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Abstract

It is well known for which gauge functions *H* there exists a flow in Z^d with finite *H* energy. In this paper we discuss the robustness under random thinning of edges of the existence of such flows. Instead of Z^d we let our (random) graph *cal* $C_\infty(Z^d, p)$ be the graph obtained from Z^d by removing edges with probability $1-p$ independently on all edges. Grimmett, Kesten, and Zhang (1993) showed that for $d \geq 3$, $p > p_c(Z^d)$, simple random walk on *C cal* $_\infty(Z^d, p)$ is a.s. transient. Their result is equivalent to the existence of a nonzero flow *f* on the infinite cluster such that the x^2 energy $\sum_e f(e)^2$ is finite. Levin and Peres (1998) sharpened this result, and showed that if $d \geq 3$ and $p > p_c(Z^d)$, then *cal* $C_\infty(Z^d, p)$ supports a nonzero flow *f* such that the x^q energy is finite for all $q > d / (d-1)$. However, for general gauge functions, there is a gap between the existence of flows with finite energy which results from the work of Levin and Peres and the known results on flows for Z^d. In this paper we close the gap by showing that if $d \geq 3$ and Z^d supports a flow of finite *H* energy then the infinite percolation cluster on Z^d also support flows of finite *H* energy. This disproves a conjecture of Levin and Peres.

Keywords

percolation, energy, electrical networks, nonlinear potential theory

Disciplines

Physics | Statistics and Probability

Energy of Flows on Percolation Clusters

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Abstract

It is well known for which gauge functions H there exists a flow on $\mathbf{Z}^d$ with finite H energy. In this paper we discuss the robustness under random thinning of edges of the existence of such flows. Instead of $\mathbf{Z}^d$ we let our (random) graph $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ be the graph obtained from $\mathbf{Z}^d$ by removing edges with probability $1-p$ independently on all edges. Grimmett, Kesten, and Zhang (1993) showed that for $d \geq 3, p > p_c(\mathbf{Z}^d)$, simple random walk on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is a.s. transient. Their result is equivalent to the existence of a nonzero flow f on the infinite cluster such that the x^2 energy $\sum_e f(e)^2$ is finite. Levin and Peres (1998) sharpened this result, and showed that if $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, then $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a nonzero flow f such that the x^q energy is finite for all $q > d/(d-1)$. However, for general gauge functions, there is a gap between the existence of flows with finite energy which results from the work of Levin and Peres and the known results on flows for $\mathbf{Z}^d$. In this paper we close the gap by showing that if $d \geq 3$ and $\mathbf{Z}^d$ supports a flow of finite H energy then the infinite percolation cluster on $\mathbf{Z}^d$ also support flows of finite H energy. This disproves a conjecture of Levin and Peres.

Keywords : percolation, energy, electrical networks, nonlinear potential theory.

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1 Introduction

We start by recalling some definitions from percolation theory (See e.g [4] for more background). Consider the following random subgraph of $\mathbf{Z}^d$. We keep each edge with probability p, and otherwise remove it. We do this independently on all edges. Properties of this so called percolation process are widely studied. It is known that for $p > p_c(\mathbf{Z}^d)$ there exists a unique infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ a.s, while for $p < p_c(\mathbf{Z}^d)$ such a cluster does not exist a.s. It is natural to ask what properties of $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ are inherited from $\mathbf{Z}^d$.

Grimmett, Kesten and Zhang proved that if $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, then simple random walk on the infinite percolation cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is a.s. transient [5]. This result is equivalent to the existence of a nonzero flow on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite x^2 energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$, where a flow and the $H(x)$ energy of a flow are defined as follows [3]. Consider each undirected edge of $\mathbf{Z}^d$ as two directed edges, one in each direction. Let vw be the directed edge from v to w. A **flow** f on $\mathbf{Z}^d$ with source 0 is an edge function such that the net flow out of any vertex $v \neq 0$ is zero: $\sum_w f(vw) - \sum_w f(wv) = 0$. Let $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ be the (unique) infinite percolation cluster of $\mathbf{Z}^d$. A flow f on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is a flow on $\mathbf{Z}^d$ which satisfies $f(e) = 0$ for any edge $e \notin \mathcal{C}_\infty(\mathbf{Z}^d, p)$. The H **energy** of a flow f on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is

$$\mathcal{E}_q(f) := \sum_{e \in \mathbf{Z}^d} H(f(e)).$$

Benjamini, Pemantle and Peres [2] gave an alternative proof of Grimmet, Kesten and Zhang's result. First they constructed “unpredictable” processes on $\mathbf{Z}$. They used these “unpredictable” processes to create a measure on $\Upsilon = \Upsilon(\mathbf{Z}^d, 0)$, the collection of infinite oriented paths in $\mathbf{Z}^d$ which emanate from 0. For $d \geq 3$ the measure μ they created has **exponential intersection tails**. That is, there exists C and a $\theta < 1$ such that for all n

$$\mu \times \mu \Big\{ (\varphi, \psi) : |\varphi \cap \psi| \geq n \Big\} \leq C\theta^n,$$

where $|\varphi \cap \psi|$ is the number of edges in the intersection of φ and ψ.

Then they showed how to use a measure on paths on $\mathbf{Z}^d$ with exponential intersection tails to generate flows of finite x^2 energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. Levin and Peres adapted this approach to show that these flows have finite x^q energy a.s. for

$q > d/(d-1)$ [8]. Since $\mathbf{Z}^d$ also supports flows of finite x^q energy if and only if $q > d/(d-1)$ (see [10]) this result is optimal.

What about general gauge functions? For any positive decreasing function h, let

$$H_h(u) := u^{d/(d-1)}/h(u)$$

for $u > 0$ and $H_h(0) = 0$. If $h(u) = [\log(1+u^{-1})]^\alpha$, then we abbreviate H_h by H_α. The main result of this paper is that for $d \geq 3$ there exists a flow of finite H_h energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ if there exists a flow of finite H_h energy on $\mathbf{Z}^d$. This implies that there exists a flow of finite H_α energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ for $\alpha > 1$.

Now we describe the functions h such that $\mathbf{Z}^d$ supports a flow of finite H_h energy. T. Lyons [9] constructed a nonzero flow f_* on $\mathbf{Z}^d$ that satisfies $f_*(e) \leq Cl^{1-d}$ for any edge e that is distance l from the origin. This implies that the flow f_* has finite H_α energy if $\alpha > 1$. This is optimal as it is easy to show that $\mathbf{Z}^d$ cannot support a flow of finite H_α energy for $\alpha \leq 1$. More generally if h satisfies

$$\sum_j^\infty \frac{1}{jh(j^{-1})} < \infty$$

then f_* has finite H_h energy. Conversely, if h does not satisfy the summation condition then it is easy to prove that $\mathbf{Z}^d$ cannot support a flow of finite H_h energy.

Levin and Peres proved a sufficient condition for $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ to support a flow of finite H_h energy. Their condition implies that if $\alpha > 2$ then $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a flow of finite H_α energy. The exact theorem that they proved is as follows.

Theorem 1.1 ([8]) *Let h be a decreasing function satisfying*

$$\sum_j^\infty \frac{1}{jh(j^{-1})} < \infty,$$

$h(x^2) \leq \kappa h(x)$ *for all* $x > 0$*, and* $H_{h^2}(u)/u$ *is concave. Then for* $p > p_c$*, there is a.s. a nonzero flow* f *on* $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ *with finite* H_{h^2} *energy.*

So for $1 < \alpha \leq 2$ there exists a flow of finite H_α energy on $\mathbf{Z}^d$, but Levin and Peres's methods are unable to construct a flow of finite H_α energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. Levin and Peres conjectured that this gap was due to the properties of percolation clusters.

Conjecture [8] For $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, with probability one there does not exist a nonzero flow f on the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite H_2 energy.

In this paper we show that there is no gap and the conjecture is false. Thus for $\alpha > 1$ and $d \geq 3$ there exists a flow of finite H_α energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. More generally we prove the following theorem.

Theorem 1.2 *Let $d \geq 3$ and h be a decreasing function satisfying*

$$\sum_j^\infty \frac{1}{jh(j^{-1})} < \infty$$

and $H_h(u)/u$ is concave. Then for $p > p_c$, there is a.s. a nonzero flow on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite H_h energy.

Now we sketch the main idea of the paper. The flows that Levin and Peres construct use only a small portion of $\mathbf{Z}^d$. Specifically their flows put a fixed fraction of their strength on $(l/\log l)^{d-1}$ of the Cl^{d-1} edges that are distance l from the origin. This is why they are unable to construct flows with finite H_α energy with $1 < \alpha \leq 2$. We get around this problem in the following manner. We construct a large class of flows. Then to construct the flow F that has finite $H_{1+\epsilon}$ energy we integrate all of the flows in this family. Each of these flows in this family is actually concentrated on a smaller portion of the graph than are the flows constructed by Levin and Peres. The flow F has lower energy than any of the original flows because any two flows, f_1 and f_2, in this family have the following property. For all sufficiently large l the set of edges at distance l from the origin where $f_1(e) > 0$ is disjoint from the set of edges at distance l from the origin where $f_2(e) > 0$. The construction is all done so that F is spread out over all of $\mathbf{Z}^d$ fairly evenly. Jensen's inequality will allow us to make all of this rigorous. The main difficulty in the proof is altering the time reversal argument in the proof of Theorem 1.1 in [8] to suit our purposes.

The result of this paper extends in a straightforward manner to a family '$2 + \epsilon$' dimensional wedges and horns in $\mathbf{Z}^3$ which include the graphs $\{(x, y, z) : |z| \leq |x|^\epsilon\}$, $\{(x, y, z) : |z| \leq (\log(1 + |x|))^\alpha\}$, $\{(x, y, z) : |y|, |z| \leq |x|^{1/2}(\log(1 + |x|))^\beta\}$, where $\epsilon > 0, \alpha > 2$, and $\beta > 1/2$. However, in order to prove that the result holds for some wedge (horn) we must know that the wedge (horn) admits exponential intersection tails. Therefore, our results do not extend to $\mathbf{Z}^2$ or even to transient wedges such as, $\{(x, y, z)||z| \leq \log(1 + |x|)^\alpha\}, 1 < \alpha < 2$, since the methods of [2] and [7] give no proof that these graphs admit exponential intersection tails.

2 Paths with Exponential Intersection Tails

In this section we introduce some definitions and recall some results about paths with exponential intersection tails.

Definitions.

1. Let $\Upsilon_1 = \Upsilon_1(\mathbf{Z}^d, 0) \subset \Upsilon$ be the set of **paths with unit speed**, those paths for which the n^{th} vertex is at distance n from 0.

2. Let $\Upsilon_{1/K}$ be the set of all paths ψ such that for all k, $|\{t : |\psi(t)| = k\}| \leq K$.

3. The percolation cluster containing a vertex v is called $\mathcal{C}(v)$.

4. For a sequence of random variables $S = \{S_n\}_{n\geq 0}$ taking values in $\mathbf{Z}^d$, we define its **predictability profile** $\{\mathsf{PRE}_S(k)\}_{k\geq 1}$ by

   $$\mathsf{PRE}_S(k) = \sup \mathbf{P}[S_{n+k} = x \mid S_0, \ldots, S_n] \, ,$$

   where the supremum is over all $x \in \mathbf{Z}^d$, all $n \geq 0$ and all histories $S_0, \ldots, S_n$.

Benjamini, Pemantle, and Peres constructed nearest neighbor process on $\mathbf{Z}$ with low predictability profiles. They used these processes to show that there exist probability measures μ on $\Upsilon(\mathbf{Z}^d, 0)$, $d \geq 3$, with exponential intersection tails.

Lemma 2.1 ([2]) *Let $\{\Gamma_n\}$ be a sequence of random variables taking values in $\mathbf{Z}^d$. If the predictability profile of Γ satisfies $\sum_{k=1}^{\infty} \mathsf{PRE}_\Gamma(k) < \infty$, then there exist C and $\theta < 1$, such that for any sequence $\{v_n\}_{n\geq 0}$ in $\mathbf{Z}^d$ and all $m \geq 1$,*

$$\mathbf{P}[\#\{n \geq 0 \, : \, \Gamma_n = v_n\} \geq m] \leq C\theta^m \, .$$

Benjamini et. al. then used the existence of measures on Υ_1 with exponential intersection tails to prove transience of infinite percolation clusters on $\mathbf{Z}^d$, $d \geq 3$.

Lemma 2.2 ([2]) *Consider percolation with parameter p on $\mathbf{Z}^d$ and let μ be a probability measure on $\Upsilon_1(\mathbf{Z}^d, 0)$ that has exponential intersection tails. Then for every $p > p_c$ there is a.s. a vertex v in $\mathbf{Z}^d$ such that the cluster $\mathcal{C}(v)$ is transient.*

This next lemma is a refinement of the previous. Levin and Peres used it in [8] to prove Theorem 1.1.

Lemma 2.3 ([8]) *Let μ be a probability measure on Υ_1, the set of paths with unit speed. Suppose that there exists $p \in (0,1)$, $\gamma > 1$ and $C < \infty$, so that for any fixed path ψ containing edge e_l at distance l from v_0,*

$$\int_{\Upsilon_1} p^{-|\varphi \cap \psi|} \mathbf{1}_{\{\varphi \ni e_l\}} d\mu(\varphi) \leq C l^{-\gamma}. \tag{1}$$

If $|\mathcal{C}(0)| = \infty$ then $\mathcal{C}(0)$ supports a nonzero flow f with

$$\mathbf{E}_p \sum_{|e|=l} H_h(f(e)) \leq \frac{C}{l^{\gamma(1/d-1)} h(l^{-\gamma})}.$$

For our purposes it is better to use a variation of Lemma 2.3 which has the same proof.

Lemma 2.4 *Let $K \geq 1$ be a constant. Let $\Upsilon_{1/K}$ be the set of all paths ψ such that for all k, $|\{t : |\psi(t)| = k\}| \leq K$. Let μ be a probability measure on $\Upsilon_{1/K}$, such that that there exists $p \in (0,1)$, $\gamma > 1$ and $C_1 < \infty$, so that for any fixed path ψ containing edge e_l at distance l from v_0,*

$$\int_{\Upsilon_{1/K}} p^{-|\varphi \cap \psi|} \mathbf{1}_{\{\varphi \ni e_l\}} d\mu(\varphi) \leq C_1 l^{-\gamma}. \tag{2}$$

If $|\mathcal{C}(0)| = \infty$ then $\mathcal{C}(0)$ supports a nonzero flow f with

$$\mathbf{E}_p \sum_{|e|=l} H_h(f(e)) \leq \frac{C_2}{l^{\gamma(1/d-1)} h(l^{-\gamma})}.$$

where C_2 depends only on C_1 and K.

3 Construction of the Flow

For notational purposes we consider the case where $d = 3$. Thus $H_h(u) = u^{3/2}/h(u)$. In order to prove the existence of the flow we need to introduce the time reversal of a process $X = \{X_n\}_{n=1}^{\infty}$. The **time reversal** $\overleftarrow{X}$ of X up to time l, started at y, is

$$\overleftarrow{X}_k := y + X_{l-k} - X_l \text{ for } k \in [0, l].$$

Note that each path X corresponds to one path $\overleftarrow{X}$.

Lemma 3.1 *There exists a NN (nearest neighbor) process on* $\mathbf{Z}$ $\{S_t\}_{t=1,\infty}$ *such that for all* t, $|S_t| \leq \max(1, \frac{t^{3/4}}{2})$, *for all* k,

$$\mathsf{PRE}_S(k) \leq Ck^{-3/4}, \tag{3}$$

and for all n, *and* $k \leq n/2$,

$$\mathsf{PRE}_{\{\overleftarrow{S}|S_0,\ldots,S_{n/2}\}}(k) \leq Ck^{-3/4}, \tag{4}$$

where $\overleftarrow{S}$ *is the time reversal of* S *up to time* n *started at some point* y.

Proof: The existence of a NN process S_t which satisfies $|S_t| \leq \max(1, \frac{t^{3/4}}{2})$ and (3) is proven in Proposition 1.5 of [7]. Here we sketch a proof that the processes constructed there also satisfies (4). The processes in [7] was constructed in the following way. We define $\{B(n)\}_{n=0}^{\infty}$ (B as in Boundary) by letting

$$B(n) = 2^j \quad \text{where} \quad j = \begin{cases} 0 & \text{if } f(n) \leq 1 \\ \max\{i \in \{1,2,\ldots\} : 2^i \leq \max(1, \frac{n^{3/4}}{2})\} & \text{otherwise.} \end{cases}$$

and we construct a NN process $S_n = \sum_{i=1}^{n} X_i$ on $\mathbf{Z}$ such that for all $n \geq k$

$$\max_{x,(S_0,\ldots,S_n)} \mathbf{P}[(\sum_{i=n+1}^{n+k} X_i) \bmod B(k) = x \,|\, X_0,\ldots,X_n] \leq Ck^{-3/4}. \tag{5}$$

Next, it is shown that if S is such a process and we take $S^*(n)$ to be the processes obtained from S by reflecting S at time n at the boundary $B(n)$ then S^* satisfies (3). More formally we use an auxiliary $\{-1,1\}$ valued process $\{Z_n\}_{n=0}^{\infty}$ and define inductively $Z_0 = 1, S_0^* = 0$, and

$$\begin{cases} S_n^* = S_{n-1}^* + Z_{n-1}(S_n - S_{n-1}) \\ Z_n = Z_{n-1} \end{cases}$$

unless $S_{n-1}^* + Z_{n-1}(S_n - S_{n-1})$ happens to fall outside of the range $\{-B(n),\ldots,B(n)\}$, in which case we instead let

$$\begin{cases} S_n^* = S_{n-1}^* - Z_{n-1}(S_n - S_{n-1}) \\ Z_n = -Z_{n-1}. \end{cases}$$

The proof of (5) given in [7] extends immediately to show that

$$\max_{x,(S_0,\ldots,S_n)} \mathbf{P}[(\sum_{i=n+1}^{n+k} X_i) \bmod B(k) = x \,|\, X_0,\ldots,X_n,X_{n+k+1},\ldots] \leq Ck^{-3/4}. \tag{6}$$

Now (6) implies (4), as (5) implied (3). □.

Now we use the one dimensional processes to construct a 3-dimensional process.

Lemma 3.2 *For every point $(x,y) \in B = [-.025, .025] \times [-.025, .025]$ there exists a $\mathbf{Z}^3$ valued process $\Phi = \Phi_{(x,y)}$ such that $\mathsf{PRE}_\Phi(k) \le Ck^{-3/2}$ for all k, and for all $n, k \le n/2$,*

$$\mathsf{PRE}_{\{\overleftarrow{\Phi}|\Phi_1,\ldots,\Phi_{n/2}\}}(k) \le Ck^{-3/2}, \tag{7}$$

where $\overleftarrow{\Phi}$ is a time reverse of Φ up to time n started at some point. In addition for all $t \ge T = 100$,

$$\Phi(t) \in [xt - t^{3/4}, xt + t^{3/4}] \times [yt - t^{3/4}, yt + t^{3/4}] \times [\lfloor \frac{2t}{3} \rfloor], \tag{8}$$

and Φ generates a measure on paths μ that is supported on $\Upsilon_{1/6}$.

Proof. Let S be the process constructed in Lemma 3.1. We define $\{S_t^x\}_{t=0}^\infty$ for each $x \in [-.15, .15]$. This is done based on S. Let $\tilde{S}$ be a (deterministic) process such that for all t,

1. $\tilde{S}_{2t} = \tilde{S}_{2t-1}$,
2. $\tilde{S}_{2t+1} - \tilde{S}_{2t} \in \{-1, 1\}$, and
3. $|\tilde{S}_t - 2xt| \le 2$.

(Such a process clearly exists.) Now, we define $S_0^x = 0$ and for all t, $S_{2t+1}^x - S_{2t}^x = \tilde{S}_{2t+1} - \tilde{S}_{2t}$ and $S_{2t}^x - S_{2t-1}^x = S_t - S_{t-1}$. Clearly $S_t^x \in [xt - t^{3/4} + 2, xt + t^{3/4} - 2]$ for all $t \ge 16$, and $\mathsf{PRE}_{S^x}(k) < 2Ck^{-3/4}$.

Define clocks

$$t_1(n) := \lfloor \frac{n+5}{6} \rfloor, t_2(n) := \lfloor \frac{n+2}{6} \rfloor, \text{ and } t_3(n) := \lfloor \frac{2n}{3} \rfloor.$$

Let S' and S'' be independent copies of S and let S^{6x} and S^{6y} be generated from S' and S'' respectively. Write

$$\Phi_{(x,y)}(n) = (S^{6x}_{t_1(n)}, S^{6y}_{t_2(n)}, t_3(n)).$$

It is then easy to see that

$$\mathsf{PRE}_{\Phi_{(x,y)}}(k) \le \left[\mathsf{PRE}_{S^x}(\lfloor \frac{k}{6} \rfloor)\right]^2 \le \left[6Ck^{-3/4}\right]^2 \le C_2 k^{-3/2}.$$

and similarly that Φ satisfies (7). It is clear that the measure Φ generates is supported on $\Upsilon_{1/6}$ and that (8) holds for $t \ge 96$. □

Now we are almost ready to define the the flow F with finite $H_{1+\epsilon}$ energy. If Γ is a collection of edges let $I(\Gamma)$ be the indicator of the event that all the edges in Γ are open in the percolation. Also let $J_e(\Gamma)$ be the indicator of the event $\{e \in \Gamma\}$. Write

$$f_{(x,y),N}(e) = \int_{\Upsilon_{1/6}} p^{-N} I(\varphi_N) J_e(\varphi_N) d\mu_{(x,y)}(\varphi)$$

for edges directed away from 0. The random variables $\{f_{(x,y),N}(e)\}_{N \geq |e|}$ form a martingale and converge almost surely to $f_{(x,y)}(e)$. ¿From arguments in [2] and [7] we see that $f_{(x,y)}(e)$ are positive with positive probability for some e. Finally define

$$F(e) = \int_B f_{(x,y)}(e) dm.$$

This is the flow which allows us to prove Theorem 1.2.

Next we verify the second hypotheses of Lemma 2.4 for $\mu_{(x,y)}$ with $\gamma = 3/2$.

Lemma 3.3 *There exists a constant C_1 such that for any (x,y) the process $\Phi_{(x,y)}$ defines a measure on paths, $\mu_{(x,y)}$, that satisfies the condition in line 2 in Lemma 2.4 with $\gamma = 3/2$.*

PROOF: Fix a path $\psi \in \Upsilon_1$, and let $(e_1, e_2, e_3 \ldots)$ be its constituent edges. Denote by ψ_l the first l edges of ψ and let $\psi(l) = e_l$. Note that $|e_t| < |e_{t+6}|$ for any t, so it suffices to prove (2) for e_l. For a pair of paths write $U(\varphi, \psi, l) := |\varphi \cap \psi| - |\varphi_l \cap \psi|$. Let $\Phi = \Phi_{(x,y)}$ be one of the processes constructed in Lemma 3.2. By Lemma 2.1, Φ has the property that, given the history of the first l steps, the number of subsequent intersections with a fixed trajectory has an exponential tail:

$$\mu[\varphi : U(\varphi, \psi, l) > n \mid \mathcal{F}_l] \leq C_1 \theta^n , \tag{9}$$

where $\mathcal{F}_l$ is the σ-field generated by the random variables $\{\mathbf{1}_{\{e \in \varphi\}} : e(3) \leq 2l/3\}$, and $e(3)$ of an edge that connects (x,y,z) and (x',y',z') is the larger of z and z'.

Our goal is to verify

$$\int_{\Upsilon_{1/6}} p^{-|\varphi \cap \psi|} \mathbf{1}_{\{\varphi \in e_l\}} d\mu(\varphi) \leq C l^{-3/2},$$

for p sufficiently close to 1. We can rewrite the left hand side as

$$E_\mu \left[p^{-|\varphi_l \cap \psi|} \mathbf{1}_{\{\varphi \ni e_l\}} E_\mu[p^{-U(\varphi,\psi,l)} \mid \mathcal{F}_l] \right] .$$

By (9), this is bounded by

$$\frac{C_1}{1-p^{-1}\theta}\sum_{m=1}^{\infty} p^{-m}\mu[|\varphi_l\cap\psi| = m \text{ and } \varphi\ni e_l]. \tag{10}$$

Now we define $B := \{|\varphi_{3l/4}\cap\psi|\geq 3m/4\}$, and $D := \{|(\varphi_l\setminus\varphi_{3l/4})\cap\psi|\geq m/4\}$. Then we have

$$\mu[|\varphi_l\cap\psi| = m \text{ and } \varphi\ni e_l]\leq \mu\left[B\cap\{\varphi\ni e_l\}\right]+\mu\left[D\cap\{\varphi\ni e_l\}\right]. \tag{11}$$

By Lemma 2.1, $\mu[B]\leq C_1\theta^{3m/4}$, and by Lemma 3.2, $\mu[\varphi\ni e_l\mid B]\leq C_2 l^{-3/2}$. Thus

$$\mu\left[B\cap\{\varphi\ni e_l\}\right]=\mu[B]\cdot\mu[\varphi\ni e_l\mid B]\leq C_3\theta^{3m/4}l^{-3/2},$$

It is more difficult to bound $\mu[D\cap\{\varphi\ni e_l\}]$. First we define $D_e = D\cap\{\varphi\ni e\}$. From this we get the equations

$$D\cap\{\varphi\ni e_l\}=\bigcup_{e:e(3)=\lfloor l/3\rfloor} D_e\cap\{\varphi\ni e_l\},$$

and

$$\mu[D_e\cap\{\varphi\ni e_l\}]=\mu[D_e\cap\{\varphi\ni e_l\}\mid\{\varphi\ni e\}]\cdot\mu[\{\varphi\ni e\}].$$

In order to bound the probability of the last event let $\overleftarrow{\varphi}$ be the time-reversal of φ started at e_l. Let $\overleftarrow{D}_e$ be the event that $\{\overleftarrow{\varphi}\}_{t=0,\ldots,l/2}$ intersects the vertices determined by $\psi-\psi_{3l/4}$ at least $m/4$ times, and it intersects e. Since the number of edge intersections is bounded by the number of vertex intersections, we have:

$$\mu[D_e\cap\{\varphi\ni e_l\}\mid\{\varphi\ni e\}]\leq\mathbf{P}[\overleftarrow{D}_e].$$

However, from (7), Lemma 2.1 and Lemma 3.2 we have

$$\mathbf{P}[\overleftarrow{D}_e]\leq C\theta^{m/4}l^{-3/2}.$$

Thus

$$\begin{aligned}
\mu[D\cap\{\varphi\ni e_l\}] &\leq \sum_{e(3)=\lfloor l/3\rfloor}\mu[D_e\cap\{\varphi\ni e_l\}\mid\{\varphi\ni e\}]\cdot\mu[\{\varphi\ni e\}]\\
&\leq \sum_{e(3)=\lfloor l/3\rfloor}\mathbf{P}[\overleftarrow{D}_e]\cdot\mu[\{\varphi\ni e\}]\\
&\leq (C_7{\theta_1}^{m/4}l^{-3/2})\sum_{e(3)=\lfloor l/3\rfloor}\mu[\{\varphi\ni e\}]\\
&\leq 3C_7{\theta_1}^{m/4}l^{-3/2},
\end{aligned}$$

where we have used the fact that (8) implies

$$\sum_{e(3)=\lfloor l/3 \rfloor} \mu[\{\varphi \ni e\}] \le 3.$$

This implies that the right hand side of (11) is bounded by $C_8 {\theta_1}^{m/4} l^{-3/2}$. Thus for $p > {\theta_1}^{1/4}$, the sum (10) is bounded by $Cl^{-3/2}$, and (2) follows. □

PROOF OF THEOREM 1.2: In order to show that F has finite H_h energy we need to show that

$$\mathbf{E}_p \sum_{e \in E(\mathbf{Z}^3)} H_h(F(e)) < \infty.$$

First we sum only over edges e so that $|e| = l \ge T$.

$$\begin{aligned} \mathbf{E}_p \sum_{|e|=l} H_h(F(e)) &= \mathbf{E}_p \sum_{|e|=l} H_h\left(\int_B f_{(x,y)}(e)dm\right) \\ &= \mathbf{E}_p \sum_{|e|=l} H_h\left(\int_{B_e} f_{(x,y)}(e)dm\right), \end{aligned} \tag{12}$$

where B_e is defined as follows. For each edge e let (x_e, y_e) be the point so that the endpoint of e is $(x_e e(3), y_e e(3), e(3))$. Define

$$B_e = [x_e - e(3)^{-1/4}, x_e + e(3)^{-1/4}] \times [y_e - e(3)^{-1/4}, y_e + e(3)^{-1/4}]$$

Line 12 is valid because $f_{(x,y)}(e) = 0$ for any point $(x,y) \notin B_e$.

Jensen's inequality generates the inequality

$$H_h\left(\int_X f d\mu\right) \le (\mu(X))^{1/2}\left(\int_X H_h(f)d\mu\right).$$

Combining this with the fact that for each edge e which F can assign a positive value to and is at distance l from the origin $m(B_e) \le 16l^{-1/2}$ we get that

$$\begin{aligned} \mathbf{E}_p \sum_{|e|=l} H_h(F(e)) &\le \mathbf{E}_p \sum_{|e|=l} (16l^{-1/2})^{1/2}\left(\int_{B_e} H_h(f_{(x,y)}(e))dm\right) \\ &\le \mathbf{E}_p \sum_{|e|=l} 4(l^{-1/4})\left(\int_B H_h(f_{(x,y)}(e))dm\right) \\ &\le 4(l^{-1/4}) \int_B \mathbf{E}_p \sum_{|e|=l} H_h(f_{(x,y)}(e))dm \\ &\le 4(l^{-1/4}) \int_B \frac{C}{h(l^{-3/2})l^{3/4}} dm \qquad (13) \\ &\le \frac{4C}{h(l^{-1})l}. \qquad (14) \end{aligned}$$

Line 13 follows from Lemmas 2.4 and 3.3. Line 14 is true because h is decreasing.

Summing up over l we see that

$$\mathbf{E}_p \sum_e H_h(F(e)) \leq (2T)^3 \max_{t\in[0,p^{-T}]} H_h(t) + \sum_l \frac{2C}{h(l^{-1})l} < \infty$$

and F has finite expected H_h energy. Thus there exists a flow of finite H_h energy on percolation clusters in $\mathbf{Z}^3$ with positive probability. The event that there exists a flow of finite H_h energy does not depend on the status of any finite collection of edges, and hence by Kolmogorov's zero–one law, has probability one.

This concludes the proof for p near 1. The general case $p > p_c$ is reduced to the case of p near 1 by the renormalization argument used in Corollary 2.1 of [2]. This relies on techniques of [6], [1], and [11]. The renormalization also requires a result of Soardi and Yamasaki [12], who proved that the existence of a flow of finite H_h energy is invariant under rough isometries. □

References

[1] P. Antal and A. Pisztora (1996). On the chemical distance in supercritical Bernoulli percolation. *Ann. Probab.* **24** 1036–1048.

[2] I. Benjamini, R. Pemantle and Y. Peres (1998). Unpredictable paths and percolation. *Ann. Probab.*, to appear.

[3] P. G. Doyle and E. J. Snell (1984). *Random walks and electrical networks.* Carus Math. Monographs **22**, Math. Assoc. Amer., Washington, D. C.

[4] G.R. Grimmett (1989). *Percolation*. Springer, New York.

[5] G. R. Grimmett, H. Kesten and Y. Zhang (1993). Random walk on the infinite cluster of the percolation model. *Probab. Th. Rel. Fields* **96**, 33–44.

[6] G. R. Grimmett and J. M. Marstrand (1990). The supercritical phase of percolation is well behaved. *Proc. Royal Soc. London Ser. A* **430**, 439–457.

[7] O. Häggström and E. Mossel (1998). Nearest-neighbor walks with low predictability profile and percolation in $2 + \epsilon$ dimensions. *Ann. Probab.*, to appear.

[8] D. Levin and Y. Peres (1998). Energy and cutsets in infinite percolation clusters. *Proceedings of the Cortona Workshop on Random Walks and Discrete Potential Theory*, M. Picardello and W. Woess (Editors), Cambridge Univ. Press, to appear.

[9] Lyons, T. (1983). A simple criterion for transience of a reversible Markov chain, *Ann. Probab.* **11**, 393–402.

[10] F. Y. Maeda (1977). A remark on the parabolic index of infinite networks. *Hiroshima J. Math.* **7**, 147–152.

[11] A.. Pisztora (1996). Surface order large deviations for Ising, Potts and percolation models. *Probab. Th. Rel. Fields* **104**, 427–466.

[12] P. M. Soardi and M. Yamasaki (1993). Parabolic indices and rough isometries. *Hiroshima J. Math* **23**, 333–342.