Potts-model formulation of the random resistor network

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The randomly diluted resistor network is formulated in terms of an n-replicated s-state Potts model with a spin-spin coupling constant J in the limit when first n, then s, and finally $1/J$ go to zero. This limit is discussed and to leading order in $1/J$ the generalized susceptibility is shown to reproduce the results of the accompanying paper where the resistor network is treated using the xy model. This Potts Hamiltonian is converted into a field theory by the usual Hubbard-Stratonovich transformation and thereby a renormalization-group treatment is developed to obtain the corrections to the critical exponents to first order in $\epsilon=6-d$, where d is the spatial dimensionality. The recursion relations are shown to be the same as for the xy model. Their detailed analysis (given in the accompanying paper) gives the resistance crossover exponent as $\phi_1=1+\epsilon/42$, and determines the critical exponent, t for the conductivity of the randomly diluted resistor network at concentrations, p, just above the percolation threshold: $t=(d-2)\nu+\phi_1$, where ν is the critical exponent for the correlation length at the percolation threshold. These results correct previously accepted results giving $\phi=1$ to all orders in ϵ. The new result for ϕ_1 removes the paradox associated with the numerical result that $t>1$ for $d=2$, and also shows that the Alexander-Orbach conjecture, while numerically quite accurate, is not exact, since it disagrees with the ϵ expansion.

I. INTRODUCTION

The properties of the randomly diluted resistor network has received considerable attention over the past several years.[1-8] In the usual model one associates a finite nonzero conductance σ with occupied bonds and zero conductance with unoccupied bonds. Each bond is randomly occupied with probability p and unoccupied with probability $1-p$. Consider the resistance $R(\mathbf{x},\mathbf{x}')$ between two terminals at the sites $\mathbf{x}$ and $\mathbf{x}'$. If the sites $\mathbf{x}$ and $\mathbf{x}'$ are in different clusters, this quantity will be infinite. Even if the two sites are in the same cluster, $R(\mathbf{x},\mathbf{x}')$ will still be described by a (conditional) probability distribution. For p near the percolation threshold, p_c, this distribution function may be characterized by an infinite set of crossover exponents ϕ_k, $k=1, 2, 3, \ldots$. In mean-field theory, i.e., for spatial dimension $d>6$, $\phi_k=1$ for all k. For $d=6-\epsilon\lesssim 6$ these exponents may be calculated in two apparently very different formalisms.[9,10] In the accompanying paper[11] we give the details of such calculations[12] based on consideration of a random xy model, following the technique introduced by Stephen.[10] Here we describe the calculations based on a modification of the connection found by Kasteleyn and Fortuin[13] between the resistor network and the $s\to 0$ limit of the s-state Potts model. We show that the renormalization-group recursion relations obtained using the Potts model are identical to those obtained and fully analyzed in the accompanying paper for the xy model. To avoid undue repetition their analysis is not reproduced here. The results of this paper were summarized previously.[14]

A brief discussion placing these calculations in the proper context is given in the introduction of the accompanying paper, and will not be repeated here. However, some comments specific to the technique based on the Potts model are in order. In the original analysis of Kasteleyn and Fortuin[13] and in our succeeding work, use has been made of the connection between the resistance between two terminals in a network and the generating function for spanning trees on that network. Of course, in the presence of random dilution, one obtains the proper quenched average by considering the $n\to 0$ limit of the appropriate n-replicated Hamiltonian. The most important modification[9] of the development from that presented by Kasteleyn and Fortuin,[13] is to replace the $\sqrt{s}$ in their formulation by suitable limiting operations on s and on the coupling constant J appearing in the Hamiltonian. These changes facilitate the development of a field-theoretic Hamiltonian upon which the appropriate limiting processes can readily be performed.

Briefly, this paper is organized as follows. In Sec. II we discuss in some detail the formalism whereby the appropriate quenched average of the arbitrary moments of the resistance between two terminals can be obtained using the replica formalism for the s-state Potts model. In particular, we show that it is important to take the limits in the order, $n\to 0$ first, then $s\to 0$, and finally $J\to\infty$. In Sec. III this model is converted into a field theory using a Hubbard-Stratonovich transformation. The recursion relation from which the crossover exponents are obtained are derived in Sec. IV and are shown to be equivalent, for $J\to\infty$ to those in the accompanying paper,[11] where their consequences are analyzed in detail. Finally, Sec. V contains discussion and concluding remarks.

II. FORMULATION

In this section we derive the Potts-model Hamiltonian from which one can obtain the resistance between two sites in a random resistor network. This formulation[9] is a modification in two respects of that given originally by Kasteleyn and Fortuin.[13] First of all, we introduce the replicated Hamiltonian to treat the quenched disorder in the resistor network. Secondly, our treatment of the s-state Potts model in the limit $s \to 0$ is more suitable for setting up a field theory than was the original formulation.

We start by discussing the Hamiltonian of a Potts model in which there is associated with each site $\mathbf{x}$ a set of vectors $\{\mathbf{e}^\tau : \tau = 1,2,\ldots,s\}$ corresponding to the s different directions from the center of the $(s-1)$-dimensional simplex (or regular multihedron) to its s vertices. For later convenience these vectors are normalized to have length $(s-1)^{1/2}$ and therefore obey the relations[15]

$$\mathbf{e}^\tau \cdot \mathbf{e}^{\tau'} = s\delta_{\tau,\tau'} - 1 , \tag{2.1}$$

where δ is the Kronecker delta. The interaction between two sites $\mathbf{x}$ and $\mathbf{x}'$, denoted $h_{\mathbf{x},\mathbf{x}'}$ is given by

$$h_{\mathbf{x},\mathbf{x}'} = -J\sigma_{\mathbf{x},\mathbf{x}'} \mathbf{e}^{\tau(\mathbf{x})} \cdot \mathbf{e}^{\tau(\mathbf{x}')} \tag{2.2}$$

$$= -J\sigma_{\mathbf{x},\mathbf{x}'} \left[s\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')} - 1 \right] . \tag{2.3}$$

As we will discuss below, J^{-1} is an expansion parameter and $\sigma_{\mathbf{x},\mathbf{x}'}$ is the conductance between sites $\mathbf{x}$ and $\mathbf{x}'$ in the associated resistor network. The interaction of Eq. (2.3) is such that a pair of neighboring spins can have either of two energies: if they are in the same state, their energy is $J\sigma_{\mathbf{x},\mathbf{x}'}(1-s)$, and if they are in different states, their energy is $J\sigma_{\mathbf{x},\mathbf{x}'}$. Thus the energy gap between the ground state of a pair of spins and the excited state is $sJ\sigma_{\mathbf{x},\mathbf{x}'}$. In what follows we will see that the existence of a gap *for all* $s \neq 0$ has crucial effects on the cross-over phenomena to be expected at the percolation threshold. Roughly speaking, the existence of a gap is typified by Ising symmetry, whereas the nonexistence of a gap is typified by xy or Heisenberg symmetry. The fact that this difference is correctly reflected in our mathematical formulation is an indication that our procedures are valid.

Since we will be interested in the partition function and thermal correlation functions of this Potts model, we note the following relations

$$e^{-h_{\mathbf{x},\mathbf{x}'}} = e^{-J\sigma_{\mathbf{x},\mathbf{x}'}} e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}} \tag{2.4a}$$

$$= e^{-J\sigma_{\mathbf{x},\mathbf{x}'}} [1 + (e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}} - 1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}] . \tag{2.4b}$$

Now we consider some *arbitrary* assembly Γ of conductances. In general, this network will consist of independent clusters (denoted $\gamma_1, \gamma_2, \ldots$, etc.) of sites, such that voltages and currents in one cluster are completely independent of those in other clusters. This situation is obviously relevant for a discussion of the randomly dilute resistor network. The Potts model associated with this network can be written as

$$H(\Gamma) = \sum_{\gamma \in \Gamma} H(\gamma) , \tag{2.5}$$

where the Hamiltonian for the cluster γ is

$$H(\gamma) = -J \sum_{\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma} \sigma_{\mathbf{x},\mathbf{x}'} \mathbf{e}^{\tau(\mathbf{x})} \cdot \mathbf{e}^{\tau(\mathbf{x}')} , \tag{2.6}$$

where $\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma$ means that the bond connecting site $\mathbf{x}$ and $\mathbf{x}'$ is in the set γ. For an isolated site, i.e. for a cluster consisting of a single site, the sum in Eq. (2.6) is empty and $H(\gamma) = 0$ in this case. We now consider the correlation function

$$\langle \mathbf{e}^{\tau(\mathbf{y})} \cdot \mathbf{e}^{\tau(\mathbf{y}')} \rangle_{H(\Gamma)} \equiv g_\Gamma(\mathbf{y},\mathbf{y}') , \tag{2.7}$$

where $\langle \cdots \rangle_H$ denotes a statistical average with weight $\exp(-H)$. There are two cases to consider, depending on whether or not $\mathbf{y}$ and $\mathbf{y}'$ are in the same cluster. If they are not, then the vectors $\mathbf{e}^{\tau(\mathbf{y})}$ and $\mathbf{e}^{\tau(\mathbf{y}')}$ are uncorrelated and $g_\Gamma(\mathbf{y},\mathbf{y}')$ vanishes. If $\mathbf{y}$ and $\mathbf{y}'$ are in the same cluster γ, then the presence of other clusters is irrelevant and we have

$$g_\Gamma(\mathbf{y},\mathbf{y}') = \langle \mathbf{e}^{\tau(\mathbf{y})} \cdot \mathbf{e}^{\tau(\mathbf{y}')} \rangle_{H(\gamma)} = g_\gamma(\mathbf{y},\mathbf{y}') . \tag{2.8}$$

Explicitly, this is

$$g_\gamma(\mathbf{y},\mathbf{y}') = \frac{\mathrm{Tr} e^{-H(\gamma)} \mathbf{e}^{\tau(\mathbf{y})} \cdot \mathbf{e}^{\tau(\mathbf{y}')}}{\mathrm{Tr} e^{-H(\gamma)}} . \tag{2.9}$$

Using Eqs. (2.2) we may write this as

$$g_\gamma(\mathbf{y},\mathbf{y}') = \frac{\mathrm{Tr} \prod_{\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma} [1 + (e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}} - 1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}](s\delta_{\tau(\mathbf{y}),\tau(\mathbf{y}')} - 1)}{\mathrm{Tr} \prod_{\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma} [1 + (e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}} - 1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}]} . \tag{2.10}$$

In order to discuss Eq. (2.10), we write it as

$$g_\gamma(\mathbf{y},\mathbf{y}') = \frac{C_\gamma(\mathbf{y},\mathbf{y}') - Z(\gamma)}{Z(\gamma)} , \tag{2.11}$$

where $Z(\gamma)$, the partition function for the cluster γ, is

$$Z(\gamma) = \mathrm{Tr} \prod_{\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma} [1 + (e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}} - 1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}] , \tag{2.12a}$$

and

$$C_\gamma(\mathbf{y},\mathbf{y}') = \mathrm{Tr} \prod_{\langle \mathbf{x},\mathbf{x}' \rangle \in \gamma} [1 + (e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}} - 1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}] \times s\delta_{\tau(\mathbf{y}),\tau(\mathbf{y}')} . \tag{2.12b}$$

We now interpret $Z(\gamma)$ and $C_\gamma(\mathbf{y},\mathbf{y}')$ graphically. To do this, we consider bond percolation on the lattice defined by the cluster γ, which by construction is a connected cluster. Thus, if γ consists of $n_B(\gamma)$ bonds, we

consider the $2^{n_B(\gamma)}$ graphs G which can be formed by either taking (as occupied) or rejecting (as vacant) each bond in γ. There is obviously a one-to-one correspondence between each graph G and the $2^{n_B(\gamma)}$ terms one obtains by expanding the product for $Z(\gamma)$ in Eq. (2.12a), $\prod[1+(e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}}-1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}]$, in powers of $(e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}}-1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}$. In this correspondence we identify the factor $\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}$ with an occupied bond between sites $\mathbf{x}$ and $\mathbf{x}'$ and absence of this factor with a vacant bond. Each graph G (on γ) can be decomposed into subclusters $\alpha_1, \alpha_2, \ldots, \alpha_{n_{\rm cl}(G)}$, of the cluster γ. A subcluster is defined as a set of site(s) on γ connected (for the graph G) with respect to occupied bonds. Each graph G will, of course, give rise to its own system of $n_{\rm cl}(G)$ subclusters α, as shown in Fig. 1. Now we interpret Eq. (2.12a) for $Z(\gamma)$ in this graphical manner. We regard $Z(\gamma)$ as being given by the sum over contributions from each of the $2^{n_B(\gamma)}$ graphs G. In the graphical interpretation each bond in G will carry a factor $e^{sJ\sigma}-1$ and since the τ's within each subcluster α of G, consisting of one or more sites, are constrained by the δ functions to be the same, the trace will give a factor s for each subcluster. Thus if G has $b_{\rm occ}(G)$ occupied bonds and consists of $n_{\rm Cl}(G)$ clusters, we have

$$Z(\gamma)\equiv\sum_G \mathrm{Tr}\prod_{\langle\mathbf{x},\mathbf{x}'\rangle\in G}[(e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}}-1)\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}]$$
$$=\sum_G s^{n_{\rm Cl}(G)}\prod_{\rm occ}(e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}}-1)\ , \tag{2.13}$$

where $\prod_{\rm occ}$ indicates that the product is over the $b_{\rm occ}(G)$ bonds which are occupied in the graph G. Now we consider the limit $s\to 0$, so that

$$e^{sJ\sigma_{\mathbf{x},\mathbf{x}'}}-1\approx sJ\sigma_{\mathbf{x},\mathbf{x}'},\quad s\to 0\ . \tag{2.14}$$

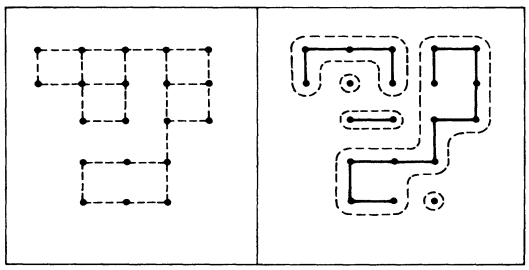

FIG. 1. Graphical interpretation of $Z(\gamma)$, where γ is the cluster comprising the set of bonds (indicated by dashed lines) shown on the left. Note that a bond connecting two sites in the cluster γ need not be in γ. On the right we show a particular graph G on the cluster γ. The subclusters α_i for this G are enclosed by dashed lines. For the case shown here the total number of bonds in γ is $n_B(\gamma)=25$, the total number of sites in γ is $n_s(\gamma)=20$, the number of bonds occupied for the graph G is $b_{\rm occ}(G)=15$, and the number of clusters for the graph G is $n_{\rm cl}(G)=5$. We show here a graph G with no loops, so that $b_{\rm occ}+n_{\rm cl}=n_s(\gamma)$.

From Eq. (2.13) we see that the s-dependence of the term corresponding to G in this limit is of the form $s^{n_{\rm cl}(G)+b_{\rm occ}(G)}$. Clearly, in the sum over G we only need to consider those graphs which correspond to the smallest power of s, i.e. to the minimum value of $n_{\rm cl}(G)+b_{\rm occ}(G)$. If G contains no bonds, each site forms a cluster and this exponent assumes the value $n_s(\gamma)$, the total number of sites in γ. Adding bonds, as long as no loops are formed, does not alter this exponent, because each added bond increases $b_{\rm occ}(G)$ by unity and decreases $n_{\rm cl}(G)$ by the same amount. Diagrams with loops correspond to larger values of the exponent, $[n_{\rm cl}(G)+b_{\rm occ}(G)]$, and therefore are to be discarded. Thus the lowest power of s in Eq. (2.11) is associated with graphs which consist of one or more trees. A "tree" is defined to be a connected graph with no loops. Now we consider the limit $J\gg 1$ [but still within the requirement of Eq. (2.14) that $sJ\to 0$]. Since from Eq. (2.14) each bond carries a factor of J, we see that for large J the sum over G is restricted to those graphs which consist of trees having the maximum number of bonds. For these maximal, or spanning, trees, the number of bonds is simply equal to $n_s(\gamma)-1$. If $\sum_{\rm ST}$ denotes a sum over G restricted to be spanning trees, then we have

$$Z(\gamma)=s^{n_s(\gamma)}J^{n_s(\gamma)-1}\sum_{\rm ST}\prod_{\rm occ}\sigma_{\mathbf{x},\mathbf{x}'}\ . \tag{2.15}$$

We can similarly analyze the term $C_\gamma(\mathbf{y},\mathbf{y}')$ in the numerator in Eq. (2.10). The effect of the additional factor not present in $Z(\gamma)$, $s\delta_{\tau(\mathbf{y}),\tau(\mathbf{y}')}$, is to restrict the sum over G to graphs which would be spanning trees if the points $\mathbf{y}$ and $\mathbf{y}'$ were imagined to be connected. We shall indicate this restricted sum by $\sum_{\rm ST'}$. In comparison to the denominator we will have one less factor of sJ but an additional factor of s. Consequently, we may write Eq. (2.10) as

$$g_\Gamma(\mathbf{y},\mathbf{y}')=\frac{s^{n_s(\gamma)}J^{n_s(\gamma)-2}\sum_{\rm ST'}\prod_{\rm occ}\sigma_{\mathbf{x},\mathbf{x}'}-Z(\gamma)}{Z(\gamma)} \tag{2.16a}$$
$$=-1+\frac{\sum_{\rm ST'}\prod_{\rm occ}\sigma_{\mathbf{x},\mathbf{x}'}}{J\sum_{\rm ST}\prod_{\rm occ}\sigma_{\mathbf{x},\mathbf{x}'}}=-1+\frac{R(\mathbf{y},\mathbf{y}')}{J}\ . \tag{2.16b}$$

In Eq. (2.16b) appears the celebrated formula[13,16] for the resistance between two sites $\mathbf{y}$ and $\mathbf{y}'$ in a resistor network in terms of spanning trees. To illustrate this formula we give the resistance, $R_{1,2}$, between the points 1 and 2 in Fig. 2 in terms of the conductances shown:

$$R_{1,2}=\frac{ac+ad+ae+bd+cd+bc+be+ce}{abc+abd+abe+ace+ade+bcd+bde+cde}\ . \tag{2.17}$$

This result can be verified using the equivalent circuit of resistances shown in Fig. 2c. We can incorporate the results for the cases when the points may or may not be in the same cluster by writing

$$\chi_\Gamma(\mathbf{y},\mathbf{y}')\equiv\lim_{s\to 0}(s-1)^{-1}g_\Gamma(\mathbf{y},\mathbf{y}')$$
$$=\nu(\mathbf{y},\mathbf{y}')[1-R(\mathbf{y},\mathbf{y}')/J], \tag{2.18}$$

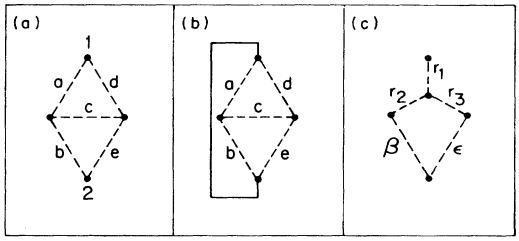

FIG. 2. A simple circuit to illustrate the formula for the resistance between two nodes, 1 and 2, in a network in terms of spanning trees. (a) shows the network of *conductances* a, b, c, d, and e, over which the sum of all spanning trees gives the denominator in Eq. (2.17). (b) The numerator in Eq. (2.17) is obtained as the sum of all spanning trees which include the connection shown between nodes 1 and 2. (c) *Resistances* in the equivalent circuit obtained via the star-triangle transformation on the conductances, a, c, and d of the original network. Here $r_1=\alpha\delta/(\alpha+\gamma+\delta)$, $r_2=\alpha\gamma/(\alpha+\gamma+\delta)$, and $r_3=\gamma\delta/(\alpha+\gamma+\delta)$, where $\alpha=a^{-1}$, $\beta=b^{-1}$, $\gamma=c^{-1}$, $\delta=d^{-1}$, and $\epsilon=e^{-1}$.

where $v(\mathbf{y},\mathbf{y}')$ is unity if the sites $\mathbf{y}$ and $\mathbf{y}'$ are connected, and vanishes otherwise, and vR is interpreted to be zero if v is zero. Note that it is essential that the limits $s\to 0$ and $J\to\infty$ be taken in such a way that sJ is small and Eq. (2.14) holds. That is, we should first let $s\to 0$ and then let $J\to\infty$.

We now extend the above formulation to random networks in which each conductance is a random variable and averages over this distribution of conductances are indicated by $[\ ]_\sigma$. For this purpose we introduce the n-replicated Hamiltonian $H^{(n)}$ defined by

$$\exp(-H^{(n)})=\left[\exp\left[\frac{J}{2}\sum_{\mathbf{x},\mathbf{x}'}\sum_{\alpha=1}^{n}\sigma_{\mathbf{x},\mathbf{x}'}\right]\mathbf{e}_\alpha^{\tau(\mathbf{x})}\cdot\mathbf{e}_\alpha^{\tau(\mathbf{x}')}\right]_\sigma . \tag{2.19}$$

We note that the partition function associated with $H^{(n)}$ is

$$\mathrm{Tr}[\exp(-H^{(n)})]=[Z^n]_\sigma . \tag{2.20}$$

Following the usual formulation in terms of replicas we have the desired result:

$$\chi(\mathbf{y},\mathbf{y}')\equiv\lim_{s\to 0}\lim_{n\to 0}(s-1)^{-1}\langle\mathbf{e}_\alpha^{\tau(\mathbf{y})}\cdot\mathbf{e}_\alpha^{\tau(\mathbf{y}')}\rangle_{H^{(n)}}$$

$$=\lim_{s\to 0}\lim_{n\to 0}\frac{[\langle-\mathbf{e}^{\tau(\mathbf{y})}\cdot\mathbf{e}^{\tau(\mathbf{y}')}\rangle_{H(\Gamma)}Z^n]_\sigma}{[Z^n]_\sigma} \tag{2.21a}$$

$$=\lim_{s\to 0}[\langle-\mathbf{e}^{\tau(\mathbf{y})}\cdot\mathbf{e}^{\tau(\mathbf{y}')}\rangle_{H(\gamma)}]_\sigma \tag{2.21b}$$

$$=[v(\mathbf{x},\mathbf{x}')-J^{-1}v(\mathbf{x},\mathbf{x}')R(\mathbf{x},\mathbf{x}')]_\sigma+O(J^{-2}) . \tag{2.21c}$$

Here again vR is interpreted to be zero if v vanishes. Note that it is important to take the limit $n\to 0$ *before* $s\to 0$ because only then will the factor Z^n which is of order $s^{nn_s(\Gamma)}$ be unity. Also, it is necessary to take the limit $s\to 0$ *before* allowing $J\gg 1$ in order that Eq. (2.12) be valid. Thus it is essential that the order of limits be such that first n, then s, and finally J^{-1} be taken to zero. In particular, this will be reflected in our calculation when we encounter the factor s^n which will be unity in the required limit.

More generally, we can define arbitrary-order two-point correlation functions:

$$\chi^{(l)}(\mathbf{y},\mathbf{y}')\equiv\lim_{s\to 0}\lim_{n\to 0}\langle[(s-1)^{-1}\mathbf{e}_1^{\tau_1(\mathbf{y})}\cdot\mathbf{e}_1^{\tau_1(\mathbf{y}')}][(s-1)^{-1}\mathbf{e}_2^{\tau_2(\mathbf{y})}\cdot\mathbf{e}_2^{\tau_2(\mathbf{y}')}]\cdots[(s-1)^{-1}\mathbf{e}_l^{\tau_l(\mathbf{y})}\cdot\mathbf{e}_l^{\tau_l(\mathbf{y}')}]\rangle_{H^{(n)}} \tag{2.22a}$$

$$=\lim_{s\to 0}[(\langle-\mathbf{e}^{\tau(\mathbf{y})}\cdot\mathbf{e}^{\tau(\mathbf{y}')}\rangle_{H(\Gamma)})^l]_\sigma \tag{2.22b}$$

$$=\left[v(\mathbf{y},\mathbf{y}')\left[1-\frac{1}{J}R(\mathbf{y},\mathbf{y}')+\frac{1}{J^2}S(\mathbf{y},\mathbf{y}')\cdots\right]^l\right]_\sigma , \tag{2.22c}$$

where $S(\mathbf{y},\mathbf{y}')$ is a correlation function for tree connections between $\mathbf{y}$ and $\mathbf{y}'$. This correlation function is not related in any obvious way to resistance properties and its form will not be needed in what follows. From Eq. (2.22c) we see that

$$\chi^{(l)}(\mathbf{y},\mathbf{y}')=\left[v(\mathbf{y},\mathbf{y}')\left[1-\frac{l}{J}C^{(1)}(\mathbf{y},\mathbf{y}';J)+\frac{l(l-1)}{2J^2}C^{(2)}(\mathbf{y},\mathbf{y}';J)\cdots\right]\right]_\sigma , \tag{2.23}$$

where $C^{(k)}(\mathbf{y},\mathbf{y}';J)\to R^k(\mathbf{y},\mathbf{y}')$ as $J\to\infty$. Thus, if we calculate $\chi^{(l)}$ as a power series in l and keep only the leading contributions as $J\to\infty$ to the coefficient of l^k, we have

$$\chi^{(l)}(\mathbf{y},\mathbf{y}')=[v(\mathbf{y},\mathbf{y}')\exp\{-lR(\mathbf{y},\mathbf{y}')/J\}]_\sigma . \tag{2.24}$$

However, this result reproduces the susceptibility for the xy model analyzed in detail in the accompanying paper if the identification is made that l/J in Eq. (2.24) corresponds to $\frac{1}{2}k^2$ in the xy model. Consequently, we expect that to leading order in $1/J$ the field theory and the recursion relations should be identical for both models. This identity will be shown in Sec. IV.

III. FIELD THEORY

In this section we will derive a continuum field theory[15,17–19] for the replicated s-state Potts model of Eq. (2.19). Each step in this procedure is straightforward and

well established, but is rendered quite complex by the large number of variables and the subtleties of the limiting procedures as discussed in the preceding section. We begin by performing the average over occupied and unoccupied bonds implicit in Eq. (2.19). For each nearest-neighbor bond $\langle \mathbf{x},\mathbf{x}' \rangle$ we assume that the conductance, $\sigma_{\mathbf{x},\mathbf{x}'}$ takes the values 0 with probability $1-p$ and σ with probability p, as usual for percolation. Thus we obtain

$$\exp(-H^{(n)})=\prod_{\langle x,x' \rangle}\left[1-p+p\exp\left[sJ\sigma\sum_{\alpha}(\delta_{\tau^{\alpha}(x),\tau^{\alpha}(x')}-1)\right]\right], \tag{3.1}$$

or

$$H^{(n)}=\sum_{\langle \mathbf{x},\mathbf{x}' \rangle} H^{(n)}_{\langle \mathbf{x},\mathbf{x}' \rangle} \tag{3.2a}$$

with

$$H^{(n)}_{\langle \mathbf{x},\mathbf{x}' \rangle}=-\ln\left[1-p+p\exp\left[sJ\sigma\sum_{\alpha}(\delta_{\tau^{\alpha}(\mathbf{x}),\tau^{\alpha}(\mathbf{x}')}-1)\right]\right]. \tag{3.2b}$$

This expression can be developed in terms of products over different replicas of the unit vectors **e**. In Appendix A, we derive the following expression for $H^{(n)}$:

$$H^{(n)}=-\sum_{\langle x,x' \rangle}\sum_{(T)} A_t S^{(T)}(\mathbf{x})S^{(T)}(\mathbf{x}') , \tag{3.3}$$

where

$$S^{(T)}(\mathbf{x})=e_{l_1}^{\tau^{\alpha_1}(\mathbf{x})}\cdots e_{l_t}^{\tau^{\alpha_t}(\mathbf{x})} \tag{3.4}$$

and

$$A_t=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l\left[1+\frac{1}{lJ\sigma}\right]^{-t}, \tag{3.5}$$

where $v=p/(1-p)$. Here $(T)\equiv(t,\boldsymbol{\alpha},\mathbf{l})$ is a composite index specifying the number of replicas t, the specific set of replicas $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_t)$, with $\alpha_1<\alpha_2,\ldots,<\alpha_t$, and the specific components $\mathbf{l}=(l_1,l_2,\ldots l_t)$. The state with no replicas $t=0$ is *not* included in (T). The sum over (T), therefore, includes the sum over all possible distinct groups of t different replicas from the ensemble of n possible replicas, as well as over the number of replicas t. Note that A depends only on t and not on $\boldsymbol{\alpha}$ or $\mathbf{l}$.

In the expression for A_t, we have taken the limits in the correct order, viz. $n\to 0$ first and then $s\to 0$, although, as noted in Appendix A these limits commute with regard to the evaluation of A_t. The sensitivity of the calculation to this order of limits will become apparent in the renormalization-group (RG) recursion relations to be derived in the next section. To obtain the limit $J\to\infty$, we can expand A_t in powers of J^{-1} as

$$A_t=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{(t+k-1)!}{t!}(lJ\sigma)^{-k}\right]=\ln(1-p)^{-1}-\frac{t}{J\sigma}C_1+\frac{1}{2(J\sigma)^2}t(t+1)C_2\cdots, \tag{3.6}$$

where

$$C_k=\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{k+1}}v^l . \tag{3.7}$$

The terms in this series can be regrouped in the form of a power series in t:

$$A_t=\ln(1-p)^{-1}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!(J\sigma)^k}C_k\left[\frac{1}{J\sigma}\right]t^k, \tag{3.8}$$

where $C_k(z)\to C_k$ as $z\to 0$.

Note that A_t is independent of t in the limit $J^{-1}=0$, and $H^{(n)}$ becomes

$$H^{(n)}\Big|_{J=\infty}=-\sum_{\langle x,x' \rangle}\ln(1-p)^{-1}\sum_{(T)}[S^{(T)}(\mathbf{x})S^{(T)}(\mathbf{x}')] , \tag{3.9a}$$

$$=-\sum_{\langle x,x' \rangle}\ln(1-p)^{-1}\left[\prod_{\alpha=1}^{n}\left[1+\sum_{l}e_l^{\tau^{\alpha}(\mathbf{x})}e_l^{\tau^{\alpha}(\mathbf{x}')}\right]-1\right], \tag{3.9b}$$

$$\equiv-\sum_{\langle x,x' \rangle}\ln(1-p)^{-1}\left[\left[\prod_{\alpha=1}^{n}s\delta^{\tau^{\alpha}(\mathbf{x}),\tau^{\alpha}(\mathbf{x}')}\right]-1\right] . \tag{3.9c}$$

The first term in square brackets in Eq. (3.9c) is $s^n P_0(\mathbf{x},\mathbf{x}')$ where $P_0(\mathbf{x},\mathbf{x}')$ is the projection operator onto states with $\tau_\alpha(\mathbf{x})=\tau_\alpha(\mathbf{x}')$ for every α. In other words, Eq. (3.9c) is an s^n-state Potts model, which in the limit $n\to 0$ before $s\to 0$ becomes a one-state model. Since such a model is known to be a representation of the percolation problem,[13] we indeed do recover percolation for $J^{-1}=0$.

The replicated Hamiltonian is a quadratic form whose partition function can be expressed as an integral over fields $\Psi^{(T)}(\mathbf{x})$ via the Hubbard-Stratonovich transformation. Thereby we obtain

$$Z^{(n)} \equiv \mathrm{Tr} e^{-H^{(n)}} = \int D\Psi^{(T)} \exp\left[\tfrac{1}{2} \sum_{\mathbf{x},\mathbf{x}',(T)} \gamma^{-1}(\mathbf{x},\mathbf{x}') A_t^{-1} \Psi^{(T)}(\mathbf{x}) \Psi^{(T)}(\mathbf{x}')\right] \mathrm{Tr} \exp\left[\sum_{\mathbf{x},(T)} \Psi^{(T)}(\mathbf{x}) S^{(T)}(\mathbf{x})\right], \tag{3.10}$$

where $\gamma(\mathbf{x},\mathbf{x}')$ is a matrix in the scripts $\mathbf{x}$ and $\mathbf{x}'$ which is unity if $\mathbf{x}$ and $\mathbf{x}'$ are nearest neighbors and zero otherwise, $D\Psi^{(T)}$ denotes integration over all fields $\Psi^{(T)}$, and (T) again denotes the multiple index $(t,\alpha,\mathbf{l})$.

Expanding Eq. (3.10) in powers of $\Psi^{(T)}$ and retaining only terms relevant to the long-wavelength limit, we obtain

$$Z^{(n)} = \int D\Psi^{(T)} e^{-H(\Psi^{(T)})}, \tag{3.11}$$

where

$$H(\Psi^{(T)}) = \tfrac{1}{2} \sum_{(T)} \int_{\mathbf{q}} (r_t + c_t q^2) \Psi^{(T)}(\mathbf{q}) \Psi^{(T)}(-\mathbf{q}) - \frac{1}{3!} u \sum_{(T)(T')(T'')} \int_{\mathbf{q}_1,\mathbf{q}_2} F_{(T)(T')(T'')} \Psi^{(T)}(\mathbf{q}_1) \Psi^{(T')}(\mathbf{q}_2) \Psi^{(T'')}(-\mathbf{q}_1-\mathbf{q}_2), \tag{3.12}$$

where (T') and (T'') are composite indices like (T),

$$\Psi^{(T)}(\mathbf{q}) = \sum_{\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}} \Psi^{(T)}(\mathbf{x}) , \tag{3.13a}$$

$$r_{(T)} = 1 - \frac{1}{zA_t} \equiv r_t, \tag{3.13b}$$

$$c_t = a^2 q^2 / (z^2 A_t) , \tag{3.13c}$$

and

$$F_{(T)(T')(T'')} = \frac{1}{s^n} \sum_{\tau_1,\ldots,\tau_n} e_{l_1}^{\tau_1} \cdots e_{l_t}^{\tau_t} e_{l_1'}^{\tau_1} \cdots e_{l_t'}^{\tau_t} e_{l_1''}^{\tau_1} \cdots e_{l_t''}^{\tau_t} . \tag{3.14}$$

In writing Eqs. (3.13b) and (3.13c) we used the long-wavelength expansion of the Fourier transform $\gamma(\mathbf{q}) = z - a^2 q^2$, where a is the lattice constant. As usual, we will rescale the fields $\Psi^{(T)}(\mathbf{q})$ so that $c_t = 1$. This leads to rescalings of r_t and u which, since they do not affect any of our results, we will ignore. Just as in the case of A_t, r_t can be expanded in powers of t:

$$r_t = r + \sum_k w_k t^k, \tag{3.15}$$

where $r \sim (p_c - p)$ and $w_k \sim (J\sigma)^{-k}$ in the limit $J \to \infty$. Note that r_t becomes independent of t for $J = \infty$, in which case the model becomes an s^n-state Potts model describing percolation[20] in the limit $n \to 0$, as we showed above.

The vertex $F_{(T),(T'),(T'')}$ requires further comment. We consider the effect of the normalized trace operation, $s^{-1}\mathrm{Tr}$ within each replica space. A given replica, say α, can appear in all three of the indices, (T), (T'), and (T''), in any two of them, in only one of them, or not at all. If α appears in only one index, the normalized trace over τ_α will give zero. Such a term will therefore not appear in $F_{(T),(T'),(T'')}$. If α appears in no index, the normalized trace, i. e., the sum over τ_α divided by s, will give unity. If α appears in all three indices, the normalized trace in this replica will contribute a factor

$$\frac{1}{s} \sum_{\tau} e_l^{\tau} e_{l'}^{\tau} e_{l''}^{\tau} \tag{3.16}$$

to F, where l is the vector index associated with α in (T) and similarly for l' and l''. Finally, if α appears in (T) and (T'), but not in (T''), it will contribute a factor $\delta_{l,l'}$ to F. Accordingly, we let n_1 be the number of replicas common to (T), (T'), and (T''), n_2 the number common only to (T) and (T'), n_3 the number common only to (T) and (T''), and n_4 the number common only to (T') and (T''). Then

$$t = n_1 + n_2 + n_3 , \tag{3.17a}$$

$$t' = n_1 + n_2 + n_4 , \tag{3.17b}$$

$$t'' = n_1 + n_3 + n_4 , \tag{3.17c}$$

and

$$F_{(T)(T')(T'')} = \prod_{i \in n_1} \left[\frac{1}{s} \sum_{\tau} e_{l_i}^{\tau} e_{l_i'}^{\tau} e_{l_i''}^{\tau}\right] \prod_{j \in n_2} \delta_{l_j,l_j'} \prod_{k \in n_3} \delta_{l_k,l_k''} \prod_{m \in n_4} \delta_{l_m',l_m''} , \tag{3.18}$$

where the product over i, j, k, m is over indices belonging to the respective sets of $n_1, \ldots, n_4$ replicas, described above. A diagrammatic representation of this vertex is useful. Each incoming line breaks up into three parts which flow either through both of the other legs or through only one of the other legs as shown in Fig. 3. Thus each diagram is a sort of "direct product" of diagrams labelled within each replica.[21,22]

IV. ϵ EXPANSION

As discussed in Secs. I and II, the average resistance between sites $\mathbf{x}$ and $\mathbf{x}'$ can be obtained from

$$\frac{\partial}{\partial J^{-1}} G(\mathbf{x},\mathbf{x}') \Big|_{J^{-1}=0} .$$

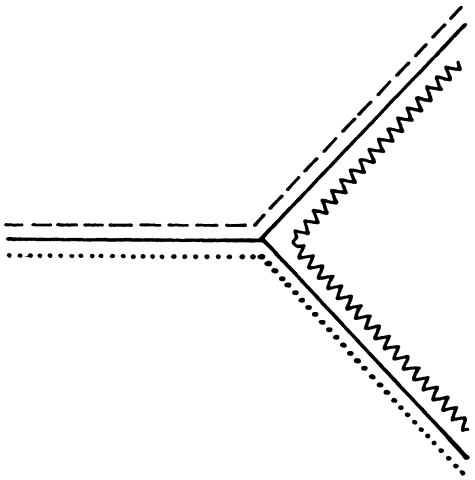

FIG. 3. Schematic representation of the three-point vertex described by Eq. (3.18). The replica indices of each leg are divided into three sets: one set shared by both the other legs and two sets shared by only one of the other legs. Replica indices that are the same in two or more legs are associated with the same kind of line, either solid, dashed, dotted, or wavy.

We are, therefore, interested in crossover with respect to J^{-1} in the vicinity of the percolation critical point where $J^{-1}=0$. In this section, we will study this crossover, and the crossover with respect to higher powers of J^{-1}, via an ϵ expansion about six dimensions to first order in $\epsilon=6-d$.

As discussed in the preceding section, the model under consideration is identical to percolation when $J^{-1}=0$. We can, therefore, use well-established results for the ϵ expansion for percolation as a basis for the calculation of the crossover with respect to powers of J^{-1}. In particular, at the percolation critical point, we have[17–19]

$$K_d(u^*)^2 \equiv g^* = 2\epsilon/7 , \tag{4.1a}$$

$$\eta = -\epsilon/21 , \tag{4.1b}$$

$$\nu = \tfrac{1}{2} + 5\epsilon/84 , \tag{4.1c}$$

where $K_d=\Omega_d/(2\pi)^d$, where Ω_d is the area of a sphere of unit radius in d dimensions, and η and ν are, respectively, the anomalous dimension and correlation length exponents for percolation.

Recursion relations for r_t can be obtained in the usual way by integrating out degrees of freedom with wave number in the annulus $b^{-1}\Lambda < q < \Lambda = 1$, where Λ is a cutoff determined by the lattice constant a such that $a\Lambda \sim$ unity, and rescaling fields via

$$\Psi^{(t)}(\mathbf{q}/b) \rightarrow b^{(d-2+\eta)/2}\Psi^{(t)}(\mathbf{q}) .$$

Eliminating an infinitesimal shell at each iteration with $b=e^{\delta l}$, we obtain the differential recursion relation[23] for the quadratic coefficient, r_t, of Eq. (3.12) as

$$\frac{dr_t}{dl} = (2-\eta)r_t - \tfrac{1}{2}g^*\Pi_t , \tag{4.2}$$

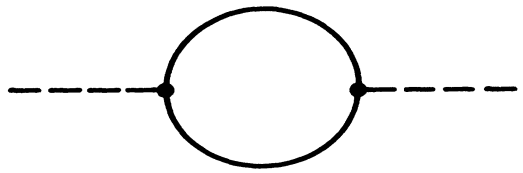

FIG. 4. Diagram which contributes to Σ_t of Eq. (4.2). The momentum shell integration is over the internal propagators indicated by solid lines.

where we set $g=g^*$ and Π_t is the contribution from the diagram in Fig. 4. It can be expressed formally as

$$\Pi_t = \sum_{(T')(T'')} F^2_{(T)(T')(T'')} G_{t'} G_{t''} , \tag{4.3}$$

where $G_t=(1+r_t)^{-1}$ is the bare propagator evaluated at $q=\Lambda=1$. Throughout this development we have used the fact that $r_{(T)}$ depends only on t, the number of replicas. This same property is shared also by Π and G. To evaluate Π_t, we could evaluate the sums in Eq. (4.3) directly using the explicit forms of $F_{(T)(T')(T'')}$ given in the preceding section. A somewhat more direct evaluation, which is generalized elsewhere[21] to higher order diagrams, can be obtained as a result of the following observations: Each replica α which enters a graph for Π on the left must follow a continuous path to emerge on the right. There are three ways a replica can run continuously through the one loop graph shown in Fig. 4: It can run through either both legs in the loops or else through only the top or only the bottom leg of the loop. These possibilities are shown in Figs. 5(a) – 5(c). In addition, if t replicas enter, then any of the $n-t$ of the remaining replicas can cover both the top and bottom legs of the loop as shown in Fig. 5(d). Thus, Π_t is obtained by summing over all possible ways each replica can appear, either as in Figs. 5(a) – 5(c) if it appears in an external propagator, or not at all as in Fig. 5(d) if it does not appear in an external propagator. However *at least one replica must appear in each leg of the graph* for it to actually correspond to the diagram of Fig. 4. The matrix element for Fig. 4 is thus constructed as

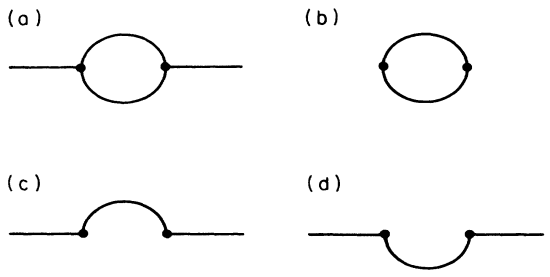

FIG. 5. Ways in which a single replica can "cover" the diagram of Fig. 4.

the product of matrix elements for each replica. Each replica as in Fig. 5(a) contributes a factor $s-2$, each as in Fig. 5(b) or 5(c) a factor unity, and for each replica as in Fig. 5(d) a factor $s-1$. Let n_i, where i assumes the values a, b, c, or d, be the number of replicas corresponding to Figs. 5(a)—5(d), respectively. Then $t=n_a+n_b+n_c$, $t'=n_a+n_b+n_d=t+n_d-n_c$, and $t''=n_a+n_c+n_d=t+n_d-n_b$. We therefore obtain

$$\Pi_t^{(n)}=-2G_0G_t+\sum_{n_b,n_c,n_d}(s-2)^{t-n_b-n_c}(s-1)^{n_d}\left[\frac{t!}{n_b!n_c!(t-n_b-n_c)!}\right]\left[\frac{(n-t)!}{n_d!(n-t-n_d)!}G_{t+n_d-n_c}G_{t+n_d-n_b}\right], \tag{4.4}$$

where $G_0=(1+r_{t=0})\equiv(1+r)^{-1}$, and where we have added the superscript (n) to indicate that the $n\to 0$ limit has not yet been taken. The second term in Eq. (4.4) contains an unrestricted sum over all values of n_a, n_b, and n_d and includes the cases $t=0$ and $t'=0$ in which one of the legs of the bubble are covered by no replicas. These unwanted contributions are removed by the first term in Eq. (4.4). It is convenient to write Eq. (4.4) as

$$\Pi_t^{(n)}=G_0^2-2G_0G_t+\delta\Pi_t^{(n)}, \tag{4.5}$$

where we have separated Π into a "normal" part, $G_0^2-2G_0G_t$, which gives all crossover exponents, ϕ_t, equal to unity, and an "anomalous" part, $\delta\Pi$ given by

$$\delta\Pi_t^{(n)}=-G_0^2+\sum_{k,l,m}(s-2)^{t-k-l}(s-1)^m \times C_k^t C_l^{t-k}C_m^{n-t}G_{t+m-k}G_{t+m-l}\ . \tag{4.6}$$

where C_k^t is the binomial factor $t!/[k!(t-k)!]$ In Appendix B we show that Eq. (4.6) can be put into the form

$$\delta\Pi_t=\sum_{r=1}^{t}(-1)^r C_r^t\sum_{p=0}^{\infty}C_p^{p+r-1}(1-s)^p(\Delta^r G_p)^2\ , \tag{4.7}$$

where Δ^r is the rth-order finite-difference operator:

$$\Delta^r G_p=\sum_{l=0}^{r}(-1)^l C_l^r G_{p+l}\equiv(e^{d/dh}-1)^r G_h\,|_{h=p}\ . \tag{4.8}$$

To see the meaning of this result, note that $\delta\Pi_t$ is clearly zero when all of the w_k's are zero, since $\Delta G_p=0$ in this case. Secondly, since ΔG_p is at least of linear order in the w's, each term in the sum over p is at least of quadratic order in the w's provided that this sum converges, as it does for $0<s<2$. (This conclusion can no doubt be extended to $s\geq 2$ by further analysis.) Thus $\delta\Pi_t$ does not contribute to the crossover exponents for w_k for any $s>0$. Consequently, for $s>0$

$$\Sigma_t=-2G_0G_t+G_0^2\sim -1+2r_t \tag{4.9}$$

and

$$\frac{dr_t}{dl}=(2-\eta-2g^*)r_t+g^*, \tag{4.10}$$

from which we determine the stability exponent $\lambda_t=1/\nu$ and a crossover exponent $\phi_t\equiv\lambda_t\nu=1$ for all t in agreement with the calculation to first order in ϵ of Stephen and Grest[24] (for the Ising model, $s=2$) and with the proofs of Wallace and Young[25] (for the general s-state Potts model), and of Coniglio[26] (which is expected to hold for discrete-spin models with an energy gap).

As mentioned above, the energy gap vanishes as $s\to 0$, and this phenomenon reflects itself in our calculation: when $s\to 0$ the factor $(1-s)^p$ no longer ensures the convergence of the integrals in Eq. (4.7), and a more careful analysis is necessary. To see what Eq. (4.9) implies for $s=0$, we write it more explicitly as

$$\delta\Pi_t=-t\sum_{p=0}^{\infty}(\Delta^1 G_p)^2+\frac{t(t-1)}{2}\sum_{p=0}^{\infty}(p+1)(\Delta^2 G_p)^2 +\frac{t(t-1)(t-2)}{2(3)}\sum_{p=0}^{\infty}\frac{(p+1)(p+2)}{2}(\Delta^3 G_p)^2+\cdots \tag{4.11a}$$

$$\equiv\sum_{k=1}^{\infty}\frac{(-1)^k t!}{(t-k)!k!}X_k\ . \tag{4.11b}$$

As in Eq. (2.23) to treat resistance properties we should keep only the leading dependence on J within a given power of t. Corrections of relative order $(1/J)^j$ for $j>0$ do not affect the moments of the resistance discussed in Sec. II. Recall that $w_t\sim J^{-t}$, so that $G_p=(1+r+\sum_k w_k p^k)^{-1}$ is a function of (p/J). Therefore the sums will be dominated by values of p which are of order J. Thus to leading order in $1/J$ we set $(p+k-1)!/p!=p^{k-1}$ and replace the sums by integrals, so that

$$X_k=\sum_{p=0}^{\infty}\frac{(p+k-1)!}{p!(k-1)!}(\Delta^k G_p)^2 =\frac{1}{(k-1)!}\int_0^{\infty}p^{k-1}\left[\frac{\partial^k G_p}{\partial p^k}\right]^2 dp\ . \tag{4.12}$$

Since G_p is a function of p/J, one sees from this form that $X_k\sim J^{-k}$. When it is expanded in powers of t, Eq. (4.11b) becomes

$$\delta\Pi_t=\sum_k(-1)^k\frac{t^k}{k!}[X_k+a_k^{(1)}X_{k+1}+a_k^{(2)}X_{k+2}\cdots]\ . \tag{4.13}$$

Within the square brackets of this equation the leading dependence on $1/J$ comes from the first term, X_k. Dropping higher order in $1/J$ corrections we may therefore write

$$\delta\Pi_t=\sum_{j=1}^{\infty}\delta\Pi^{(j)}t^j\ , \tag{4.14}$$

with

$$\delta\Pi^{(k)}=\frac{1}{k!}\frac{(-1)^k}{(k-1)!}\int_0^{\infty}y^{k-1}\left[\frac{\partial^k G_y}{\partial y^k}\right]^2 dy\ . \tag{4.15}$$

The above result is identical to that obtained in the accompanying paper [Eq. (4.25)] for the diluted xy model. We therefore, stop the analysis here and refer the reader to the accompanying paper[11] for the calculation of the crossover exponents and scaling functions. The main additional result is an explicit calculations of a class of universal amplitude ratios which can be formed by the susceptibilities associated with the moments of the resistance. The reader perhaps should also be alerted to the fact that the results for the subsidiary crossover exponents, which describe corrections to scaling, differ from those previously announced.[12,14]

V. DISCUSSION AND CONCLUSION

The principle purpose of this paper was to derive the scaling properties of the probability distribution of the two-point resistance $R(\mathbf{x},\mathbf{x}')$ between sites $\mathbf{x}$ and $\mathbf{x}'$ on the same cluster at the percolation threshold. We showed here in detail that the critical properties (in particular the ϵ-expansion recursion relations) are the same as for the randomly diluted xy model treated in the accompanying paper.[11] The detailed calculations of the crossover exponents, the scaling functions, and the universal amplitude ratios which are obtained via the ϵ-expansion recursion relations in the accompanying paper[11] could therefore just as well be obtained from the Potts-model formulation presented here. The conclusions mentioned in the abstract follow from the analysis in the accompanying paper and are discussed there.

ACKNOWLEDGMENT

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APPENDIX A: DERIVATION OF THE COEFFICIENTS A_t

In this appendix, we will derive Eq.(3.3). To do so, we introduce an Abelian representation[27] for the s-state Potts model. Let

$$\psi_k(\mathbf{x})=e^{2\pi ik\tau(\mathbf{x})/s}\ . \tag{A1}$$

where $k=0,\ldots,s-1$ is an integer. Then

$$\frac{1}{s}\sum_{k=0}^{s-1}\psi_k(\mathbf{x})\psi_k^*(\mathbf{x}')=\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}\ , \tag{A2}$$

$$\frac{1}{s}\mathrm{Tr}\psi_k(\mathbf{x})\equiv\frac{1}{s}\sum_{\tau=1}^{s}e^{2\pi ik\tau/s}=\delta_{k,0}\ , \tag{A3}$$

and

$$\sum_{k=1}^{s-1}\psi_k(\mathbf{x})\psi_k^*(\mathbf{x}')=\mathbf{e}^{\tau(\mathbf{x})}\cdot\mathbf{e}^{\tau(\mathbf{x}')}=(s\delta_{\tau(\mathbf{x}),\tau(\mathbf{x}')}-1)\ . \tag{A4}$$

The Hamiltonian of Eq.(3.2) can be expanded in powers of $v\equiv p/(1-p)$:

$$H^{(n)}_{\mathbf{x},\mathbf{x}'}=-\ln(1-p)+\sum_{l=1}^{\infty}\frac{(-1)^l}{l}v^l e^{sl\sigma J\sum_{\alpha}(\delta_{\tau^\alpha(\mathbf{x}),\tau^\alpha(\mathbf{x}')}-1)}\ . \tag{A5}$$

The exponential factor in this equation can be expanded in terms of

$$\psi_{\mathbf{k}}\equiv\prod_{\alpha=1}^{n}\psi_{k_\alpha} \tag{A6}$$

with $\mathbf{k}\equiv(k_1,k_2,\ldots,k_n)$. We obtain

$$\exp\left[sl\sigma J\sum_{\alpha}(\delta_{\tau^\alpha(\mathbf{x}),\tau^\alpha(\mathbf{x}')}-1)\right]=\sum_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}^*(\mathbf{x}')F_{\mathbf{k}}\ . \tag{A7}$$

Since both sides are functions of $\tau^\alpha(\mathbf{x})-\tau^\alpha(\mathbf{x}')$, we determine $F_{\mathbf{k}}$ by multiplying both sides of Eq. (A7) by $\exp\{-2\pi i\mathbf{k}\cdot[\tau(\mathbf{x})-\tau(\mathbf{x}')]/s\}$ and summing over $\tau(\mathbf{x})-\tau(\mathbf{x}')\equiv\mathbf{y}$. Thereby we obtain

$$\begin{aligned}F_{\mathbf{k}}&=\sum_{\mathbf{y}}s^{-n}e^{-2\pi i\mathbf{k}\cdot\mathbf{y}/s}e^{sl\sigma J\sum_{\alpha}(\delta_{y^\alpha,0}-1)}\\&=\prod_{\alpha}\left[\frac{1}{s}+\frac{1}{s}\sum_{y^\alpha\neq 0}e^{-2\pi ik_\alpha y^\alpha/s}e^{-sl\sigma J}\right],\\&=\prod_{\alpha}\left[\frac{1}{s}[1+(s-1)e^{-sl\sigma J}]\delta_{k_\alpha,0}+\frac{1}{s}(1-e^{-sl\sigma J})\right],\\&=\sum_{t=0}^{n}s^{-n}(1-e^{-sl\sigma J})^t[1+(s-1)e^{-sl\sigma J}]^{n-t}P^{(t)}\ ,\end{aligned} \tag{A8}$$

where

$$P^{(t)}=\sum\prod_{i}^{t}(1-\delta_{k_{\alpha_i},0})\prod_{j=t+1}^{n}\delta_{k_{\alpha_j},0} \tag{A9}$$

is the projection operator onto the state with any t of the n components of $\mathbf{k}$ nonzero and the other $n-t$ components zero. The sum in the last equation is over all of the C_t^n ways of selecting t of n components to be nonzero. Taking the limit $n\to 0$ and using Eq. (A8) in Eq. (A5), we obtain

$$H^{(n)}_{\mathbf{x},\mathbf{x}'}=-\ln p-\sum_{t=1}^{n}A_t\sum_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}^*(\mathbf{x}')P^{(t)} \tag{A10}$$

where

$$A_t=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l\left[\frac{1-e^{-sl\sigma J}}{1+(s-1)e^{-sl\sigma J}}\right]^t\rightarrow\left[\frac{1}{l\sigma J}\right]^{-t}\quad\text{as } s\rightarrow 0\ . \tag{A11}$$

Note that

$$\sum_{\mathbf{k}}P^{(t)}\psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}^{*}(\mathbf{x}')=\sum_{\{\alpha\},\{l\}}S^{(T)}(\mathbf{x})S^{(T)}(\mathbf{x}')\ . \tag{A12}$$

Thus, apart from the $\ln p$ term which we drop, Eq. (A10) is identical to Eq. (3.3).

APPENDIX B: DERIVATION OF Eq. (4.7)

In this appendix we derive Eq. (4.7) starting from Eq. (4.6). In order to perform the sums over k, l, and m in Eq. (4.6), we use

$$G_k=\int_0^{\infty}dh\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_h e^{i(h-k)\omega}\ , \tag{B1}$$

$$\sum_{k,l}C_k^t C_l^{t-k}x^k y^l=(1+x+y)^t\ , \tag{B2}$$

and

$$\lim_{n\rightarrow 0}\sum_m C_m^{n-t}x^m=(1+x)^{-t} \tag{B3}$$

to obtain

$$\delta\Pi_t=\int_0^{\infty}dh_1\int_0^{\infty}dh_2\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}G_{h_1}G_{h_2}\left[\left[1-\frac{(e^{i\omega_1}-1)(e^{i\omega_2}-1)}{e^{i(\omega_1+\omega_2)}+s-1}\right]^t-1\right]e^{i(h_1\omega_1+h_2\omega_2)}\ . \tag{B4}$$

We denote the term within square brackets in this equation by X. Expanding X we obtain

$$X=\sum_{r=1}^{\infty}(-1)^r C_r^t(e^{i\omega_1}-1)^r(e^{i\omega_2}-1)^r(e^{i(\omega_1+\omega_2)}+s-1)^{-r} \tag{B5a}$$

$$=\sum_{r=1}^{\infty}\sum_{p=0}^{\infty}C_r^t(e^{i\omega_1}-1)^r(e^{i\omega_2}-1)^r C_p^{-r}(s-1)^p e^{i(\omega_1+\omega_2)(-r-p)}\ . \tag{B5b}$$

Now use $C_p^{-r}=(-1)^p C_p^{p+r-1}$, whence

$$X=\sum_{r=1}^{\infty}\sum_{p=0}^{\infty}e^{-ip\omega_1}(1-e^{-i\omega_1})^r e^{-ip\omega_2}(1-e^{-i\omega_2})^r(1-s)^p C_r^t C_p^{p+r-1}(-1)^r\ . \tag{B6}$$

Now the integrals over h_1, h_2, ω_1, and ω_2 can be performed using Eq. (B1) to obtain Eq. (4.7):

$$\delta\Pi_t=\sum_{r=1}^{\infty}(-1)^r C_r^t\sum_{p=0}^{\infty}C_p^{p+r-1}(1-s)^p(\Delta^r G_p)^2\ , \tag{B7}$$

where Δ^r is the rth-order finite difference operator:

$$\Delta^r G_p=\sum_{l=0}^{r}(-1)^l C_l^r G_{p+l}\ . \tag{B8}$$

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