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New Discrete States of Strings Near a Black Hole

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We find the physical spectrum for the "black hole" solution recently proposed by E. Witten as an exact string background with a 2-dimensional target space. The spectrum contains some of the states found in studies of $d = 1$ noncritical string theory but, in addition, has *new* states not previously found in the $d = 1$ noncritical string. Along the way we find a remarkable "stringy" symmetry of the spectrum relating massive states to massless ones.

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1. Introduction

String propagation in a Lorentzian-signature spacetime must be described by a *nonunitary* world-sheet conformal field theory. For example, in flat space the oscillators for the time coordinate, X^0, create states of negative norm. All is not lost, however. By imposing a suitable set of physical state conditions or in a more modern language, by imposing BRS-invariance, one finds a physical subspace of the full Hilbert space in which all states have positive semidefinite norm. Furthermore, one finds that all of the zero-norm states are BRS-exact (corresponding to null-states in the older language). The *BRS cohomology*, the quotient of the physical Hilbert space (*ker* Q) by the space of zero-norm states (*im* Q), is then a positive-definite Hilbert space. Indeed, the BRS-exact states decouple from correlation functions of physical operators, which behave sensibly — just as if one were working with a unitary conformal field theory. This is called the *No-Ghost Theorem* [1] [2][3].

As originally stated, the no-ghost theorem was proven for strings propagating in flat spacetime. However, it is not too hard to generalize the proof to the case where some or all of the transverse dimensions are replaced by a more general *unitary* conformal field theory, perhaps corresponding to strings propagating in some curved background. However, it is crucial in all existing proofs of the no-ghost theorem that at least two directions (X^0, X^1) be flat, uncompactified $\mathbb{R}^{1,1}$. This is something of a deficiency, in that one would like to consider strings propagating in more general spacetime backgrounds — black hole solutions, cosmological solutions, *etc.* — in which there are no flat light-cone directions which can be singled out.

There have been a few previous attempts to study string propagation in spacetimes which are non-flat in the above sense. Mostly, they have centered on string propagation on noncompact group manifolds such as $SU(1,1)$, since here we can hope to use methods of current algebra to construct the nonunitary CFT [4]. This theory is somewhat complicated by the infinite degeneracy of states at a given mass level. A much cleaner class of theories are the coset models discussed by Bars and Nemeschansky [5], who considered string propagation on $SO(d,2)/SO(d,1)$. None of the above authors went so far as to prove a no-ghost theorem[1]. The simplest of the models of [5], $d=1$, has recently received a flurry of interest because of the observation by E. Witten [7] that it corresponds to a two-dimensional spacetime with a black hole[2]. Given this interpretation, it is all the more interesting to see if the resulting theory is unitary.

But there are numerous difficulties inherent in such a program. Many key properties of compact current algebras are absent in the noncompact case, for example the uniqueness of highest-weight representations used in [8]. The crucial fact that one may truncate the spectrum of compact current algebra representations to the unitary ones without spoiling modular invariance is no longer assured. More generally the meaning of the Knizhnik-Zamolodchikov equations is unclear; it has even been suggested in the context of complex groups that we must abandon holomorphy of the conformal blocks [9].

Due to all this uncertainty one may wonder whether it is possible to go beyond the semiclassical analysis of [7]. Indeed, in this paper we will not actually construct the black hole field theory explicitly. We will find, however, that very general considerations of representation theory can tell us much about the structure of the theory. Specifically we will use the representation theory of the coset model currents to find all the physical states which *could* enter the full coset-model string theory. As an unexpected bonus we will also find a new "stringy" symmetry of the spectrum of physical states, an isometry which exchanges massive and massless states.[3] We will also see that for Minkowski signature the no-ghost theorem is true in a rather trivial way. For the Euclidean

[1] But see [6], where positivity is claimed for the $SU(1,1)$ string, provided one appropriately truncated the allowed spectrum of j. It is not clear that this truncation is respected by the operator product algebra.

[2] $SL(2,\mathbb{R})/SO(1,1)$ gives the Minkowski signature black hole, while $SL(2,\mathbb{R})/U(1)$ gives Euclidean signature. We will study the BRS cohomology for both theories.

[3] Whether this symmetry is reflected in the *dynamics* of the theory is beyond the scope of our analysis.

solution we will find examples where the cancellation of negative-norm states is quite nontrivial, but we have no general theorem to this effect.

One of the conjectures that Witten made about the black hole theory is that the amplitudes for scattering off the extremal black hole are identical to the scattering amplitudes of $d = 1$ noncritical string theory. We will not calculate any scattering amplitudes in this theory, but we will compare its physical spectrum with that of the $d = 1$ noncritical string. The massless spectrum, the "tachyon" of the $d = 1$ string, is identical. In addition to the tachyon, one finds in the $d = 1$ string a set of states at higher mass levels at certain discrete values of the momenta [10]. We indeed find the analogs of these "discrete states" in the physical Hilbert space of the coset theory. However, this does not complete the list of the physical spectrum of the coset theory. There are additional discrete states, as well as states at nontrivial ghost number. Though the massless spectra agree, the spectrum at higher levels is much richer in the coset theory. Of course it is possible that the true spectrum is a subset of the states allowed by representation theory, so that the two theories really are equivalent. A definitive statement on possible equivalence of these two theories will have to await a direct calculation of the scattering amplitudes in the coset model.

2. The coset model

We consider an $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ current algebra whose currents obey the operator product relations

$$J^a(z)J^b(w) = \frac{\frac{1}{2}k\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{abc}\eta_{cd}J^d}{z-w} \tag{2.1}$$

where $\eta_{ab} = diag(1,-1,1)$ The Sugawara stress tensor of the $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ current algebra is

$$T_{\mathfrak{sl}(2,\mathbb{R})} = \frac{\eta_{ab}}{k-2} : J^a J^b := \frac{1}{k-2} : (J^1)^2 - (J^2)^2 + (J^3)^2 : \quad . \tag{2.2}$$

We will be interested in two important subalgebras of this current algebra: the $\widehat{\mathfrak{u}}(1)$ current algebra generated by J^2 whose stress tensor is

$$T_{\mathfrak{u}(1)} = -\frac{1}{k} : (J^2)^2 : \tag{2.3}$$

and the $\widehat{\mathfrak{so}}(1,1)$ subalgebra generated by J^3, whose stress tensor is

$$T_{\mathfrak{so}(1,1)} = \frac{1}{k} : (J^3)^2 : \quad . \tag{2.4}$$

What representations of the current algebra will be relevant to us? To analyze the Euclidean black hole, it is convenient to diagonalize $L_0^{\mathfrak{sl}(2,\mathbb{R})}$ and J_0^2. The representations are generated by acting on base states $|j,m\rangle$ with the modes of the currents where the base states satisfy

$$\begin{aligned} L_0^{\mathfrak{sl}(2,\mathbb{R})}|j,m\rangle &= \frac{-j(j+1)}{k-2}|j,m\rangle \\ J_0^2|j,m\rangle &= m|j,m\rangle \\ L_n|j,m\rangle &= J_n^a|j,m\rangle = 0, \qquad n>0 \quad . \end{aligned}$$

It is convenient to define the ladder operators $J^{\pm}(z) = J^3(z) \pm iJ^1(z)$, which have simple commutation relations with $J^2(z)$:

$$[J_n^2, J_m^{\pm}] = \pm J_{m+n}^{\pm} \quad .$$

The physics of the WZW model dictates that the representations are endowed with an inner product (which is nondegenerate on the base) and that currents are realized as *Hermitian* operators with respect to this inner product. We may as well require that our representation be an irreducible representation of ordinary $SL(2,\mathbb{R})$ at the base, since we can always decompose it until this is true. We will call a representation in which the currents are Hermitian and the inner product is nondegenerate at the base (but not necessarily positive-definite), a *Hermitian representation*, reserving the term *unitary representation* for the case in which the inner product is positive-definite. Having a Hermitian representation is a serious restriction on the allowed values of j, m. In particular, $j(j+1)$ must be real. Since we demand that the representation be *irreducible* at the base, then m must also be real. The Hermitian inner product [4] on the highest weight states

$$\langle j, m' \mid j, m\rangle = \pm\delta_{m,\overline{m}'} \tag{2.5}$$

is only nondegenerate if for every $m \in spec(J_0^2)$, $\overline{m}$ is also in the spectrum of J_0^2. For an irreducible representation, $spec(J_0^2) \subset m_0 + \mathbb{Z}$, for some fixed $m_0 \in \mathbb{C}$. So a representation is Hermitian if and only if m_0 is real.

The types of Hermitian representations which can occur are then classified by $spec(J_0^2)$ on the base. A highest-weight representation has $spec(J_0^2)|_{\text{base}}$ bounded above, a lowest-weight representation has $spec(J_0^2)|_{\text{base}}$ bounded below, and a double-sided representation has $spec(J_0^2)|_{\text{base}}$ bounded on both sides (Fig. 1). For a continuous representation, $spec(J_0^2)|_{\text{base}}$ is unbounded.

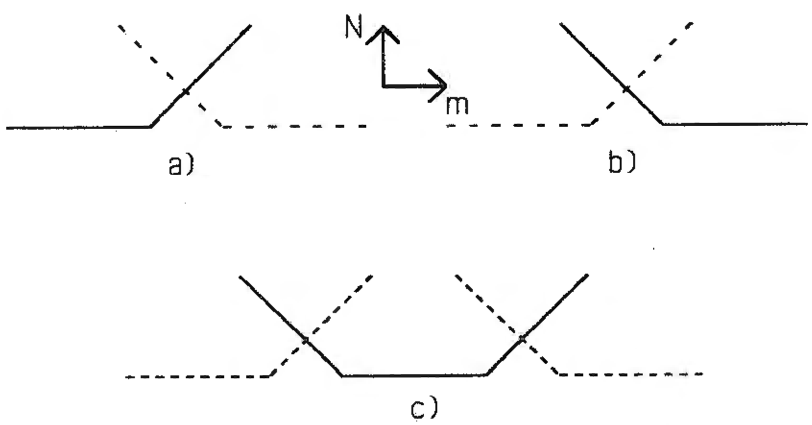


Fig. 1: Decomposing a reducible "continuous" representation:
a) a highest-weight representation
b) a lowest-weight representation
c) a double-sided representation

[4] A unitary representation would have all + signs.

Since the representations are really labeled, not by j, but by the Casimir $-j(j+1)$, we see that j and $-j-1$ are really *the same* representation. To avoid this trivial redundancy, we restrict $j \geq -\frac{1}{2}$ for $j \in \mathbb{R}$. For j in the *principal continuous series*, $j = -\frac{1}{2} + i\lambda$, we should also in principle restrict the sign of the allowed values of λ. However, we will argue later that, in the context of the Minkowski black hole, both signs of λ are in a certain sense physical.

To analyze the Minkowski black hole, we need to construct representations on which J_0^3 can be diagonalized. Only then can we perform the coset construction. These representations behave very differently from those on which J^2 can be diagonalized. We will denote the highest weight eigenstates of J_0^3 by $|j,\mu)$

$$J_0^3|j,\mu) = \mu|j,\mu) \quad . \tag{2.6}$$

In this basis, the currents are Hermitian with respect to the inner product:

$$(j,\mu' \mid j,\mu) = \pm\delta_{\mu,\overline{\mu}'} \quad . \tag{2.7}$$

The difference now is that $spec(J_0^3) \subset \mu_0 + i\mathbb{Z}$. This is because the ladder operators $\widetilde{J}^{\pm} = J^2 \mp J^1$ have the commutation relations:

$$[J_n^3, \widetilde{J}_m^{\pm}] = \pm i\widetilde{J}_{m+n}^{\pm} \quad .$$

So we see that a Hermitian representation has either **a)** $\mu_0 \in \mathbb{R}$ or **b)** $\mu_0 \in \frac{i}{2}+\mathbb{R}$ (*mod* $i\mathbb{Z}$).

The stress tensor of the coset theory is the difference of the stress tensor of $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ and that of the subalgebra (2.3) or (2.4):

$$\begin{aligned} \text{Euclidean:} \quad & T = T_{\mathfrak{sl}(2,\mathbb{R})} - T_{\mathfrak{u}(1)} \\ \text{Minkowski:} \quad & T = T_{\mathfrak{sl}(2,\mathbb{R})} - T_{\mathfrak{so}(1,1)} \quad . \end{aligned} \tag{2.8}$$

In either case, the central charge of the coset model is $c = 2 + \frac{6}{k-2}$. If we wish to build a critical string theory out of the coset model alone, we should take $k = 9/4$, so that $c = 26$. This is interpreted as a string propagating in a $1+1$-dimensional target space. More generally, we can consider $9/4 \leq k \leq \infty$ and take some unitary conformal field theory with central charge $24 - \frac{6}{k-2}$ to represent the "transverse" degrees of freedom of a string propagating in a higher dimensional space.

Finally, we introduce the conformal ghosts $b(z), c(z)$ and a BRS operator

$$Q = \frac{1}{2\pi i}\oint cT + bc\partial c \quad . \tag{2.9}$$

As in any critical string theory, $Q^2 = 0$ for $c = 26$, which in our case means $k = 9/4$. However, there is now one subtlety in our way of treating the coset model. By construction, $J^0(z)$ (where J^0 denotes J^2 or J^3, depending on whether we are discussing the Euclidean- or the Minkowski-signature coset model) commutes with $T(z)$, and hence with Q. Thus if we simply took the cohomology of Q with no further restrictions, we would get an infinite number of isomorphic copies of the same physical cohomology, created by the action of the raising operators J^0_{-n}. As in any situation where we have an additional symmetry acting, we should consider instead the *equivariant BRS cohomology.* Our physical state conditions, then, are:

$$Q|\Psi\rangle = 0 \tag{2.10a}$$

$$b_0|\Psi\rangle = L_0|\Psi\rangle = 0 \tag{2.10b}$$

$$J^0_n|\Psi\rangle = 0 \qquad n > 0 \tag{2.10c}$$

and two physical states $|\Psi_{1,2}\rangle$ are considered to be equivalent (cohomologous) if $|\Psi_1\rangle = |\Psi_2\rangle + Q|\Lambda\rangle$, for some $|\Lambda\rangle$ satisfying (2.10b,c).

The equivariance condition (2.10b) is familiar from usual string theory [3], [11], and is a consequence of correctly treating the ghost zero-modes. The inner product in which we want to prove positivity is

$$(|\psi_1\rangle, |\psi_2\rangle) = \langle\psi_1^* \mid c_0 \mid \psi_2\rangle \quad . \tag{2.11}$$

Q is not Hermitian with respect to this inner product, but on the subspace satisfying (2.10b), it *is* Hermitian. (For another motivation for imposing (2.10b), from the point of view of obtaining well-defined string scattering amplitudes, see [12][13] [14].)

We can usefully rephrase this procedure by taking seriously the Lagrangian formulation of this theory as a *gauged* WZW model coupled to gravity. What we have just said amounts to constructing the coset model first as a conformal field theory on a fixed Riemann surface. Imposing the highest weight condition (2.10c) can be thought of as calculating the cohomology of the BRS operator $Q_{\widehat{\mathrm{u}(1)}}$ associated to the gauged $U(1)$. We then couple the coset model to gravity and calculate the cohomology of the BRS operator Q associated to the diffeomorphism symmetry. In other words, we are considering an *iterated* cohomology problem. Alternatively, one could consider calculating the cohomology of $Q + Q_{\widehat{\mathrm{u}(1)}}$. *A priori*, these two procedures could yield *different* answers for the cohomology, since the former corresponds to considering the coset model as a background for the ordinary bosonic string while the latter corresponds to considering a sort of $U(1)$-extended string theory. Fortunately we are spared this choice. We will see in sect. 8 that both problems yield the same answer for the cohomology.

In either formulation, the problem is now simply stated: calculate the equivariant BRS cohomology of the coset model and prove that the inner product (2.11) on the cohomology is positive definite.

3. The cohomology (Euclidean case)

Before introducing any heavy machinery to compute the BRS cohomology, we can simply look for solutions to (2.10) and see what we find. Let us look for physical states of the form $|\Psi\rangle = c_1|\psi\rangle$, where $|\psi\rangle$ is some state in the coset theory.

First consider the case $|\psi\rangle = |j, m\rangle$, a Kač-Moody primary state. This will be Q-closed provided $|j, m\rangle$ satisfies the mass-shell condition

$$(L_0 - 1)|j, m\rangle = \left(\frac{-j(j+1)}{k-2} + \frac{m^2}{k} - 1\right)|j, m\rangle = 0 \quad . \tag{3.1}$$

This quadratic equation has two roots

$$m = \pm(3j + 3/2) \quad . \tag{3.2}$$

A priori, m can be any real number and $-j(j+1)$ must be real. We *do not*, however, make any other restrictions on the allowed values of j, m in the hope that the resulting parafermion module will be unitary (as in [15]). We expect unitarity to emerge (if at all) only *after* taking the BRS cohomology. Indeed, that is precisely what happens here. The physical states that we have found

$$c_1|j, \pm(3j+3/2)\rangle, \qquad j \in \mathbb{R}, \quad j \geq -1/2$$

cannot be written as Q of something and (trivially) have positive norm.

Can we find some other states? Let us try a state of the form

$$|\psi_+^N\rangle = (J_{-1}^+)^N |j, m-N\rangle \quad .$$

This state is automatically annihilated by L_n and J_n^2 for $n \geq 2$. We must, however, impose by hand the conditions

$$L_1|\psi_+^N\rangle = J_1^2|\psi_+^N\rangle = 0 \quad .$$

Both conditions will be satisfied if

$$J_0^+|j, m-N\rangle = 0 \quad . \tag{3.3}$$

This yields a quadratic equation

$$-j(j+1) + (m-N)(m-N+1) = 0 \quad . \tag{3.4}$$

which, together with the mass-shell condition

$$\frac{-j(j+1)}{k-2} + \frac{m^2}{k} + N = 1 \tag{3.5}$$

determine the allowed values of j, m. We find two solutions

$$m = -3/8 + 3N/2, \qquad j = N/2 - 3/8 \tag{3.6a}$$

$$m = 3/4\,(N-1), \qquad j = (N-1)/4 \quad . \tag{3.6b}$$

Identical reasoning shows that the states

$$|\Psi_-^N\rangle = c_1(J_{-1}^-)^N|j, m+N\rangle \tag{3.7}$$

are physical states for

$$m = 3/8 - 3N/2, \qquad j = N/2 - 3/8 \tag{3.8a}$$

$$m = -3/4\,(N-1), \qquad j = (N-1)/4 \quad . \tag{3.8b}$$

We will see in sect. 7 that these states are but the first entries in a series of physical states which are labeled by a *pair* of positive integers r, s. There is one series (containing the states (3.6a), (3.8a)) which occurs for

$$\widetilde{\mathcal{D}}^{\pm}: \qquad m = \pm\frac{3}{8}(2s-4r-1), \qquad j = \frac{1}{8}(2s+4r-5) \tag{3.9a}$$

and another (containing the states (3.6b), (3.8b)) for

$$\mathcal{D}^{\mp}: \qquad m = \pm\frac{3}{4}(s-2r+1), \qquad j = \frac{1}{4}(s+2r-3) \quad . \tag{3.9b}$$

Finally, there is a series of physical states with

$$\mathcal{C}: \qquad m = \frac{3}{2}(s-r), \qquad j = \frac{1}{2}(s+r-1) \tag{3.10}$$

which come from "continuous series" representations.

One easily calculates that the norms of the states found above. For example, the states in (3.6), $|\Psi_+^N\rangle = c_1|\psi_+^N\rangle$,

$$\begin{aligned}(|\Psi_+^N\rangle, |\Psi_+^N\rangle) =& \langle j, m-N \mid (J_1^-)^N (J_{-1}^+)^N \mid j, m-N\rangle \\ =& N!\frac{\Gamma(k+2m-N)}{\Gamma(k+2m-2N)}\langle j, m-N \mid j, m-N\rangle\end{aligned}$$

have the same sign as that of the highest weight state $\langle j, m-N \mid j, m-N\rangle$ for (3.6a). For (3.6b), the relative sign alternates, depending on whether N is even or odd.

If there were only one state in the BRS cohomology from any given Hermitian representation, we could simply *declare* it to be positive norm. Unfortunately, there are, in general, several states in the BRS cohomology from a given representation. It is nontrivial that their norms should be of the same sign.

Let us look at a simple example. Consider the states in the highest weight representation with $j = 1/2$. There are 2 states in the cohomology, $c_1|\psi_1\rangle$ and $c_1|\psi_2\rangle$, where

$$
\begin{aligned}
|\psi_1\rangle &= (J^{+}_{-1})^3|1/2,-3/2\rangle \\
|\psi_2\rangle &= -\frac{\sqrt{15}}{12}(J^{+}_{-1})^3|1/2,-9/2\rangle \\
&\quad + \left(J^{2}_{-1}(J^{+}_{-1})^2 + \frac{21}{8}J^{+}_{-2}J^{+}_{-1}\right)|1/2,-7/2\rangle \\
&\quad + \sqrt{2}\left(-9J^{+}_{-3} - 68J^{2}_{-1}J^{+}_{-2} + 180J^{2}_{-2}J^{+}_{-1} + \frac{656}{3}(J^{2}_{-1})^2J^{+}_{-1}\right)|1/2,-5/2\rangle \\
&\quad + \sqrt{6}\left(\frac{535}{4}J^{-}_{-2}J^{+}_{-1} - \frac{107}{4}J^{-}_{-1}J^{+}_{-2} + \frac{428}{3}J^{2}_{-1}J^{-}_{-1}J^{+}_{-1} - \frac{2627}{18}J^{2}_{-3}\right. \\
&\qquad \left. - \frac{1868}{9}J^{2}_{-2}J^{2}_{-1} + \frac{1600}{81}(J^{2}_{-1})^3\right)|1/2,-3/2\rangle \quad .
\end{aligned}
\tag{3.11}
$$

When we calculate the norms of these states, we find

$$
\begin{aligned}
\langle\psi_1 \mid \psi_1\rangle &= -\frac{45}{32}\langle 1/2,-3/2 \mid 1/2,-3/2\rangle \\
\langle\psi_2 \mid \psi_2\rangle &= -\frac{22541396057}{41472}\langle 1/2,-3/2 \mid 1/2,-3/2\rangle \quad .
\end{aligned}
$$

Clearly, we can make both of these positive by defining $\langle 1/2,-3/2 \mid 1/2,-3/2\rangle$ to be negative. We have checked a few low-lying representations and find that the same seems to hold true: all of the states in the BRS cohomology have the same sign for their norms and it is consistent to choose them to be all positive. We suspect that this holds true generally, but unfortunately, we have no proof.

In any case, what we really want is something slightly weaker. We don't really care about the norms of the chiral states that we have found. What we really want is that the norms of the nonchiral states obtained by tensoring together in the left- and right-moving BRS cohomologies should be positive norm.

We do not have a complete proposal for tensoring together left and right to obtain the full CFT, but it will suffice to know how to tensor together left- and right-movers from a *given* representation. A physically reasonable proposal

in this regard is discussed in [16]. The prescription is simply that the left- and right-mover lie in the same representation and $m, \bar{m}$ must be related by

$$
m = \frac{1}{2}(n + kn'), \qquad \bar{m} = \frac{1}{2}(-n + kn') \tag{3.12}
$$

for some choice of integers n, n'. In general, an arbitrary tensor product of a left- and a right-moving state will not satisfy (3.12). We only need to check positivity for those which do.

In the above example (3.11), regardless of the common sign of the norms of the chiral states, the nonchiral states one forms out of them have positive norm. Naively, one might think that we would obtain four nonchiral states by tensoring the two chiral states together in all possible ways. In fact, only two states are allowed by the condition (3.12),

$$
c_1\bar{c}_1|\psi_1\rangle \otimes |\bar{\psi}_2\rangle \quad \text{and} \quad c_1\bar{c}_1|\psi_2\rangle \otimes |\bar{\psi}_1\rangle \quad . \tag{3.13}
$$

These are positive so long as the norms of $|\psi_1\rangle, |\psi_2\rangle$ have the same sign. More generally, the condition (3.12) may play an important role in removing troublesome would-be states from the spectrum. Indeed, it appears that the spectrum of states of the Euclidean model satisfying (3.12) may be positive-semidefinite even before imposing BRS invariance but, again, we have no proof.

There are also states at other ghost numbers. The simplest case occurs at $j = m = 0$. Here, in addition to the states that we have already found, $c_1J^{+}_{-1}|0,-1\rangle$ and $c_1J^{-}_{-1}|0,1\rangle$, there are also the states $|0,0\rangle$ at ghost number zero and $c_1c_{-1}|0,0\rangle$ at ghost number 2 which are obviously BRS cohomology. Exactly the same states occur in the BRS cohomology of the bosonic string and (as discussed in the previous section) are a crucial ingredient in constructing the physical dilaton state.

Similarly, for each of the other states in (3.9),(3.10), we find a state at ghost number 0 or 2 at precisely the same values of j, m.

4. Liouville interpretation

In the $d = 1$ noncritical string theory, we have a free scalar field, X, and the Liouville field, ϕ. The dispersion relation for the "tachyon" vertex operator (the gravitationally dressed $e^{ip_x X}$) is

$$p_\phi = \pm p_x - \sqrt{2} \quad . \tag{4.1}$$

Comparing this with the dispersion relation (3.2) of the coset model, we can identify

$$p_\phi = 2\sqrt{2} j, \qquad p_x = \frac{2\sqrt{2}}{3} m \quad . \tag{4.2}$$

This is not to say that the states $|j, m\rangle$ are plane waves; they are honest states of the coset conformal field theory. At best, they may approach plane waves asymptotically on the target space manifold. Note that our restriction on the allowed values of j ($j \geq -\frac{1}{2}$) translates into $p_\phi \geq -\sqrt{2}$, which, on the basis of semiclassical reasoning [17] has been argued to be necessary for normalizable states in Liouville theory.

The identification of the m with p_x could have been anticipated by considering the geometry of the target space of the coset theory. It is obtained by taking the quotient of the group manifold $SL(2, \mathbb{R})$ by the $U(1)$ action

$$g \to h\, g\, h \qquad h = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in H = U(1) \quad .$$

The quotient space has a Killing symmetry generated by the $U(1)$ subgroup $H' \subset SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ which commutes with the above action of H, namely $g \to hgh^{-1}$. The $d = 1$ string also has a Killing symmetry — translation in the X direction. If we choose our conventions so that in the conformal field theory H is generated by $J_0^0 - \overline{J}_0^0$, then the Killing symmetry of the coset theory is generated by $J_0^0 + \overline{J}_0^0$ which one should naturally identify with p_x, the generator of X-translations in the $d = 1$ string.

In addition to the tachyon, it is also known in the $d = 1$ string that there exist a series of "discrete states" at quantized values of p_x[10]. In the coset

model, too, we have found a series of discrete states. Translated into Liouville units, the discrete states that we found occur at

$$\tilde{\mathfrak{D}}^{\pm}: \qquad p_x = \pm\frac{2s - 4r - 1}{2\sqrt{2}}, \qquad p_\phi = \frac{2s + 4r - 5}{2\sqrt{2}} \tag{4.3a}$$

$$\mathfrak{D}^{\mp}: \qquad p_x = \pm\frac{s - 2r + 1}{\sqrt{2}}, \qquad p_\phi = \frac{s + 2r - 3}{\sqrt{2}} \tag{4.3b}$$

$$\mathfrak{C}: \qquad p_x = \frac{2(s - r)}{\sqrt{2}}, \qquad p_\phi = \frac{2(s + r - 1)}{\sqrt{2}} \quad . \tag{4.3c}$$

The discrete states in the $d = 1$ string occur at $\sqrt{2}p_x = u - v$, $\sqrt{2}p_\phi = (u + v - 2)$ for $u, v \in \mathbb{Z}^+$ (Fig. 2).

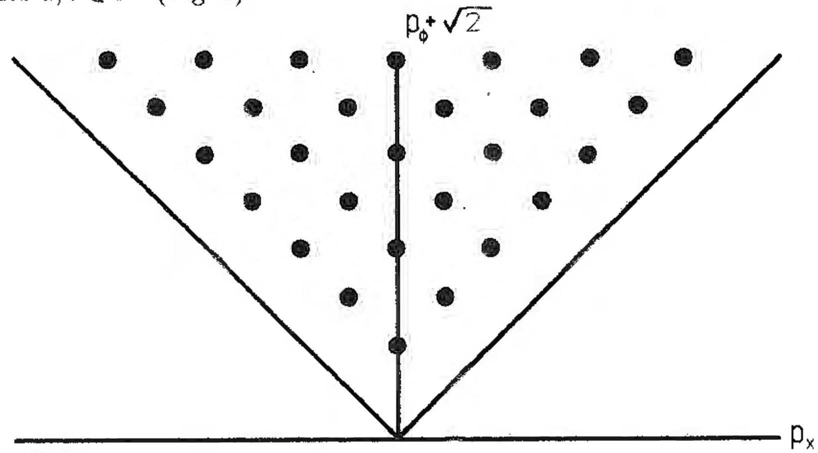
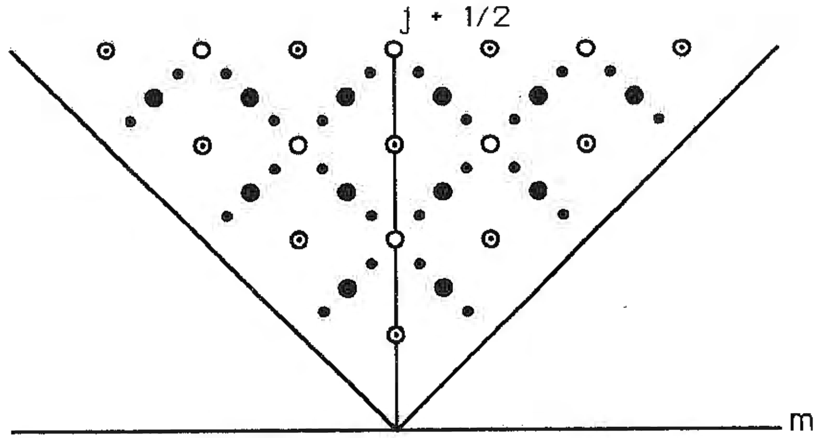


Fig. 2: Physical states of the $d = 1$ noncritical string

For each choice of sign in (4.3b), the states correspond with *half* of the discrete states of Liouville theory ($u = s$, $v = 2r + 1$, or vice versa). The states with u, v both odd are thus doubly occupied. The states of Liouville theory with u, v both even correspond to the states (4.3c).

In addition to the states with u, v odd, for which there are two states of the coset theory for each state of Liouville, the states in (4.3a) are completely new and seem to have no counterpart in $d = 1$ noncritical string theory. The situation is summarized in (Fig. 3).



Physical states of the Euclidean black hole

Fig. 3: ○ $\mathcal{C}$ ⊙ $\mathcal{D}$ (double-occupied) ● $\mathcal{D}$ • $\tilde{\mathcal{D}}$

5. The cohomology (Minkowski case)

Much of the analysis of sect. 3 carries over straightforwardly to the $SL(2,\mathbb{R})/SO(1,1)$ coset model. The mass shell condition now reads

$$\frac{-j(j+1)}{k-2}-\frac{\mu^2}{k}+N=1 \quad , \tag{5.1}$$

where μ is the eigenvalue of J_0^3 and N is the oscillator number. Kač-Moody primary states $|j,\mu\rangle$ (recall the notation of (2.6)) are on mass shell provided j is in the *Principal continuous series*,

$$j=-\tfrac{1}{2}+i\lambda, \qquad \lambda\in\mathbb{R}$$

and

$$\mu=\pm 3\lambda \quad .$$

This is the dispersion relation for a free massless scalar field in two dimensions, the "tachyon" of the Minkowski signature theory. μ is the energy, and the two signs of μ correspond to incoming or outgoing plane waves. λ corresponds

asymptotically to the spatial momentum of the wave, and the sign of λ determines whether the wave is left-moving or right-moving. The Minkowski black hole has two causally disconnected regions. In one, incoming waves (which eventually hit the singularity) are necessarily left-moving and outgoing waves are necessarily right-moving. In the other region, the reverse is true. To account for waves propagating in both regions, we should allow all four possible signs of m and λ.

What about the discrete states? We will learn in sect. 8 that the condition for the existence of a discrete state at level $N=n_+n_-$ is

$$2j+1\pm 2i\mu/3=n_\pm\in\mathbb{Z} \quad .$$

For the principal continuous series, $j=-1/2+i\lambda$, there are no solutions (except for the tachyon found above). However, if we take j to be real and μ to be pure imaginary, we have in principle (a subset of) the solutions found in sect. 3 for the Euclidean theory, with m replaced by $i\mu$.

It is crucial, however, that the inner product in the two theories is different. In the Minkowski theory, the inner product pairs μ and $\overline{\mu}$. In the Euclidean case, for j,m both integer or half-integer, we can use the invariant inner product to decompose a continuous representation into a direct sum of a highest weight, a lowest weight, and a "double-sided" representation. Since in the Minkowski case, the inner product is inherently off-diagonal for imaginary μ, we *can* perform an orthogonal decomposition into (double-sided representation)$\oplus$(the rest), but we *cannot* further decompose "the rest" into (highest weight)$\oplus$(lowest weight) representations. Rather, it forms a single Hermitian representation[5].

With the appropriate inner product (2.7), we can ship over from the Euclidean theory all of the discrete states that we found with $m\in\frac{1}{2}\mathbb{Z}$ to become discrete states of the Minkowski theory.

Of particular interest are the states at level 1,

$$|\Psi^\pm\rangle=c_1\tilde{J}^\pm_{-1}|0,\mp i\rangle \quad . \tag{5.2}$$

[5] Recall that $Im(\mu)\in\frac{1}{2}\mathbb{Z}$ for a Hermitian representation.

These both have $\mu = 0$. These are analogs of the graviton-trace in the flat space string. Correspondingly, at ghost numbers $0, 2$ we have the states $|0\rangle$ and $c_1 c_{-1}|0\rangle$.

We will defer a discussion of the norms of the discrete states to sect. 10.

6. The $k \to \infty$ limit

We have been considering the coset model with $k = 9/4$ as our string background. However, we could equally well imagine tensoring the coset model with a unitary conformal field theory with $c = 24 - 6/(k-2)$, and considering that as our string background. We might expect that a no-ghost theorem should go through in this more general case. Indeed, in the limit $k \to \infty$, the central charge of the "transverse" theory, $c \to 24$, and we expect that our theory should approach that of the *critical* bosonic string with a *flat* light-cone. How does this come about?

First of all, we should rescale the currents

$$\widetilde{J}^i(z) = \sqrt{2/k}\, J^i(z)$$

so that, in terms of the $\widetilde{J}^i$'s, the current algebra (2.1) has a well-defined limit as $k \to \infty$. In this limit, the current algebra becomes abelian and we recognize $\widetilde{J}^{\pm}$ as the light-cone currents $i\partial X^{\pm}$. We also need to rescale the quantum numbers j, m which label the states. Let

$$p_1 = i\sqrt{2/(k-2)}(j+1/2), \qquad \begin{cases} p_0 = \sqrt{2/k}\; m & \text{(Euclidean)} \\ p_0 = \sqrt{2/k}\; \mu & \text{(Minkowski)} \end{cases}$$

so that, for instance,

$$L_0 = \tfrac{1}{2}p_1^2 \pm \tfrac{1}{2}p_0^2 + \frac{1}{4(k-2)} + N + L_0^{transverse}$$

(where N is the light-cone oscillator contribution, and the $+$ $(-)$ sign is for the Euclidean (Minkowski) theory). In the $k \to \infty$ limit, this becomes precisely the Virasoro generator for flat space. For the *Principal continuous series*, $j = -1/2 + i\lambda$, and so p_1 is real. This is exactly what one expects for a flat light-cone. In our analysis of the $k = 9/4$ theory, the "transverse" theory had only one state — the identity. We found that for any fixed value of $p^+ = p_1 + p_0$, there was one state in the BRS cohomology, which we called the tachyon. More generally, we expect that for every state in the transverse theory, we obtain one state in the BRS cohomology. In sect. 7 we will use a "vanishing theorem" to show that there are no other states than this. As $k \to \infty$ this theorem, which we will prove in sect. 8, indeed becomes precisely the vanishing theorem of [3].

Unfortunately, the vanishing theorem which we will prove does not apply to the discrete states, and we should examine their fate in the $k \to \infty$ limit. The answer is very illuminating. The vanishing theorem does not apply when (see sect. 8)

$$\begin{aligned} ip_1 \pm p_0 &= \frac{1}{\sqrt{2(k-2)}}\, n_{\pm} & \text{(Euclidean)} \\ i(p_1 \pm p_0) &= \frac{1}{\sqrt{2(k-2)}}\, n_{\pm} & \text{(Minkowski)} \end{aligned} \tag{6.1}$$

for some choice of integers $n_{\pm} \in \mathbb{Z}$. We then get extra states in the BRS cohomology at light-cone oscillator level

$$N = n_+ n_- > 0 \tag{6.2}$$

provided we can satisfy the mass-shell condition

$$L_0 - 1 = \left(1 - \frac{1}{4(k-2)}\right)(N-1) = 0 \quad . \tag{6.3}$$

For $k > 9/4$, the right-hand side of (6.3) vanishes if and only if $N = 1$. But the $N = 1$ states are precisely the "extra" states $c_1 \alpha_{-1}^{\pm}|0\rangle$ which one finds in the BRS cohomology at zero momentum of the bosonic string [3]. The rest of the discrete states disappear for $k > 9/4$. Qualitatively, at least, the behavior of the coset model in the $k \to \infty$ limit is just what we expect for strings whose light-cone becomes flat in this limit.

7. Systematic analysis of the cohomology

So far we have written various *ansatze* for the physical states. This approach has the advantage of being concrete, but leaves us wondering whether we have really found all the states. In this section and the next we will employ much more abstract means to verify that we haven't missed any states in the BRS cohomology.

Our strategy of proof will be as follows. First we will compute the *index* of the BRS complex. This will give the alternating sum of the dimensions of the BRS cohomology groups, graded by their ghost-number. The states found in the previous sections actually saturate the index. Thus to show that they exhaust the cohomology it suffices to show by some other means that the cohomology is trivial for all but one value of ghost-number. In other words we need a *vanishing theorem*, similar to the one used in the corresponding bosonic string problem [2][3]. Just as in the bosonic string [3], the vanishing theorem holds only for generic values of j, m. Thus we will have to treat a few special cases separately. To simplify the discussion in this section we will simply state the vanishing theorem and apply it, deferring the somewhat technical proof to the next section.

Let us begin by restating the problem. In this section we will only consider the Euclidean black hole. Thus we begin with a module for the $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(2)$ parafermion algebra. We have argued that we wish to consider modules induced from *Hermitian* representations of the full current algebra, (see the definition above (2.5)).[6] Our space of states is now the tensor product of the parafermion algebra with the standard b, c system, subject to the physical state conditions (2.10). To get the physical state space we finally need to pass to the BRS cohomology of the operator Q in (2.9).

The first step is to compute the index of the BRS complex. Let $\hat{\mathcal{V}}_{j,m_0}$ be the Verma module of the current algebra built by acting on a Kač-Moody

[6] In particular our module could contain null vectors; we are not going to remove them by hand since there is no physical mechanism to do so. Instead we are to investigate whether the BRS cohomology procedure removes them automatically.

primary state $|j, m_0\rangle$ with the modes of the currents J^i_{-n} $(n \geq 0)$. This module is (almost) freely generated. The only relation we impose is the one coming from the global algebra:

$$J_0^{\pm}|j,m\rangle = \sqrt{m(m\pm 1) - j(j+1)}\,|j, m\pm 1\rangle \quad .$$

Let $\mathcal{V}_{j,m}$ be the subspace of $\hat{\mathcal{V}}_{j,m_0}$ consisting of states which satisfy

$$\begin{aligned} J_l^2|\psi\rangle &= 0 \qquad l > 0 \\ J_0^2|\psi\rangle &= m|\psi\rangle \quad . \end{aligned}$$

The direct sum $\oplus_m \mathcal{V}_{j,m}$ forms a module for the $SL(2,\mathbb{R})$ parafermion algebra. We will, loosely, refer to $\mathcal{V}_{j,m}$ as the parafermion Verma module; even though it is not itself freely generated by the parafermion currents, it is a subspace of $\hat{\mathcal{V}}_{j,m_0}$ which *is* freely generated by all the currents.

Tensoring $\mathcal{V}_{j,m}$ with the ghost Fock space built by acting with the b_{-n} $(n > 1)$ and c_{-n} $(n > -2)$ on the SL_2-invariant vacuum, we obtain the module $\hat{C}_{j,m}$ of the (parafermion)$\otimes$(ghost) system. $\hat{C}_{j,m}$ has a submodule $C_{j,m}$ of states satisfying

$$b_0|\Psi\rangle = 0 \quad .$$

$C_{j,m}$ is naturally graded by ghost-number, and we denote the eigenspace with ghost-number n by $C^n_{j,m}$ (where the "ground state" $c_1|j,m\rangle$ has ghost-number 1). $C^n_{j,m}$, in turn, has a finite dimensional subspace $E^n_{j,m}$ of states satisfying the mass-shell condition

$$L_0|\Psi\rangle = 0.$$

The *equivariant BRS complex* is simply the complex

$$\cdots \xrightarrow{Q} E^{-1}_{j,m} \xrightarrow{Q} E^0_{j,m} \xrightarrow{Q} E^1_{j,m} \xrightarrow{Q} \cdots \tag{7.1}$$

and by the Euler-Poincaré principle, the index of the BRS operator (the alternating sum of the dimensions of its cohomology groups) is equal to the alternating sum of the dimensions of the E^n

$$Index_Q \equiv -\sum_n (-1)^n dim\ \mathrm{H}^n = -\sum_n (-1)^n dim\ E^n_{j,m} \quad . \tag{7.2}$$

To compute this, we define the character-valued "index"

$$\begin{aligned} Ind(q) &= -\sum_n Tr|_{C^n_{j,m}}(-1)^n q^{L_0} \\ &= -\,Tr|_{C_{j,m}}(-1)^G q^{L_0} \end{aligned} \tag{7.3}$$

where G is the ghost-number. Then $Index_Q$ is simply the q^0 term in $Ind(q)$. But the ghost contribution to $Ind(q)$ is easily computed:

$$Ind(q) = -(1-q^{-1})(1-q)\prod_2^\infty (1-q^n)^2 \chi_{j,m}(q) \quad , \tag{7.4}$$

where $\chi_{j,m}(q)$ is the character ($Tr\ q^{L_0}$) of the $SL(2,\mathbb{R})/U(1)$ parafermion module $\mathcal{V}_{j,m}$.

One of the fundamental tools for studying these parafermion modules is the fact that they admit a construction in terms of free bosons coupled to background charges [18][19] [20]. Introduce three free bosons with the stress tensor

$$T = \tfrac{1}{2}\partial\phi\partial\phi - \tfrac{1}{2}(\partial\sigma\partial\sigma - a\partial^2\sigma) - \tfrac{1}{2}\partial\phi'\partial\phi'$$

where $a = \sqrt{2/(k-2)}$. Then the $\mathfrak{sl}(2,\mathbb{R})$ current algebra can be represented as

$$\begin{aligned} J^+(z) &= e^{-i\alpha(\phi-\phi')}(i\partial\sigma/a + \partial\phi'/\alpha) \\ J^3(z) &= i\partial\phi/\alpha \\ J^-(z) &= e^{i\alpha(\phi-\phi')}(i\partial\sigma/a - \partial\phi'/\alpha) \end{aligned} \tag{7.5}$$

where $\alpha = \sqrt{2/k}$. The Kač-Moody primary states $|j,m\rangle$ are created by the (chiral) vertex operators

$$V^j_m = \Gamma(j+m+1)^{1/2}\Gamma(j-m+1)^{1/2} e^{-im\alpha\phi} e^{-aj\sigma} e^{im\alpha\phi'} \quad . \tag{7.6}$$

To obtain the $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{u}(1)$ parafermion theory, we simply drop all dependence on the boson ϕ. The stress tensor of the coset model is $T = -\frac{1}{2}(\partial\sigma\partial\sigma - a\partial^2\sigma) - \frac{1}{2}\partial\phi'\partial\phi'$. The parafermion currents ψ_1, $\psi_1^\dagger$ are obtained by dropping the exponentials of ϕ in the expressions for $J^\pm$ and the vertex operators for the coset model by dropping the corresponding exponentials of ϕ in (7.6) [18]. Normally,

the free-field representation is used to study the *irreducible* parafermion module either by taking the Felder cohomology [21], [18] or by removing the null vectors by hand[20]. As mentioned above, however, we are interested in the parafermion Verma module, rather than its irreducible quotient.

For generic j,m, the parafermion Verma module $\mathcal{V}_{j,m}$ is obtained by acting on the primary state $|j,m\rangle$ with the bosonic oscillators σ_{-n}, ϕ'_{-n}, where we expand

$$i\partial\sigma(z) = \sum_n \sigma_n z^{-n-1}, \qquad i\partial\phi'(z) = \sum_n \phi'_n z^{-n-1} \quad .$$

The parafermion Verma module $\mathcal{V}_{j,m}$ is, in fact, a Fock module — freely generated by two sets of bosonic oscillators. Thus the character is

$$\mathcal{C}: \qquad \chi_{j,m} = q^{h_{j,m}} \prod_1^\infty (1-q^n)^{-2} \quad . \tag{7.7}$$

Plugging this into (7.4), one finds

$$Ind(q) = q^{h_{j,m}-1} \quad . \tag{7.8}$$

Thus the index $Index_Q = 1$ provided j,m satisfy the mass-shell condition $h_{j,m} = 1$ and vanishes otherwise.

When j,m lie in a discrete representation, however, the parafermion Verma module *is not* isomorphic to the free boson Fock module. For highest weight representations (such as those which contain the physical states (3.6)), the characters are

$$\widetilde{\mathcal{D}}^-: \qquad \chi_{j,m} = q^{h_{j,m}} \prod_1^\infty (1-q^n)^{-2} \sum_{s=0}^\infty (-1)^s q^{s(s-2(m-j)+1)/2} \tag{7.9a}$$

for the representations of type (3.6a) and

$$\mathcal{D}^-: \qquad \chi_{j,m} = q^{h_{j,m}} \prod_1^\infty (1-q^n)^{-2} \sum_{s=0}^\infty (-1)^s q^{s(s-2(m+j+1)+1)/2} \tag{7.9b}$$

for the representations of type (3.6b). These representations are characterized by the fact that the highest weight state at the base has $m = j$ (for $\widetilde{\mathcal{D}}^-$) or

$m = -j-1$ (for $\mathcal{D}^-$). Since they are not of the form (7.7), the free boson construction does not directly give these modules. Similarly, for the lowest weight representations of type (3.8a, b), the characters are, respectively,

$$\widetilde{\mathcal{D}}^+ : \qquad \chi_{j,m} = q^{h_{j,m}} \prod_1^\infty (1-q^n)^{-2} \sum_{s=0}^\infty (-1)^s q^{s(s+2(m+j)+1)/2} \tag{7.10a}$$

$$\mathcal{D}^+ : \qquad \chi_{j,m} = q^{h_{j,m}} \prod_1^\infty (1-q^n)^{-2} \sum_{s=0}^\infty (-1)^s q^{s(s+2(m-j-1)+1)/2} \tag{7.10b}$$

What is happening? For special values of j, m the Fock module may be reducible at the base; indeed this happens when j, m describe a discrete representation of $\mathfrak{sl}(2,\mathbb{R})$. In terms of the free boson representation, $\widetilde{\mathcal{D}}^\pm$ are submodules of the free bosonic Fock module, characterized by [18][20]

$$\widetilde{\mathcal{D}}^\pm = \ker(\mathcal{S}^\pm) \tag{7.11}$$

where

$$\mathcal{S}^\pm = \oint e^{\sigma/\alpha \pm i\phi'/\alpha} \quad . \tag{7.12}$$

(For generic j, m the current in (7.12) is not local with respect to the vertex operators which create the base states, and so $\mathcal{S}$ is not defined.) Hence the free boson module is bigger than the parafermion Verma module $\widetilde{\mathcal{D}}^\pm$, and this is why (7.9), (7.10) differ from (7.7). We summarize the situation by the pictures Fig. 1.

There is just one other possibility. Namely the submodules $\widetilde{\mathcal{D}}^\pm$ may themselves be reducible at the base. This can happen when *both* the operators $\mathcal{S}^\pm$ are well defined on the Fock module, so that we can define a submodule $\mathcal{U}$ by

$$\mathcal{U} = \ker(\mathcal{S}^+) \cap \ker(\mathcal{S}^-) \quad . \tag{7.13}$$

This happens when $j \in \frac{1}{2}\mathbb{Z}$, $j - m \in \mathbb{Z}$. In this case $\mathcal{F}$ contains a *lowest-weight* vector at the base in addition to the highest-weight vector, and we get a "double-sided" parafermion module. See Fig. 1c. The character is then

$$\mathcal{U} : \qquad \chi_{j,m} = \prod_1^\infty (1-q^n)^{-2} \left[1 - \sum_{s=0}^\infty (-1)^s q^{s(s-2j-1)/2} (q^{ms} + q^{-ms}) \right] \quad . \tag{7.14}$$

Thus the modules $\widetilde{\mathcal{D}}^\pm$ and $\mathcal{U}$ are submodules of the free boson Fock module. The modules $\mathcal{D}^-, \mathcal{D}^+$ are best characterized as the *quotient* of the free boson Fock module by $\widetilde{\mathcal{D}}^+ = \ker(\mathcal{S}^+)$ and $\widetilde{\mathcal{D}}^- = \ker(\mathcal{S}^-)$ respectively. The reader may recall that we have argued against considering the quotient of parafermion Verma modules, since these are the primary physical spaces arising from the path integral in the coset model and there is no mechanism other than BRS to project to a quotient. No such scruples apply to to the boson Fock spaces, however. They are simply an alternate representation of the primary object, the parafermion Verma modules. If we need to take a quotient to reproduce the latter in a particular case, then we may do so without further ado.

We now have all the characters we need in (7.9)-(7.10), (7.14). Plugging these characters into (7.4), we find that $Index_Q \neq 0$ for certain values of j, m labeled by a pair of positive integers r, s.

$Index_Q$	Representation	j	m
$+1$	$\widetilde{\mathcal{D}}^-$	$\frac{1}{8}(2s+4r-5)$	$-\frac{3}{8}(2s-4r-1)$
$+1$	$\mathcal{D}^-$	$\frac{1}{4}(s+2r-3)$	$\frac{3}{4}(s-2r+1)$
$+1$	$\widetilde{\mathcal{D}}^+$	$\frac{1}{8}(2s+4r-5)$	$\frac{3}{8}(2s-4r-1)$
$+1$	$\mathcal{D}^+$	$\frac{1}{4}(s+2r-3)$	$-\frac{3}{4}(s-2r+1)$
-1	$\widetilde{\mathcal{D}}^-$	$\frac{1}{4}(2s+2r-3)$	$-\frac{3}{4}(2s-2r+1)$
-1	$\mathcal{D}^-$	$\frac{1}{8}(2s+4r-5)$	$\frac{3}{8}(2s-4r-1)$
-1	$\widetilde{\mathcal{D}}^+$	$\frac{1}{4}(2s+2r-3)$	$\frac{3}{4}(2s-2r+1)$
-1	$\mathcal{D}^+$	$\frac{1}{8}(2s+4r-5)$	$-\frac{3}{8}(2s-4r-1)$
-2	$\mathcal{U}$	$\frac{1}{2}(s+r-2)$	$\frac{3}{2}(s-r)$

This completes the first step of our strategy to compute the BRS cohomology. As mentioned earlier, the second step involves a vanishing theorem, whose proof we defer to the next section. Here we simply state the result. Recall that a parafermion module is always a Virasoro representation by the Sugawara construction, and hence also a module for the raising part Vir^- of Virasoro.

Vanishing Theorem. *a) Whenever a parafermion module $\mathcal{M}$ is a* free Vir^--*module, the BRS cohomology vanishes identically:* $H^*_Q(\mathcal{M}) = 0$. *The only exception is when there is a highest weight state and it satisfies the mass-shell condition; then* $H^1_Q(\mathcal{M}) = \mathbb{C}$. *b) In particular, when* $j \notin \frac{1}{4}\mathbb{Z}$, *then the free boson module* $\mathcal{F}$ *satisfies the condition in (a). c) When* $j \in \frac{1}{4}\mathbb{Z}$, $m \in \frac{3}{4}\mathbb{Z}$ *then* $H^1_Q(\mathcal{F}) = H^0_Q(\mathcal{F}) = \mathbb{C}$.

We will prove (a), (b) in the next section; for proofs of (b), (c) see [22] . We will now combine these results with our index analysis.

Just about everything is known about the BRS cohomology of the free boson system. In fact, this is precisely the system studied in [22] in the context of $d = 1$ noncritical string theory. For generic j, m we have seen that the free boson cohomology is just what we want. Even for the special cases (3.9*a*–*b*) we can still use the free boson construction, however. For example, the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \widetilde{\mathcal{D}}^{\pm} \to \mathcal{F} \to \mathcal{D}^{\mp} \to 0$$

tells us about the cohomology of $\mathcal{D}, \widetilde{\mathcal{D}}$ starting from the cohomology of $\mathcal{F}$. When $j = \frac{1}{8}(2s+4r-5)$ (*i.e.* for the series (3.9*a*)) the vanishing theorem applies and so $H^*_Q(\mathcal{F}) = 0$. The long exact sequence then implies $H^{n+1}_Q(\widetilde{\mathcal{D}}^{\pm}) \simeq H^n_Q(\mathcal{D}^{\mp})$. But for these values of j, $\widetilde{\mathcal{D}}$, being a submodule of a free Vir^--module, is itself a free Vir^--module. Hence the vanishing theorem holds for it as well:

$$H^n_Q(\widetilde{\mathcal{D}}) = 0, \qquad n \neq 1 \quad .$$

Since the index in the above table is nonzero, H^1 must in fact be nonzero. Using the long exact sequence, we finally conclude

$$H^1_Q(\widetilde{\mathcal{D}}^{\pm}_{j,m}) = H^0_Q(\mathcal{D}^{\mp}_{j,m}) = \begin{cases} \mathbb{C} & j = \frac{1}{8}(2s+4r-5),\ m = \pm\frac{3}{8}(2s-4r-1) \\ 0 & \text{otherwise} \quad . \end{cases}$$

The next case to consider is $j = \frac{1}{4}(s+2r-3)$, $m = \pm\frac{3}{4}(s-2r+1)$ (the series (3.9*b*)). Here the vanishing theorem tells us that $H^1_Q(\mathcal{F}) = H^0_Q(\mathcal{F}) = \mathbb{C}$. Let us begin with the subcase where $\widetilde{D}^{\pm}$ is irreducible at the base, *i.e.* s is even. We notice that any state annihilated by $\mathcal{S}^{\pm}$ can actually be written as $\mathcal{S}^{\pm}$ acting on something. Thus, since it commutes with BRS, $\mathcal{S}^{\pm}$ provides an isomorphism of BRS cohomologies

$$\mathcal{S}^{\pm} : H^*_Q(\mathcal{D}^{\mp}_{j,m}) \xrightarrow{\sim} H^*_Q(\widetilde{\mathcal{D}}^{\pm}_{j-1/8, m\pm 9/8}) \quad .$$

But we can use the previous technique on the right hand side. Thus we learn that

$$H^*_Q(\widetilde{\mathcal{D}}^{\pm}_{\frac{1}{4}(s+2r-3), \pm\frac{3}{4}(s-2r+1)}) = H^*_Q(\mathcal{D}^{\mp}_{\frac{1}{8}(2s+4r-5), \pm\frac{3}{8}(2s-4r-1)}) \quad .$$

From this, and the long exact sequence, we conclude that the index is again saturated by $H^1_Q(\mathcal{D}^{\mp}) = \mathbb{C}$ and $H^0_Q(\widetilde{\mathcal{D}}^{\pm}) = \mathbb{C}$.

We have still to consider the series (3.9*b*) for $j = \frac{1}{2}(s'+r'-2)$, $m = \frac{3}{2}(s'-r')$ (the case where s above is odd, so we get a double-sided parafermion representation (7.13)). We have the exact sequence

$$0 \to \mathcal{U} \to \mathcal{F} \to \mathcal{D}^{+} \oplus \mathcal{D}^{-} \to 0 \quad .$$

As before, the isomorphism

$$\mathcal{S}^{\pm} : H^*_Q(\mathcal{D}^{\mp}_{j,m}) \xrightarrow{\sim} H^*_Q(\widetilde{\mathcal{D}}^{\pm}_{j-1/8, m\pm 9/8})$$

tells us that $H^1_Q(\mathcal{D}^{+}) = H^1_Q(\mathcal{D}^{-}) = \mathbb{C}$, and from the long exact sequence, we learn that $H^0_Q(\mathcal{U}) = H^2_Q(\mathcal{U}) = \mathbb{C}$.

We have now exhausted all cases where the free boson module is reducible. As we mentioned at the outset, in other cases the vanishing theorem may be used directly. We need only remember the exceptional case. This occurs when $j \in \frac{1}{4}\mathbb{Z}$ but is not one of the reducible values studied above, *i.e.* for

$$j = \frac{1}{2}(s+r-1), \qquad m = \frac{3}{2}(s-r)$$

(the series (3.9*c*)). In this case, even though the index vanishes, there *is* cohomology: the third part of the vanishing theorem says $H^1_Q(\mathcal{F}) = H^0_Q(\mathcal{F}) = \mathbb{C}$.

Our use of the free boson representation has been little more than a formal trick to get at the relations between the BRS cohomologies associated to the various representations. However, it does shed some indirect light on the use of the free boson representation to, for instance, calculate correlation functions. The free boson representation is obviously applicable to correlation functions of continuous representations. However, it is not obviously meaningful to calculate the correlation functions which involve, say, combinations of $\mathfrak{D}^+$ and $\widetilde{\mathfrak{D}}^-$ operators. For in the free boson representation, the operators in $\mathfrak{D}^+$ are represented as the *equivalence class* of the free boson operator modulo operators in $\widetilde{\mathfrak{D}}^-$. If that representation is to be consistent, then operators in $\widetilde{\mathfrak{D}}^-$ must be equated to zero. In the free boson representation, we cannot simultaneously have operators of both types being nontrivial. This is not to say that operators of both types don't exist, merely that the free boson representation cannot simultaneously represent them both nontrivially.

8. The vanishing theorem

We will now prove parts (a), (b) of the vanishing theorem enunciated in the previous section. We will again make use of the free boson representation for parafermion modules. Since as we will see the bosonic representation proves to be a rather clumsy tool for this task, we will also comment on how part (a) may be obtained using the powerful theorem of Lian and Zuckerman [23] . To this end we will need to relate our Q cohomology to that of $Q+Q_{\widehat{\mathrm{u}}(1)}$, as promised earlier.

8.1. Free-boson proof

When the free boson module is irreducible at the base, there is a direct method of proof, which more or less follows the proof of Kato and Ogawa for the flat space bosonic string [2]. Let us decompose the BRS operator in the usual way

$$Q = c_0 L_0^{tot} + b_0 M + \widetilde{Q} \quad ,$$

where $\widetilde{Q}$ contains neither c_0 nor b_0. The states in the equivariant BRS complex are all annihilated by L_0^{tot} and b_0, so on the equivariant BRS complex, $\widetilde{Q}=Q$, and $\widetilde{Q}^2 = -ML_0^{tot} = 0$.

Let us consider a 1-parameter family of deformations of $\widetilde{Q}$

$$\begin{aligned}\widetilde{Q}(\beta) &= \sum_{m\neq 0} \tfrac{1}{2}(p^- + ina/2)c_{-m}(\sigma_m + i\phi'_m) \\ &\quad + \beta\Big[\sum_{m\neq 0} c_{-m}N_m + (\text{three ghost terms})\Big] \\ &\quad + \beta^2 \sum_{m\neq 0} \tfrac{1}{2}p^+ c_{-m}(\sigma_m - i\phi'_m) \\ &\equiv Q_0 + \beta Q_1 + \beta^2 Q_2 \quad .\end{aligned} \tag{8.1}$$

where[7] (recall that $a=\sqrt{2/(k-2)}$, $\alpha=\sqrt{2/k}$)

$$p^{\pm} = \sigma_0 + \frac{ia}{2} \pm i\phi'_0 = i[a(j+1/2) \pm \alpha m] \tag{8.2}$$

and

$$N_m = \sum_{\substack{n>0\\ n\neq m}} (\sigma_{m-n}\sigma_n + \phi'_{m-n}\phi'_n) \quad .$$

Now, $\widetilde{Q}(\beta)^2 = -\beta^2 ML_0^{tot} = 0$ on the equivariant complex, so we can consider its equivariant cohomology for any value of the deformation parameter β. Also $\widetilde{Q}(\beta)$ interpolates between $\widetilde{Q}(\beta=0)=Q_0$ and $\widetilde{Q}(\beta=1)=\widetilde{Q}$. Following Kato and Ogawa [2], we assume that the states in the cohomology of $\widetilde{Q}(\beta)$ can be expanded in the form

$$|\psi(\beta)\rangle = \sum_{n\geq 0} \beta^n |\Psi_n\rangle \quad . \tag{8.3}$$

(In fact this is guaranteed, as we will briefly indicate below using a more sophisticated point of view.) Equating powers of β, the state $|\Psi_0\rangle$ is in the cohomology

[7] For one of the two tachyon representations, p^- actually vanishes. To analyze that case, we can consider the opposite deformation, in which the roles of p^+ and p^- are reversed.

of Q_0 and the states $|\Psi_n\rangle$ are determined by the recursion relations

$$\begin{aligned} Q_0|\Psi_0\rangle &= 0 \\ Q_0|\Psi_1\rangle &= -Q_1|\Psi_0\rangle \\ Q_0|\Psi_n\rangle &= -Q_1|\Psi_{n-1}\rangle - Q_2|\Psi_{n-2}\rangle \quad . \end{aligned} \tag{8.4}$$

Our ability to calculate the cohomology of Q_0 (and, coincidentally, to show that there is no obstruction to solving the recursion relations (8.4)) rests on the following observation. Consider the operator

$$R = \sum_{n\neq 0} \frac{1}{p^- + ina/2} b_n(\sigma_{-n} - i\phi'_{-n}) \quad . \tag{8.5}$$

For generic j, *i.e.* when $2j+1-\frac{2\alpha m}{a}$ not an integer, R is defined, preserves the equivariance conditions and satisfies

$$\{Q_0, R\} = \mathcal{N} \quad ,$$

where

$$\mathcal{N} = 1 + \sum_{n>0}(\sigma_{-n}\sigma_n + \phi'_{-n}\phi'_n) - c_1 b_{-1} + c_{-1}b_1 + \sum_{n>1} n(b_{-n}c_n + c_{-n}b_n) \quad .$$

$\mathcal{N}$ is a positive semi-definite operator. It annihilates *only* the ground state $c_1|j,m\rangle$ of the Fock module at ghost number 1. For all other states, $\mathcal{N} > 0$. Imagine we have some state $|\Psi\rangle$, other than this ground state, which is annihilated by Q_0. We now see that in fact $|\Psi\rangle$ is exact,

$$|\Psi\rangle = Q_0|\chi\rangle$$

where

$$|\chi\rangle = \frac{1}{\mathcal{N}} R|\Psi\rangle \quad .$$

Thus the cohomology of Q_0 is empty, unless the ground state $c_1|j,m\rangle)$ is physical. Since we have just inverted Q_0, we can now solve the recursion relations (8.4) to show that the same is true of the cohomology of $\widetilde{Q}$. This is the desired vanishing result.

Essentially, what we are doing here is using the spectral sequence of a filtered complex [24] to calculate the cohomology of $\widetilde{Q}$, starting with the cohomology of the simpler operator Q_0. The filtration is by the lowest power of β appearing in the formal power series expansion of $|\Psi\beta\rangle$ (8.3). Indeed, provided $p^- + ina/2 \neq 0$, $\mathrm{H}^m_{Q_0} = 0$ for $m \neq 0$. In that case, this spectral sequence degenerates at $E_1 = \mathrm{H}^*_{Q_0}$. This proves that $\mathrm{H}^*_{\widetilde{Q}} \simeq \mathrm{H}^*_{Q_0}$, as we have seen explicitly above.

Parenthetically we note that Q_0 defined in (8.1) itself looks like a BRS operator, where the role of the Virasoro generators is played by the *linear* Fock raising operators. The Fock module is of course free with respect to the latter operators. Deforming Q_0 to $\widetilde{Q}$ we see that whenever (8.4) can be solved then $\mathcal{F}$ is a free Vir^--module as claimed in the vanishing theorem.

The crucial step in the above analysis was constructing the "chain homotopy" operator R. Clearly R fails to exist if $2ip^-/a \in \mathbb{Z}$. However, we could equally well have deformed Q in the opposite way (exchanging the roles of p^+ and p^-). Then the obstruction to finding an operator R would be $2ip^+/a \in \mathbb{Z}$. Thus we get the vanishing result unless *both* conditions hold:

$$-2ip^{\pm}/a = 2j + 1 \pm \frac{2}{3} m \in \mathbb{Z} \quad .$$

Putting this together with the mass shell condition,

$$L_0^{tot} = \tfrac{1}{2}p^+p^- + \mathcal{N} = 0$$

we find the solutions labeled by a pair of integers r, s (both positive, or both negative).

$$j = \frac{1}{4}(r+s-2), \qquad m = \frac{3}{4}(r-s) \quad .$$

For these values of j, m the cohomology can be nontrivial, a refinement of the claim in the vanishing theorem part (b).

A slightly more sophisticated analysis [22] allows us to find the cohomology of the free boson module in these cases as well. The answer is that for $r, s > 0$, $\mathrm{H}^1(\mathcal{F}) = \mathrm{H}^0(\mathcal{F}) = \mathbb{C}$. For $r, s < 0$, $\mathrm{H}^1(\mathcal{F}) = \mathrm{H}^2(\mathcal{F}) = \mathbb{C}$. This is the claim in the vanishing theorem part (c).

We stated the vanishing theorem for the Euclidean $SL(2,\mathbb{R})/U(1)$ case, but the proof can be rather trivially adapted to the Minkowski $SL(2,\mathbb{R})/SO(1,1)$ case. We can equally well represent the $\mathfrak{sl}(2,\mathbb{R})$ current algebra as

$$\begin{aligned} \widetilde{J}^{+}(z) &= e^{\alpha(\phi-\phi')}(\partial\sigma/a - \partial\phi/\alpha) \\ J^3(z) &= i\partial\phi'/\alpha \\ \widetilde{J}^{-}(z) &= e^{-\alpha(\phi-\phi')}(\partial\sigma/a + \partial\phi/\alpha) \quad . \end{aligned} \tag{8.6}$$

The primary states $|j,\mu\rangle$ are created by the (chiral) vertex operators

$$V^j_\mu = \Gamma(j+i\mu+1)^{-1/2}\Gamma(j-i\mu+1)^{-1/2}e^{-i\mu\alpha\phi}e^{-aj\sigma}e^{i\mu\alpha\phi'} \quad . \tag{8.7}$$

Again, for generic j,μ, the parafermion Verma module $\mathcal{V}_{j,\mu}$ is isomorphic to the bosonic Fock module generated by the bosonic oscillators σ_{-n}, and ϕ_{-n} acting on the Fock ground state $|j,\mu\rangle$, where we expand

$$i\partial\sigma(z) = \sum_n \sigma_n z^{-n-1}, \qquad i\partial\phi(z) = \sum_n \phi_n z^{-n-1} \quad .$$

Following exactly the same steps, we prove the vanishing theorem in the Minkowski case. We deform the BRS operator $\widetilde{Q}$

$$\begin{aligned} \widetilde{Q}(\beta) &= \sum_{m\neq 0} \tfrac{1}{2}(p^- + ian/2)c_{-m}(\sigma_m + \phi_m) \\ &+ \beta\Big[\sum_{m\neq 0} c_{-m}N_m + (\text{three ghost terms})\Big] \\ &+ \beta^2 \sum_{m\neq 0} \tfrac{1}{2}(p^+ + ian/2)c_{-m}(\sigma_m - \phi_m) \\ &\equiv Q_0 + \beta Q_1 + \beta^2 Q_2 \end{aligned} \tag{8.8}$$

where

$$p^{\pm} = \sigma_0 + \frac{ia}{2} \pm \phi_0 = ia(j+1/2) \pm \alpha\mu \tag{8.9}$$

The chain-homotopy operator $R = \sum_{n\neq 0} \frac{1}{p^- + ian/2} b_n(\sigma_{-n} - \phi_{-n})$ again satisfies $\{Q_0, R\} = \mathcal{N} \geq 0$ which leads directly to the desired vanishing theorem.

Presumably, the free-field representation of the $\mathfrak{sl}(2,\mathbb{R})$ current algebra can be used in proving a vanishing theorem for the BRS cohomology of strings

propagating on the group manifold itself. In that case, however, it is known that the BRS cohomology is not sufficient to remove all of the negative norm states, and one must *by hand* restrict the allowed representation content of the theory. Also, one has the technical headache of an infinite degeneracy of states at each mass level (since L_0 is independent of m). Thus more care is needed in order to apply the index arguments that we have used here.

8.2. Another proof

The proof in the previous subsection is adequate for the applications in sect. 7. As stated, however, the vanishing theorem part (a) makes no reference to the free boson representation. In this subsection we will briefly indicate how to obtain part (a) from the general result of ref. [23].

The first issue we must contend with is the fact that the analysis of [23] applies only to a single complex, while we have been discussing the iterated cohomology $\mathrm{H}^*_Q(\mathrm{H}^*_{Q_{\widehat{u}(1)}})$ (see the discussion at the beginning of sect. 7). One can readily show, however, that $\mathrm{H}^n_{Q_{\widehat{u}(1)}} = 0$, $n \neq 1$. This means that in the double complex associated to Q and $Q_{\widehat{u}(1)}$, all the rows are exact with the exception of just one entry at $n=1$. Standard arguments[8] then imply that the desired iterated cohomology is just $\mathrm{H}^*_Q(\mathrm{H}^*_{Q_{\widehat{u}(1)}}) = \mathrm{H}^*_{Q+Q_{\widehat{u}(1)}}$. Thus as claimed earlier these are completely equivalent cohomology problems. We can now use the result of [23] to compute the cohomology of $Q + Q_{\widehat{u}(1)}$.

We start with the $SL(2,\mathbb{R})$ WZW model coupled to the $U(1)$ gauge field. In covariant gauge, the gauge field becomes a conformal field ($\overline{\partial}A(z) = 0$), with the OPE

$$A(z)A(w) = \frac{2/k}{(z-w)^2} \quad .$$

In choosing the covariant gauge, we introduce a set of $U(1)$ ghosts and the BRS operator

$$Q_{\widehat{u}(1)} = \frac{1}{2\pi i}\oint \xi(z)(J^2(z) + \frac{k}{2}A(z)) \tag{8.10}$$

[8] One says that the spectral sequence of the double complex degenerates at the E_2 term [24].

where the $U(1)$ ghosts satisfy

$$\xi(z)\eta(w) = \frac{1}{z-w} \quad .$$

We are to consider the cohomology of $Q_T = Q + Q_{\widehat{u}(1)}$,

$$Q_T = \frac{1}{2\pi i}\oint c\,(T_{\widehat{sl}(2,\mathbb{R})} + T_A + T_{\eta\xi}) + bc\partial c + \xi\,(J^2 + \frac{k}{2}A)$$

with the equivariance conditions

$$L_0|\Psi\rangle = b_0|\Psi\rangle = 0 \qquad (8.11a)$$

$$(J_0^2 + \frac{k}{2}A_0)|\Psi\rangle = \eta_0|\Psi\rangle = 0 \quad . \qquad (8.11b)$$

Lian and Zuckerman considered the equivariant BRS cohomology associated to gauging an arbitrary Lie (super)algebra $\mathcal{G}$, where the equivariance conditions are specified by an *abelian* subalgebra $\mathcal{H}_0$. We now see that our problem is of this form, with $\mathcal{G} = (\mathrm{Vir})\ltimes\widehat{u}(1)$ and $\mathcal{H}_0 = \mathbb{C}L_0 \oplus \mathbb{C}(J_0^2 + \frac{k}{2}A_0)$.

The reader may have noticed that (8.11b) does not quite agree with (2.10c)[9]. The modification is needed in order for $\mathcal{H}_0$ to be abelian. Eqn. (8.11b) reduces to (2.10c) because, when we remove states by gauging the $U(1)$ rather than by fiat, the contribution of the modes of the gauge field A_{-n} and of J^2_{-n} precisely cancel the contribution of the $U(1)$ ghosts. For the same reason the index of the BRS complex in the present formulation is exactly the same as what we calculated in sect. 7.

Lian and Zuckerman proved that the cohomology of Q_T vanishes for ghost number$\neq 1$ *provided* that we have a freely generated $\mathcal{G}^-$-module. This is the desired statement (a) of the vanishing theorem.

Once again, we find that there can be nontrivial cohomology if the null vectors generated by L_{-n}'s and $(J^2_{-n} + \frac{k}{2}A_{-n})$'s vanish identically. For generic j, m this does not happen, the module is indeed freely generated and the vanishing theorem applies. However, for $j \in \frac{1}{4}\mathbb{Z}$, $m \in \frac{3}{4}\mathbb{Z}$, there are Virasoro null vectors which vanish identically. These are precisely the locations where we have already seen that the free boson module has nontrivial cohomology.

[9] Recall that in (2.10) J^0 refers to J^2 for the present Euclidean case or J^3 for the Minkowski case.

9. Stringy symmetry

In sect. 7, we noticed that free boson representation provided a way of realizing an isomorphism of parafermion modules $\mathcal{S}^{\pm} : \mathcal{D}^{\mp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{\pm}$. In this section, we will point out some of the surprising physical consequences of that isomorphism. Recall that since the screening operator $\mathcal{S}^{\pm}$ commutes with the parafermions, this is really an isomorphism of representations of the parafermion algebra. In particular, it is an isometry of the invariant metric on these modules.

The existence of such an isomorphism is rather remarkable. $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ come from *inequivalent* $SL(2,\mathbb{R})$ modules. Nevertheless as parafermion modules, they are equivalent.

Still more surprising, the isomorphism exchanges massive states (states with nonzero "oscillator number") along the "edge" of the representation (see Fig. 1) with massless states ("tachyons"). In particular, if we write

$$L_0 = -\frac{j(j+1)}{k-2} + m^2/k + \mathcal{N}$$

we see that since $[\mathcal{S}^{\pm}, L_0] = 0$, $\mathcal{S}^{\pm}$ does not commute with the number operator. Rather,

$$[\mathcal{S}^{\pm}, \mathcal{N}] = (j \pm m)\mathcal{S}^{\pm} = \mathcal{S}^{\pm}\,(j+1\pm m) \quad . \qquad (9.1)$$

The highest (lowest) weight state of $\mathcal{D}^{\mp}$ gets carried into the lowest (highest) weight state of $\widetilde{\mathcal{D}}^{\pm}$, but the base of $\mathcal{D}$ gets carried into the edge of $\widetilde{\mathcal{D}}$, and vice versa. This is very different from the familiar "$R \to 1/R$"-type symmetries which *commute* with oscillator number. Perhaps this symmetry relating states of different spin is a peculiarity of two-dimensional target spaces, but it is worthwhile looking for other examples of this phenomenon in string theory.

Because of this "stringy" symmetry, it is perhaps redundant to include *both* $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ in our list of the BRS cohomology. If we include only $\mathcal{D}$, then the "correspondence" with $d = 1$ noncritical string theory is somewhat strengthened[10].

[10] Though the meaning of this correspondence is somewhat obscured, since j, m are no longer good quantum numbers. Note, however, that when we tensor together left and right, $\mathcal{S}\bar{\mathcal{S}}$ *does* commute with $m - \bar{m}$, which is therefore still a good quantum number.

The only remaining discrepancy is that for $j = \frac{1}{2}(s+r-2)$, $m = \pm\frac{3}{2}(s-r)$, there are two states in the coset theory (one from $\mathcal{D}^+$, and one from $\mathcal{D}^-$), whereas there is only one state in Liouville theory.

10. The norms of discrete states

Actually, the no-ghost theorem does not, as is commonly supposed, assert that all states in the BRS cohomology are positive norm. Even for the covariant quantization of the bosonic string in 26-dimensional flat space, *that* formulation of the no-ghost theorem is *false*! There are extra states in the BRS cohomology which occur only at exceptional values of the momenta. These states, a subset of which have become known in the context of $d = 1$ noncritical string theory as "discrete states" [10], do not necessarily have positive norm. In fact, as we shall see presently, half of them have positive norm and half negative norm.

Doesn't this apparent "violation" of the no-ghost theorem lead to some sort of disaster? In fact, it does not. The discrete states do not lead to unphysical poles in scattering amplitudes. Because they occur in pairs with positive and negative norm, the residue of the would-be pole vanishes. Secondly, because these states occur only at exceptional values of the momenta, there is no phase space to produce them. One can obtain a consistent (if non-covariant) theory by omitting them. This noncovariant theory is isomorphic to the light-cone gauge.

Let us illustrate this familiar, if not widely appreciated, phenomenon by enumerating the BRS cohomology occurring at $k = 0$ in the 26-dimensional bosonic string. So as to be totally explicit, we will tensor together the holomorphic and antiholomorphic sectors and we will use the inner product

$$(|\chi\rangle, |\psi\rangle) = \langle\chi \mid c_0\bar{c}_0 \mid \psi\rangle \quad . \tag{10.1}$$

We will consider states which are simultaneously in the equivariant cohomology of Q and $\bar{Q}$ (equivariant with respect to b_0 and $\bar{b}_0$)[11].

The following table contains all of the states in the equivariant BRS cohomology at $k = 0$:

Ghost #	# States			
0	1		$\|0\rangle$	
1	52		$c_1\alpha^\mu_{-1}\|0\rangle$,	$\bar{c}_1\bar{\alpha}^\mu_{-1}\|0\rangle$
2	678	$c_1\bar{c}_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}\|0\rangle$,	$c_1c_{-1}\|0\rangle$,	$\bar{c}_1\bar{c}_{-1}\|0\rangle$
3	52		$c_1c_{-1}\bar{c}_1\bar{\alpha}^\mu_{-1}\|0\rangle$,	$\bar{c}_1\bar{c}_{-1}c_1\alpha^\mu_{-1}\|0\rangle$
4	1		$c_1c_{-1}\bar{c}_1\bar{c}_{-1}\|0\rangle$	

(10.2)

Let us first consider the states at ghost number 2. At nonzero momentum (with $k^2 = 0$), there are 299 transverse gravitons, 276 transverse modes of the antisymmetric tensor field, and one physical dilaton, the state

$$[c_1\bar{c}_1\alpha_{-1}\cdot\bar{\alpha}_{-1} + (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})]|k\rangle \quad .$$

As $k \to 0$, these become 576 of the 678 states we have found. What of the remaining 102 states at zero momentum? When we calculate their norms, we find that 51 have positive norm and 51 have negative norm. There is no covariant way of distinguishing the discrete states from the $k \to 0$ limit of the physical graviton and antisymmetric tensor field states at nonzero k. Thus in a covariant treatment we must include all of them, even though some have negative norm. As mentioned before, these discrete states actually *cancel* in their contributions to a sum over intermediate states in any scattering amplitude. Thus in a noncovariant treatment (like light-cone gauge) it is possible to eliminate them in a consistent way.

[11] The list of discrete states at exotic ghost numbers is somewhat shorter if, instead, we consider the cohomology of $Q + \bar{Q}$, equivariant with respect to $b_0 - \bar{b}_0$. Though this is all that is physically necessary in order to have well-defined string scattering amplitudes, we will stick to the above definition of physical states because it is closer the chiral considerations of the rest of this paper.

The same phenomenon takes place for the states at "exotic" ($\neq 2$) ghost numbers. The 104 states at ghost numbers 1 and 3 combine[12] to form 52 positive and 52 negative norm states. Similarly, the two states at ghost numbers 0 and 4 combine to form one positive norm and one negative norm state. Again, these cancel in pairs in their contribution to sums over intermediate states.

We can also look at contributions to the one-loop partition sum. The partition sum simply counts states with a factor of $(-1)^{ghost\#}$, regardless of their norm (which cancels out when one takes the trace). The 104 states at odd ghost number cancel the contributions of the 104 extra states at even ghost number.

The moral of this story is very simple. The precise statement of what we require in a no-ghost theorem *is not* that all states in the BRS cohomology be positive norm. Rather, we should demand that if there are states of negative norm, a) they should appear only at certain quantized values of the momenta and b) they should be paired with extra positive norm states which also only occur at those same quantized values of the momenta. All *propagating* states — those which occur for continuous values of the momenta — must be positive norm.

It should come as no surprise that, if one goes back and calculates the norms of the discrete states in the Minkowski-signature theory found in sect. 5, one discovers that, just as in flat space, half are positive and half negative. Consider the states (5.2). The inner product is off-diagonal and the linear combinations which diagonalize the inner product are $|\Psi^+\rangle \pm |\Psi^-\rangle$. One linear combination is positive norm, the other is negative norm. (Exactly the same remark holds true for the other discrete states.) But that is precisely the same behavior we see for the discrete states in flat space. As we have seen, this is perfectly compatible with the no-ghost theorem. The tachyon, which is the only *propagating* state in the theory, is manifestly positive norm. Thus the no-ghost theorem is true in a rather trivial way.

12 Ghost-charge conservation forces the inner product to be pure off-diagonal in the basis of (10.2), pairing states of ghost charge 1 with states of ghost charge 3.

We should note that in the *Euclidean*-signature theory (as is commonly considered in $d = 1$ noncritical strings), the discrete states can play a more fundamental role. There, the states at ghost number 2 all have *positive* norm (except, of course, the state $(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle$). Thus, rather than cancelling in a sum over intermediate states, they add. Thus, in the Euclidean theory, the discrete states *can* lead to physical poles in scattering amplitudes, as has been seen in $d = 1$ noncritical string theory [10].

11. Conclusions

We have found all possible physical states allowed by the representation theory of the algebra of parafermion currents defining the $SL(2,\mathbb{R})/U(1)$ and $SL(2,\mathbb{R})/SO(1,1)$ coset models. The no-ghost theorem is satisfied for the Minkowskian $(SL(2,\mathbb{R})/SO(1,1))$ coset, albeit in a rather simple way. The actual spectrum of the theory could be smaller than our list, though it is not easy to see how such a truncation could be consistent with modular invariance. The spectrum we have found is similar, but not identical to that of the $d = 1$ noncritical string theory. It is also interesting that it is not quite what one would expect from a naive "Wick-rotation" of the Euclidean-signature $SL(2,\mathbb{R})/SO(2)$ coset model [16]. This is not surprising, given that there is curvature in the time direction. Indeed, the relation between Euclidean- and Minkowski-signature quantum gravity has never been obvious, for in a general spacetime background there is no notion of Wick-rotation. It will be interesting to explore further the relation between the Euclidean- and Minkowski-signature theories to see how string theory addresses this question in quantum gravity.

It would also be interesting to extend our analysis to even more general nonunitary CFT's. Roughly speaking, we expect that BRS decoupling can take care of one time-like free boson's worth of negative-norm states (in the fermionic string, one time-like boson and one time-like fermion's worth). Can this be made precise? Just how nonunitary can a nonunitary CFT be and still have a no-ghost theorem hold?

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