Toward the Regulation and Composition of Cyclic Behaviors

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Many tasks in robotics and automation require a cyclic exchange of energy between a machine and its environment. Since most environments are "under actuated" — that is, there are more objects to be manipulated than actuated degrees of freedom with which to manipulate them — the exchange must be punctuated by intermittent repeated contacts. In this paper, we develop the appropriate theoretical setting for framing these problems and propose a general method for regulating coupled cyclic systems. We prove for the first time the local stability of a (slight variant on a) phase regulation strategy that we have been using with empirical success in the lab for more than a decade. We apply these methods to three examples: juggling two balls, two legged synchronized hopping and two legged running — considering for the first time the analogies between juggling and running formally.

1 Introduction[1]

A robot is a source of programmable work. Robot programming problems arise when a mechanism designed with certain directly actuated degrees of freedom is required to exchange energy with its environment in such a fashion that some useful work — its "task," involving the imposition of specified forces over specified motions — is accomplished. Typically, the environment is not directly actuated and has its own preferred natural dynamics whose otherwise uninfluenced motions would be at least indifferent and, possibly, inimical to the task. The prior century's end has witnessed the practical triumph and emerging formal understanding of programs for information exchange and manipulation. There does not yet seem to exist a programming paradigm that can specify similarly goal oriented work exchange at any reasonable level of generality with any reasonable likelihood of successful implementation (much less, of formal verification).

For reasons we have discussed elsewhere at length [7, 21], we construe "task" to mean any behavior that can be encoded as the limit set of the closed loop dynamical system resulting from coupling the robot up to its environment. By "programming" is meant (at the very least) a means of composing from extant primitive task behaviors new, more specialized, or elaborate capabilities. A decade's research by the second author and colleagues has yielded the beginnings of a compositional methodology for tasks that can be encoded as point attractors [29, 28, 7, 14, 19]. In the present paper, we take the first steps toward a formal foundation for tasks that can be encoded as the next simplest class of steady state dynamical systems behavior — limit cycles.

1.1 Contributions of the Paper

In this paper we are able to prove for the first time the partial correctness of a (slight variant on) a phase regulation strategy that we have been using with empirical success in the lab for more than a decade [5, 31]. The object of study is a discrete dynamical control system on a co-dimension one subset of the tangent bundle over the two-torus, $\Sigma \subseteq \mathbb{T}^2$. The theoretical result is the proof of local asymptotic stability for a specified fixed point on this subset, $x^* \in \Sigma$. To illustrate the potential implications of the emerging theory, we introduce three example systems that move from juggling toward phase coordination strategies for legged machines.

We show in our first example that this discrete system corresponds to the parametrized family of return maps that arise when a one degree of freedom actuated piston strikes two otherwise unactuated one de-

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gree of freedom balls falling in gravity. This abstraction presumes sufficient control of the piston to track a suitably distorted image — a "mirror law" [6] — of the trajectory described by the two balls, along the lines of the empirical setup in [5, 30]. Under these assumptions, the torus bundle represents the *phase coordinate representation* of the two degree of freedom hybrid flow formed when the paddle repeatedly but intermittently strikes one or the other ball . The functional freedom afforded by the choice of "mirror law" yields the parametrization of the available closed loop return maps whose domain, Σ, is now interpreted as the phase condition at which an impact is made. The preliminary analysis presented here develops a set of sufficient conditions on the mirror law that guarantees the local asymptotic stability of a limit cycle corresponding to the the desired two-juggle. We suspect, but have not yet proven, that the desired limit cycle is essentially globally asymptotically stable.

We have not yet formalized the notion of behavioral composition (as we have begun to do for behaviors encoded as point attractors [7, 19]) but it represents a strong unifying theme throughout the paper. The two-juggle mirror law may be seen as a kind of informal "interleaving" of two one-juggle functions. However, because we desire a more general compositional notion not tied to the (effective but very costly in sensory effort) mirror constructions, we next apply our phase regulation method to "interleave" a very different style of individual controller. Specifically, with the appropriate notion of phase coordinates described above, we are now able to consider for the first time the analogies between juggling and running.

Our second example concerns two vertical Raibert hoppers [27], each of whose leg springs has an adjustable stiffness. Now, although both legs are partially actuated, the contact with ground is no longer instantaneous and we abandon mirror laws in favor of a Raibert style energy management strategy coordinated over repeated intermittent stance modes so as to nudge the total vertical energy toward that value which encodes the desired behavior. When the legs are decoupled from each other, arguments nearly identical to those we have developed in past work [20] yield essential global asymptotic stability of the two independent vertical "gaits". Note that the reference energy is achieved asymptotically, rather than by a deadbeat one step correction as was assumed in the first example. Nevertheless, applying the identical phase regulation scheme yields a closed loop system that exhibits in simulation the same striking coordinated behavior as we have proven to hold true (at least locally) in the case of the two-juggle. We suspect, but have not yet proven that the coordinated bipedal vertical gait is once again (essentially globally) asymptotically stable.

The final example represents our first and still rather tentative efforts to interleave constituent cyclic behaviors that arise in systems possessed of more than one degree of freedom. Raibert's running machines combined in parallel, for each leg, three independent and decoupled controllers that operated in three very strongly coupled degrees of freedom, with excellent empirical success. Moreover, he devised a notion of "virtual leg" that successfully coordinated the relative phases of the "vertical" components of the physical legs without damaging their other degrees of freedom. In this paper, we are content simply to extend our emerging notion of "phase" to a pair of two-degree of freedom pogo sticks (the "Spring Loaded Inverted Pendulum" [35]) and assume that their individual phase regulation mechanisms are deadbeat. Once again, simulations suggest that this is the appropriate generalization, but we remain cautious regarding the larger implications pending more realistic constituent models.

1.2 Motivation and Relation to Existing Literature

Coupled oscillators have long been used to model complex physical and biological settings wherein phase regulation of cyclic behavior is paramount [15]. The biological reality of neural central pattern generators (CPGs) — oscillatory signals that arise spontaneously from appropriate intercommunication between neurons — seems by now to have been conclusively demonstrated in organisms ranging from insects [26, 12] to lampreys [9]. Mathematical models proposed to explain the manner in which families of coupled dynamical systems can stimulate a sustained oscillation and stably entrain a desired phase relationship have become progressively more biologically detailed [8, 13, 16]. But the work presented here has relatively little overlap with that literature. While we are intrigued by the capabilities of purely "clock driven" systems [36, 32], it seems clear that no significant level of autonomy can be developed in the absence of perceptual feedback. The present investigation cleaves to the opposite (i.e., perceptually driven) end of the sensory spectrum in adopting the device of a "mirror law" [6] with its commitment to profligate sensory dependence [30]. In this sense, the present work bears a closer relationship to the biological literature concerned with reflex modulated phase regulation [11].

Many tasks in robotics and automation entail a cyclic exchange of energy between a machine and its environment. This is evidently the case for legged locomotion systems as well as for many less obvious examples wherein a mechanism repeatedly changes its local "shape" so as to effect some global "progress" [24]. When viewed from an appropriate geometric perspective, the recourse to repetitive self-motion may be interpreted as a means of "rectification" — exercising indirectly the unactuated degrees of freedom through the influence of the actuated degrees of freedom arising from an interaction between symmetries and constraints [2]. Because our notion of a task is so completely bound up with a closed loop dynamical interaction between the robot and its environment, this invaluable geometric control perspective provides no solution but merely a complete account of the (open loop) setting within which our search for stabilizing feedback controllers can begin. Since the dynamics in question are inevitably nonlinear, the relation between open loop controllability properties and feedback stabilizability properties is far from clear.

In our understanding, the most relevant connection to date remains the nearly two decade old observation of Brockett [4], precluding smooth feedback stabilization to a point in the face of conditions known [3] to characterize the nonholonomic constraints that appear in the present underactuated setting [22]. At the very least, this fact necessitates the appearance of hybrid controllers — feedback laws whose resulting closed loops make non-smooth transitions triggered by state — in the case of tasks encoded as point attractors [22]. In the present situation, when tasks are encoded as limit cycles, we are aware of no similar necessary conditions. Nevertheless, the feedback solutions we construct have a strong hybrid character. Since the nonholonomic constraints in our setting arise from the "under actuated" nature of the problem [21], it seems intuitively clear that the robot's work on the components of its environment must be punctuated by intermittent repeated contacts.

One last influence on the present work that bears some comment concerns the possibility of composition. Since good regulation mechanisms are hard to find, there is considerable interest in developing techniques for composing existing behaviors to get new ones. However, as the degrees of freedom increase, the burdens of high dimensionality make centralized control schemes prohibitively expensive. There is considerable interest in developing cyclic behaviors that are as decoupled as possible, promoting decentralized regulation. Our present model for pursuing this desideratum is provided by our initial work on concurrent composition of point attractors [19]. Since our reference flows have gradient-like cross-section maps, we harbor some hope that the connection may be forthcoming.

2 Preliminary Discussion

We start in Section 2.1 by defining phase coordinates that enable us to re-cast physical equations involving potential and kinetic energy as geometric equations relating progress around a circle and its velocity. In the examples at hand, the physical control variables are used to adjust the energy of the unactuated degrees of freedom upon their intermittent contacts with the actuated components. In phase coordinates, the phase velocity of each constituent subsystem is subject to control at each impact, and effects a corresponding resetting of the various relative phases.

Having arrived at a convenient model space, the torus, we next examine in Section 2.2 the notion of a "reference field" — a family of limit cycle generating vector fields on the k-torus whose return maps on the $(k-1)$-torus admit as a Lyapunov function a "Navigation Function" [23] down to the fixed point. The topologically unavoidable repellors can be identified with the application as phase pairs that are to be avoided (e.g., when both balls come down at exactly the same of time). Although our ultimate constructions appear as maps of an appropriate cross section (so the topological constraints appear to lose their force) these toral maps are classical objects and yield very convenient and workable targets.

2.1 Controlling Phase

Let $f^t : \mathbb{R} \times X \to X$ be a flow on X. We are concerned with flows that are cyclic in the sense that a global cross section can be found. Formally, a global cross section Σ is a connected submanifold of X which transversally

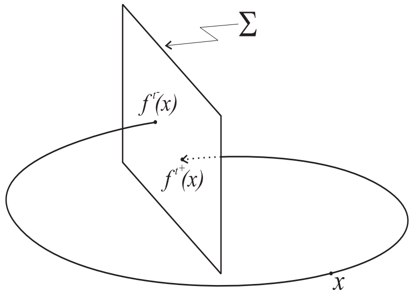


Figure 1: *The relationship of $t^{-}(x)$ and $t^{+}(x)$ to x.*

intersects every flowline. For any point $x \in X$, define the **time to return** of x to be

$$t^{+}(x) = min\{t > 0 \mid f^{t}(x) \in \Sigma\} \qquad (1)$$

and define the **time since return** of x to be

$$t^{-}(x) = min\{t \geq 0 \mid f^{-t}(x) \in \Sigma\}\ . \qquad (2)$$

The first return map, $P : \Sigma \to \Sigma$, is the discrete, real valued map given by $P(x) = f^{t^{+}}(x)$.

Let $s(x) = t^{+}(x) + t^{-}(x)$. s is the time it takes the system starting at the point $f^{t^{-}}(x) \in \Sigma$ to reach Σ again. Now, define the **phase** of a point x by

$$\phi(x) = \frac{t^{-}(x)}{s(x)} \qquad (3)$$

Notice that the rate of change of phase, $\dot{\phi}$, is equal to $1/s$. The relationship of these functions to Σ is shown in Figure 1. Therefore, $\dot{\phi}$ is constant or piecewise constant, changing only when the state passes through Σ.

In Section 3, we give a one-dimensional example (Juggling) where $h : X \to Y$, defined by $h(x, \dot{x}) = (\phi, \dot{\phi})$, is actually a change of coordinates where $Y = S^{1} \times \mathbb{R}^{+}$. We use the section $\Sigma \in X$ defined by $x = 0$ which corresponds to the set of states where the robot may contact (and thereby actuate) the system. The image of this section $h(\Sigma)$ will be given by the set $\mathcal{C} = \{(0, \dot{\phi}) \mid \dot{\phi} \in \mathbb{R}^{+}\}$. Because we consider intermittent control situations, it is only in this section that $\dot{\phi}$ may be altered by the control input u. That is, we change $\dot{\phi}$ according to a control policy u to get the return map $P' : \mathcal{C} \to \mathcal{C}$ given by $P'(0, \dot{\phi}) = (0, u(\dot{\phi}))^{2}$. We design the controller so that there is a unique and stable fixed point at some desired phase velocity $\dot{\phi}^{*} = \omega$.

Of course we really want to control the system so that the return map P has a stable fixed point at some x^{*}. Whether or not $h^{-1}(0, \omega) = (0, \dot{x}^{*})$ depends on the dimension of Σ. If $dim\ \Sigma = 1$, as it will be in the examples we supply, then the preimage of ω is indeed a point.

The main contribution of this paper concerns the composition or interleaving of two such systems. That is, we suppose that we have the system $(x_1, \dot{x}_1, x_2, \dot{x}_2) \in X^{2}$ with corresponding phase coordinates $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \in Y^{2}$. As before, system i may only be actuated when $\phi_i = 0$. In the examples we will consider, we suppose that the systems cannot be actuated simultaneously. Thus the set of states where $\phi_1 = \phi_2 = 0$ should be repelling. We will design a controller such that the attracting limit cycle is given by

$$\begin{aligned} \mathcal{G} = \{(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \quad &| \quad \phi_1 = \phi_2 + \frac{1}{2}\ (\mathrm{mod}\ 1) \\ &\wedge \quad \dot{\phi}_1 = \dot{\phi}_2 = \omega\}\ . \end{aligned} \qquad (4)$$

The constraint $\phi_1 = \phi_2 + \frac{1}{2}\ (\mathrm{mod}\ 1)$ encodes our desire to have the pair of phases as far from the situation $\phi_1 = \phi_2 = 0$ as possible. In fact, we will consider the more general case wherein the phase velocities are controlled to $(\kappa A, \kappa B)$ for some $A, B \in \mathbb{Z}$ and scaling factor κ.

To analyze and control such a system, we restrict our attention to the sections $\Sigma_1 \subseteq Y^{2}$ and $\Sigma_2 \subseteq Y^{2}$ defined by $\phi_1 = 0$ and $\phi_2 = 0$ respectively. Suppose that the flow alternates between the two sections. Let $G^{t} = H \circ F^{t} \circ H^{-1}$ be the flow in Y^{2} conjugate to the flow in X^{2} where $F = (f, f)$ and $H = (h, h)$ and $\tau_i(w) = \min\{\tau > 0 \mid H \circ F^{\tau} \circ H^{-1}(w) \in \Sigma_{3-i}\}$. Start with a point $w \in \Sigma_1$. Let $w' = G^{\tau_1}(w)$ and $w'' = G^{\tau_2}(w')$. We have $w' \in \Sigma_2$ and $w'' \in \Sigma_1$, so we have defined the return map on Σ_1. Now since G is parameterized by the control inputs u_1 and u_2 we get

$$\begin{aligned} w = (0, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \quad &\mapsto \quad w' = (\phi_1', u_1, 0, \dot{\phi}_2) \\ &\mapsto \quad w'' = (0, u_1, \phi_2', u_2)\ . \end{aligned}$$

[2] In Section 3.1, deadbeat control of the phase velocity is possible. In Section 3.2, the control of phase velocity is asymptotically stable. Our analysis in Section 4 depends on the former. We believe a similar treatment will eventually apply to the latter.

Thus, the phase velocity updates $u_1(w)$ and $u_2(w')$ must be found so that (4) is achieved. We do this with two examples in Section 3 and prove the stability of our method in Section 4.

Notice that a single phase describes a circle S^1 and two phases describe a torus $\mathbb{T}^2 = S^1 \times S^1$. In the next section, we define a "reference" vector field on the k-dimensional $\mathbb{T}^k$ which encodes the ideal behavior of the system as though it were fully actuated. Then, we show how to use the field to generate velocity updates as above.

2.2 Construction of a Reference Flow on $\mathbb{T}^k$

The problem of composing dynamical systems with point-goal attractors is relatively straightforward, due in no small part to the convenient topological fact that the product of a zero-dimensional set (a point goal) with a zero-dimensional set is again a zero-dimensional set: point-goals are well-behaved with respect to Cartesian products. This is not so for the case of systems with an attracting periodic orbit. The Cartesian product of k such continuous systems gives rise to a flow with an attracting k-torus $\mathbb{T}^k \triangleq S^1 \times \cdots \times S^1$. The desired behavior for a flow on this set is again an attracting periodic orbit; however, such mode locking can occur only if the oscillators are coupled. More unfortunately, such dynamics arise only through a relatively careful tuning of the individual systems and their mutual couplings. Baesens et al. [1] carefully explore the intricacies of this problem, illustrating the prevalence of complexity in both the dynamics and the bifurcation structures of flows on the attracting $\mathbb{T}^k$ in the (ostensibly simple) case $k = 3$.

An important measure of complexity for the dynamics of a flow on a torus $\mathbb{T}^k$ is the set of *winding vectors*. Choose a lift $\tilde{\phi}_t$ of the flow ϕ_t on $\mathbb{T}^k$ to the universal cover $\mathbb{R}^k$ of $\mathbb{T}^k$. Then, consider for each $x \in \mathbb{T}^k$ with lift $\tilde{x} \in \mathbb{R}^k$ the vector $\tilde{\phi}_t(\tilde{x})$ normalized to unit length: denote this by $w_t(\tilde{x})$. This vector lies in the unit $(k-1)$-sphere $S^{k-1} \subset \mathbb{R}^k$ of directions in $\mathbb{R}^k$. Define $w(x) \subset S^{k-1}$ to be the set limit points of $w_t(\tilde{x})$ as $t \to \infty$. This set (independent of the lift $\tilde{x}$ in the case of a nonsingular flow) defines the *winding vectors* of x. The union of $w(x)$ over all $x \in \mathbb{T}^k$ comprises the *winding set* of the flow. Winding vectors/sets are the continuous analogues of the rotation vectors/sets defined for torus homeomorphisms[3]: cf. the discussions in [1, 25] in the context of coupled oscillators and [34] for a topological generalization to arbitrary spaces.

The systems we consider have specific constraints on the winding vectors. In order to have a single mode-locked attracting periodic orbit, the winding set must consist of a unique winding vector. In Section 3.1 we present a system consisting of a piston which must vertically juggle two balls so that the first bounces A times for every B times the second bounces (See Figure 4), where A and B are integers: the winding vector is thus (A, B) (rescaled to unit length). The generalization of this situation to n juggled items requires a winding vector of integers $(A_1, A_2, \ldots, A_n)$.

Our goal is to couple systems with unique attractors satisfying the above restrictions in such a manner that the product system remains in the same dynamical class: a single attractor with appropriate winding vector. In addition, the existence of unstable invariant sets (in general forced by topological considerations) is desirable for setting up "walls" of repulsion in the configuration space. For example, in juggling it may be desirable for the configuration wherein both balls hit the paddle simultaneously to be a repellor. For both attractors and repellors, the freedom to "tune" these invariant sets geometrically is a necessity. We thus turn to a brief exposition of two appropriate classes of reference flows on the k-torus $\mathbb{T}^k$ which will serve as skeletons for the goal dynamics of the control schemes to be constructed.

The flows we consider on $\mathbb{T}^k$ will all have global cross sections Σ homeomorphic to $\mathbb{T}^{k-1}$. To obtain a unique attracting periodic orbit for the flow, we specify the appropriate dynamics on the cross-section and accordingly suspend to a flow: the flow is then determined by the dynamics of the return map and the desired winding vector.

Consider the diffeomorphism which is the time-1 map of the gradient field $-\nabla V$, where

$$V : \Sigma \to \mathbb{R} \qquad (\theta_1, \theta_2, \ldots, \theta_{k-1}) \mapsto \kappa \sum_{i=1}^{k-1} \cos(2\pi\theta_i). \tag{5}$$

Here $\Sigma \cong S^1 \times \cdots \times S^1$ is parameterized by $k-1$ angles $\theta_i \in [0, 1]$ and $\kappa > 0$ is an amplitude which controls the

[3] More specifically, for those homeomorphisms which are continuously deformable to the identity map.

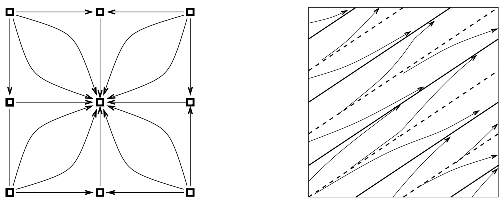


Figure 2: *(left) The ideal reference dynamics on a $\mathbb{T}^2$ cross-section to a flow on $\mathbb{T}^3$ having a single attracting fixed point. Here, the 2-torus is represented as a square with opposite sides identified. (right) A reference flow on $\mathbb{T}^2$ with winding vector $(3, 2)$. The repelling orbit passes through the origin. The appropriate cross-section here is the circle along the "diagonal" of the square.*

rate of attraction. The dynamics of this return map decouples into the cross-product of the circle maps which have the "north pole" ($\theta_i = 0$ for all i) as a repellor and the "south pole" ($\theta_i = 1/2$ for all i) as an attractor. It thus follows that (5) has exactly $(k-1)$-choose-j hyperbolic fixed points whose unstable manifold is of dimension j. There is thus a unique attracting fixed point, and V defines a navigation function [29] for the Poincaré return map of the flow. See Figure 2(left) for an illustration of the dynamics in the case $k = 3$.

The existence, quantity, and placement of the unstable invariant manifolds in the dynamics of (5) are governed by Morse-theoretic considerations (see [29] for applications of Morse theory to the design of navigation functions). Of particular interest is the forced existence of repelling unstable invariant manifolds of all dimensions. This is extremely relevant to the control problem in that the "obstacles" in the configuration space (where the "paddle" is forced to contact several distinct elements simultaneously) can be of variable dimension. The prevalence of unstable manifolds in the dynamics of (5) is a valuable resource when one wants to "tune" the dynamics on the configuration space.

Consider the problem of designing a vector field on $\mathbb{T}^k$ such that all orbits possess a unique winding vector w which points in the direction $(A_1, \ldots, A_k) \in \mathbb{Z}^k$, assuming that the A_i are all relatively prime. The cross-sectional dynamics of this system will be conjugate to (5) for an appropriate cross-section: namely, the cross-section which is the orthogonal complement to τ, where τ is an integer vector satisfying $\tau \cdot (A_1, \ldots, A_k) = 1$, see [1, App. A]. To obtain a reference flow with the

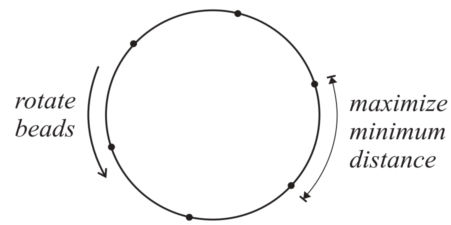


Figure 3: *Embedding distinct phases as the "beads on a circle" problem. The beads must rotate around the circle while maximally avoiding their neighbors.*

desired winding vector, we may suspend (5) to a flow on $\mathbb{T}^k$ and then change coordinates so that the attracting orbit in the new coordinates has slope $(A_1, \ldots, A_k)$. Supposing we start with the slope $(0, 0, ..., 1)$, we desire a linear map M on $\mathbb{R}^k$ (the covering space for $\mathbb{T}^k$) such that $M \cdot (0, 0, ..., 1) = (A_1, \ldots, A_k)$ and so that M, when projected onto the torus is a change of coordinates. This amounts to requiring that $M \in SL(n, \mathbb{Z})$ with its last column given by $(A_1, \ldots, A_k)$.

This construction can be tuned so that the attracting orbit does not pass through the pairwise "obstacle" where two phases are identical. However, in the resulting system the obstacles become dynamically neutral — neither attracting nor repelling. In applications where these obstacles are not physically meaningful, we may use this construction. Otherwise we must design a reference field wherein these obstacles are dynamically repelling.

One manner of generalizing (5) to a flow on $\mathbb{T}^k$ which avoids determining a complicated coordinate change and which may be suitable in applications where the obstacles are important is as follows. We imagine the phases of the system as coming from k distinct point on a circle. Each point must rotate around the circle with some velocity and the distance between any two consecutive points must be maximized, as in Figure 3. The potential function

$$\begin{aligned} V : \mathbb{T}^k &\rightarrow \mathbb{R} \\ (\phi_1, \ldots, \phi_k) &\mapsto \kappa \sum_{i<j} \cos\left(2\pi[A_i\phi_j - A_j\phi_i]\right) \end{aligned} \quad (6)$$

is used to accomplish the latter task. Here the coordinates $\phi_i \in S^1 = [0, 1]/\{0 \sim 1\}$ are angular coordinates on T^k. The function V attains a global maximum along the straight line (mod 1) through the origin tangent to the winding vector $(A_1, \ldots, A_k)$. The function attains its global minimum along a shifted parallel line (mod 1). The addition of a global drift term in the winding direction gives a realization of the desired flow.

$$\dot{\phi}_i = \kappa_1 A_i - (\nabla V)_i \tag{7}$$

See Figure 2(right) for an illustration of the two dimensional case which is used in the examples in this paper. With this field the obstacles are indeed repelling, but there are several attracting orbits, one for each of the $(k-1)!$ arrangements of k beads on a circular wire. For applications such as juggling any of these orbits will represent a viable juggling behavior. For other applications, such as controlling the gait of a legged machine, further tuning to achieve the correct order will be required. In the case of $k = 2$ that we specifically consider in this paper, the attracting orbit is unique and maximally bounded away from the origin.

3 From Juggling to Running

In this section we examine in detail the task of regulating the phases of just two cyclic processes. We will consider intermittent contact systems. For example, a ball being bounced on a controllable paddle is an intermittent contact system where the phase velocity corresponds to the energy of the ball between contacts. Another type of intermittent contact system is one that has a *stance mode*, that is, $\dot{\phi}_i$ is controllable only when $\phi_i \in [a, b] \subseteq [0, 1]$ for some a and b. A hopping robot, for example, is in its stance mode when it is touching the ground. Only in stance mode may the controller change the energy of the robot. We will not consider stance systems in general but instead show how to consider certain models of hopping robots as though their phase velocities were determined by their phase velocities at the single point $\phi_i = a = 0$, thereby reducing the problem to one apparently involving instantaneous contact.

We are concerned with regulating the two systems so that (1) the rate of change of each phase is some desired value (i.e. the first system oscillates A times for every B times the second does) and (2) the phases are maximally separated. That is, we require that

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad A\phi_2 = B\phi_1 + \frac{1}{2} \pmod 1 \tag{8}$$

where κ_1 scales the phase velocities A and B to values reasonable for the system.

We construct a reference vector field on $\mathbb{T}^2$ with this circle as a limit cycle such that $(\dot{\phi}_1, \dot{\phi}_2) = \kappa_1(A, B)$ along the cycle as described in Section 2. This field encodes the ideal behavior of the system as though it were fully actuated. The potential function is

$$V(\phi_1, \phi_2) \quad = \quad \cos(2\pi[A\phi_2 - B\phi_1]) \,. \tag{9}$$

and the field is

$$\mathcal{R}(\phi_1, \phi_2)^T = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} - \kappa_2 \nabla V(\phi_1, \phi_2), \tag{10}$$

the two dimensional instantiation of (6) and (7). Here κ_2 is an adjustable gain which controls the rate of convergence to the limit cycle. The lines $A\phi_2 = B\phi_1$ and $A\phi_2 = B\phi_1 + \frac{1}{2}$ are equilibrium orbits. The first is unstable, the second is stable.

3.1 Juggling

Consider the system wherein a paddle with position p controls a single ball with position b to bounce, repeatedly, to a prespecified apex. We suppose the paddle always strikes the ball at $p = b = 0$ and instantaneously changes its velocity according to the rule

$$\dot{b}_{new} = -\alpha \dot{b} + (1 + \alpha)\dot{p} \tag{11}$$

where α is the coefficient of restitution in a simple ball and paddle collision model. We suppose that the velocity of the paddle is unchanged by collisions. Evidently, a paddle velocity of $\dot{p} = (\alpha - 1)/(\alpha + 1)\dot{b}$ will set $\dot{b}_{new} = -\dot{b}$. Now define $\eta = \frac{1}{2}\dot{b}^2 + bg$ to be the total energy of the system. By conservation of energy, $\dot{\eta} = 0$ between collisions. Set η^* to be the desired energy (corresponding to a desired apex). Define a reference trajectory for the paddle to follow by $\mu = cb$ where

$$c = \frac{\alpha - 1}{\alpha + 1} + \kappa(\eta - \eta^*)$$

is constant between collisions. μ is called a *mirror law* because it defines a distorted "mirror" of the ball's trajectory. As the ball goes up the paddle goes down and *vice versa*. κ is a gain that adjusts how aggressively the controller minimizes the energy error. The analysis of this system in [5] proceeds, roughly, as follows. A "return map", mapping the energy just before a collision

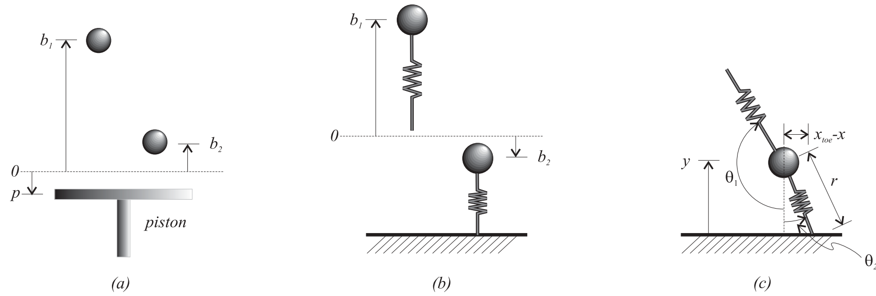


Figure 4: *Models considered in this paper: (a) two balls juggled on a piston; (b) two Raibert style hoppers, hopping out of phase; (c) two legged spring loaded inverted pendulum (SLIP) model of a biped.*

to the energy just before the next collision, is derived. It is shown that the discrete, real valued dynamic system that results is globally asymptotically stable by showing that the map is unimodal [10] with parameter κ adjusting the period of the map.

3.1.1 Phase Regulation of Two Balls

To control *two* balls to bounce on the paddle so that one hits exactly when the other is at its highest point as in Figure 4(a), we will use the reference field (10). This represents a point of departure from earlier work on juggling two balls where a *phase error* term was combined with two mirror laws somewhat informally.

We construct a mirror law for each system separately and then combine the laws into a single mirror law using an "attention function". First define the phase of a ball according to the discussion in Section 2.1. Suppose a ball rebounds from a collision with the paddle with velocity $\dot{b}_0$. By integrating the dynamics $\ddot{b} = -g$ and noting that collisions occur when $b = 0$, we obtain the time since the last impact and the time between impacts, a computationally effective instance of (1) and (2), as

$$t^- = \frac{\dot{b}_0 - \dot{b}}{g} \text{ and } s = t^- + t^+ = \frac{2\dot{b}_0}{g} \qquad (12)$$

respectively. The change of coordinates $h : (\mathbb{R}^+ \times \mathbb{R}) - (0,0) \to S^1 \times \mathbb{R}^+$ from ball coordinates to phase coordinates is given by $h(b, \dot{b}) = (\phi, \dot{\phi})$ where, following the recipe (3), we take

$$\phi = \frac{t^-}{s} = \frac{\dot{b}_0 - \dot{b}}{2\dot{b}_0} \text{ and } \dot{\phi} = \frac{g}{2\dot{b}_0}\,. \qquad (13)$$

In this manner, for a two ball system with ball positions b_1 and b_2, we obtain two phases ϕ_1 and ϕ_2. Notice that $\phi_i \in [0,1]$ and that $\dot{\phi}_i$ is constant between collisions between ball i and the paddle as established in Section 2.1. The velocity $\dot{\phi}_i$ is reset instantaneously upon collisions, corresponding to the update rule (11).

In the rest of this section we will take advantage of the fact that the flow $G^t = H \circ F^t \circ H^{-1}$, described in Section 2 and instantiated here, has the very simple form $(y_1, \dot{y}_1, y_2, \dot{y}_2) \mapsto (y_1 + \dot{y}_1 t, \dot{y}_1, y_2 + \dot{y}_2 t, \dot{y}_2)$ between collisions.

We define reference trajectories μ_1 and μ_2 based on the mirror law idea. Given any pair (ϕ_1, ϕ_2) we define a lookahead function $C_1 : [0,1]^2 \to [0,1]^2$ for ball one which gives the phase of ball two at the next ball one collision. Thus,

$$C_1(\phi_1, \phi_2) = \frac{\dot{\phi}_2}{\dot{\phi}_1}(1 - \phi_1) + \phi_2.$$

We desire that after this collision, $\dot{\phi}_{1,new} = u_1 = \mathcal{R}_1 \circ C_1(\phi_1, \phi_2)$ (i.e. the control input u_i follows $\mathcal{R}$). Since $\dot{\phi}_{1,new} = -g/\sqrt{2\eta_{1,new}}$ and $\eta_{1,new} = \frac{1}{2}\dot{b}^2_{1,new}$ (since $b_1 = 0$ at the collision), we have, using (11),

$$\dot{\phi}_{1,new} = \frac{-g}{-\alpha\dot{b}_1 + (1+\alpha)c_1\dot{b}_1} = \mathcal{R}_1 \circ C_1(\phi_1, \phi_2) \quad (14)$$

where c_1 is the coefficient in the mirror law trajectory $\mu_1 = c_1 \dot{b}_1$. Solving for c_1 and using the fact that at $b_1 = 0$, $\dot{b}_1 = \sqrt{2\eta_1}$, gives

$$c_1 = \frac{1}{(1+\alpha)\sqrt{2\eta_1}}\left[\alpha\sqrt{2\eta_1} - \frac{g}{\mathcal{R}_1 \circ C_1(\phi_1, \phi_2)}\right]. \quad (15)$$

A similar expression for c_2 can be obtained in terms of $\mathcal{R}_2 \circ C_2$. This gives us a mirror law for each ball.

However, we have only one paddle so we need an attention function $s : [0,1]^2 \to [0,1]$ and a new reference trajectory composed of μ_1 and μ_2:

$$\mu = s\mu_1 + (1-s)\mu_2.$$

Such a function is fairly easy to define for specific instances of A and B. A more complete treatment of attention functions can be found in [18].

We have simulated this system and have found it to work as expected. Figures 5(a) and (b) show a run of the system with $A : B = 1 : 1$ and Figures 5(c) and (d) show a run of the system with $A : B = 2 : 1$. In both cases the paddle regulates the phases to very near the limit cycle within two or three hits of the balls. After presenting two more examples of phase regulated systems, we give an analysis of this controller for the 1:1 case in Section 4.

3.2 Synchronized Hoppers

In this section we examine a model of a bouncing point mass reminiscent of Raibert's hopper [27]. A single, vertical hopping leg is modeled by a mass $m = 1$ attached to a massless spring leg. The hybrid dynamics has three discrete modes: flight (the leg is not touching the ground), compression (the leg has landed and is compressing) and decompression (the mass has reached its lowest point and is being pushed upward). These latter modes each have the dynamics of a linear, damped spring. Flight mode is entered again once the leg has reached its full extension. The equations of motion, borrowed and altered somewhat from [20], are as follows:

$$\ddot{x} = \begin{cases} -g & \text{if } x > 0 \quad \text{flight} \\ -\omega^2(1+\beta^2)x - 2\omega\beta\dot{x} & \text{if } x < 0 \wedge \dot{x} < 0 \quad \text{compression} \\ -\omega_2^2(1+\beta_2^2)x - 2\omega_2\beta_2\dot{x} & \text{if } x < 0 \wedge \dot{x} > 0 \quad \text{decompression} \end{cases} \tag{16}$$

where ω and β are parameters which determine the spring stiffness $\omega^2(1+\beta^2)$ and damping $2\omega\beta$. During decompression, $w_2 \triangleq \omega\tau$ and $\beta_2 \triangleq \beta/\tau$. We define τ, the thrust, by $\tau \triangleq \kappa/(1+x_b^2)$ where x_b is the lowest point reached by the mass the last time the decompression mode was entered and κ is a gain which determines the energy of the leg (and which we will use as a control input in the next section). Notice that the damping during decompression, $2\omega_2\beta_2 = 2\omega\beta$, is the same as during compression. The spring stiffness during decompression is $\omega_2^2(1+\beta_2^2) = \omega^2(\tau^2+\beta^2)$ and is thus proportional to the magnitude of the thrust. The trajectory in the position/velocity plane of a simulation of this system is shown in Figure 6 (a). That this is a stable system follows from an argument similar to that made in [20] wherein similar systems are shown to have global asymptotic stability. Thus, a leg cannot be controlled to a specified apex height in one bounce as could the balls in Section 3.1.

To obtain a definition of phase for the leg, we change coordinates to real canonical form in the compression and decompression modes. This is demonstrated in [20]. We arrive at systems $(\dot{E}_c, \dot{\theta}_c) = (-2\omega\beta E_c, -\omega)$ and $(\dot{E}_d, \dot{\theta}_d) = (-2\omega\beta E_d, -\omega\tau)$ for compression and decompression respectively. Note that the changes in the second coordinates of these systems are constant. They can therefore be used as phase variables. In compression, for example, $\dot{\theta}_c = \omega$ and θ_c varies between $\theta_{c,max}$ and 0. Thus,

$$\theta_c = \omega t = \theta_{c,max}\frac{t}{t_c} \tag{17}$$

where t_c is the duration of the compression mode. This expression is a scalar multiple of the general phase definition described in Section 2.1. However, to control two physically unconnected legs to hop in a synchronized manner as in Figure 4(b), our present constructive approach requires constant phase velocity throughout each cycle of a hop as in Figure 6(b). We must transform three piecewise linear phases so that each has the same rate of change, as assumed in the discussion of Section 2.1. We begin with the phases during compression and decompression, θ_c and θ_d and the phase during flight, $\theta_f \triangleq \frac{\dot{x}_0 - x}{2\dot{x}_0}$.

The construction of a suitable phase definition from θ_d, θ_c and θ_f is in two steps. First, apply affine transformations so that the decompression phase varies from 0 to 1/3, the flight phase from 1/3 to 2/3 and the compression phase from 2/3 to 1. This gives a piecewise linear map from interval of time connecting the time of one lowest point to the time of the next. Next, smooth this map so that each segment has the same slope.

First, note that θ_d varies between 0 and $\theta_{d,min}$, θ_f varies between 0 and 1 and θ_c varies between $\theta_{c,max}$

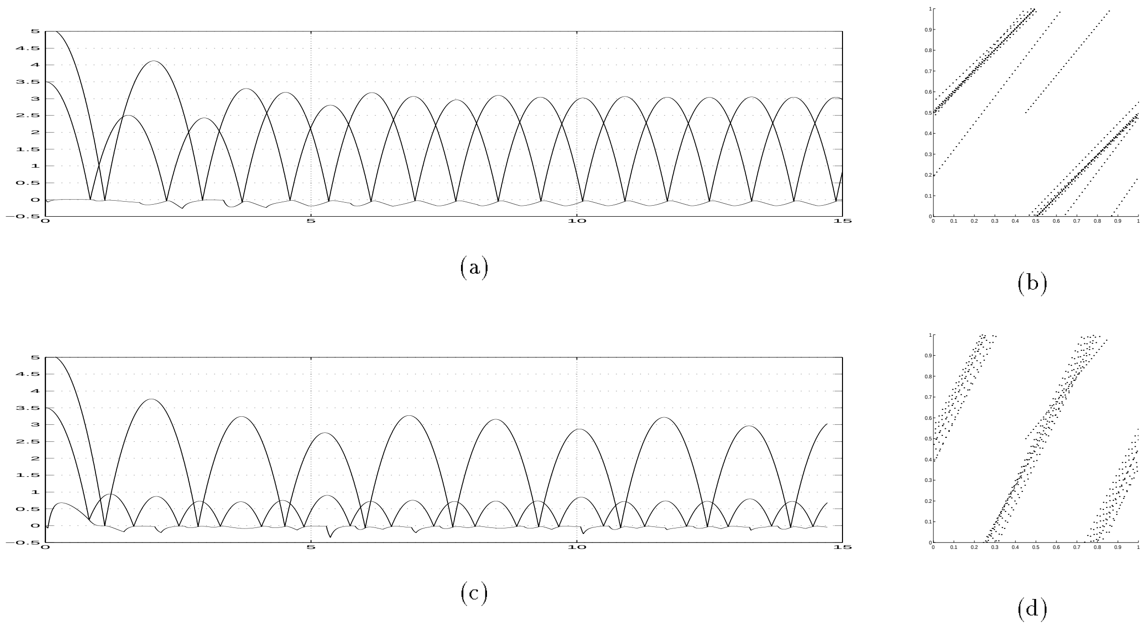


Figure 5: *(a) The positions of the two balls and the paddle vs. time for $A\!:\!B = 1\!:\!1$ starting from a randomly chosen initial condition. (b) The phase plot $\phi_1(t)$ vs. $\phi_2(t)$ for the same run. Note the limit cycle where $\phi_1 = \phi_2 + \frac{1}{2}$. (c) and (d) show the same information for $A\!:\!B = 1\!:\!2$.*

and 0. Thus,

$$\tilde{\theta}_d \triangleq -\frac{\theta_d}{3\theta_{d,min}}, \qquad \tilde{\theta}_f \triangleq \frac{\theta_f}{3} + \frac{1}{3} \quad \text{and} \quad \tilde{\theta}_c \triangleq -\frac{\theta_c}{3\theta_{c,max}} + 1 \tag{18}$$

is the first transformation.

Second, think of each of these phases as a linear homogeneous transformation $P_d : [0, t_1] \to [0, 1/3]$, $P_f : [t_1, t_2] \to [1/3, 2/3]$ and $P_c : [t_2, t_3] \to [2/3, 1]$ where t_1, t_2 and t_3 are the durations of the liftoff, touchdown and bottom modes, respectively. We desire the phase to increase from 0 to 1 linearly as time increases from 0 to t_3. The following, final definition of phase for the hopping leg system, has this property:

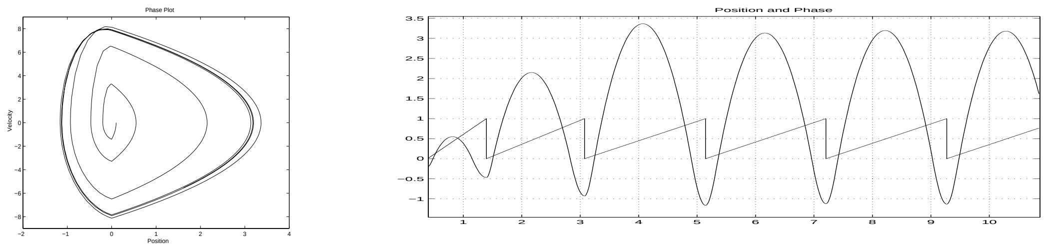
$$\theta \triangleq \begin{cases} HP_d^{-1}(\tilde{\theta}_d,\, 1) & \text{if } x < 0 \wedge \dot{x} > 0 \\ HP_f^{-1}(\tilde{\theta}_f,\, 1) & \text{if } x > 0 \\ HP_c^{-1}(\tilde{\theta}_c,\, 1) & \text{if } x < 0 \wedge \dot{x} < 0\,. \end{cases} \tag{19}$$

Here H is the transformation $(x, 1) \mapsto (x/t_3, 1)$. This results in a phase definition for a single leg that is equivalent to $t/(t_1 + t_2 + t_3)$ as in (3). Figure 6 (b) illustrates how θ changes with the position of the mass in simulation.

For a system with two physically unconnected legs modeled by (16) with states $(x_1, \dot{x}_1)$ and $(x_2, \dot{x}_2)$, let θ_1 and θ_2 be the phases of the legs. Since each leg actuates itself, there is no need for an attention function. Once again, we wish to control the legs so that they are hopping to a prespecified height and so that they are out of phase: one leg is at its lowest point when the other is at its highest. We use a reference field $\mathcal{R}$ with $A\!:\!B = 1\!:\!1$.

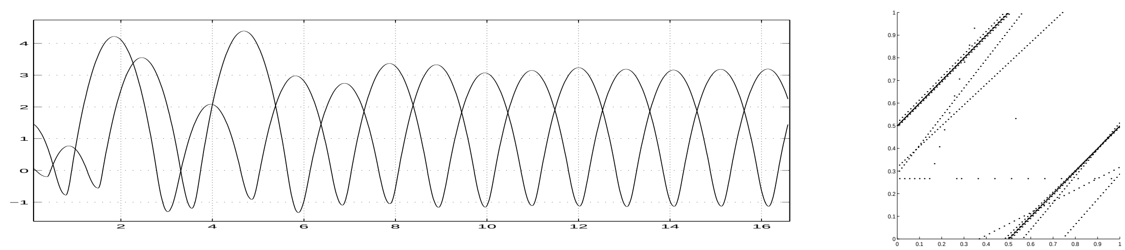
Recall that the thrust, τ_i, supplied by a leg is constant through the decompression phase. The gain for each thrust, κ_i, controls the phase velocity. A larger thrust gives a smaller phase velocity (because the leg goes higher and takes longer to return to its lowest point). Thus, we simply reset the thrust gains, κ_i to be $1/\mathcal{R}_i(\theta_1, \theta_2)$ whenever $\theta_i = 0$, $i = 1, 2$.

Figures 7(a) and (b) show simulations of the system with $A\!:\!B = 1\!:\!1$. In all runs with various reasonable values of the parameters the legs regulate their phases to very near the limit cycle within two or three stance modes.



(a) (b)

Figure 6: *(a) A plot of the velocity vs. the position of a simulated one legged hopper emphasizing the stable limit cycle. (b) A plot of the position and phase of the hopper vs. time. Between minimal points of the hop, phase velocity is constant.*



(a) (b)

Figure 7: *(a) The positions of the leg's centers of mass versus time for $A:B = 1:1$ starting from a randomly chosen initial condition. (b) The phase plot $\theta_1(t)$ vs. $\theta_2(t)$ for the same run. Note the limit cycle where $\theta_1 = \theta_2 + \frac{1}{2}$.*

3.3 Two Legged SLIP

One obvious shortcoming of the synchronized hopping model of walking is that the two legs are physically unconnected. In this section we examine the *Spring Loaded Inverted Pendulum* (SLIP) model [35] of a hopping leg which we have modified to have two legs as shown in Figure 4(c). The SLIP model has a single point mass (which we assume to be 1 in this paper) in the plane and a massless, spring loaded leg. When it is not touching the ground, its dynamics are ballistic. When the toe of the leg is touching the ground, the spring of the leg exerts force on the mass along the direction of the leg. Our slightly different model consists of a mass with two rotating, massless legs which we call the *roadrunner*. In an alternating gait, the roadrunner uses one leg to hop and then the other. When a leg is touching the ground, it is in *stance* mode – which again consists of compression and thrust phases. Between hops, a leg must swing around the mass to ready itself for the next hop. This is called the *swing phase*. The task is to construct a controller for the legs that realizes this gait.

This example is different from our previous examples in that we start with a specification of the aggregate behavior of the system and decompose it into controllers for the individual subsystems. In the juggling and hopping examples, the phase regulated systems are themselves cyclic. For example, in the $1:1$ case, the aggregate phase of a phase regulated system at equilibrium is

$$\phi_{agg}(\phi_1, \phi_2) = \begin{cases} \frac{1}{2}(\phi_1 + \phi_2 - \frac{1}{2}) \text{ if } \phi_1 < \phi_2 \\ \frac{1}{2}(\phi_1 + \phi_2 + \frac{1}{2}) \text{ otherwise .} \end{cases} \quad (20)$$

In the juggling example, ϕ_{agg} is a measurement of the phase of the paddle. In this example, ϕ_{agg} will be a measurement of the phase of the underlying SLIP model which the legs will then "service" according to the pseudo-inverse of ϕ_{agg}:

$$\phi_{agg}^{\dagger}(\phi) = (\phi, \phi + \frac{1}{2}) \pmod 1 . \quad (21)$$

That is, the phases of the legs are functions of the phase of the underlying SLIP model. It remains to define phase for the SLIP model.

As shown in Figure 4(c) the system is described by variables x, y, θ_1, and θ_2 where (x, y) is the position of the body and θ_1 and θ_2 are the angles of the legs. When in stance mode, the position of the body will also be described by the distance r from the body to the toe (given by (x_{toe}, y)) and the angle θ of the leg touching the ground. When a leg is compressing, $\dot{r} < 0$, the stiffness of the spring is k_1 and when it is decompressing, $\dot{r} > 0$, the stiffness is k_2. The spring model we will use has potential

$$U(r) = \frac{k}{2}\left(\frac{1}{r^2} - \frac{1}{l^2}\right)$$

where l is the natural length of the leg and k is the current spring stiffness. The dynamics of the system can be derived from the Hamiltonian as in [35] for the cases with or without gravity in stance. We consider only the case without gravity. We have during flight:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \text{ and } \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where g is the force of gravity and u_1 and u_2 are velocity inputs the legs. During stance, suppose that leg 1 is touching the ground and leg 2 is not so that $\theta = \theta_1$. Then we have

$$\begin{pmatrix} \ddot{r} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} r\dot{\theta}^2 - \frac{1}{m}\nabla U(r) \\ -2\dot{r}\dot{\theta}/r \end{pmatrix} \text{ and } \begin{pmatrix} \ddot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ u_2 \end{pmatrix}$$

where u_2 is a control input. The equations for when the legs are reversed are similar. We do not consider the case where both legs are touching the ground or when the mass hits the ground – situations we would like to avoid. In [33], the control of the SLIP model is discussed. We do not repeat this discussion here, but simply assume that upon liftoff that θ_{td}, k_1 and k_2 are given by the controller.

The phase of the virtual leg will once again be composed of the phases of flight, compression and decompression. In the rest of the section, variables subscripted with l represent the state at liftoff and those subscripted with td represent that state a touchdown. As in the previous example, the phase is obtained from a piecewise linear transformation on the phases during the various modes. We use the results from [35] wherein the systems are integrated to obtain the durations of the flight, compression and decompression modes, t_f, t_c and t_d respectively. For a given state w of the leg, equation (2) defines $t^-(w)$ to be the time since the last lift off. Then the phase is

$$\phi = \frac{t^-}{t_f + t_c + t_d} .$$

Notice that the phase varies between 0 and 1. Since each leg will service every other stance mode, we could define the phase of the SLIP model so that it completes two cycles between 0 and 1 instead of one cycle so that (21) makes sense in the present context. We neglect this detail here.

Now define the position of the legs during their swing phases as a function of the phase. Let θ_{top} be an angle near the middle of the swing phase, such as π. We give each leg a discrete state s_i defined by

$$s_i = \begin{cases} 0 \text{ if } \theta_l < \theta_i < \theta_{top} \\ 1 \text{ if } \theta_{top} < \theta_i < \theta_{td} \\ 2 \text{ otherwise (leg is touching the ground)} . \end{cases} \tag{22}$$

Thus, a leg is characterized by a sequence such as $\langle 0, 1, 2, 0, 1, 2 ... \rangle$. We define reference maps (functions of phase, which is in turn a function of the state of the body) $\theta_{ref,0}$ and $\theta_{ref,1}$ as functions of the leg phase which give the ideal trajectory of a leg during each of the discrete states 1 and 2. $\theta_{ref,0}$ varies between θ_l and θ_{top} as ϕ varies from 0 to 1 so that $\dot{\theta}_{ref,0}$ is equal to $\dot{\theta}$ at liftoff. $\theta_{ref,1}$ varies from θ_{top} to θ_{td} as ϕ varies from 0 to its value at touchdown. These may be smoothed in various ways to minimize, for example, the velocity of the toe relative to the ground at touchdown. There is no reference phase during stance because when a leg is in stance it is not actuated. If the discrete states of the legs are initially different, they will alternately service the stance mode of the robot.

4 Analysis of the Phase Regulation Algorithm

We have presented three examples of phase regulation that differ in several important respects. In the juggling controller, we are assured that the reference field can be followed closely because of the deadbeat nature of the ball control. That is, within the limits of the actuator, we can achieve any desired ball energy by striking it with the paddle using (11). Therefore, to analyze the stability of the control method, we need only consider the system in terms of the phase states and velocities. We do so in this section. With the synchronized hopping example, we do not have deadbeat leg control, but only asymptotic stability. Thus, to analyze the stability of the hoppers, we would need to take in to account the rate of convergence of a single leg to the reference phase velocity dictated by the reference field controller. We have not yet performed this analysis. However, because of the fast rate of convergence of the single leg controller in practice, the analysis in this section is likely appropriate. The two legged SLIP controller, in a sense, needs no further analysis. If we assume that the legs can follow the reference trajectory accurately, the model is the same as the original SLIP model [35].

4.1 Analysis

Consider the phase regulated system $(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) \in \mathbb{T}^2 \times \mathbb{R}^2$ where $\dot{\phi}_i$ is constant except for discrete jumps made when $\phi_i = 0$. These jumps are governed by the reference field (10). That is, when $\phi_i = 0$, $\dot{\phi}_i$ becomes $\mathcal{R}(\phi_1, \phi_2)$. Notice that when $A : B = 1 : 1$, then $\mathcal{R}(0, \phi_2) = \mathcal{R}(\phi_1, 0)$. To simplify notation in this section, we redefine $\mathcal{R} : S^1 \to \mathbb{R}$ to be the reference field restricted to $\phi_1 = 0$. Therefore, with $A{:}B = 1{:}1$, $\mathcal{R}(\phi) = \kappa_1 - \kappa_2 \sin(2\pi\phi)$.

To analyze the dynamics of this system, we consider the Poincaré sections Σ_1 and Σ_2 of $\mathbb{T}^2 \times \mathbb{R}^2$ given by $\phi_1 = 0$ and $\phi_2 = 0$ respectively. We suppose that adjustments to the phase velocities alternate between the two phases (i.e. the system is near the limiting behavior). We construct the return map from Σ_1 into Σ_1 as follows. Start with a point $w \in \Sigma_1$, integrate the system forward to obtain a point in Σ_2, then integrate again to get a point in $f(w) \in \Sigma_1$.

A point in Σ_1 has the form $w = (0, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$. This maps to the point $w' = (C_1, 0, \mathcal{R}(\phi_2), \dot{\phi}_2) \in \Sigma_2$ where C_1 is the phase of the first system when the trajectory of the total system first intersects Σ_2. w' in turn maps to the point $f(w) = (0, C_2, \mathcal{R}(\phi_1), \mathcal{R}(C_1))$ where C_2 is the phase of the second system when the trajectory next intersects Σ_1. The phases C_1 and C_2, which can be obtained via the point-slope formula for a line (in the ϕ_1, ϕ_2 plane), are given by

$$C_1 = \frac{\mathcal{R}(\phi_2)}{\dot{\phi}_2}(1 - \phi_2) \text{ and } C_2 = \frac{\mathcal{R}(C_1)}{\mathcal{R}(\phi_2)}(1 - C_1) . \quad (23)$$

Let $(x, y, z) = (\phi_2, \dot{\phi}_1, \dot{\phi}_2)$. Then, expanding $f(w)$, we obtain a discrete, real valued map on Σ_2 given by

$$\begin{aligned} x_{k+1} &= \frac{\mathcal{R}\left[\frac{\mathcal{R}(x_k)}{z_k}(1 - x_k)\right]}{\mathcal{R}(x_k)} \left[1 - \frac{\mathcal{R}(x_k)}{z_k}(1 - x_k)\right] \\ y_{k+1} &= \mathcal{R}(x_k) \\ z_{k+1} &= \mathcal{R}\left[\frac{\mathcal{R}(x_k)}{z_k}(1 - x_k)\right] . \end{aligned} \quad (24)$$

Since the x and z advance functions are not functions of y, we can treat y as an output of this system. Thus, analytically, it will suffice to treat (24) as an iterated map of the the variables $(x, z) \in S^1 \times \mathbb{R}^+$ given by $F(x_k, z_k) = (x_{k+1}, z_{k+1})$. We have the following fixed point conditions:

Proposition 4.1 *$F(x, z) = (x, z)$ if and only if $\mathcal{R}(x) = \mathcal{R}(1 - x) = z$.*

We omit the proof, which is straightforward algebra (note that the values of x are always taken modulo 1 since $x \in S^1$). For the reference field we are using, we have:

Corollary 4.1 *If $\mathcal{R}(\phi) = \kappa_1 - \kappa_2 \sin(2\pi\phi)$, then the only fixed points of F are $(1/2, \kappa_1)$ and $(0, \kappa_1)$.*

We wish to show that the first fixed point, $(1/2, \kappa_1)$, is stable, since it corresponds to the situation where the two subsystems are out of phase and at the desired velocity. To do this, we examine the Jacobian. Suppose that the fixed point condition we desire is $F(1/2, v) = (1/2, v)$ where v is the desired phase velocity. Then

$$J_{(\frac{1}{2}, \kappa_1)}F = \begin{pmatrix} \frac{1}{2v^2}\left(\frac{m}{2} - 1\right)m - \frac{m}{v} + 1 & \frac{1}{2v} - \frac{m}{4v^2} \\ \frac{1}{v}\left(\frac{m}{2} - 1\right)m & -\frac{m}{2v} \end{pmatrix} . \quad (25)$$

Here, $m = \mathcal{R}'(1/2)$ is the slope of $\mathcal{R}$ at $1/2$. F is stable at $(1/2, v)$ if the eigenvalues of the Jacobian lie within the unit circle. Values for m and v which guarantee this are not difficult to find. For example,

Proposition 4.2 *If $m = 2v - 2$ then the eigenvalues of $J_{(\frac{1}{2}, \kappa_1)}F$ are 0 and $\frac{2}{v^2} - 1$ which implies that $(1/2, v)$ is a stable fixed point of F whenever $v > 1$.*

Once again, the proof is just a calculation: simplify (25) using $m = 2v - 2$ and compute the eigenvalues. With the reference field we are using, $m = 2\pi\kappa_1$. Thus, for a given value of v, we set $\mathcal{R}(\phi) = v - \frac{m}{2\pi} sin(2\pi\phi)$. In practice, it is not difficult to find other parameters which make F stable. For a given v, we first choose m to be quite small and increase it slowly until the controller is aggressive, yet still stable.

5 Conclusion

In this paper we have taken the first steps toward a formal treatment of phase regulation for underactuated environments that must be repeatedly and intermittently contacted by an actuated robot. We have introduced a variant of the two-juggle controller [5, 30] and, by re-writing the system in phase coordinates, exhibited sufficient conditions for local asymptotic stability of a 1:1 mode-locked rhythm. The obvious next step concerns the extent of the domain of attraction. Here, there is a natural hybrid structure — the order of "contact events" (i.e., the sequence of balls hit) — whose desired sequences might be seen as a pattern to be regulated against disturbances. Moreover, there is a "forbidden" set in phase space — where both balls must be hit at the same time — that must be shown to be a repeller. We have also suggested the manner in which this 1:1 "juggling" framework carries over to simple problems in legged locomotion. Because the effective input enters through an additional dynamical lag in such problems, our present sufficient conditions for asymptotic stability will need to be modified in order to address them. We have not dealt at all with the problem of regulating more general A:B mode-locking, but we believe that similar methods can be used to achieve such behaviors.

Although the applications focus of this paper is limited to locomotion systems, we are intrigued by the prevalence of phase regulation problems in more abstract settings such as factory automation [19, 17] and will seek to apply these ideas in that context as well.

Acknowledgements

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