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Abstract

The Entropy/Influence conjecture, raised by Friedgut and Kalai (1996) [9], seeks to relate two different measures of concentration of the Fourier coefficients of a Boolean function. Roughly saying, it claims that if the Fourier spectrum is “smeared out”, then the Fourier coefficients are concentrated on “high” levels. In this note we generalize the conjecture to biased product measures on the discrete cube, and prove a variant of the conjecture for functions with an extremely low Fourier weight on the “high” levels.

Keywords

entropy, influence, discrete fourier analysis, probabilistic combinatorics

Disciplines

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Comments

At the time of publication, author Elchanan Mossel was affiliated with the University of California, Berkeley. Currently, he is a faculty member at the Statistics Department at the University of Pennsylvania.

A Note on the Entropy/Influence Conjecture

Nathan Keller*, Elchanan Mossel† and Tomer Schlamk‡

May 16, 2011

Abstract

The entropy/influence conjecture, raised by Friedgut and Kalai [8] in 1996, seeks to relate two different measures of concentration of the Fourier coefficients of a Boolean function. Roughly saying, it claims that if the Fourier spectrum is “smeared out”, then the Fourier coefficients are concentrated on “high” levels. In this note we generalize the conjecture to biased product measures on the discrete cube, and prove a variant of the conjecture for functions with an extremely low Fourier weight on the “high” levels.

1 Introduction

Definition 1.1. Consider the discrete cube $\{0, 1\}^n$ endowed with the product measure $\mu_p = (p\delta_{\{1\}} + (1-p)\delta_{\{0\}})^{\otimes n}$, denoted in the sequel by $\{0, 1\}_p^n$, and let $f : \{0, 1\}_p^n \rightarrow \mathbb{R}$. The Fourier-Walsh expansion of f with respect to the measure μ_p is the unique expansion

$$f = \sum_{S \subset \{1, 2, \dots, n\}} \alpha_S u_S,$$

where for any $T \subset \{1, 2, \dots, n\}$,¹

$$u_S(T) = \left(-\sqrt{\frac{1-p}{p}}\right)^{|S \cap T|} \left(\sqrt{\frac{p}{1-p}}\right)^{|S \setminus T|}.$$

In particular, for the uniform measure (i.e., $p = 1/2$), $u_S(T) = (-1)^{|S \cap T|}$. The coefficients α_S are denoted by $\hat{f}(S)$,² and the level of the coefficient $\hat{f}(S)$ is $|S|$.

Properties of the Fourier-Walsh expansion are one of the main objects of study in discrete harmonic analysis. The entropy/influence conjecture, raised by Friedgut and Kalai [8] in 1996, seeks to relate two measures of concentration of the Fourier coefficients (i.e. coefficients of the Fourier-Walsh expansion) of Boolean functions. The first of them is the *spectral entropy*.

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¹Throughout the paper, we identify elements of $\{0, 1\}^n$ with subsets of $\{1, 2, \dots, n\}$ in the natural way.

²Note that since the functions $\{u_S\}_{S \subset \{1, \dots, n\}}$ form an orthonormal basis, the representation is indeed unique, and the coefficients are given by the formula $\hat{f}(S) = \mathbb{E}_{\mu_p}[f \cdot u_S]$.

Definition 1.2. Let $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$ be a Boolean function. The spectral entropy of f with respect to the measure μ_p is

$$\text{Ent}_p(f) = \sum_{S \subset \{1, \dots, n\}} \hat{f}(S)^2 \log \left(\frac{1}{\hat{f}(S)^2} \right),$$

where the Fourier-Walsh coefficients are computed w.r.t. to μ_p .

Note that by Parseval's identity, for any Boolean function we have $\sum_S \hat{f}(S)^2 = 1$, and thus, the squares of the Fourier coefficients can be viewed as a probability distribution on the set $\{0, 1\}^n$. In this notation, the spectral entropy is simply the entropy of this distribution, and intuitively, it measures how much are the Fourier coefficients "smeared out".

The second notion is the *total influence*.

Definition 1.3. Let $f : \{0, 1\}_p^n \rightarrow \{0, 1\}$. For $1 \leq i \leq n$, the influence of the i -th coordinate on f with respect to μ_p is

$$I_i^p(f) = \Pr_{x \sim \mu_p} [f(x) \neq f(x \oplus e_i)],$$

where $x \oplus e_i$ denotes the point obtained from x by replacing x_i with $1 - x_i$ and leaving the other coordinates unchanged.

The total influence of the function f is

$$I_p(f) = \sum_{i=1}^n I_i^p(f).$$

Influences of variables on Boolean functions were studied extensively in the last decades, and have applications in a wide variety of fields, including Theoretical Computer Science, Combinatorics, Mathematical Physics, Social Choice Theory, etc. (see, e.g., the survey [10].) As observed in [9], the total influence can be expressed in terms of the Fourier coefficients:

Observation 1.4. Let $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$. Then

$$I_p(f) = \frac{1}{4p(1-p)} \sum_S |S| \hat{f}(S)^2. \quad (1)$$

In particular, for the uniform measure $\mu_{1/2}$, $I_{1/2}(f) = \sum_S |S| \hat{f}(S)^2$.

Thus, in terms of the distribution induced by the Fourier coefficients, the total influence is (up to normalization) the *expectation of the level* of the coefficients, and it measures the question whether the coefficients are concentrated on "high" levels.

The entropy/influence conjecture asserts the following:

Conjecture 1.5 (Friedgut and Kalai). Consider the discrete cube $\{0, 1\}^n$ endowed with the uniform measure $\mu_{1/2}$. There exists a universal constant c , such that for any n and for any Boolean function $f : \{0, 1\}_{1/2}^n \rightarrow \{-1, 1\}$,

$$\text{Ent}_{1/2}(f) \leq c \cdot I_{1/2}(f).$$

The conjecture, if confirmed, has numerous significant implications. For example, it would imply that for any property of graphs on n vertices, the sum of influences is at least $c(\log n)^2$ (which is tight for the property of containing a clique of size $\approx \log n$). The best currently known lower bound, by Bourgain and Kalai [5], is $\Omega((\log n)^{2-\epsilon})$, for any $\epsilon > 0$.

Another consequence of the conjecture would be an affirmative answer to a variant of a conjecture of Mansour [13] stating that if a Boolean function can be represented by a DNF formula of polynomial size in n (the number of coordinates), then most of its Fourier weight is concentrated on a polynomial number of coefficients (see [14] for a detailed explanation of this application). This conjecture, raised in 1995, is still wide open.

In this note we explore the entropy/influence conjecture in two directions:

Biased measure on the discrete cube. We state a generalization of the conjecture to the product measure μ_p on the discrete cube:

Conjecture 1.6. *There exists a universal constant c , such that for any $0 < p < 1$, for any n and for any Boolean function $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$,*

$$\text{Ent}_p(f) \leq cp \log(1/p) \cdot \text{I}_p(f).$$

We prove that Conjecture 1.6 follows from the original Entropy/Influence conjecture, and that it is tight for the graph property of containing a clique of fixed size (at the critical probability). This answers a question raised by Kalai [11].

Functions with a low Fourier weight on the “high” levels. We consider a weaker version of the conjecture stating that if “almost all” the Fourier weight of a function is concentrated on the lowest k levels, then its entropy is at most $c \cdot k$. We prove this statement in an extreme case:

Proposition 1.7. *Let $f : \{0, 1\}_{1/2}^n \rightarrow \{-1, 1\}$ be a Boolean function such that all the Fourier weight of f is concentrated on the first k levels. Then all the Fourier coefficients of f are of the form*

$$\hat{f}(S) = a(S) \cdot 2^{-k}$$

where $a(S) \in \mathbb{Z}$. In particular, $\text{Ent}_{1/2}(f) \leq 2k$.

Then we cite a stronger unpublished result of Bourgain and Kalai [6] which shows that if the Fourier weight beyond the k th level decays exponentially, then the spectral entropy is bounded from above by $c \cdot k$.

Finally, we suggest that if one could generalize the result of Bourgain and Kalai to *some* slower rate of decay, this would lead to a proof of the entire entropy/influence conjecture, using a *tensorisation* technique.

This note is organized as follows. In Section 2 we consider the generalization of the entropy/influence conjecture to the biased measure on the discrete cube. Functions with a low Fourier weight on the high levels are discussed in Section 3. We conclude the paper with an easy proof of a weaker upper bound on the entropy, and with a connection between the entropy/influence conjecture and Friedgut’s characterization of functions with a low total influence [7] in Section 4.

2 Entropy/Influence Conjecture for the Product Measure μ_p on the Discrete Cube

In this section we consider the space $\{0, 1\}_p^n$, for $0 < p < 1$. First we formulate a variant of the entropy/influence conjecture for the biased measure and prove that it follows from the original conjecture. Then we show that it is tight for the graph property of containing a copy of a complete graph K_r as an induced subgraph, for random graphs distributed according to the model $G(n, p)$, at the critical probability p_c .

Proposition 2.1. *Assume that the entropy/influence conjecture holds. Then there exists a universal constant c such that for any $0 < p < 1$, for any n and for any $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$, we have*

$$\text{Ent}_p(f) \leq cp \log(1/p) \cdot \text{I}_p(f).$$

Our proof is based on a standard reduction from the biased measure μ_p to the uniform measure $\mu_{1/2}$ first considered in [4]. Let $p \leq 1/2$, and assume that $p = t/2^m$.³ For any function $f : \{0, 1\}_p^n \rightarrow \mathbb{R}$ we define a function $\text{Red}(f) = g : \{0, 1\}^{mn} \rightarrow \mathbb{R}$ as follows: each $y \in \{0, 1\}^{mn}$ is considered as a concatenation of n vectors $y^i \in \{0, 1\}^m$, and each such vector is translated to a natural number $0 \leq \text{Bin}(y^i) < 2^m$ through its binary expansion (i.e., $\text{Bin}(y^i) = \sum_{j=0}^{m-1} 2^j \cdot y_{m-j}^i$). Then, for any $y \in \{0, 1\}^{mn}$,

$$g(y) = g(y^1, y^2, \dots, y^n) := f(h(y^1), h(y^2), \dots, h(y^n)),$$

where $h : \{0, 1\}^m \rightarrow \{0, 1\}$ is given by

$$h(y^i) = \begin{cases} 1, & \text{Bin}(y^i) \geq 2^m - t \\ 0, & \text{Bin}(y^i) < 2^m - t. \end{cases}$$

We use two simple properties of the reduction. The first, proved by Friedgut and Kalai [8], relates the total influence of g (w.r.t. $\mu_{1/2}$) to that of f (w.r.t. to μ_p).

Lemma 2.2 (Friedgut and Kalai). *Let $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$, and let $g = \text{Red}(f)$. Then*

$$\text{I}_{1/2}(g) \leq 6p \lceil \log(1/p) \rceil \text{I}_p(f). \quad (2)$$

The second property relates the Fourier coefficients of f (w.r.t. μ_p) to corresponding coefficients of g (w.r.t. $\mu_{1/2}$).

Lemma 2.3. *Let $f : \{0, 1\}_p^n \rightarrow \mathbb{R}$, and let $g = \text{Red}(f)$. For any $S \subset \{1, 2, \dots, mn\}$, denote $S_i = S \cap \{(i-1)m+1, (i-1)m+2, \dots, im\}$, and for $S' \subset \{1, 2, \dots, n\}$, let*

$$V(S') = \{S \subset \{1, 2, \dots, mn\} : \{i : |S_i| > 0\} = S'\}.$$

Then:

$$\sum_{S \in V(S')} \hat{g}(S)^2 = \hat{f}(S')^2. \quad (3)$$

³It is clear that there is no loss of generality in assuming that p is dyadic, as the results for general p follow immediately by approximation.

Proof: For each $S' \subset \{1, 2, \dots, n\}$, let $f_{S'} : \{0, 1\}_p^n \rightarrow \mathbb{R}$ be defined by $f_{S'} = \hat{f}(S')u_{S'}$. We claim that

$$\text{Red}(f_{S'}) = \sum_{S \in V(S')} \hat{g}(S)u_S. \quad (4)$$

This claim implies the assertion, as by the Parseval identity, Equation (4) implies:

$$\sum_{S \in V(S')} \hat{g}(S)^2 = \|\text{Red}(f_{S'})\|_2^2 = \|f_{S'}\|_2^2 = \hat{f}(S')^2.$$

(The first and third equalities use the Parseval identity, and the middle equality holds since by the structure of the reduction, it preserves all L_p norms.)

In order to prove Equation (4), we use Proposition 2.2 in [12] that describes the exact relation between the Fourier coefficients of $\text{Red}(f)$ and the corresponding coefficients of f . By the proposition, for all $S \in V(S')$,

$$\widehat{\text{Red}(f)}(S) = c(S, p) \cdot \hat{f}(S'),$$

where $c(S, p)$ depends on S and p but not on f . Hence, for all $S \in V(S')$, we have $\widehat{\text{Red}(f_{S'})}(S) = \widehat{\text{Red}(f)}(S)$ (since both are determined by S, p , and $\hat{f}(S')$). Similarly, for all $S \notin V(S')$, $\widehat{\text{Red}(f_{S'})}(S) = 0$, since $\hat{f}_{S'}(S'') = 0$ for all $S'' \neq S'$. Therefore, the Fourier expansion of $\text{Red}(f_{S'})$ is:

$$\text{Red}(f_{S'}) = \sum_{S \in V(S')} \widehat{\text{Red}(f)}(S)u_S,$$

as asserted. \square

Now we are ready to prove Proposition 2.1.

Proof: Let $f : \{0, 1\}_p^n \rightarrow \{-1, 1\}$, and let $g = \text{Red}(f)$. By Equation (3),

$$\begin{aligned} \text{Ent}_{1/2}(g) &= \sum_{S \subset \{1, 2, \dots, mn\}} \hat{g}(S)^2 \log \frac{1}{\hat{g}(S)^2} = \sum_{S' \subset \{1, 2, \dots, n\}} \sum_{S \in V(S')} \hat{g}(S)^2 \log \frac{1}{\hat{g}(S)^2} \\ &\geq \sum_{S' \subset \{1, 2, \dots, n\}} \sum_{S \in V(S')} \hat{g}(S)^2 \log \frac{1}{\hat{f}(S')^2} = \sum_{S' \subset \{1, 2, \dots, n\}} \hat{f}(S')^2 \log \frac{1}{\hat{f}(S')^2} = \text{Ent}_p(f). \end{aligned} \quad (5)$$

Combining Equation (5) with Equation (2) and applying the entropy/influence conjecture to g , we get:

$$\text{Ent}_p(f) \leq \text{Ent}_{1/2}(g) \leq c \cdot \text{I}_{1/2}(g) \leq c \cdot 6p[\log(1/p)]\text{I}_p(f),$$

and therefore,

$$\text{Ent}_p(f) \leq c'p \log(1/p)\text{I}_p(f),$$

as asserted. \square

Consider the random graph model $G(n, p)$. Recall that in this model, the probability space is $\{0, 1\}_p^N$, where $N = \binom{n}{2}$, the coordinates correspond to the edges of a graph on n vertices, and each edge exists in the graph with probability p , independently of the other edges. It is well-known that for the graph property of containing the complete graph K_r as an induced subgraph, there exists a threshold at $p_t = \Theta(n^{-2/(r-1)})$. This means that if $p \ll n^{-2/(r-1)}$ then

$\Pr[K_r \subset G | G \in G(n, p)]$ is close to zero, and if $p \gg n^{-2/(r-1)}$ then $\Pr[K_r \subset G | G \in G(n, p)]$ is close to one. We choose a value p_0 in the critical range, consider the characteristic function f of this graph property in $G(n, p_0)$, and show that the assertion of Proposition 2.1 is tight for f . In order to simplify the computation, we choose p_0 such that the expected number of copies of K_r in $G(n, p_0)$ is “nice”. However, the same argument holds for any value of p in the critical range.

Proposition 2.4. *Let n, r be integers such that $r < \log n$. Consider the random graph $G(n, p_0)$ where p_0 is chosen such that $\binom{n}{r} \cdot p_0^{\binom{r}{2}} = 1/2$. Let f be defined by:*

$$f(G) = 1 \Leftrightarrow G \text{ contains a copy of } K_r \text{ as an induced subgraph,}$$

and $f(G) = 0$ otherwise. Then

$$\text{Ent}_{p_0}(f) \geq c \cdot p_0 \log(1/p_0) \cdot \mathbb{I}_{p_0}(f),$$

where c is a universal constant.

Proof: The result is a combination of an upper bound on $\mathbb{I}_{p_0}(f)$ with a lower bound on $\text{Ent}_{p_0}(f)$.

In order to bound $\mathbb{I}_{p_0}(f)$ from above, note that a necessary (but not sufficient) condition for an edge $e = (v, w)$ to be pivotal for f at a graph G ⁴ is that there exists a set S of r vertices including v and w such that all $\binom{r}{2}$ edges inside S except for e appear in G . Hence, a simple union bound yields that for any edge e ,

$$\mathbb{I}_e^{p_0}(f) \leq \binom{n-2}{r-2} \cdot p_0^{\binom{r}{2}-1} = \frac{r(r-1)}{n(n-1)p_0} \cdot \binom{n}{r} p_0^{\binom{r}{2}} = \frac{r(r-1)}{2n(n-1)p_0},$$

and thus,

$$\mathbb{I}_{p_0}(f) = \sum_e \mathbb{I}_e^{p_0}(f) \leq \frac{1}{p_0} \cdot \frac{r(r-1)}{4}. \quad (6)$$

In order to bound $\text{Ent}_{p_0}(f)$ from below, we show that at least a constant portion of the Fourier weight of f is concentrated on coefficients that correspond to copies of K_r in $\{0, 1\}^N$. Concretely, we show that if S corresponds to a copy of K_r , then:

$$\hat{f}(S)^2 \geq c' \cdot \binom{n}{r}^{-1}. \quad (7)$$

As the number of such coefficients is $\binom{n}{r}$, it will follow that:

$$\text{Ent}_{p_0}(f) \geq \sum_{\{S: S \text{ is a copy of } K_r\}} \hat{f}(S)^2 \log \left(\frac{1}{\hat{f}(S)^2} \right) \geq c' \cdot \log \binom{n}{r} \geq c'' \cdot r \log(n), \quad (8)$$

where the rightmost inequality holds since $r < \log n$. Finally, a combination of Equation (8) with Equation (6) will imply:

$$\text{Ent}_{p_0}(f) \geq c'' \cdot r \log(n) \geq c'' \cdot \frac{r(r-1)}{2} \cdot \log(1/p_0) \geq c'' \cdot p_0 \log(1/p_0) \cdot \mathbb{I}_{p_0}(f),$$

⁴An edge e is pivotal for the property f at a graph G if $f(G) = 1$ and $f(G \setminus \{e\}) = 0$.

as asserted.

To prove Equation (7), consider a specific copy H of K_r and denote its set of edges by $S = E(H)$. By the definition of the Fourier coefficients, we have:

$$\begin{aligned}
\hat{f}(S) &= \sum_{T \in \{0,1\}^N} \mu_{p_0}(T) \left(-\sqrt{\frac{1-p_0}{p_0}} \right)^{|S \cap T|} \left(\sqrt{\frac{p_0}{1-p_0}} \right)^{\binom{r}{2} - |S \cap T|} f(T) \\
&= \sum_{T \in \{0,1\}^N} \mu_{p_0}(T \setminus S) \cdot (p_0(1-p_0))^{\binom{r}{2}/2} (-1)^{|S \cap T|} f(T) \\
&= (p_0(1-p_0))^{\binom{r}{2}/2} \sum_{T \in \{0,1\}^N} \mu_{p_0}(T \setminus S) (-1)^{|S \cap T|} f(T),
\end{aligned} \tag{9}$$

where $\mu_{p_0}(T \setminus S)$ denotes the induced measure of the graph $T \setminus S$. Note that the total contribution to $\hat{f}(S)$ of $\{T \in \{0,1\}^N : S \subset T\}$ is

$$(-1)^{|S|} \cdot (p_0(1-p_0))^{\binom{r}{2}/2}, \tag{10}$$

since $f(T) = 1$ for all $T \supset S$. On the other hand, if $f(T) = 1$ and $T \supsetneq S$, then T contains a copy of K_r , in which $k \leq r-1$ vertices are included in $V(H)$, and the remaining $r-k$ vertices are not included in $V(H)$. Hence, the total contribution to $\hat{f}(S)$ of $\{T \in \{0,1\}^N : S \subsetneq T\}$ is bounded from above (in absolute value) by:

$$(p_0(1-p_0))^{\binom{r}{2}/2} \cdot \sum_{k=0}^{r-1} \binom{n}{r-k} \binom{r}{k} p_0^{\binom{r}{2} - \binom{k}{2}} = (p_0(1-p_0))^{\binom{r}{2}/2} \cdot (1/2 + o_n(1)), \tag{11}$$

since for our choice of p_0 , the term corresponding to $k=0$ equals $\binom{n}{r} p_0^{\binom{r}{2}} = 1/2$, and the other terms are negligible. Combining estimates (10) and (11), we get:

$$\hat{f}(S)^2 \geq (1 - 1/2 - o_n(1))^2 (p_0(1-p_0))^{\binom{r}{2}} \geq c p_0^{\binom{r}{2}} = (c/2) \cdot \binom{n}{r}^{-1}. \tag{12}$$

This completes the proof. \square

We conclude this section by noting that if p is inverse polynomially small as a function of n , then one can easily prove a statement which is only slightly weaker than the entropy/influence conjecture. In [14] it was shown that with respect to the uniform measure, we have

$$\text{Ent}_{1/2}(f) \leq (\log n + 1) \text{I}_{1/2}(f) + 1,$$

for any Boolean f . The statement generalizes easily to a general biased measure μ_p , and yields the following:

Claim 2.5. *There exists a universal constant c such that for any $0 < p < 1$, for any n and for any $f : \{0,1\}_p^n \rightarrow \{-1,1\}$, we have*

$$\text{Ent}_p(f) \leq cp(1-p) \log(n) \cdot \text{I}_p(f).$$

Proof: Assume w.l.o.g. that $p \leq 1/2$. As shown in [14], we have:

$$\text{Ent}_p(f) \leq (\log n + 1) \sum_S |S| \hat{f}(S)^2 + \epsilon \log(1/\epsilon) + 2\epsilon,$$

where $1 - \epsilon = \hat{f}(\emptyset)^2$. (Note that this part of the proof of Theorem 3.2 in [14] holds without any change for the biased measure). In order to bound the term $\epsilon \log(1/\epsilon) + 2\epsilon$, it was shown in Proposition 3.6 of [14] that by the edge isoperimetric inequality on the cube, it is bounded from above by $2I_{1/2}(f)$. By Equation (2), this implies that for the measure μ_p , we have

$$\epsilon \log(1/\epsilon) + 2\epsilon \leq 12p \lceil \log(1/p) \rceil I_p(f)$$

(since the reduction from the biased measure to the uniform measure preserves the expectation). Thus, by Equation (1),

$$\text{Ent}_p(f) \leq (\log n + 1) \cdot 4p(1 - p)I_p(f) + 12p \lceil \log(1/p) \rceil I_p(f) \leq cp(1 - p) \log(n)I_p(f),$$

as asserted. \square

For p that is inverse polynomially small in n , the statement of Claim 2.5 differs from the assertion of the entropy/influence conjecture only by a constant factor.

3 Functions with a Low Fourier Weight on the High Levels

In this section we consider the uniform measure $\mu_{1/2}$ on the discrete cube, and study Boolean functions with a low Fourier weight on the high levels. In order to simplify the expression of the Fourier expansion, we replace the domain by $\{-1, 1\}^n$. As a result, the characters are given by the formula

$$u_{\{i_1, \dots, i_r\}}(x) = x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_r},$$

and thus, the Fourier expansion of a function is simply its representation as a multivariate polynomial.

Proposition 3.1. *Let $f : \{-1, 1\}_{1/2}^n \rightarrow \mathbb{Z}$, such that all the Fourier weight of f is concentrated on the first k levels. Then all the Fourier coefficients of f are of the form*

$$\hat{f}(S) = a(S) \cdot 2^{-k},$$

where $a(S) \in \mathbb{Z}$. In particular, $\text{Ent}_{1/2}(f) \leq 2k$.

Proof: The proof is by induction on k . The case $k = 0$ is trivial. Assume that the assertion holds for all $k \leq d - 1$, and let f be a function of Fourier degree d (i.e., all its Fourier coefficients are concentrated on the d lowest levels). For $1 \leq i \leq n$, let f^i be the discrete derivative of f with respect to the i th coordinate, i.e.,

$$f^i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \frac{f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_n)}{2}.$$

It is easy to see that if $f = \sum_S \hat{f}(S)u_S$, then the Fourier expansion of f^i is given by:

$$f^i = \sum_{S \subset (\{1, 2, \dots, n\} \setminus \{i\})} \hat{f}(S \cup \{i\})u_S. \quad (13)$$

Hence, f^i is of Fourier degree at most $d - 1$. Note that by the definition of f^i , we have $2f^i(x) \in \mathbb{Z}$ for all $x \in \{-1, 1\}^{n-1}$, and thus by the induction hypothesis, the Fourier coefficients of f^i satisfy $2\hat{f}^i(S) = a(S) \cdot 2^{-d+1}$, where $a(S) \in \mathbb{Z}$. This holds for any $1 \leq i \leq n$, and therefore, by Equation (13), all the Fourier coefficients of f (except, possibly, for $\hat{f}(\emptyset)$), are of the form $\hat{f}(S) = a(S) \cdot 2^{-d}$, where $a(S) \in \mathbb{Z}$. Finally, $\hat{f}(\emptyset)$ must be also of this form, since otherwise $f(x)$ cannot be an integer. This completes the proof. \square

In an unpublished work [6], Bourgain and Kalai obtained a stronger result:

Theorem 3.2. *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, and assume that there exist $c_0 > 0$, $0 < a < 1/2$, and k , such that for all t ,*

$$\sum_{\{S:|S|>t\}} \hat{f}(S)^2 \leq e^{c_0 k} \cdot e^{-at},$$

then for any $\alpha > 1$, there exists a set B_α , such that:

1. $\log |B_\alpha| \leq C \cdot \alpha k$, where C depends only on a and c_0 .
2. $\sum_{S \notin B_\alpha} \hat{f}(S)^2 \leq n^{-\alpha}$.

The theorem asserts that if the Fourier weight of f beyond the k th level decays exponentially, then most of the Fourier weight of f is concentrated on $\exp(Ck)$ coefficients, and thus, $\text{Ent}_{1/2}(f) \leq C'k$ (for an appropriate choice of C'). The proof uses the d th discrete derivative of f (like our proof above), and the Bonami-Beckner hypercontractive inequality [2, 1]. We note that the exact dependence of C on a (i.e., the rate of the exponential decay) in the assertion of the theorem, which is important if a is allowed to be a function of n , is of order $C = \Theta(a^{-1} \log(a^{-1}))$.

A tensorisation technique. In [11], Kalai observed that the entropy/influence conjecture tensorises, in the following sense. For $f : \{-1, 1\}_{1/2}^l \rightarrow \{-1, 1\}$ and $g : \{-1, 1\}_{1/2}^m \rightarrow \{-1, 1\}$, define $f \otimes g : \{-1, 1\}_{1/2}^{l+m} \rightarrow \{-1, 1\}$ by:

$$f \otimes g(x_1, \dots, x_{l+m}) = f(x_1, \dots, x_l) \cdot g(x_{l+1}, \dots, x_{l+m}).$$

Furthermore, let

$$f^{\otimes N} = f \otimes f \otimes \dots \otimes f,$$

where the tensorisation is performed N times. It is easy to see that $\text{I}_{1/2}(f^{\otimes N}) = N \cdot \text{I}_{1/2}(f)$ and $\text{Ent}_{1/2}(f^{\otimes N}) = N \cdot \text{Ent}_{1/2}(f)$. Hence, proving the entropy/influence conjecture for any “tensor power” of f is equivalent to proving the conjecture for f itself. This observation was used in [14] to deduce that it is sufficient to prove a seemingly weaker version of the conjecture: $\text{Ent}_{1/2}(f) \leq c\text{I}_{1/2}(f) + o(n)$, where n is the number of variables.

We observe that tensorisation can be used to enhance the rate of decay of the Fourier coefficients. By the Law of Large Numbers, as $N \rightarrow \infty$, the level of the Fourier coefficients of $f^{\otimes N}$ is concentrated around its expectation, which is $N \cdot \text{I}_{1/2}(f)$, and the rate of decay above that level, i.e., $\sum_{|S|>t} \widehat{f^{\otimes N}}(S)^2$ becomes “almost” inverse exponential in t . This holds even if the rate of decay of the Fourier coefficients of f is much slower (like the Majority function, for which $\sum_{|S|>t} \widehat{MAJ}(S)^2 \approx t^{-1/2}$). Therefore, if one obtains a result similar to Bourgain-Kalai’s Theorem 3.2 for a slower rate of decay, e.g., under the weaker assumption $\sum_{|S|>t} \hat{f}(S)^2 \leq e^{c_0 \sqrt{k}} \cdot e^{-a\sqrt{t}}$, then the result can be enhanced to any rate of decay, by tensoring the function to itself until its rate of decay reaches $e^{-a\sqrt{t}}$. However, we weren’t able to find such generalization of the Bourgain-Kalai result.

4 Concluding Remarks

We conclude this paper with two remarks related to the entropy/influence conjecture.

A weaker upper bound on the entropy that can be proved easily. As mentioned in Section 2, it was shown in [14] that with respect to the uniform measure, one can easily prove the following weaker upper bound on the entropy of any Boolean function:

$$\text{Ent}_{1/2}(f) \leq (\log n + 1)\text{I}_{1/2}(f) + 1.$$

We provide an independent proof of a slightly stronger claim.

Claim 4.1. *For any n and for any $f : \{0, 1\}_{1/2}^n \rightarrow \mathbb{R}$, we have*

$$\text{Ent}_{1/2}(f) \leq \sum_{i=1}^n h(\text{I}_i^{1/2}(f)) \leq 2\text{I}_{1/2}(f)(\log n - \log \text{I}_{1/2}(f)),$$

where

$$h(x) = -x \log x - (1 - x) \log(1 - x).$$

Proof: As the proof deals only with the uniform measure on the discrete cube, we write $\text{Ent}(f)$ and $\text{I}(f)$ instead of $\text{Ent}_{1/2}(f)$ and $\text{I}_{1/2}(f)$ during the proof.

Let $S \subset \{1, 2, \dots, n\}$ be chosen according to the Fourier distribution (i.e., $\Pr[S = S_0] = \hat{f}(S_0)^2$), and let $X_i = 1_{\{i \in S\}}$. Then by the basic rules of entropy,

$$\text{Ent}(f) = H(S) = H(X_1, \dots, X_n) \leq \sum_{i=1}^n H(X_i) = \sum_{i=1}^n h(\text{I}_i(f)),$$

thus obtaining the first inequality. Note that if $\text{I}_i(f) \geq 0.5$, then $h(\text{I}_i(f)) \leq 2\text{I}_i(f)$, and otherwise, $h(\text{I}_i(f)) \leq -2\text{I}_i(f) \log \text{I}_i(f)$. Therefore,

$$\begin{aligned} \frac{1}{2}\text{Ent}(f) &\leq \text{I}(f) + \sum_{i=1}^n \text{I}_i(f)(-\log \text{I}_i(f)) \\ &= \text{I}(f) \left(1 + \sum_{i=1}^n \frac{\text{I}_i(f)}{\text{I}(f)} \cdot \left(-\log \frac{\text{I}_i(f)}{\text{I}(f)} - \log \text{I}(f) \right) \right). \end{aligned}$$

We note that the expression $\sum_{i=1}^n \text{I}_i(f)/\text{I}(f)(-\log \text{I}_i(f)/\text{I}(f))$ is the entropy of the random variable Y defined by $\Pr[Y = i] = \text{I}_i(f)/\text{I}(f)$ which is supported on $\{1, 2, \dots, n\}$, and is therefore bounded by $\log n$. We thus conclude that

$$\frac{1}{2}\text{Ent}(f) \leq \text{I}(f)(1 + \log n - \log \text{I}(f))$$

as asserted. \square

It is easy to see that the bound using the entropy is stronger in some cases, in particular when there is variability in the influences of different coordinates.

We note that the proof does not use the fact that f is Boolean and indeed it could not provide a proof of the Entropy/Influence conjecture, as can be seen, e.g., for the majority function, where $\text{I}_{1/2}(f)$ is of order \sqrt{n} while $\sum_{i=1}^n h(\text{I}_i^{1/2}(f))$ is of order $\sqrt{n} \log n$.

Relation to Friedgut’s characterization of functions with a low influence sum. In [7], Friedgut showed that any Boolean function $f : \{0, 1\}_p^n \rightarrow \{0, 1\}$ essentially depends on at most $C(p)^{I(f)}$ coordinates, where $C(p)$ depends only on p . The main step of the proof is to show that most of the Fourier weight of the function is concentrated on sets that contain one of these coordinates. A stronger claim one may hope to prove is that most of the Fourier weight is concentrated on at most $C(p)^{I(f)}$ coefficients. Formally, we raise the following conjecture that resembles the assertion of Bourgain-Kalai’s theorem:

Conjecture 4.2. *For any $0 < p < 1$, there exists a constant $C(p) > 0$ such that for any $\epsilon > 0$, for any n and for any $f : \{-1, 1\}_p^n \rightarrow \{-1, 1\}$, there exists a set $B_\epsilon \subset \{0, 1\}^n$ such that:*

1. $\log |B_\epsilon| \leq C(p) \cdot I(f)$, and
2. $\sum_{S \notin B_\epsilon} \hat{f}(S)^2 < \epsilon$.

This conjecture is clearly stronger than Friedgut’s theorem and even implies a variant of Mansour’s conjecture [13] (since as shown in [3], if a Boolean function f can be represented by an m -term DNF, then $I_{1/2}(f) = O(\log m)$), but it still does not imply the entropy/influence conjecture, since the remaining Fourier coefficients (whose total Fourier weight is at most ϵ) can still contribute $n \cdot \epsilon$ to $\text{Ent}_{1/2}(f)$.

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