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GROWTH RATES FOR MONOTONE SUBSEQUENCES

A. DEL JUNCO AND J. MICHAEL STEELE

Abstract. The growth rate of the largest monotone subsequence of a uniformly distributed sequence is obtained. For $a_n = na \mod 1$ with $a$ algebraic irrational the exponent of growth is found to be precisely the same as for a random sequence.

1. Introduction. A well-known result of Erdős and Szekeres [1] states that any sequence of $n$ real numbers contains a monotone subsequence with at least $n^{1/2}$ elements. More recently, Hammersley [2] proved that if $I_n = I_n(a_1, a_2, \ldots, a_n)$ is the order of the largest increasing subsequence of $a_1, a_2, \ldots, a_n$, and the $a_i$ are chosen independently with the uniform distribution on $[0, 1]$, then

$$\lim_{n \to \infty} n^{-1/2} I_n = C,$$  \hspace{1cm} (1)

where $C$ denotes a constant and the convergence is in probability. This result was strengthened by Kesten [4] to provide almost sure convergence, and Logan and Shepp [6] proved that $C > 2$. Our objective here is to provide results like (1) for sequences which are uniformly distributed in $[0, 1]$, but which are not random. Of particular interest to us is the sequence $a_n = na \mod 1$ where $a$ is an algebraic irrational.

2. Uniformly distributed sequences. We will denote by $1_{[a,b)}(x)$ the indicator function of the interval $[a, b)$ and will say a sequence $(a_n)$ is uniformly distributed in $[0, 1]$ provided for all $0 \leq a < b \leq 1$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_{[a,b)}(a_i) = b - a.$$

The best one can say about the growth rate of $I_n$ for a general uniformly distributed sequence is the following:

Theorem 1. If $(a_n)$ is uniformly distributed, then

$$\lim_{n \to \infty} n^{-1} I_n = 0.$$  \hspace{1cm} (2)

Proof. Let $A$ and $n$ be positive integers and for $0 \leq i < A - 1$ and
By \(|S_j|\) we denote the cardinality of \(S_j\) and we set \(g(n) = \max_{j,j'}|S_{j'}|\). If \(n\) tends to infinity along the subsequence \(n = \gamma n, \gamma = 1, 2, \ldots\), then \(g(n)/n\) is easily seen to converge to \(A^{-2}\) by the uniform distribution of \((a_n)\).

Next let \(S = \{i_1 < i_2 < \cdots < i_j\}\) be any subsequence of \(1, 2, \ldots, n\) such that \(a_{i_1} < a_{i_2} < \cdots < a_{i_j}\). We note that \(S\) intersects at most \(2A - 1\) of the \(S_{ij}\). (One can identify \(a_1, a_2, \ldots, a_n\) with its graph in \(\{1, 2, \ldots, n\} \times [0, 1]\) and view the \(S_{ij}\) as “boxes.”) This observation yields the inequality \(\sum_{j} |S_{ij}| < 2Ag(n)\), and since \(\sum_{j} |S_{ij}| < |S|\) we have \(\lim_{n \to \infty} l_j/n < 2/A\) provided the limit is taken along the subsequence \(n = kA\).

For \(kA < n < (k + 1)A\) we note that

\[
I(a_1, a_2, \ldots, a_n) \leq I(a_1, a_2, \ldots, a_{Ak}) + I(a_{Ak+1}, \ldots, a_n) \\
\leq I(a_1, a_2, \ldots, a_{Ak}) + A.
\]

This proves

\[
\lim_{n \to \infty} \frac{l_j}{n} \leq \lim_{k \to \infty} \frac{(kA + A)}{kA} \leq \frac{2}{A},
\]

which completes the proof of (1), since \(A\) was an arbitrary positive integer.

3. Results concerning \((na)\). To show that \(l_n = o(n)\) is best possible we do not have to go out of the class of sequences \(a_n = na \mod 1\).

**Theorem 2.** Let \(C_n\) be a sequence of real numbers such that \(C_n \to 0\) as \(n \to \infty\); then there is a transcendental \(\alpha\) such that for \(a_n = na \mod 1\) we have

\[
n^{-1}l_n > C_n \text{ for infinitely many } n.
\]

**Proof.** The proof depends on an elementary lower estimate for \(l_n\) in terms of the denominators \(q_k\) of the convergents \(p_k/q_k\) of \(\alpha\). First we assume \(n = q_{k+1}\) and that \(\{q_k\alpha\} > 0\), where \(\{x\} = x - \lfloor x + 1/2\rfloor\). For \(j = S_{q_k}\) the sequence \(ja\) with \(S = 1, 2, \ldots, [q_{k+1}/q_k]\) can be viewed as making small positive steps, so we have the lower bound

\[
l_n > \min(1/\{q_k\alpha\}, q_{k+1}/q_k).\]

By the standard theory of continued fractions (e.g., [3, p. 9]) we have \(|[a_{q_{k+1}}]| < 1/q_{k+1}\), so (4) implies \(l_n > q_{k+1}/q_k\). Since \(C_n \to 0\) we can choose \(q_k\) which go to infinity as rapidly as we like such that \(1/q_k > C_t\) for \(t = q_{k+1}\). In particular, we may require \(q_k\) to grow rapidly enough to ensure that \(\alpha\) is transcendental. Finally, we note that if the condition \(\{q_k\alpha\} > 0\) is not met by infinitely many \(k\), we need only replace \(\alpha\) by \(1 - \alpha\). This will then complete the proof.

There is a more precise result which can be proved if \(\alpha\) is algebraic. To state it succinctly, we let \(l'_n\) denote the order of the largest monotone
THEOREM 3. If \( a_n = n\alpha \mod 1 \) where \( \alpha \) is an algebraic irrational, then
\[
\lim \frac{(\log l'_n)}{(\log n)} = 1/2. \tag{5}
\]

PROOF. We must obtain quantitative versions of the estimates used in Theorem 1. To begin, for \( 0 \leq i \leq n - 1 \) and \( 0 \leq j \leq n - 1 \) we let
\[
S_y = \{a_k: i/n < a_k < (i+1)/n, jn+1 < k < (j+1)n\}
\]
and observe that
\[
\max_{i,j} |S_y| \leq \max_{0 \leq j \leq n-1} \{1 + 2nD'_j\}, \tag{6}
\]
where
\[
D'_j = \sup_{0 < x < 1} \left| n^{-1} \sum_{k = jn+1}^{(j+1)n} l_{(0,x)}(a_k) - x \right|.
\]
Also, if \( S = \{a_1, a_2, \ldots, a_n\} \) is any monotone subsequence of \( \{a_1, a_2, \ldots, a_n\} \), we know \( S \) intersects at most \( 2n - 1 \) of the \( S_y \). Thus, we have
\[
n \leq l''_n \leq 2n \max_{i,j} |S_y|, \tag{7}
\]
where the first inequality follows from the Erdős-Szekeres theorem mentioned in the introduction.

Since the sets \( \{(jn+1)\alpha, (jn+2)\alpha, \ldots, (j+1)n\alpha\}, j = 0, 1, \ldots, n-1, \) are translates of \( \{\alpha, 2\alpha, \ldots, n\alpha\} \), we have
\[
\max_{0 < j < n-1} D'_j = O(D'_1). \tag{8}
\]
By the Thue-Siegel-Roth theorem [5, pp. 122-124] we know that \( D_n = D'_1 = O(n^{-1+\epsilon}) \) for all \( \epsilon > 0 \). This fact, with (7) and (8), yields
\[
\lim_{n \to \infty} \frac{(\log l''_n)}{(\log n)} = 1. \tag{9}
\]
For the final step choose \( n \) so that \( n^2 < j < (n + 1)^2 \) and note \( l''_n < l'_j < l''_n + 2n \). By the bounds on \( j \) and the limit in (9), one completes the proof with a brief computation.

There are two corollaries of the proof of Theorem 3.

COROLLARY 1. If \( \alpha \) is an irrational for which \( D_n = O(n^{-1+\epsilon}) \) for all \( \epsilon > 0 \), then (5) holds. In particular, this is the case if \( \alpha \) is of finite type 1.

COROLLARY 2. For all \( \alpha \) except a set of measure 0, one has (5).

The proof of Corollary 2 depends only on the fact that \( D_n = O(n^{-1+\epsilon}) \) for all \( \epsilon > 0 \) and almost every \( \alpha \). (For more precise results on \( D_n \), see Niederreiter [7].)

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