma; [A]; [A]}{\vdash \Gamma; [A]} \text{ mult-C} \quad \frac{\vdash \Gamma; \mathbf{A}, B, B}{\vdash \Gamma; \mathbf{A}, B} \text{ add-C}$$

Logic

$$\frac{\vdash B_1, \ldots, B_n; A}{\vdash [B_1], \ldots, [B_n]; !A} \text{ of-course}_{n>0} \qquad \frac{\vdash \Gamma; A}{\vdash \Gamma; ?A} \text{ why-not}$$

$$\frac{\vdash C^1_1, \ldots, C^1_{p_1}; \ldots; C^n_1, \ldots, C^n_{p_n}; A_1; \ldots; A_m}{\vdash [C^1_1]; \ldots; [C^1_{p_1}]; \ldots; [C^n_1]; \ldots; [C^n_{p_n}]; \S A_1; \ldots; \S A_m} \text{ neutral}$$

$$\frac{\vdash \Gamma; C \quad \vdash \Gamma; D}{\vdash \Gamma; C \,\&\, D} \text{ with} \qquad \frac{\vdash \Gamma; C}{\vdash \Gamma; C \oplus D} \text{ L-plus} \quad \frac{\vdash \Gamma; D}{\vdash \Gamma; C \oplus D} \text{ R-plus}$$

$$\frac{\vdash \Gamma; C \quad \vdash \Delta; D}{\vdash \Gamma; \Delta; C \otimes D} \text{ tensor} \qquad \frac{\vdash \Gamma; C; D}{\vdash \Gamma; C \parr D} \text{ par}$$

$$\frac{\vdash \Gamma; A}{\vdash \Gamma; \forall X.A} \text{ forall} \qquad \frac{\vdash \Gamma; A[B/X]}{\vdash \Gamma; \exists X.A} \text{ exists}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma; \bot} \text{ false} \qquad \frac{}{\vdash 1} \text{ one} \qquad \frac{}{\vdash \Gamma; \top} \text{ true}$$

Figure 3: Girard's LLL