HETEROGENEOUS AGENT MODEL WITH REAL BUSINESS CYCLE WITH APPLICATION IN OPTIMAL TAX POLICY AND SOCIAL WELFARE REFORM

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A DISSERTATION

in

Applied Mathematics and Computational Science

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2019

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Acknowledgments

I would like to thank Penn Wharton Budget Model (PWBM) for dissertation grant and graduate research grant. I am grateful to Dirk Krueger, Alexander Ludwig, Victor Preciado and Kent Smetters for guidance and advice. I would like to thank Wharton Research Data Service for CSRP data and Wharton HPCC for High Performance Computing support.

ABSTRACT

HETEROGENEOUS AGENT MODEL WITH REAL BUSINESS CYCLE WITH APPLICATION IN OPTIMAL TAX POLICY AND SOCIAL WELFARE REFORM

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In this paper, we develop a dynamic stochastic general equilibrium (DSGE) model with financial friction and incomplete risk-sharing among overlapping-generation (OLG) heterogeneous households. The economy is embedded with taxation system and social security system calibrated to current U.S. economy and tax policy, as well as elastic labor supply. Our baseline model can match wealth-income disparity and moment conditions in financial market as well as macroeconomic variables. In baseline setting, the mean risk-free rate is 1.36% per year, the unlevered equity premium is 4.08%, and Gini coefficient for labor earning and total income is 0.65 and 0.51 respectively. The equity risk premium is driven by incomplete risk sharing among household and participation barrier to equity market. Furthermore, our model can act as workhorse model for policy experiment including debt policy, wealth tax reform, capital income tax reform and social security system reform. This paper could be beneficial to policy maker to understand the impact of policy change to macroeconomy and household-level behavior.

Contents

Chapter 1

Introduction

After the Great Recession in 2008, we have witnessed unprecedented phenomena in financial market as well as wealth-income disparity on household level. The real interest rate fluctuated at extremely low level but, simultaneously, the U.S. federal debt-GDP ratio soared to 46% (held by domestic investors). The correlation between debt-GDP ratio and real interest rate is near zero or even negative in more recent years. The real per-capita GDP deviated from its long-run trend significantly. On household level, the distribution of income and, especially, of wealth are highly concentrated and skewed to the right. Across the United States, the share of wealth owned by the richest 1% of households reaches 37.8% and share of income earned by the top 1% households reaches 24.4% from Survey of Consumer Finances (2016). Our questions are: what does it matter to macroeconomy from debt policy if there exists zero or negative correlation between real interest rate and debt-GDP ratio; what leads to such a correlation from fiscal policy perspective; and what leads to such unprecedented wealth-income concentration? In this paper, we provide a workhorse model that could cast light on the joint effect of aggregate macroeconomic uncertainty and idiosyncratic risk on individual household level to financial market; and inter-generational as well as intra-generational wealth-income disparity. With realistic taxation system and social security system calibrated to current U.S. legislation, our model can be used to conduct policy experiment on fiscal policy, tax policy and social security system reform.

The fundamental building block of our baseline model is a dynamic stochastic general equilibrium (DSGE) model with overlapping-generations (OLG) and heterogeneous agent, which is the most persuasive framework for analyzing fiscal policy change, tax policy change and social security system reform. Unlike the canonical representative agent model with infinite horizon, the heterogeneous-agent OLG model accommodates life-cycle property into labor-leisure choice, consumption-saving choice and portfolio choice. It allows for inter-generational and intra-generational heterogeneity among households, which is necessary to generate disparity in labor earning, total income and wealth distribution. On the top of heterogeneous-agent OLG model, we introduce aggregate uncertainty, which could influence total factor of productivity (TFP) and depreciation rate of risky capital (equity) in the economy, which enters the one-period optimization problem of representative firm on production side of the general equilibrium. With this setup, factor prices including economy-wide wage, interest rate and equity return are stochastic.

Households are heterogeneous along several different dimensions: First, they have to bare uninsurable labor income shock, which influences their individual working ability. Second, we have a OLG life-cycle model in which young agent, mid-life agent and retiree with different age choose different optimal consumption, labor-leisure and portfolio choice. Third, the initial wealth of agents within same age can be different to reflect intra-generational wealth disparity. Fourth, agent's social security benefits after retirement depend on agent-specific average historical earning. During working age, agent can use their labor endowment elastically. Households are price taker and have same rational expectation about factor prices. In our baseline model, there are more than 330000 types of household.

One primary goal of our baseline model is to solve equity risk premium puzzle and risk-free rate puzzle as well as matching moment conditions of macroeconomic variables in the U.S. economy. As for the interaction between factor prices and macroeconomic variables, our baseline model is designed to match negative correlation between real interest rate and debt-GDP ratio, which implies a non-Ricardian economy. Our model is essentially an incomplete market model with heterogeneous market participants. Market incompleteness results from both aggregate and uninsurable idiosyncratic working ability shock, combined with borrowing constraint of household and participation barrier to equity market. The source of aggregate uncertainty in our model includes aggregate productivity shock and shock to the capital depreciation rate. This stochastic depreciation shock is the key building block to match moment conditions in financial market.

Solving a heterogeneous-agent OLG model with aggregate uncertainty can be computationally challenging. There are technically infinitely many heterogeneous agents in the model economy, and we need to solve their optimization problems for many periods recursively. To solve "curse of dimensionality", we accommodate approximation equilibrium first presented in Krusell and Smith (1998). To improve the robustness of approximation equilibrium, we present application of Artificial Neural Network (ANN) approximation approach, introduced by Fernández-Villaverde et al. (2018), in large-scale OLG model. We also accommodate high-performance multi-thread computing techniques to solve our model more efficiently since household optimization problems within each age cohort are independent with each other. Multi-thread distributed computing is accessible through AWS, Microsoft Azure or NSF PSC XSEDE cluster. We present two solutions: CPU-based hybrid computing with MPI and OpenMP, and later GPU-based computing with Nvidia GPU on AWS.

The rest of the paper is organized as follows: Chapter 2 provides a brief review of the existing literature and applications of heterogeneous-agent OLG model. Chapter 3 describes our baseline stylized heterogeneous-agent OLG model with a progressive income tax and a realistic Social Security system. Chapter 4 explains the computational algorithms in detail to solve the optimization problem of heterogeneous agents and to solve the overall model economy for a competitive equilibrium, Chapter 5 presents the calibration of the baseline economy to the U.S. economy with empirical data from several resources. Chapter 6 presents the result of our baseline economy. Chapter 7 focuses on policy experiments including fiscal policy change and tax reform. Chapter 8 discusses the role of some minor building blocks of our model. Chapter 9 makes concluding remark.

Chapter 2

Literature Review

2.1 Heterogeneous Agent Model

Our article is built on heterogeneous agent model with finite life-cycle, which is also known as overlapping-generation (OLG) model. However, in this type of model, the aggregate economy and, further, factor price, remains deterministic in the sense that there is no aggregate uncertainty in the economy. In these models, the household, in general, still has to bear idiosyncratic working ability shock, which is the reason for precautionary savings. The related work includes Bewley (1986) , Laitner (1992), Huggett (1993), Aiyagari (1994), Ros-Rull (1999), Imrohoroglu et al. (1995), Conesa and Krueger (1999, 2006), Storesletten et al. (2001, 2004), Domeij and Heathcote (2004), Nishiyama and Smetters (2005, 2007, 2013), Conesa et al. (2009). Most of these models solve stationary steady-state equilibrium. This type of model also allows us to analyze the transition path of some policy change and the social welfare effect of fiscal policy change.

2.2 Dynamic Stochastic General Equilibrium Model

This type of model accommodates aggregate uncertainty to macroeconomic variables that results in stochastic factor prices. It can be applied to investigate the effects of real business cycle (RBC), the term structure of debt, and optimal risk sharing across generations. The computational challenges, however, are significant because the underlying size of the state space is too large for standard dynamic programming technique. This set of literature mainly starts from Kydland and Prescott (1982). The assumptions of DSGE model include perfect competition in all markets, rational expectation of all households and infinitely lived identical price-taking households. The related work includes Smets and Wouters (2002), Christiano et al. (2005), Smets and Wouters (2007), Bloom (2009), Christiano et al. (2011) and Jermann and Quadrini (2012).

2.3 Asset pricing model

One of the major goal of this paper is to provide a realistic explanation for equity risk premium puzzle and risk-free rate puzzle in asset pricing literature. This set of literature mainly starts from Mehra and Prescott (1985). Investing in equity looks like such a great deal when considering the equity risk premium representative agent would require compared with empirical risk premium. Constantinides and Duffie (1996) shows that risk premium will increase if the idiosyncratic shock becomes more volatile during economic contraction, which is the intuition for asymmetry in idiosyncratic shock in our baseline model. Storesletten et al. (2007) adds two important ingredients to this type of model: the life cycle (OLG model), and capital accumulation. Gomes and Michaelides (2007) extends this type of model to include a production side and inelastic labor supply. An essential question to answer behind the equity risk premium puzzle is the reason for limit participation to equity market. In Basak and Cuoco (1998), the author blocks explicitly some agents from participating in equity market and this model is in continuous-time setting. The closest article to our paper is Gomes and Michaelides (2007), in which they model limit participation from life-cycle perspective endogenously and explicit participation cost only plays a moderate role in equity risk premium puzzle and participation ratio in empirical data. However, Gomes and Michaelides (2007) and later Gomes et al. (2012) heavily replies on preference heterogeneity and Epstein-Zin preference to generate difference in participation behavior.

2.4 Advanced numerical solution to dynamic OLG model

With aggregate uncertainty, most dynamic models do not have an analytic closed-form solution and one need to use numerical solution to approximate optimal solution. Therefore, solving competitive equilibrium in OLG model becomes a large-scale optimization problem that requires advanced solution method generally coming from numerical solution to canonical PDE.

The first canonical method for dynamic model is Perturbation methods introduce by by Judd and Guu (1997). Perturbation methods rely on first order or higher order Taylor series expansion of the household policy functions around the stationary steady state of the economy and a perturbation parameter for optimal policy functions. Although it's powerful in some computational macroeconomic models, it's still a local method and could fail with binding constraint or large-scale aggregate shock.

Krusell and Smith (1998) introduces an approximation equilibrium, in which household uses lower moment of wealth distribution and other aggregate variables to predict factor prices in the future. If the economy does no have sufficient insurance mechanism for household towards both aggregate shock and idiosyncratic risk, Krusell-Smith may fail in terms of small R^2 of law of motion of aggregate state variables. Understanding the distortion in consumption-saving behavior of poor household is the key to the robustness of Krusell Smith method.

To provide a global and robust solution to dynamic OLG model with realistic aggregate uncertainty, Krueger and Kubler (2004, 2006) introduce Smolyak method with sparse grid to restrict computation to key points in the high-dimensional state space to get rid of "curse of dimensionality". There is a single agent per generation and the only asset available for trade is risky capital. However, with current computing power, this type of projection method can only be applied to OLG model with around 40 state variables. The reason is that the growth speed of number of key points reduces to polynomial growth instead of exponential growth with regard to dimension. In Brumm and Scheidegger (2017), they extend this idea and add adaptiveness into sparse grid. At the same time. they choose piece-wise linear hierarchical "hat" function instead of Chebyshev polynomials as basis functions. The reason is that global polynomials like Chebyshev polynomials cannot capture the local behavior of policy functions that are not sufficiently smooth. Piece-wise hat function can handle local behavior adaptively. Introducing hierarchical surplus and a so-called refinement threshold allow us to add points only to region where steepness of policy functions is large enough since adding point to region where policy functions have extremely small steepness contributes little to improve numerical error. Although Adaptive Sparse Grid (ASG) has many promising features, it can only solve dynamic OLG model with up to 100 continuous state variables.

Another promising method that attempts to solve "curse of dimensionality" problem in dynamic OLG model is introduced by Judd et al. (2009, 2011) and based on simulation on ergodic set of the economy. From Judd et al. (2011), the ergodic set takes the form of an "oval shape" and most of the rectangular area in Sparse Grid based method that sits outside of the ovals boundaries is never visited. When solving high-dimensional state space problem, this could reduce computational cost dramatically. However, equilibrium conditions outside this set might then be substantially violated. Hasanhodzic and Kotlikoff (2013) applies the idea of Generalized Stochastic Simulation Algorithm (GSSA) in Judd et al. (2011) to large-scale OLG model (80-period).

Finally, advanced numerical method in OLG model can always be parallelized by applying hybrid computing with MPI and OpenMP or CUDA computing with GPU. The primary reason is that optimization problem of household within specific age cohort is independent with other households in the same age cohort. Parallelization can be achieved with multi-thread computing based on CPU or GPU. For example, Aldrich et al. (2011) shows how to build algorithms that use GPU installed in most modern computers to solve dynamic equilibrium models in economics. Brumm and Scheidegger (2017) use hybrid supercomputer architecture with MPI and OpenMP to apply Adaptive Sparse Grid to International Real Business Cycle (IRBC) model.

Chapter 3

Model

The discrete-time economy consists of a large number of overlapping-generation (OLG) heterogeneous households, a representative firm with constant-return-to-scale (CRS) technology and a government with commitment technology. The heterogeneity of household includes initial individual wealth, average historical earning, portfolio choice, individual productivity and age. Individual productivity exposes to uninsurable idiosyncratic labor income risk in the setup of Bewley (1986), Huggett (1993), Aiyagari (1994), Carroll (1997). Household consumption, labor-leisure and portfolio choice are influenced by aggregate uncertainty through the channel of factor prices (risk-free return, equity return, economy-wide wage rate) over real business cycle. After retirement, households receive social security benefit which is determined by their average historical earning over working age. Households can invest in two types of financial asset: a claim to the risky capital stock (equity) and a risk-free bond issued by government. Firm is perfectly competitive, price-taking, and combines risky capital and labor using a constant returns to scale (CRS) technology, to produce a non-durable consumption good. The financial market is incomplete with financial friction. Time is discrete and one period in the model represents one year in real economy. We assume a labor-augmenting productivity growth rate μ. In the following model description, individual variables other than working hours are thus adjusted by $(1+\mu)$ to reflect aggregate productivity growth.

3.1 Household's problem

The heterogeneous agent's type can be described by tuple $\mathbf{s} = (a, b, d, e, i)$, where $a \in [0, a_{\max}]$ is the beginning-of-period initial wealth consisting positions in risky equity, which represents ownership to representative firm, and risk-free bond issued by government; $b \in [b_{\min}, b_{\max}]$ is the average historical earning over past working era up to current age, $d \in [0,1]$ is the share of risk-free bond in agent's portfolio, e the individual working ability (individual productivity), i, age of agent. The agent enters the economy and begins to work at age $i = 1$, which corresponding to real age 21, retires at I_R and live at most up to age I. The average historical earning, b, is calculated to approximate the average indexed monthly earning (AIME) to determine individual social security benefit after retirement. In every period, the household receives an realization of idiosyncratic working-ability shock and they choose consumption-saving, labor-leisure and portfolio choice to maximize their expected lifetime utility.

3.1.1 Utility function

In our baseline model, households have constant relative risk aversion (CRRA) utility function defined over consumption of a single non-durable good, c_t, and leisure time, $(1-l_t)$, of each period:

$$u(c_t, l_t) = \frac{(c_t^{\alpha}(1-l_t)^{1-\alpha})^{1-\gamma}}{1-\gamma} \tag{3.1.1}$$

where γ is the coefficient of relative risk aversion, α the consumption share parameter.

In some canonical asset pricing paper, recursive preference like Epstein-Zin preference is a powerful choice to match household's behavior on portfolio choice or to generate participation barrier endogenously. Although the intuition of Epstein-Zin preference in DSGE model is unclear, introducing Epstein-Zin preference adds freedom to model calibration and preference heterogeneity. The recursion utility function is defined as:

$$V_{i,t} = \{(1-\beta)((c_t^{\alpha}(1-l_t)^{1-\alpha}))^{1-1/\psi} + \beta(E_t[\phi_i V_{i+1,t+1}^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}\}^{\frac{1}{1-1/\psi}} \tag{3.1.2}$$

where ψ is the elasticity of intertemporal substitution (EIS), which measures responsiveness of the consumption change to the real interest rate. To introduce preference heterogeneity, we can allow γ and ψ to vary among different group of household.

3.1.2 Labor endowment

Each period, agents are endowed with one unit of labor, which agents can use elastically. Suppose l_t is the hour of working agent chooses at current period, t, then $1 - l_t$ is the leisure time, which agent cherishes. The labor income is: $w_t e_t l_t$, where w_t is economy-wide wage rate per efficiency unit of labor, e_t individual working ability, which is subject to idiosyncratic risk in working age. Therefore, there is a trade-off between leisure time and labor income, which is governed mainly by α in utility function. Later, we will show this trade-off is also influenced by retirement benefit. We assume labor productivity of retired household is zero.

Idiosyncratic risk is an essential component for heterogeneous agent model to generate precautionary savings. From Mankiw (1986) and later Constantinides and Duffie (1996), aggregate shocks and the volatility of idiosyncratic shocks are negatively related, which is defined as counter-cyclical cross-sectional variance, or CCV risk. Idiosyncratic risk is also time-consistent shock, which means household with good realization of idiosyncratic risk is more possible to receive another good realization.

3.1.3 Financial market, portfolio choice and financial friction

Following growing literature studying asset pricing in a production economy, there are two types of financial asset household can invest: one-period risk-free bond issued by government and risky investment opportunity in equity. Equity reflects the ownership to representative firm in the economy. Risk-free bond acts as safe asset for household to insure themselves against both aggregate uncertainty and idiosyncratic risk. The risk-free return of current period t is denoted by $r_{f,t}$ and risky equity return of current period is denoted by r_t. The primary trade-off between investing in these two financial assets is higher investment return (equity risk premium) and higher risk exposure taken for stockholder. In our baseline model and later model with rare shock to the economy, we assume there exists a small positive fixed cost to participate in equity market such that agent with initial wealth below some level, $\underline{k}$, cannot enter equity market for the next period. Therefore, agent's participation to equity market is endogenously influenced by agent's consumption-saving choice even if $\underline{k}$ is a calibration target to match empirical participation ratio.

However, we still introduce the potential form of variable participation cost to equity market in our model since it has some microeconomic foundation as described in Gomes and Michaelides (2007), and we simply want to verify that variable cost is not the major reason for equity risk premium puzzle. The participation cost is one-time entry cost, which takes form $\kappa w_t e_t l_t$, where κ is the fixed rate on agent's labor income at the period agent chooses to participate in equity market. The intuition for participation cost is that agent has to spend time and effort to, for example, obtain knowledge on equity market, open a trading account and manage their initial portfolio actively. Once entering the equity market, agent does not need to pay any participation cost if they have positive position in equity market and receive r_t on their position in equity market. In empirical data, equity market provides a risk premium over risk-free bond market and therefore agent has to make another trade-off between participation cost and risk premium. At each period t, household can choose d_{t+1}, their bond demand as a percentage of their portfolio value for the next period.

3.1.4 Tax on household

In each period, three types of taxation are levied on households: consumption tax, progressive individual income tax and a Social Security payroll tax. Consumption tax is a flat rate, $\tau_{C,t}$, tax levied on agent's consumption, $c_t \tau_{C,t}$. The second type of tax is Social Security payroll tax, which takes form:

$$\tau_{P,t}(w_t e_t l_t) = \bar{\tau}_{P,t} \min(w_t e_t l_t, \vartheta_{\max}) \tag{3.1.3}$$

where $\bar{\tau}_{P,t}$ is the flat Old-Age and Survivors Insurance (OASI) tax rate that includes the employers portion. Income beyond $\vartheta_{\max}$ is exempted from The payroll tax. Only household in their working age is subject to this type of tax. The third type of tax is progressive individual income tax. The individual income tax function follows Gouveia and Strauss (1994):

$$\tau_{I,t}(y_t) = \varphi_t[y_t - (y_t^{-\varphi_1} + \varphi_2)^{-1/\varphi_1}] \tag{3.1.4}$$

where y_t is sum of labor income and capital income of period t with fixed deduction:

$$y_t = \max\{r_t a_t(1 - d_t) + r_{f,t} a_t d_t + w_t e_t l_t - d_{\text{exemp}}, 0\} \tag{3.1.5}$$

where φ_t the tax rate limit, φ_1 the curvature coefficient, φ_2 the scale coefficient, d_{exemp} the exemption.

In later policy experiment, we introduce wealth tax into our baseline model. In 2020 presidential election, wealth tax is a popular topic under debate. European countries currently levy wealth tax on household's asset beyond some threshold. Type of asset could include bank deposits, real estate, ownership of unincorporated businesses, financial securities, and personal trusts. Marginal wealth tax rates ranged from 0.25% to 1.5%. In general, wealth tax is proposed to mitigate wealth-income inequality around the world. For simplicity, in policy experiment, we apply a flat wealth tax on household's asset in policy experiment section.

3.1.5 Social security benefits

After retirement, agents with age $i \geq I_R$ can obtain retirement benefits. We propose a social security benefit system similar to Nishiyama and Smetters (2013), in which government tracks agent's average historical earning and distributes social security benefits based on agent's average historical earning. This creates another incentive for agent to work in their working age instead of simply labor income. Then, the social security benefit function is kinked and equal to:

$$\begin{aligned} tr_{SS,t}(i, b_t) =& \mathbf{1}_{i \geq I_R} \psi_t \{0.90 \min(b_t, \vartheta_1) + \\ & 0.32 \max[\min(b_t, \vartheta_2) - \vartheta_1, 0.00] + 0.15 \max(b_t - \vartheta_2, 0.00)\} \end{aligned} \tag{3.1.6}$$

where ϑ_1 and ϑ_2 are the thresholds for the three replacement rate brackets, 90%, 32%, and 15%, that calculate the Social Security benefit based on the average historical earnings, and ψ_t is the benefit adjustment factor to balance the budget of payroll tax revenue and social security benefit payment. b_t is the average historical earning over past working era up to current age. It is approximated by:

$$b_{t+1} = \mathbf{1}_{i < I_R} \frac{1}{i}[(i-1)b_t + \min(w_t e_t l_t, \vartheta_{\max})] + \mathbf{1}_{i \geq I_R} b_t \tag{3.1.7}$$

each period in household's working era.

However, this type of setting does not accommodate economy-wide wage growth since the calculation of b_t does not take wage growth into consideration. An alternative method to take wage growth into consideration is introduced by Li and Smetters (2011). Although our baseline model is detrended with regard to productivity growth, the economy-wide aggregate wage rate is still influenced by aggregate uncertainty. Following the notation in Li and Smetters (2011), suppose household retires at age $M = I_R$. Then, the average historical earning over working age is defined as:

$$\bar{Y}_{t,M} = \frac{Y_{t,1}\frac{Y_M^A}{Y_1^A} + Y_{t,2}\frac{Y_M^A}{Y_2^A} + \ldots + Y_{t,M-1}\frac{Y_M^A}{Y_{M-1}^A}}{M-1} \tag{3.1.8}$$

where Y_i^A the economy-wide average wage level at age i. In numerical part, a similar way with Li and Smetters (2011) is to update b_{t+1} taken aggregate-economy-level wage rate, w_t, into consideration only in the last period of working age $I_R - 1$, since economy-wide wage rate in $Y_{t,i}$ and Y_i^A is canceled out.

3.1.6 Aggregate state

Since our model has both idiosyncratic risk and aggregate uncertainty, agent's decision on consumption, labor choice, portfolio choice depends not only on individual states but also on aggregate state at each time period. The vector for aggregate states is defined as $\Phi_t = (z_t, \delta_t, \Omega_t, r_{f,t}, r_{f,t+1})$, where z_t is Total Factor of Productivity (TFP) of current period, δ_t the depreciation rate of risky capital, Ω_t the wealth distribution across different type of heterogeneous agent, $r_{f,t}$ the risk-free return of government bond of current period and $r_{f,t+1}$ the risk-free return of government bond of next period. The dimension of Ω_t is more than 330000, which is not feasible for computation purpose. In later section, we will introduce approximation equilibrium to reduce computational cost with lower moment of Ω_t. The approximation vector for aggregate states is $\Psi_t = (z_t, \delta_t, K_t, r_{f,t}, r_{f,t+1})$, where K_t is the amount of aggregate capital stock.

Need to clarify why we need to choose $r_{f,t}$ and $r_{f,t+1}$ as state variables here. First, agent need to know $r_{f,t}$ to make optimal choice since their current-period risk-free return from bond as well as tax burden is affected by $r_{f,t}$. Since our model is a general equilibrium model, $r_{f,t+1}$ is pined down by market clear condition on risk-free bond market at current period. In equilibrium, $r_{f,t+1}$ need to be tweaked to match debt-GDP target of current period. Need to clarify the timing of these two returns on risk-free asset. $r_{f,t}$ is determined in the general equilibrium of last period and $r_{f,t+1}$ in the general equilibrium of current period. However, $r_{f,t+1}$ only enters household's budget constraint for the next period.

TFP shock z_t follows an AR(1) process, which will be discretized to a two-state Markov chain, which has a period that matches period of real business cycle in empirical data. The two state is denoted by $z_t \in \{z_L, z_H\}$. Depreciation shock δ_t can take several values with different probability to match the moment conditions of equity return in empirical data. δ_t is identically independently distributed. Our goal here is to match moment conditions for equity return as well as keeping our model simple without ad hoc assumption. Then, the law of motion of these aggregate state variables are:

$$K_{t+1} = \Gamma_K(\Phi_t), r_{f,t+2} = \Gamma_f(\Phi_{t+1}), L_t = \Gamma_L(\Phi_t) \tag{3.1.9}$$

As for the second law of motion, household need to use it to predict $r_{f,t+2}$ since $r_{f,t+2}$ is a state variable of household' value function for the next period. The reason for the third law of motion is that agent need it to predict the aggregate labor supply of current period since we have elastic labor supply in our model, which changes over real business cycle and depends on the realization of aggregate state. Need to clarify the timing here: K_{t+1} is determined at the end of period t and therefore only depends on aggregate state variables at period t; $r_{f,t+2}$ is determined in the general equilibrium of period $t+1$ and therefore depends on aggregate state variables at period $t+1$; L_t is determined in general equilibrium of period t.

3.1.7 Rare risk in aggregate state

Beyond the assumption on TFP shock in our baseline model, we also introduce the concept of rare shock to TFP in OLG model in later part. Rare risk to macroeconomy is inspired by asset-pricing puzzles, which is a collapse that is infrequent and large in magnitude. Barro (2006) suggests a disaster probability of 1.52 percent per year with a distribution of decline in per capita GDP ranging between 15 percent and 64 percent. The data for decline in real per capita GDP is from twenty OECD countries in the twentieth century. This group of countries covers major economies of Western Europe, Australia, Japan, New Zealand, and the United States. In 20th and 21st century, the well-known rare disasters in U.S. include World War I, the Great Depression, World War II and more recent Great Recession in 2008, which features a shape, large and global decline in terms of GDP from its long-run trend. However, the deviation from long-run GDP growth trend in 2008 Great Recession is not as large as the first three disasters in the U.S. We think this is due to fiscal policy during the process of 2008 Great Recession. Then, we modify the TFP shock as $z_t \in \{z_L, z_H, z_R\}$, and keep stochastic depreciation shock same as baseline model. We calibration z_R to match average decline in per capita GDP in U.S. same as Barro (2006).

3.1.8 Optimization problem

The agent's optimization problem with type (a, b, d, e, i) is:

$$V(a,b,d,e,i;\Phi) = \max_{c,l,d'}\{\frac{(c^{\alpha}(1-l)^{1-\alpha})^{1-\gamma}}{1-\gamma} + \beta\phi_i E[V(a',b',d',e',i+1;\Phi')|\Phi]\} \qquad (3.1.10)$$

subject to the constraints for the decision variables:

$$c > 0, \quad 0 \leq l < 1, \quad a' \geq 0 \tag{3.1.11}$$

with law of motion of the individual state variables:

$$\begin{aligned} a' =& \frac{1}{1+\mu}[(1+r)a(1-d) + (1+r_f)ad + weh - \\ & \tau_I - \tau_P + tr_{SS}(i,b,a) - (1+\tau_C)c] \end{aligned} \tag{3.1.12}$$

$$b' = \mathbf{1}_{i<I_R}\frac{1}{i}[(i-1)b + \min(wel, \vartheta_{\max})] + \mathbf{1}_{i \geq I_R} b \tag{3.1.13}$$

From the setup of optimization problem above, the utility of household goes to minus infinity as consumption goes to zero. This introduces some problem in numerical solution part. Instead of introducing a lower bound for the value of utility function, we introduce a mechanism called "minimum-consumption", $\underline{c}$. The maximum amount for household to consume is

$$c_{\max} = (1+r)a(1-d) + (1+r_f)ad + weh - \tau_I - \tau_P + tr_{SS}(i,b,a) \tag{3.1.14}$$

If $c_{\max} < \underline{c}$, the government will provide $\underline{c}$ amount to this specific household to consume in a lump-sum manner. Household cannot save out of this amount. In later policy experiment part, we apply different $\underline{c}$ to understand the implication of minimum consumption to macroeconomic variables and wealth-income inequality. For the "non-minimum consumption" case, $\underline{c}$ is set to be zero and we assign minimum negative value in compiler for real number to utility function when consumption is zero.

3.2 Representative firm's problem

For the production side, we choose a representative firm with standard Cobb-Douglas production technology. The firm is owned totally by household and hire labor force from household. In each period, the representative firm chooses the capital input, $\tilde{K}_t$, and the labor input, $\tilde{L}_t$, to maximize its current period profit, taking factor prices, r_t and w_t, as given. The firm's production function is:

$$F(\tilde{K}_t, \tilde{L}_t) = Az_t\tilde{K}_t^{\theta}\tilde{L}_t^{1-\theta} \tag{3.2.1}$$

The firm's profit maximizing problem is:

$$\max_{\tilde{K}_t, \tilde{L}_t} F(\tilde{K}_t, \tilde{L}_t) - (r_t + \delta_t)\tilde{K}_t - w_t\tilde{L}_t \tag{3.2.2}$$

The firm's profit maximizing conditions are:

$$F_K(\tilde{K}_t, \tilde{L}_t) = r_t + \delta_t, F_L(\tilde{K}_t, \tilde{L}_t) = w_t \tag{3.2.3}$$

In a closed economy, the factor markets for aggregate capital and aggregate labor are clear when:

$$K_t = \tilde{K}_t, L_t = \tilde{L}_t \tag{3.2.4}$$

Therefore, the gross domestic product, which is identical to gross national product, Y_t, is determined by:

$$Y_t = F(K_t, L_t) = (r_t + \delta_t)K_t + w_t L_t \tag{3.2.5}$$

As an alternative solution to stochastic depreciation shock, we introduce adjustment cost to investment into representative firm's problem. We assume the adjustment cost to investment takes form:

$$\frac{1}{2}\eta(\frac{I_t - \delta_t K_t}{K_t})^2 K_t \tag{3.2.6}$$

and then the one-period optimization problem of representative firm becomes:

$$\max_{\tilde{K}_t, \tilde{L}_t} Az_t \tilde{K}_t^{\theta} \tilde{L}_t^{1-\theta} - (r_t + \delta_t)\tilde{K}_t - w_t \tilde{L}_t - \frac{1}{2}\eta(\frac{I_t - \delta_t \tilde{K}_t}{\tilde{K}_t})^2 \tilde{K}_t \tag{3.2.7}$$

From Gomes (2001), the investment-capital ratio in empirical data is 14.5%. In our baseline model, η is set to zero.

3.3 Government's problem

The role of government in our model is to collect tax and other revenue (accidental bequest), redistribute it to the household and make decision on government debt policy target. In our baseline model, we assume the government's tax policy is time-invariant.

The government's income tax tax revenue T_I is:

$$T_{I,t}(\varphi) = \sum_{i=1}^{I} \int_{A\times B\times E\times D} \tau_{I,t}(a, b, d, e; \varphi, \varphi_1, \varphi_2) dX_{i,t}(a, b, d, e) \tag{3.3.1}$$

while the consumption tax revenue T_C is:

$$T_{C,t}(\tau_C) = \sum_{i=1}^{I} \int_{A\times B\times E\times D} \tau_{C,t}(a,b,d,e;\tau_C) dX_{i,t}(a,b,d,e) \tag{3.3.2}$$

and the Social Security (OASI) payroll tax revenue is:

$$T_{P,t}(\bar{\tau}_P) = \sum_{i=1}^{I} \int_{A\times B\times E\times D} \tau_{P,t}(a,b,d,e;\bar{\tau}_P) dX_{i,t}(a,b,d,e) \tag{3.3.3}$$

and the Social Security benefit expenditure is:

$$TR_{SS,t}(\psi) = \sum_{i=I_R}^{I} \int_{A\times B\times E\times D} tr_{SS,t}(a,b,d,e;\psi) dX_{i,t}(a,b,d,e) \tag{3.3.4}$$

The budget constraint of government is:

$$C_t^G + r_{f,t} B_t = B_{t+1} - B_t + T_{I,t} + T_{C,t} + T_{P,t} - TR_{SS,t} \tag{3.3.5}$$

where C_t^G is the government consumption expenditure, B_t the amount of risk-free bond issued by the government, $T_{I,t}$ the income tax revenue, $T_{C,t}$ the consumption tax revenue, $T_{P,t}$ the payroll tax revenue, $TR_{SS,t}$ the social security payment.

For a general equilibrium model, the market for risk-free bond is cleared every period and government need to specify policy target for risk-free bond supply. In order to match the correlation of debt-GDP ratio with risk-free rate and equity return, we choose dynamic target for debt-GDP ratio based on aggregate state instead of zero-net supply or positive fixed debt-GDP ratio. The next consideration is the volatility of risk-free rate. One of the major challenge of asset pricing in DSGE model with risk-free asset and risky equity is the huge empirical difference in the volatility of returns from these two assets. Our dynamic target on debt-GDP ratio, which has significant influence on volatility of risk-free asset, should match the empirical data. Therefore, our dynamic debt-GDP target is chosen to be:

$$(D_{t+1}/Y_t) = (D_t/Y_{t-1})\frac{1}{\mathrm{TFP}_t} \tag{3.3.6}$$

Apart from goal to match empirical correlation between macro variables and factor prices, the intuition for this dynamic debt-GDP target is from empirical data of TFP process and debt-GDP ratio series. In Figure 5.5, empirical data for $(D_{t+1}/Y_t)/(D_t/Y_{t-1}) - 1$ and $1/\mathrm{TFP}_t - 1$ is plotted jointly to justify this rule. A long theoretical literature attempted to demystify the links between government debt to macroeconomic dynamics. Ricardian Equivalence claims that government debt has no effect in a frictionless model in Barro et al. (2014). However, with incomplete market and heterogeneous agent, this is not the case anymore. Our dynamic debt-GDP target is essentially counter-cyclical policy.

3.3.1 Accidental Bequests

As Nishiyama and Smetters (2014), we assume that the government collects remaining wealth held by deceased households at the end of period t and distributes it to working-age household in a lump-sum manner in the same period. Since there exists aggregate shock in the economy, the government cannot perfectly predict the sum of accidental bequests during each period. We assume the government's policy is to keep the mean of net revenue of accidental bequests zero and accidental bequest account balances in long-run. Each working-age household receives an accidental bequest q with constant probability η:

$$q_t = \frac{\sum_{i=1}^{I}(1-\phi_i)\int_{E\times A\times B\times D} a' dX_{i,t}(a,b,d,e)}{\sum_{i=1}^{I}(1-\phi_i)\int_{E\times A\times B\times D} dX_{i,t}(a,b,d,e)} \tag{3.3.7}$$

$$\eta = \frac{\sum_{i=1}^{I}(1-\phi_i)\int_{E\times A\times B\times D} dX_{i,t}(a,b,d,e)}{\sum_{i=1}^{I_R-1}\phi_i\int_{E\times A\times B\times D} dX_{i,t}(a,b,d,e)} \tag{3.3.8}$$

where q_t is the average wealth left by deceased households, and η is the ratio of deceased households to surviving working-age households. On top of that, we assume the household cannot predict the realization of bequest lottery so as to internalize this bequest into the optimization problem.

3.4 Recursive Competitive Equilibrium

We first clarify the timing of our model. The household starts current period with pre-specified initial wealth (consisting of risk-free bond and risky equity), which is determined at the end of last period. Then, the aggregate shock is realized, including TFP and depreciation rate to equity. The household is price taker of factor prices including risk-free return, wage rate and risky equity return of current period as well as representative firm. The consumption, labor-leisure choice and portfolio choice are functions of individual state variables and aggregate state variables Ψ. Heterogeneous household and representative firm makes optimal choice simultaneously, with factor markets clear. Finally, a demographic distribution on household defined on $A \times B \times D \times E \times I$ for next period is determined.

The equilibrium consists of endogenously determined factor prices (risk-free rate, economy-wide wage rate and risky equity return), a set of cohort-specific value functions, policy functions, and rational expectations about the law of motion of the endogenously determined aggregate variables, such that:

1. Representative firm maximizes its one-period profit by equating marginal products of risky capital and labor to their respective marginal cost.

2. Heterogeneous agents choose consumption, labor and asset allocation to solve their recursive optimization problem taking factor prices as given.

3. Market clear for risk-free asset, risky capital and labor market.

4. The government budget is balanced every period to sustain a given ratio of government-debt to output (can be different across different aggregate states or follow some dynamic target chosen by government).

5. Laws of motion of aggregate variables are verified in equilibrium.

The proof for the existence of competitive equilibrium in OLG model is extremely difficult and is limited to linear tax functions, which is not realistic in real economy. In our case, the existence of recursive competitive equilibrium is verified in numerical solution part.

Chapter 4

Computation Method

To solve the recursive competitive equilibrium defined in last chapter, we need to use time iteration method. With a pre-specified error threshold, ε_0, of time iteration, our algorithm keeps track of the difference between Krusell-Smith coefficients of two consecutive time iterations in the sense of sup-norm and stops when this difference is below this specific threshold. Inside each time iteration, it can be divided into two parts: optimization and simulation. In optimization part, our algorithm first uses Krusell-Smith coefficient to form rational expectation of factor prices conditional on current aggregate state, and then solves the households optimal policy functions, taking factor prices as given, recursively from age $i = I$ to age 1, since the agent with age I will die with probability one and therefore has no continuation value in their optimization problem. For agent with age other than $i = I$, we can use the value function of agent at age $i + 1$ as given. Then, with optimal policy functions, our algorithm simulates the economy for specific number of periods. The final step is to run OLS regression on simulated time series for aggregate variables to update Krusell-Smith coefficients.

4.1 State discretization

The first step of our numerical solution method is to discretize the individual state variables as follows: initial wealth space of individual household, $A = [0, a_{\max}]$ into J nodes, $\hat{A} = \{a_1, a_2, \ldots, a_J\}$; average historical earning space, $B = [b_{\min}, b_{\max}]$ into N nodes, $\hat{B} = \{b_1, b_2, \ldots, b_N\}$; and L nodes for working ability space by Tauchen and Hussey method; M nodes for portfolio choice space, $\hat{D} = [d_1, d_2, \ldots, d_M]$. Then, we discretize the aggregate state variables: aggregate amount of risky equity $[K_{\min}, K_{\max}]$ and risk-free rate $[r_f^{\min}, r_f^{\max}]$. We choose linear grid for both of these spaces. Need to mention that we choose same grid for r_f and r'_f since they are both state variables that determine household's value function and we need to interpolate on it in optimization part. A general rule of thumb is to choose interval for individual state variable to cover its ergodic set in simulation. This can be achieved with trail-and-error.

Here, we assume idiosyncratic shock to individual productivity is transitory and follows an AR(1) process. The continuous-valued AR(1) process is usually replaced by a discrete state-space Markov chain in dynamic programming problem. Tauchen's method in Tauchen (1986) or the quadrature-based method developed in Tauchen and Hussey (1991) require the persistency of AR(p) to be relatively small. The error term in AR(p) is assumed to be Gaussian white noise with variance σ_ϵ^2. However, their performance or efficiency deteriorates when the persistency approximates to 1. With large persistency coefficient or near unit root, Rouwenhorst method has the best performance among these three methods. Another advantage is that we do not need to make any distributional assumption on error term. Symmetric distribution for individual productivity shock cannot generate distortion observed in empirical data.

4.2 Krusell-Smith Algorithm

Another important ingredient for the numerical solution of this model is the law of motion for aggregate state variables. It's the core of Krusell-Smith algorithm. Household need it to form their own expectation about factor prices for the next period. Although each household in this economy is a price taker, their policy functions will influence the factor prices in general equilibrium. Instead of choosing wealth distribution over all types of agent as state vector (which can be larger than 330000 dimension), which is infeasible even with some advanced technique like Adaptive Sparse Grid, our solution method chooses to accommodate the approximation equilibrium methodology in Krusell and Smith (1998), in which agent only need to know a finite-dimensional vector (lower moments) to represent the law of motion for all the aggregate state variables. In another word, household can use lower moments of wealth distribution to approximate factor prices.

Our solution method is built on den Haan (1997), Krusell and Smith (1998), Storesletten, Telmer, and Yaron (2007) and Gomes and Michaelides (2007). It expands the idea of lower moment approximation behind these approximation equilibrium solutions. For endowment economy with only equity, household need simply law of motion for aggregate capital to form conditional rational expectation. With elastic labor supply from household and risk-free asset market, our solution method need two extra laws of motion for risk-free rate and aggregate labor supply. The functional form for this set of law of motion in our baseline model is:

$$\ln K' = q_{1,0} + q_{1,1} \ln K \tag{4.2.1}$$

$$\ln (1 + r_f'') = q_{2,0} + q_{2,1} \ln (1 + r_f') + q_{2,2} \ln K' + q_{2,3} \ln (1 + r_f) \tag{4.2.2}$$

$$\ln L = q_{3,0} + q_{3,1} \ln (1 + r'_f) + q_{3,2} \ln K + q_{3,3} \ln (1 + r_f) \tag{4.2.3}$$

The parameters in law of motion for aggregate state variables and factor prices are contingent on type of aggregate shock. Need to clarify the timing assumption here. The predicted amount of aggregate risky capital K' is determined with current realization of aggregate shock and current state variables. The predicted aggregate labor supply for the current period, L, is determined by the realization of aggregate shock, and state variables of current period K, r_f, r'_f. r'_f is determined in the general equilibrium of current period. Household need to predict r''_f, which is a state variable of their value function for the next period, and r''_f is determined in the general equilibrium of next period.

In Krusell-Smith algorithm, R^2 of law of motion is the major concern. In general, we want the law of motion that household uses to predict factor prices to match the law of motion of aggregate state variables from simulation part. The rule of thumb is that the minimum of R^2 in our model should be larger than 95% since we have three law of motions compared with just one in Krusell and Smith (1998). The intuition is law of motion of state variable should explain all variance in state variable. If we cannot go beyond this threshold, we find adding higher moments of state variables could increase R^2 dramatically. However, in our baseline model and later variations of our baseline model, insurance mechanism like accidental bequest, minimum consumption and, most importantly, social security system diminish most distortion in policy functions of extremely poor household and therefore the law of motion we present could achieve relatively large R^2.

From Krueger and Kubler (2004), the reason for Krusell-Smith algorithm to be valid or have sufficiently large R^2 is that, apart from the (small number of) agents right at the borrowing constraint, all agents in the economy have approximately the same marginal propensity to save out of current wealth. Although consumption-saving choice of poor household could be quite different from less-constrained household to the right, their contribution to aggregate consumption-saving choice can be nearly ignored since their initial wealth is so small. However, this is not the general case for macro-finance model with aggregate uncertainty. One of the major intuitions for method based on sparse grid or adaptive sparse grid is that R^2 is small for some heterogeneous agent models without enough insurance. The consumption-saving choice for extremely poor household could be highly distorted even up to some less constrained household. There is a recent paper by Fernández-Villaverde et al. (2019), in which they use neural network with one hidden layer to approximate law of motion of aggregate state variables. This approach can get rid of R^2 issue in approximation equilibrium approach. Another advantage of this method is that the training data for neural network is from simulation, which means the training of large neural network will not suffer lack of data problem.

4.3 Numerical solution method outline

As a general solution to large-scale OLG model, the numerical sequence works as follows:

1. Specify a set of coefficients for K-S forecasting equations of aggregate variables as initial guess. Except the first time iteration, this set of coefficients is the output of OLS of last time iteration with damping.

2. Solve the optimization problem of individual household, backward from age I to age 1, taken factor prices as given, and using the forecasting equations of aggregate variables

to form conditional rational expectation on factor prices for the next period.

3. Given the policy functions of household, with an initial guess on demographic distribution over different type of household, simulate the economy for T periods (discarding the observations of first T_{pre} periods), while keeping bond market, labor market and risky equity market clear period-by-period. Here, the bond market is cleared by tweaking r'_f, which is a state variable in agent's policy functions.

4. Use the simulated times series of risky equity return, risk-free return, aggregate risky capital stock and aggregate labor supply to update the forecasting equations using OLS.

5. Repeat the last four steps with new coefficients of forecasting equations from OLS until convergence. We have two convergence criteria:

   - Stable coefficients in the forecasting equations for aggregate state variables.
   - Forecasting equations with OLS regression R^2 above some pre-specified threshold.

4.4 Individual optimization

Given the set of coefficients of forecasting equations of aggregate variables (K, L, r_f, r'_f), we need to solve optimal consumption, labor choice and portfolio choice of each type of household in the economy. The first step is to replace Φ with Ψ with lower moment approximation. Portfolio choice for the next period is the outer-loop, which means that we need to solve optimal consumption and labor choice given specific portfolio choice for next period and find maximum value function across difference portfolio choice. The inner loop is to optimize consumption and labor-leisure choice given portfolio choice for the next period. To solve optimization problem in inner loop, we present three methods here. The first one is to choose 2-dimensional discrete grid on feasible space of consumption and labor choice $[0, c_{\max}] \times [0, 1]$ and choose the consumption and labor that maximizes value function on this 2-dimensional grid. The main issue of this solution method is the relatively large Euler error in terms of consumption unit. However, the Euler error could be reduced with well-chosen grid.

The second method is Hybrid Newton method based on the Euler equations of individual household. Then, combining with the law of motion of the state variables, the first-order conditions for an interior solution are:

$$f_c = u_c(c, l) - \frac{\beta\phi_i(1+\tau_C)}{1+\mu} E[v_a(s'; \Psi')|\Psi] = 0 \tag{4.4.1}$$

$$\begin{aligned} f_l =& u_l(c, l) + we[1 - \tau_{I,2} - \tau_P'] \frac{u_c(c, l)}{1+\tau_C} \\ &+ \mathbf{1}_{\{i<I_R, wel<\vartheta_{\max}\}} \frac{we}{i} \beta\phi_i E[v_b(s'; \Psi')|\Psi] = 0 \end{aligned} \tag{4.4.2}$$

With inequality constraints for consumption choice and labor choice, the Karush-Kuhn-Tucker (KKT) conditions of the individual optimization problem are:

$$f_c = 0 \ \text{ if } \ 0 < c < c_{\max}, \quad f_c > 0 \ \text{ if } \ c = c_{\max} \tag{4.4.3}$$

$$f_l = 0 \ \text{ if } \ 0 < l < l_{\max}, \quad f_l > 0 \ \text{ if } \ l = l_{\max} \tag{4.4.4}$$

Then, we combine KKT condition with boundary conditions using method in Billups

(2000, 2002) and Miranda and Fackler (2002):

$$\mathrm{CP}(c,l) = \min\left\{\max\left[\begin{pmatrix} f_c/u_c \\ \\ f_l/u_l \end{pmatrix}, \begin{pmatrix} \epsilon - c \\ \\ \epsilon - l \end{pmatrix}\right], \begin{pmatrix} c_{\max} - c \\ \\ l_{\max} - l \end{pmatrix}\right\} = 0 \tag{4.4.5}$$

where ϵ is a small positive number. In numerical solution, max function and min function will also be approximated with continuous functions. We can solve the above problem by using a Newton-type nonlinear equation solver, NEQNF, of the IMSL FORTRAN Numerical Library in FORTRAN. This library subroutine uses a modified Powell hybrid algorithm and a finite difference approximation to the Jacobian matrix. In theory. this solution method could control on-grid Euler error in terms of consumption unit at the cost of much larger computational cost since vanilla initial guess on optimal consumption-labor may be out of convergence disk.

The third method is to discretize labor choice with a grid on $[0,1]$ and solve consumption choice using Brent's method. In numerical analysis, Brent's method is a root-finding algorithm combining the bisection method, the secant method and inverse of quadratic interpolation. It has the reliability of bisection but it can be as quick as some of the less-reliable methods. The third method allows us to control on-grid Euler error for consumption.

4.5 High-Performance Computing

Although we use Krusell-Smith method to avoid using vector of whole wealth distribution across different types of agent, which could result in "curse of dimensionality" problem, the optimization part of each time iteration is still computationally costly. The reason for this computational cost is: the large number of heterogeneous household, risk-free asset market and aggregate shock. However, within specific age cohort, the optimization problem of different types of agent within this cohort can be perfectly parallelized due to the independence property since continuation value for next period is known and taken as given. To conduct parallel computing, we first use Message Passing Interface (MPI) technique. We then apply Hybrid Computing with MPI and OpenMP to further improve the performance of our code.

Message Passing Interface (MPI) is a standardized and portable message-passing standard designed by researchers from academia and industry to function on a wide variety of parallel computing architectures based on CPU. The standard defines the syntax and semantics of a core of library routines useful to a wide range of users writing portable message-passing programs in C, C++, and FORTRAN. There are several well-tested and efficient implementations of MPI, many of which are open-source or in the public domain. OpenMP (Open Multi-Processing) is an application programming interface (API) that supports multi-platform shared memory multiprocessing programming in C, C++, and Fortran. OpenMP is an implementation of multi-threading, a method of parallelizing where a master thread (a series of instructions executed consecutively) forks a specified number of slave threads and the system divides a task among them. The threads then run concurrently, with the runtime environment allocating threads to different processors.

The reason for us to introduce Hybrid Computing with OpenMP and MPI is that the RAM usage with simple MPI application is too large for our High Performance Computing platform since we need to define policy functions on each thread of each CPU and most High Performance Computing (HPC) clusters limit the sum of RAM usage on each thread. With Hybrid Computing, we can just define policy functions on the shared memory of CPU (node), where each thread (core) of this specific CPU (node) can have access. The idea is that: optimization problem within age cohort is first divided with regard to CPU. Within each CPU, different thread works on subsection of optimization problem. Then, we use MPI to collect solution on each CPU to master node and propagate policy functions from master node to each CPU.

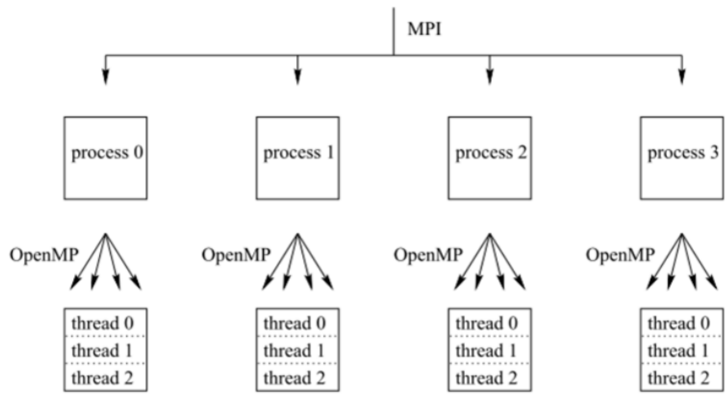


Figure 4.1: Hybrid computing with OpenMP and MPI. MPI is implemented to communicate among different CPU; OpenMP is implemented to manage shared memory on each CPU with multiple threads. Policy functions are only defined on memory of each CPU.

To further accelerate solving optimization part in time iteration, we accommodate GPU-based High-Performance Computing technique thanks to Amazon Web Service (AWS). CUDA is a parallel computing platform and programming model developed by Nvidia for general computing on its own GPUs. The advantage of GPU-based HPC solution, compared with CPU-based on, is the number of computing thread to run parallel jobs is much larger. For economic-efficiency Nvidia K80 graphic card, it has 4996 Nvidia CUDA cores, compared with only 14 thread on Intel Haswell (E5-2695 v3) CPU at PSC XSEDE.

4.6 Method to accelerate calibration

Although we have applied high-performance computing technique like MPI, OpenMP or CUDA to accelerate optimization problem within each time iteration, simulation part within each time iteration still costs considerable computing time. A common setup for simulation part is to simulate the economy for 10000 period and discard the first 5000 period.

To accelerate simulation part, we present two methods: the first one is "better initial guess on demographic distribution"; the second is that we let each CPU simulate based on CPU-specific historical path for aggregate state and take the mean of those new Krusell-Smith coefficients from OLS on each CPU. The intuition of the first method is that, in the common setup, the first 5000 period simulation is conducted to guarantee that the OLS calculation is based on simulation starting from some realistic demographic distribution. If we write the final demographic distribution from last time iteration into a binary file and use it as the initial guess of demographic distribution in the new time iteration, we can significantly reduce the number of time period of this pre-simulation part. The second method is based on intuition from law of large number. On different CPU, the simulation is based on CPU-specific historical path of aggregate state, which is a realization of same underlying aggregate transition matrix. With these two methods, computing time for simulation part can be dramatically reduced.

Chapter 5

Calibration

In this section, we describe how we map our baseline economy to the U.S. economy. Since we focus on the interaction between aggregate uncertainty and idiosyncratic risk, we calibrate the model to annual-frequency data collected from several resource as described below.

5.1 Household

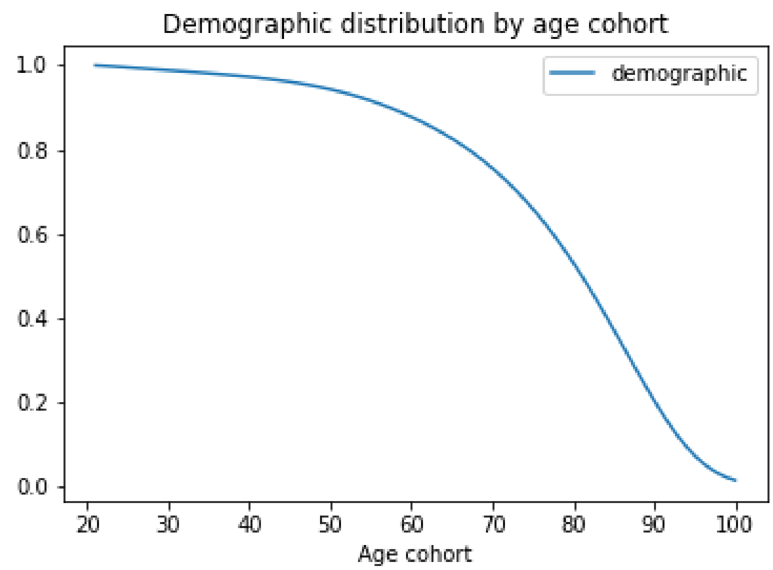
Household is assumed to enter the economy at the beginning of age $i = 1$ (real life age 21) with zero initial wealth. The household lives up to $I = 80$ (real life age 100) and retirement age starts from $I_R = 45$ (real life age 65) and they will die with probability 1 after age $I = 80$, which makes our model finite life-cycle model. The labor productivity growth rate, μ, is set at 1.8%, which is an approximation to the average growth rate, 1.8%, of real per-capita GDP from the year 1971 through 2011. Compared with Nishiyama and Smetters (2013), there is no population growth in our baseline model. Table 5.1 shows the value of main parameter of the household sector.

Table 5.1: Main parameters of baseline model

Maximum possible age	I	80
Retirement age	I_R	45
Productivity growth rate	μ	0.018
Household survival rates	ϕ_i	Social Security Administration (2013)
Consumption Share	α	0.24
Time discount factor	β	0.998
Risk aversion (CRRA)	γ	3.0
Auto-correlation parameter of log wage	ρ	0.95
Standard deviation of log wage shocks	σ	0.22
Median working ability	$\bar{e}_i$	Estimated by OLS

The conditional survival rate, ϕ_i, at the end of age i, is collected from Table 4.C6 2007 Period Life Table in Social Security Administration (2013), same as Nishiyama and Smetters (2013). The conditional survival rate at end of age I is set to zero. Figure 5.1 shows the density of demographic distribution with regard to age cohort under conditional survival probability in SSA (2013) described above. The population of agent with age 21 is normalized to be unit 1. Risk aversion parameter is chosen at the middle of canonical macro-finance literature with CRRA utility function. The share parameter of consumption, α is chosen to be 0.24 to pin down 1/3 average hour of working of all working-age household.

Figure 5.1: Demographic distribution by age cohort



5.1.1 Individual working ability process

The working ability, e_i, of an age $i < I_R$ household in the model economy is assumed to satisfy:

$$\ln e_i = \ln \bar{e}_i + \ln z_i \quad \text{for} \quad i = 1, \ldots, I_R - 1 \tag{5.1.1}$$

where $\bar{e}_i$ is the median wage rate at age i, and z_i is the persistent shock, which reflects idiosyncratic risk. The persistent shock follows an AR(1) process:

$$\ln z_i = \rho \ln z_{i-1} + \epsilon_i \quad \epsilon_i \sim N(0, \sigma^2) \tag{5.1.2}$$

The median working ability of age i, $\bar{e}_i$, for ages 21-64 is constructed from the 2010 male-worker earnings from Table 4.B6 Median Earnings of Workers by Age in Social Security Administration (2013). The working ability profile by age is approximated by a cubic function of age i, and estimated by OLS regression over the median earnings of individuals between age 25 and 61 from Nishiyama and Smetters (2013). The autocorrelation parameter, ρ, of the log persistent shock is set at 0.95. The log persistent shock, $\ln z_i$, is discretized into 9-state Markov chain using Tauchen and Hussey's method, $\{\zeta_1, \zeta_2, \ldots, \zeta_9\}$. The major concern here is to create high individual productivity household to match labor earning inequality, which is source for other inequality. With equation (5.1.1), $\{e_{i,j}\}$ is calculated as combination of both permanent individual productivity and transitory productivity. On top of that, $e_{i,1} = 0$ when the economy hits a bad TFP shock. This reflects asymmetry in idiosyncratic risk with regard to real business cycle, inspired by Constantinides and Duffie (1996). The prevalent choice for ρ is between $[0.90, 0.98]$. We assume working ability after retirement is zero and force retired household not to participate in labor market. In Figure 5.2, we report the transition matrix for the Markov chain to approximate the idiosyncratic shock.

	1	2	3	4	5	6	7	8	9
1	0.831666	0.168319	1.57E-05	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2	1.87E-02	0.850148	0.131158	7.68E-06	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
3	1.49E-07	2.73E-02	0.872576	0.100089	3.67E-06	0.00E+00	0.00E+00	0.00E+00	0.00E+00
4	1.59E-16	3.45E-07	3.91E-02	0.886145	7.48E-02	1.71E-06	0.00E+00	0.00E+00	0.00E+00
5	1.84E-29	5.91E-16	7.78E-07	5.47E-02	0.890685	5.47E-02	7.78E-07	0.00E+00	1.84E-29
6	0.00E+00	1.11E-28	2.14E-15	1.71E-06	7.48E-02	0.886145	3.91E-02	3.45E-07	1.59E-16
7	0.00E+00	0.00E+00	6.52E-28	7.58E-15	3.67E-06	0.100089	0.872576	2.73E-02	1.49E-07
8	0.00E+00	0.00E+00	0.00E+00	3.74E-27	2.62E-14	7.68E-06	0.131158	0.850148	1.87E-02
9	0.00E+00	0.00E+00	0.00E+00	0.00E+00	2.09E-26	8.80E-14	1.57E-05	0.168319	0.831666

Figure 5.2: Transition matrix for idiosyncratic shock

In canonical literature, Tauchen method is the first method to discretize AR(p) process to Markov process. With larger persistency coefficient ρ, the efficiency and accuracy of Tauchen method deteriorates dramatically. Tauchen and Hussey method is built on Gaussian quadrature and can handle large ρ efficiently. As for highly persistency AR(p) process, Rouwenhost method is the optimal choice and it makes no Gaussian assumption on the error term. Non-Gaussian assumption is useful in the future to capture right-skewness of wealth-income inequality without adding entrepreneur.

5.1.2 Household consumption

In general equilibrium with participation barrier to some household, equity return is priced by the stochastic discount factors (SDF) of stockholder, which can be expressed as a function of aggregate consumption growth in stochastic pricing kernel. In Campbell (1999) sample, the covariance between aggregate consumption growth and equity return is just 0.0037. With more recent annual data from NIPA, this covariance is only 0.0004. This moment condition is important for asset pricing literature since realistic equity risk premium can be achieved with abnormally large covariance between aggregate consumption growth and equity return. One of the major goal of our paper is to show that equity risk premium is driven by incomplete market and financial friction not abnormal covariance. The standard deviation of aggregate consumption growth is 3.3% in the 1890-1997 sample (Campbell (1999)). However, in 1982-2004 sample from Malloy, Moskowitz, and Vissing-Jorgensen (2006), the implied standard deviation of aggregate consumption growth is 1.7%. Malloy, Moskowitz, and Vissing-Jorgensen (2006) data is from more recent (1982-2004) Consumer Expenditure Survey (CES). On aggregate level, the aggregate consumption as a share of aggregate output is 59.5% from Gomes, Michaelides and Polkovnichenko (2013). The data is from BEA National Accounts and they classify 75% of durable consumption expenditures as investment and the remaining 25% as consumption.

5.1.3 Wealth, earning and income distribution

One of the major goals for this paper is to match a realistic wealth, labor earning and income disparity and that's why we introduce both inter-generational and intra-generational heterogeneity. The Survey of Consumer Finances (SCF) is considered the best dataset for the financial wealth of U.S. families. The SCF is conducted every three years. It contains the balance sheet, pension, income and other demographic characteristics of families in the United States as well as information on the use of financial institutions. The latest available results are those of the 2016 SCF survey. From SCF, the inequality in labor earning, income and wealth has increased significantly since the 1970s after several decades of stability, which means the share of the nation's income received by high income households has increased a lot over past several decades. This trend is evident among wealth, earning and income, as shown in Table 5.2. However, due to some issue about top-coding in empirical data, past literature fails to account for wealth, earning and income disparity as measured by the SCF. In Table 5.3, we report the Gini coefficients from SCF (2016) and FRED, which suffers top-coding problem. Gini index in FRED is based on primary household survey data obtained from government statistical agencies and World Bank country departments. We also report canonical result implied by model in Aiyagari (1994), which is one of the most influential paper in heterogeneous agent model.

Table 5.2: Gini coefficients for income, earning and wealth of U.S. household

Year	Income	Earning	Wealth
1989	0.540232	0.604490	0.790158
1992	0.500777	0.605726	0.785937
1995	0.515152	0.606360	0.790651
1998	0.530136	0.588915	0.799965
2001	0.562329	0.610183	0.805427
2004	0.540069	0.612579	0.809248
2007	0.574159	0.633011	0.816157
2010	0.549490	0.650248	0.845663
2013	0.574237	0.654349	0.850482
2016	0.597706	0.679937	0.859562

Table 5.3: Gini coefficients for wealth, earning and income

	SCF 2016	FRED	Aiyagari (1994)
Wealth Gini	0.8596	N/A	0.38
Earning Gini	0.6799	N/A	0.10
Income Gini	0.5977	0.415 (2016)	N/A

Based on data in SCF (2016), we then plot share of wealth-by-wealth, earning-by-earning and income-by-income by age bucket. To be comparable with our model, we consider household with age 21-100. Then, we divide household into 16 buckets with 5-year range, to smooth survey result. When dividing household into bucket by wealth, income, and labor earning measure, the sorting process happens on households within this specific age bucket instead of the total population. For earning result, we only report household that is in working age. Th result is in Appendix A.

5.1.4 Aggregate shock

In our model, aggregate uncertainty could influence both TFP and depreciation rate of risky capital, which partially determine factor prices in the economy. Households are price taker. The stochastic process for TFP shock is assumed to follow a two-state, $[1.01, 0.99]$, Markov process and its transition matrix is chosen to match the duration of real business cycle, which is around 5.5 years in empirical data. The probability of remaining in the current state is 2/3. Then, the transition matrix of TFP shock is:

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \tag{5.1.3}$$

As for model with rare shock similar to Barro (2006), we assume TFP shock follows a three-state, $[1.01, 0.99, 0.85]$ Markov process with transition matrix:

$$\begin{bmatrix} 2/3 - 0.01 & 1/3 - 0.01 & 0.02 \\ 1/3 - 0.01 & 2/3 - 0.01 & 0.02 \\ 1/3 - 0.01 & 2/3 - 0.01 & 0.02 \end{bmatrix} \tag{5.1.4}$$

In our baseline model, we assume the realization of depreciation shock is independent of the realization of TFP shock although it can be straightforward to add some correlation. The depreciation shock is independently identically distributed. The probability distribution of depreciation shock and amount of depreciation in each state are chosen mainly to match the first four moments of equity return in empirical data. However, calibration on TFP shock and depreciation shock is conducted simultaneously to match moment conditions for some aggregate variables, including standard deviation of aggregate output and aggregate consumption growth, correlation between debt-GDP ratio and risk-free rate, covariance between aggregate consumption growth and equity return. In our baseline model, the realization of depreciation shock is $[-0.06, 0.10, 0.34]$ with probability $[0.17, 0.66, 0.17]$. The systematic way $(p_1, \delta_1, \delta_2)$ to pin down this calibration on depreciation shock is:

1. Start from probability distribution $[1/3, 1/3, 1/3]$ and state $[0.0, \delta, 2 * \delta]$. Change the distribution of depreciation shock to match kurtosis of equity return. The coefficient to pick is p_1 and the probability distribution of depreciation shock is $[1/3 - p_1, 1/3 + 2p_1, 1/3 - p_1]$.

2. Choose distance coefficient δ_1 to match standard deviation of equity return. The state of depreciation shock becomes $[\delta - \delta_1, \delta, \delta + \delta_1]$, which is still symmetric at this step.

3. Add parallel shifting δ_2 to state 1 and 3 of depreciation shock to match skewness. The state of depreciation shock becomes $[\delta - \delta_1 + \delta_2, \delta, \delta + \delta_1 + \delta_2]$. Since the skewness of equity return is not large in empirical data, this shifting in state 1 and 3 has small influence to calibration on standard deviation of equity return.

Another systematic way to calibrate depreciation shock is to set up a three dimensional grid on $(p_1, \delta_1, \delta_2)$ and choose the calibration that can match the moment conditions of risky equity return. The calibration of depreciation shock in our baseline model indicates that there is 17% probability for the economy to lose 34% of its risky capital. This abnormally large standard deviation in depreciation shock is to match volatility of equity return.

5.2 Representative Firm

In our baseline model, the representative firm has the familiar Cobb-Douglas production function with share parameter of capital stock coefficient 0.36. From the production side, our calibration goal includes investment over capital ratio at 14.5% indicated by Gomes (2001). In a closed economy, the final output of the representative firm is the GDP of the economy. The U.S. GDP data is based on annual NIPA, 1939-2017. The national income and product accounts (NIPA) are part of the national accounts of the U.S.. They are produced by the BEA of the Department of Commerce. Empirical data is filtered using an Hodrick-Prescott filter with smoothing parameter of 400, implied by Storesletten et al. (2007). The difference is that Storesletten et al. (2007) chooses NIPA from 1929-2007 and we choose more recent data.

5.3 Government

The government also need to choose tax rates to finance social security system and government consumption program. Tax rates are chosen to match ratios of revenue of different types of tax over GDP as well as to fund social security program. Table 5.4 is a summary of tax rates we choose. All the tax policy is time-invariant in baseline model.

Table 5.4: Calibration: Tax rate and social security system

Individual income tax	φ_t	0.2650
	φ_1	0.8483
	φ_2	1.2343
Deduction and exemptions	d_{exemp}	0.1066
Cons. tax	$\tau_{C,t}$	0.052
Payroll tax	$\bar{\tau}_{p,t}$	0.10
OASI benefit adjustment factor	ψ_t	0.62
Maximum taxable earnings	$\vartheta_{\max}$	1.0487
Replacement rate	ϑ_1	0.0883
	ϑ_2	0.5322

The progressive individual income tax function follows the formula in Gouveia and Strauss (1994). The parameters that control progressivity is borrowed from Nishiyama and Smetters (2013) to match the curvature. We tweak φ_t to match the ratio of the income tax revenue to GDP, which is 10% in the baseline economy. The OASI payroll tax rate is 5.3% for an employee and 5.3% for an employer. The payroll tax rate, $\bar{\tau}_{P,t}$, for earnings below the maximum taxable earnings is set at 0.10, which is approximately equal to $(5.3+5.3)/(100+5.3) = 0.1007$. The consumption tax rate is set at 5%.

5.4 Financial market

5.4.1 Bond market

The debt-GDP ratio data is from Federal Reserve Economic Data (FRED).[1] FRED is a database maintained by the Research division of the Federal Reserve Bank of St. Louis. In FRED dataset, it has two series of historical data about federal debt in the U.S.: federal debt held by public as a percentage of GDP and federal debt held by foreign and international investors as a percentage of GDP. We focus on the difference of these two historical data as shown in Figure 5.3 since our model is essentially a closed economy instead of small open economy in the literature. From Figure 5.3, debt-GDP ratio is not a constant even stationary over time. That's why we believe canonical assumption of fixed debt-GDP ratio is invalid. In Figure 5.4, time series for TFP change[2] also from FRED is plotted jointly with empirical debt-GDP ratio from 1970-2017. The key moment we consider here includes the correlation between debt-GDP ratio and risk-free rate as well as equity return.

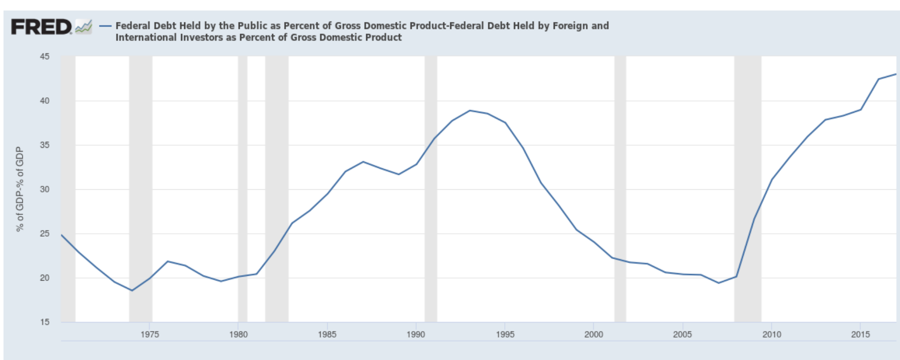


Figure 5.3: Federal debt held by domestic investor as a percentage of GDP (1970-2017)

[1] https://fred.stlouisfed.org/series/FYGFGDQ188S
[2] https://fred.stlouisfed.org/series/RTFPNAUSA632NRUG

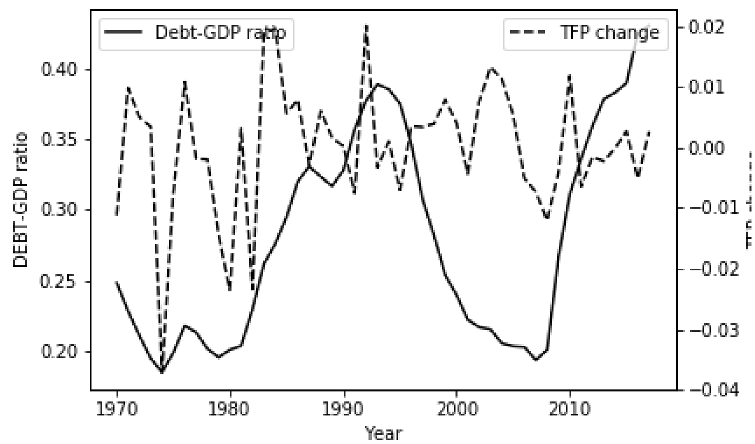


Figure 5.4: Debt-GDP ratio and TFP change (1970-2017)

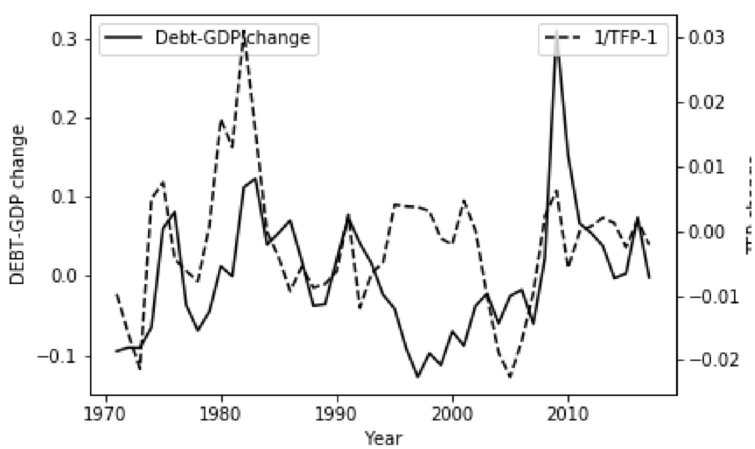


Figure 5.5: Debt-GDP change and TFP reciprocal (1970-2017)

As for the proxy for riskfree asset in the model, Storesletten et al. (2007) chooses one month US treasury bill. Nominal risk-free returns are deflated using the GDP deflator. All returns are expressed as annual percentage. In this case, the mean of riskfree return is around 1.3% and standard deviation around 1.88%. Campbell (1999) and later Gomes and Michaelides (2007) choose 3-month T-bill as proxy for riskfree rate. They report 1.58%

as the mean and 5.33% as standard deviation. In more recent paper, Bansal et al. (2012) reports mean and standard deviation as 0.57% and 2.86% respectively. In consistent with later part of our calibration, we choose annualized return of 90 day T-Bill from CSRP US Treasury and Inflation Indexes (1970-2017).[3]. Based on this dataset, the mean and standard deviation of risk-free rate is 1.31% and 2.57%. We also report moment conditions for one-month T-bill, 0.82% and 2.27% respectively; one-year T-bond as 1.85% and 3.51% respectively.

5.4.2 Equity market

In our baseline model, households are not allowed to short-sell against risk-free asset or risky equity. Therefore, our asset pricing target is on unlevered equity return as described in Storesletten et al. (2007). In order to calculate unlevered Beta, we assume the effective corporate tax is 23% and estimated market debt-equity ratio is 2/3 by Storesletten et al. (2007). Based on these assumptions, the unlevered Beta is 1.513. The proxy for equity return is the same with Storesletten et al. (2007). We choose Value Weighted Stock Index with Distribution (1956-2017) from CSRP dataset, which is slightly different from Storesletten et al. (2007), where they choose same index from 1956 to 2007. The participation ratio of U.S. household to equity market is borrowed from Gomes and Michaelides (2007), which is from SCF.

[3] https://wrds-web.wharton.upenn.edu/wrds/ds/crsp

Chapter 6

Results

With calibration as described in last section, we report moment conditions in asset pricing as well as macroeconomic variables, indicated by our baseline model. As for risk-free asset, key moments are mean and standard deviation at annual frequency. As for equity return, calibration target takes one step further to include skewness and kurtosis, which measures the shape of its simulated distribution. As for macro variables, we choose a set of moments and correlations in macro finance literature to quantify the performance of our baseline model. The main concern here is to justify equity risk premium is not driven by abnormal covariance but incomplete market with financial friction. Beyond moment conditions, we also report life-cycle profile for consumption, labor and initial wealth. The next step is earning, income and wealth inequality indicated by the baseline model. The final consideration is the relationship between interest rate and debt-GDP target.

6.1 Asset pricing implication

Table 6.1 reports the main asset pricing moments indicated by the baseline model, along with their empirical U.S. counterparts as described in calibration part. This is achieved with moderate risk aversion coefficient ($\gamma = 3$) compared with some canonical asset pricing literature that chooses extremely large γ, for instance, $\gamma = 10$ with Epstein-Zin preference. To match the standard deviation, kurtosis and skewness of equity return, we focus on choosing corresponding realization of stochastic depreciation shock. The mean and standard deviation of risk-free rate implied by our baseline model can match their empirical counterparts. However, the empirical mean of risk-free rate is time-variant and subject to proxy for risk-free asset one chooses. Our baseline model implies an unlevered equity risk premium of 4.08%.

Table 6.1: Asset pricing moments

Variable	Riskfree return		Equity return				
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt	Sharpe ratio
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585	0.3337
Baseline	1.36%	2.52%	5.44%	11.79%	-0.6405	2.8980	0.3112

Table 6.2 reports moment conditions for macroeconomic variables, including GDP, aggregate consumption, market participation and aggregate investment. Participation ratio implied by our baseline model is slightly higher than empirical value and this can be solved by adding variable participation cost. Participation ratio is also a time-variant variable in SCF. The empirical mean of debt-GDP ratio we report here includes post Great Recession period, when debt-GDP ratio in U.S. soared. Therefore, the mean debt-GDP ratio implied

by our model matches empirical mean excluding recession period. The main issue here is the standard deviation of aggregate consumption. The correlation between debt-GDP ratio and risk-free rate is also time variant. The number we report here is based on data of more recent year (1980-2017). If we contains post 1980 data, this correlation is -0.006.

Table 6.2: Macroeconomic variable moments

Variable	Moment	Baseline	Data
Log Agg. Output	Std. Dev.	5.60%	5.41%
Log Agg. Cons.	Std. Dev.	5.78%	3.52%
Debt-GDP,risk-free rate	Correlation	-0.1091	-0.2415
Debt-GDP,equity return	Correlation	0.2835	0.2609
Consumption,equity return	Covariance	0.0003	0.0004
Participation ratio	Mean	0.5820	0.5190
Consumption/Output	Mean	0.5852	0.5950
SS Benefit/GDP	Mean	5.00%	5.00%
Investment/Agg. Capital	Mean	14.30%	14.50%
Working hour (unit)	Mean	0.3006	0.3333

We feel obliged to mention hitting boundary phenomenon in simulation part of each time iteration. The method we apply to pin down bond market condition is to choose r'_f, the risk-free rate for next period determined in general equilibrium at current period, as state variable. With a grid discretizing the continues space of r'_f, there is a corresponding interval for debt-GDP ratio on demand side. However, if our debt-GDP target from supply side is not in this interval, we choose two boundary of r'_f interval as the risk-free rate for next period, which means the existence of mismatch between bond demand and bond supply. Hitting lower bound means extra demand and hitting upper bound means extra supply. This method is similar as placing some rigidity on risk-free rate since we never witness extreme real risk-free rate in empirical data. When hitting boundary happens, we force the debt supply to match debt demand to clear the bond market. We think this is a problem associated with financial friction setup. In real world, participation barrier for some U.S. household is far complicated than assuming a simple barrier based on initial wealth without variable participation cost.

In later Chapter 8, we present a potential solution for this artificial boundary with Epstein-Zin preference and well-chosen preference parameters. In this case, the hitting boundary case is reconciled and moment conditions matches their empirical counterparts. This Epstein-Zin model shows similar relationship between interest rate and debt-GDP ratio in terms of correlation we show above and regression analysis in later part.

6.2 Consumption, labor and portfolio choice

In this part, we plot the life-cycle profiles of consumption, initial wealth, and labor earning for the average household within age cohort in Figure 6.1. The hump shape in initial wealth with regard to age cohort is mainly due to average working ability profile described in calibration part. Consumption is smoothed compared with labor earning.

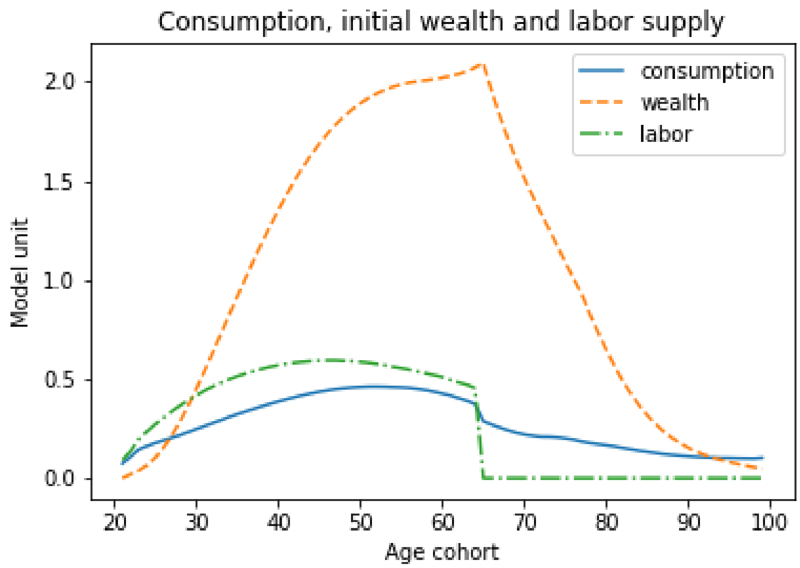


Figure 6.1: Consumption, Wealth and labor by age cohort

In our baseline model, we model participation barrier as a small positive cost such that agent with initial wealth below some threshold cannot enter equity market. Figure 6.2 and 6.3 plot life-cycle portfolio decision. The hump-shape in both participation ratio profile and portfolio choice profile is due to wealth accumulation over life-cycle since household need to accumulate some initial wealth to overcome participation barrier. The difference between participation ratio by age and portfolio choice by age reflects the fact that poor people are the main participant in bond market in this type of participation barrier model.

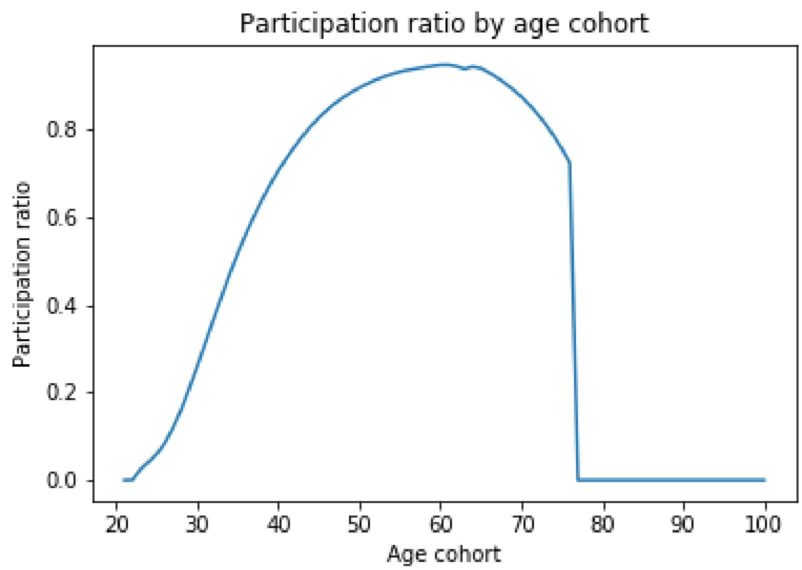


Figure 6.2: Participation ratio by age cohort

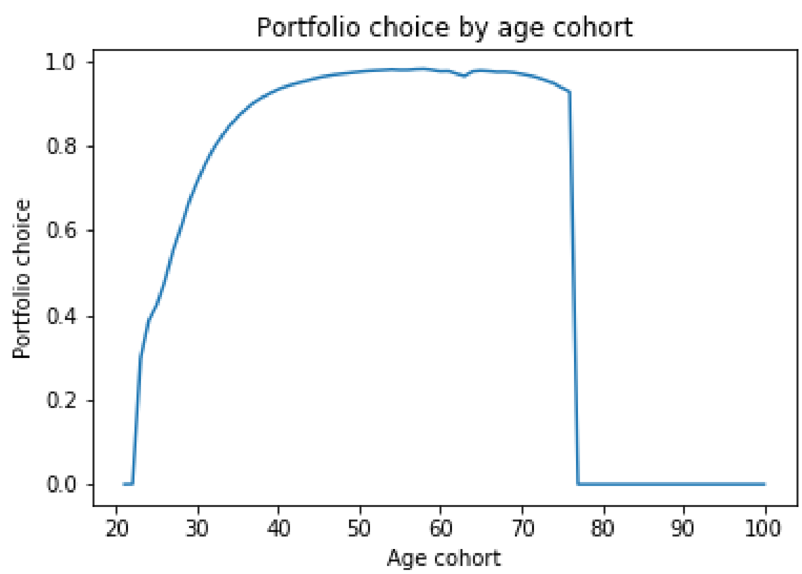


Figure 6.3: Portfolio choice by age cohort

In our baseline model, we discretize idiosyncratic risk into a 9-state Markov process. Therefore, in Table 6.3, we report participation behavior, portfolio choice and average working hour by different working ability group. Labor earning is the source for wealth-income inequality in heterogeneous agent model without entrepreneur. Need to mention here that households are not evenly distributed across these 9 working types but following stationary distribution of the Markov process for idiosyncratic risk. Type 1 household has lowest individual productivity and Type 9 household has highest productivity. Participation ratio is defined as share of population to enter equity market (strictly positive position) and portfolio choice is define as share of asset to invest in equity market. A clear trend is that participation in equity market and labor-leisure choice both increases with regard to individual productivity.

Table 6.3: Result based on working ability type

	A. Participation ratio								
	1	2	3	4	5	6	7	8	9
Baseline	0.262	0.307	0.391	0.511	0.638	0.740	0.814	0.865	0.899
	B. Portfolio choice								
	1	2	3	4	5	6	7	8	9
Baseline	0.729	0.727	0.738	0.764	0.795	0.822	0.854	0.883	0.908
	C. Working hour								
	1	2	3	4	5	6	7	8	9
Baseline	0.097	0.232	0.263	0.280	0.299	0.315	0.325	0.329	0.333

6.3 Wealth, earning and income disparity

In Table 6.4, we report Gini coefficient implied by our baseline model. Compared with other dataset like Current Population Survey (CPS), SCF does not suffer from top-coding problem, which prevents researchers from obtaining additional information about top 10 % or even top 1% household. Gini coefficient for earning and income is a little bit lower than SCF data. Our definition of earnings includes only labor earning. In SCF, negative earnings contribute to higher Gini coefficient compared to other measure provided in the literature. However, in our model, there is no possibility for household to have negative labor income. One of the potential reason for negative earning in the data is self-employment, which is beyond the topic of this paper. With similar reason, non-negative income is the cause for small income Gini coefficient. However, there is still huge difference between wealth Gini coefficient in the data and from the model. The reason is that our model does not have entrepreneur or time-preference heterogeneity.

Then, we report earning, income and wealth share by different bucket sorted by household's labor earning, income and wealth. This is a much more challenging task than simply Gini coefficient. The main finding is that the skewness to the right is smaller in baseline model compared with real U.S. economy from SCF (2016), which results in smaller Gini coefficients compared with empirical counterparts. We believe adding entrepreneur into our baseline model could solve this or we can model asymmetric idiosyncratic individual productivity shock, which is also known as Michael Jordan shock. In empirical data, bottom 20% household sorted by wealth level on average has negative wealth but our baseline model assumes household cannot be in debt.

Table 6.4: Distributions of Earnings and Wealth in the U.S. Economy

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1.0	Gini
	A. Distributions of Earning								
Baseline	0.000	0.026	0.108	0.227	0.213	0.139	0.170	0.117	0.6508
Data	0.000	0.016	0.106	0.216	0.178	0.132	0.185	0.167	0.6799
	B. Distributions of Income								
Baseline	0.017	0.070	0.167	0.277	0.191	0.109	0.112	0.059	0.5088
Data	0.026	0.061	0.101	0.166	0.132	0.100	0.169	0.244	0.5977
	C. Distributions of Wealth								
Baseline	0.000	0.062	0.178	0.302	0.197	0.108	0.103	0.050	0.5013
Data	-0.005	0.006	0.029	0.086	0.109	0.117	0.280	0.379	0.8595

The next step is to divide household into 5-year age bucket and calculate labor earning, income and wealth share by different bucket sorted by household's earning, income and wealth within this age bucket. Dividing household into 5-year age bucket is mainly to smooth survey result. The result indicated by baseline model is in Appendix B. The empirical counterparts from SCF (2016) is in Appendix A.

6.4 Regression analysis

The sensitivity of interest rates with respect to changes in the level of debt is vitally important in budget projections of federal government since the level of federal debt to GDP is projected to rise dramatically over the next decade. In Gamber and Seliski (2019) at Congressional Budget Office (CBO), it shows that the average long-run effect of debt on interest rates ranges from about 2 to 3 basis points for each increase of 1 percentage point in debt as a percentage of GDP. They estimated the effect of expected federal debt on expected interest rates using the following reduced-form regression:

$$ {}_t i_{t+5}^{(10)} = \beta_0 + \beta_{1t}\pi_{t+5} + \beta_{2t}D_{t+5} + \beta_3 X_t + \epsilon_t \tag{6.4.1} $$

where ${}_tD_{t+5}$ is the projected five-year-ahead debt-to-GDP ratio published by CBO at time t, ${}_ti_{t+5}^{(10)}$ the five-year-ahead 10-year rate from the yield curve observed at the end of the month in which the CBO report containing the projected debt-to-GDP ratio was published. The regression residual is represented by ϵ_t. ${}_t\pi_{t+5}$ a measure of expected inflation five-period-ahead at period t. The vector X_t contains additional control variables suggested by economic theory and used in previous studies. In Gamber and Seliski (2019), dividend yield and trend real GDP growth are chosen as control variables, to control the effect of equity return and real consumption growth on interest rate.

In our baseline model, every factor price is in real term and, therefore, inflation effect is removed. The linear equation we estimate using OLS, at period t, is:

$$ i_{t+5} = \beta_0 + \beta_1 D_{t+5} + \beta_2 K_t + \beta_3 \Delta Y_t \tag{6.4.2} $$

where K_t is aggregate capital and ΔY_t the growth rate in GDP. To run OLS on above linear model, we can use time series of factor prices from simulation part of our numerical solution. i_{t+5} is realized five-period-ahead risk-free return; D_{t+5} is realized five-period-ahead debt-GDP ratio. Based on realized data, we also estimate linear relationship between one-period-ahead interest rate and debt-GDP ratio. Table 6.5 reports result of the reduced-form

OLS regression. Based on realized data, there exists statistically significant negative relation between interest rate and debt-GDP target.

Table 6.5: Regression based on realized simulation data

	A. $i_{t+5} = \beta_0 + \beta_1 D_{t+5} + \beta_2 K_t + \beta_3 \Delta Y_t$				
Baseline	β_0	β_1	β_2	β_3	R^2
	0.0613	-0.2072	-0.0049	0.0090	0.03
	(6.172)	(-3.574)	(-0.865)	(0.523)	
	B. $i_{t+1} = \beta_0 + \beta_1 D_{t+1} + \beta_2 K_t + \beta_3 \Delta Y_t$				
Baseline	β_0	β_1	β_2	β_3	R^2
	0.0694	-0.2255	-0.0078	-0.1126	0.10
	(6.905)	(-4.713)	(-1.675)	(-6.982)	

Using realized data from simulation does not match the intuition of CBO regression analysis, which focuses on expectation. Beyond simulated time series, we can also use binomial-tree model to form expected factor prices five-period ahead. In our baseline model, household has full knowledge about transition matrix of aggregate state. Debt-GDP ratio target and household's expectation on factor prices only depend on aggregate state. Therefore, household use rational expectation on aggregate state variables to form rational expectation about factor prices five-period-ahead.

$$E_t[i_{t+5}] = \beta_0 + \beta_1 E_t[D_{t+5}] + \beta_2 K_t + \beta_3 \Delta Y_t \tag{6.4.3}$$

From Table 6.6, there exists negative relationship between interest rate and debt-GDP ratio. This negative relationship is statistically significant and 1% increase in expected debt-GDP target introduce around 1 basis decrease in expected interest rate, given other variables fixed.

Table 6.6: Regression based on rational expectation

	A. $E_t[i_{t+5}] = \beta_0 + \beta_1 E_t[D_{t+5}] + \beta_2 K_t + \beta_3 \Delta Y_t$				
Baseline	β_0	β_1	β_2	β_3	R^2
	0.0363	-0.0115	-0.0163	0.0168	0.68
	(29.504)	(-2.004)	(-29.992)	(8.844)	

Chapter 7

Policy Experiment

In our baseline model, we focus on dynamic debt-GDP rule to match correlation among macro variables and factor prices. Meanwhile, a realistic taxation system calibrated to current U.S. policy is levied on household to fund social security program and government consumption plan. With general equilibrium in hand, understanding the implication of several types of policy change is feasible here. The policy change we consider includes fiscal policy change, wealth tax reform and capital income tax reform. Average change to financial market factor prices and macro variables is reported as well as transition path for those variables. Transition path analysis is almost always conducted in stationary model without aggregate uncertainty. Since our model has aggregate uncertainty that will affect factor prices in equilibrium, we use bootstrapping technique to obtain mean transition path and 95% empirical distribution band.

This section explains three examples of policy experiments based on baseline model. The economy is assumed to be in a stochastic steady-state equilibrium at the period when new policy is announced. After the policy change, the economy responds and eventually approaches a new stochastic steady-state equilibrium. The general assumption here is that household cannot predict when government announces new policy and household's rational expectation about factor prices in terms of Krusell-Smith coefficients will not change during transition path of policy change. This means that household cannot re-optimize their policy functions against policy change in advance.

The simulation for transition path starts from solving general equilibrium in our baseline model and obtain optimal policy functions without policy change. Then, we simulate the baseline economy for T period and use the demographic distribution at the end of the simulation period T as initial demographic distribution for transition path simulation, which is in stochastic steady-state as we mention above. The length of simulation for transition path is T_{tp} and we simulate N_{tp} sample path with bootstrapping technique, which is one realization of transition matrix for aggregate state in our baseline setup. The first half of transition path simulation is under same policy as our baseline model. At the end of period $T_{tp}/2$, new policy change takes effect and participants in baseline economy re-optimize their policy functions on consumption, labor-leisure and portfolio choice. Need to clarify here that, when re-optimizing household's policy functions, we use the same Krusell-Smith coefficient as that in the baseline competitive equilibrium without policy change. The intuition behind this is that household's conditional rational expectation on factor prices and aggregate variables cannot change instantly and household cannot predict when new policy is exerted by the government. With new optimal policy functions, we simulate the economy for another $T_{tp}/2$ period with consistent demographic distribution and debt-GDP target at the end of period $T_{tp}/2$.

To extract the net effect of policy change to the economy, for each sample path, we also simulate transition path for T_{tp} period without policy change at the end of period $T_{tp}/2$. Then, we calculate period-by-period change between simulation with policy change and simulation without policy change. For factor prices including risk-free return and equity return, we plot the change in absolute value. For macroeconomic variables, we plot the change rate instead of change in absolute value.

7.1 Debt policy change

In the proposition of Ricardian equivalence, households are forward looking so as to internalize the government's budget constraint when making their consumption decisions. What's the influence to aggregate variables and individual household choice if the government suddenly flows large amount of risk-free government debt into the economy? In this policy experiment, the length of simulation of transition path is T_{tp} period and we simulate N_{tp} sample path. Suppose the debt-GDP ratio at the end of period $T_{tp}/2$ is $(D/Y)_0$. Then, at $t = T_{tp}/2 + 1$, the debt-GDP target becomes $(\nu + 1)(D/Y)_0/z_t$, where $\nu > 0$. We follow the method above to calculate period-by-period change in aggregate variables and factor prices. The assumptions still are: no prediction or internalization on policy change; no change in belief on conditional rational expectation on factor prices when change happens; instant re-optimization on policy functions when change happens. Factor prices change is still in absolute value and aggregate variables in change rate.

7.1.1 Transition path

In the simulation for transition path with debt policy change, we focus on the crowding-out effect of more federal debt to equity market and macroeconomic variables. Figure

7.1, 7.2 and 7.3 plot the transition path for aggregate variables as well as factor prices when government chooses to flow 50% more debt into the system. Figure 7.4, 7.5 and 7.6 plot the case with 100% more debt. From the period-by-period change, flowing more debt into financial system will crowd out risky equity. Aggregate consumption decreases at first and then wealth effect in risk-free return dominates, which leads to recovery in aggregate consumption level. Aggregate output has negative response to flowing more debt into financial system, both from aggregate labor supply and aggregate capital stock. Economy-wide wage rate decreases by -0.0092 and -0.021 respectively. Here, we choose $N_{tp} = 1600$.

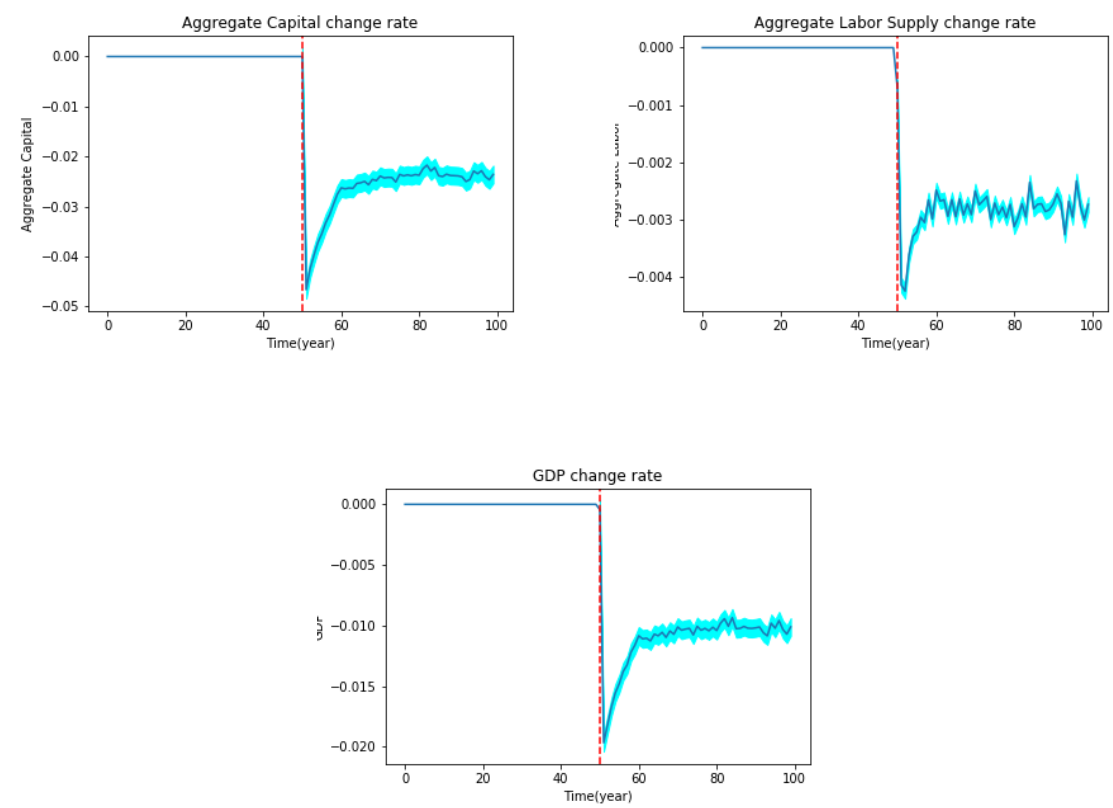


Figure 7.1: Change for aggregate variables, $\nu = 0.5$

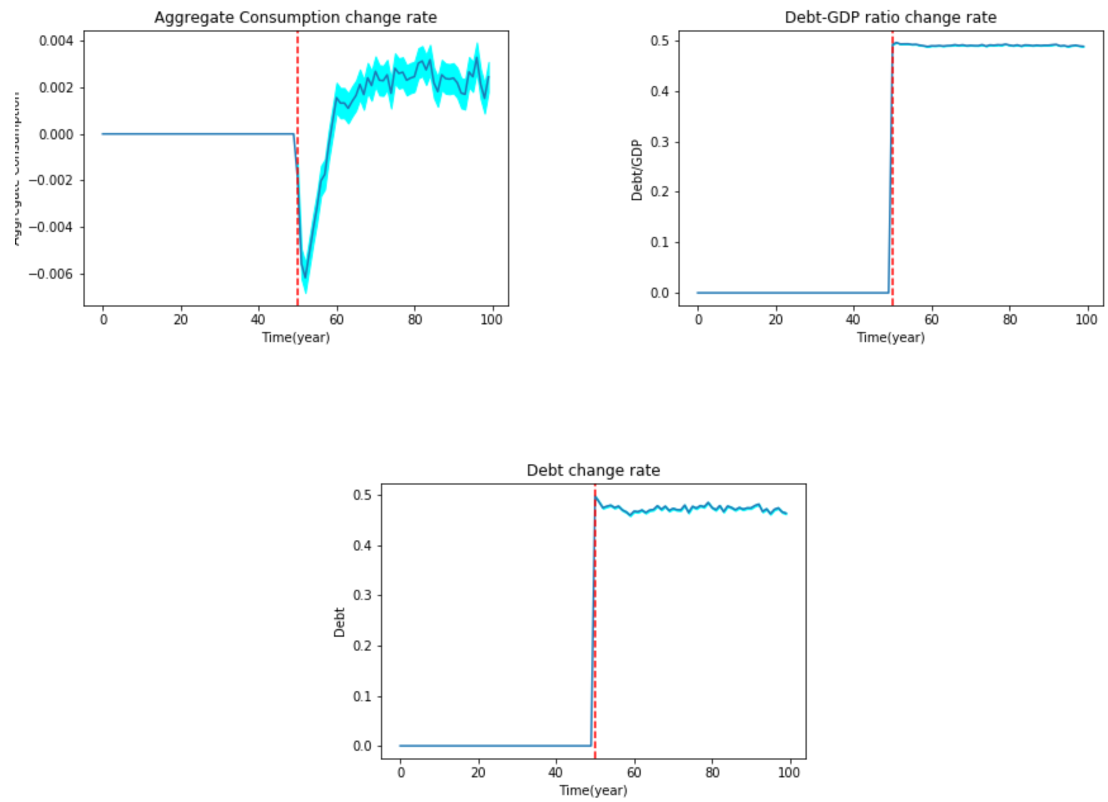


Figure 7.2: Change for aggregate variables, $\nu = 0.5$

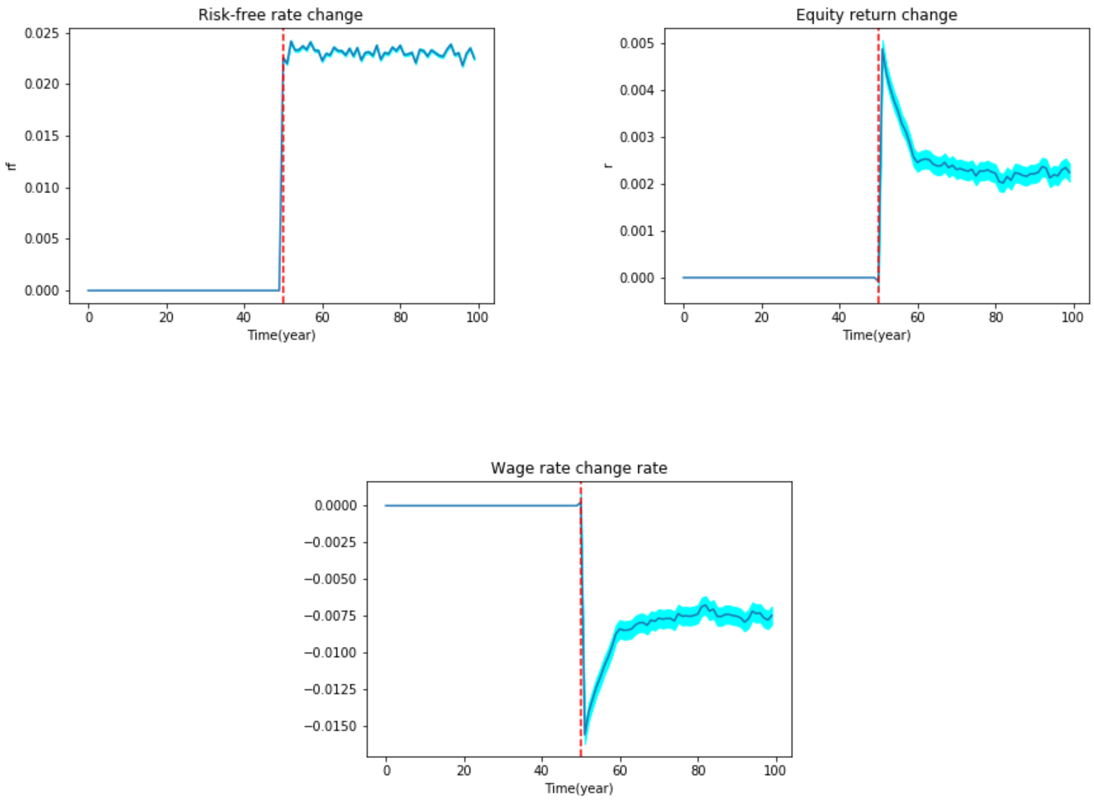


Figure 7.3: Change for factor prices, $\nu = 0.5$

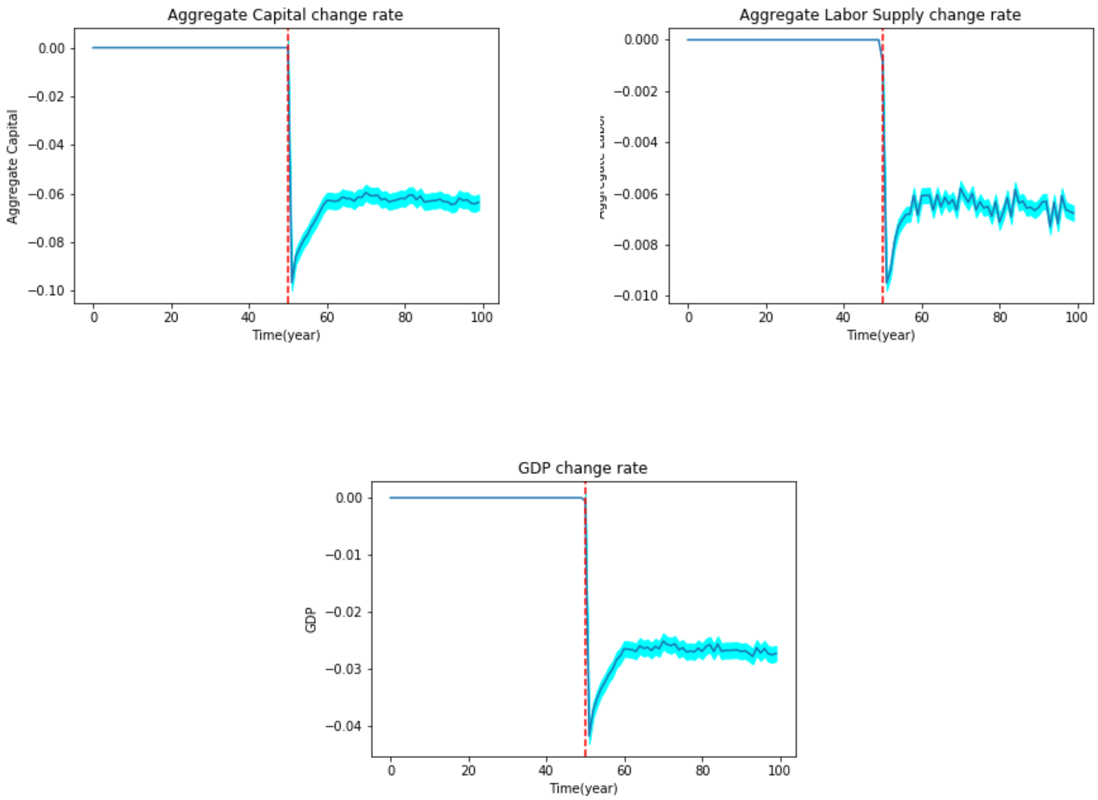


Figure 7.4: Change for aggregate variables, $\nu = 1.0$

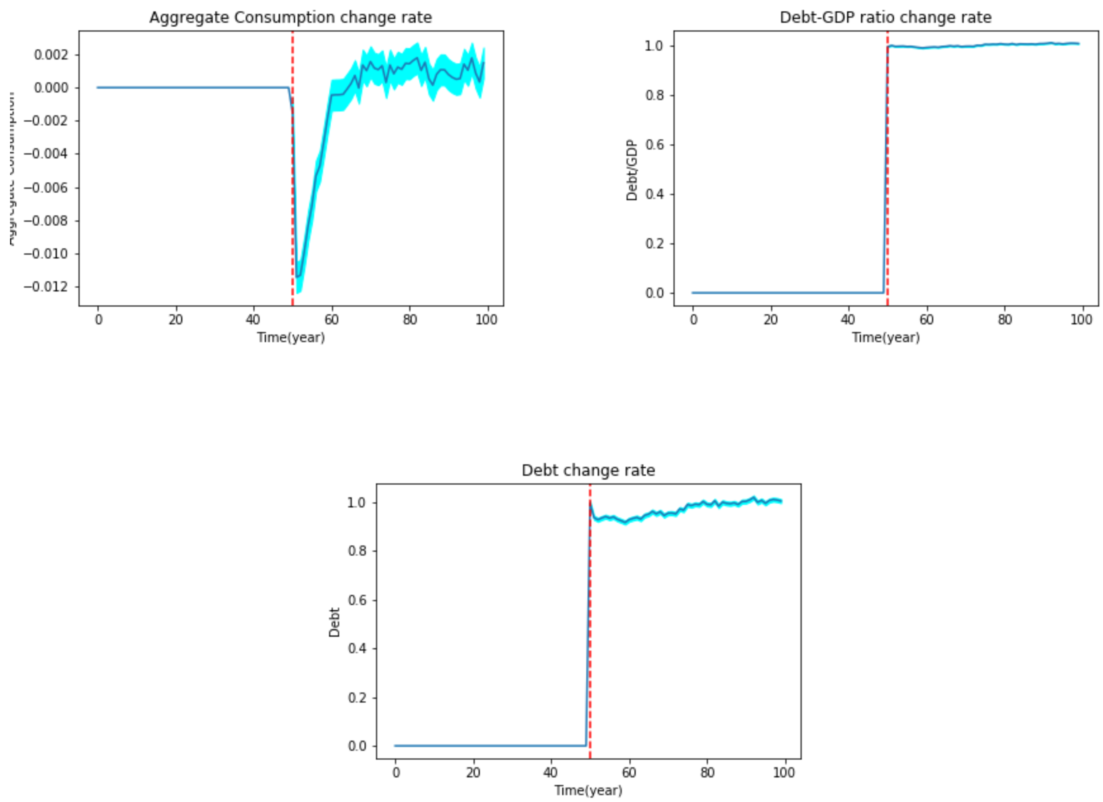


Figure 7.5: Change for aggregate variables, $\nu = 1.0$

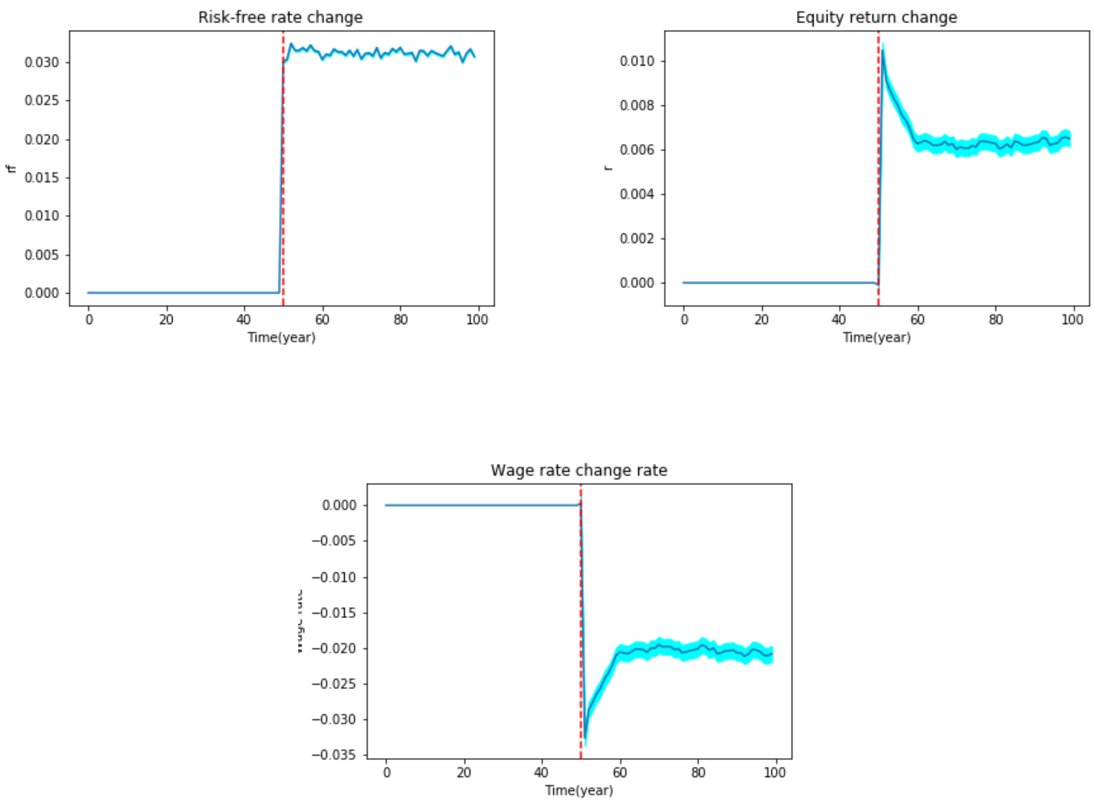


Figure 7.6: Change for factor prices, $\nu = 1.0$

7.1.2 Change in moment conditions

In this section, we focus on the average effect to macroeconomic variables when the government increases debt-GDP ratio by certain amount, 50%, 100%. In Gomes, Michaelides and Polkovnichenko (2012), an increase in the ratio of government debt to GDP of approximately 25% causes a permanent reduction in the capital stock of 2.1%. In our model setup, 50% increase in debt-GDP ratio causes an average reduction in capital stock of 2.58%; 100% increase in debt-GDP ratio causes an average reduction in capital stock of 6.54%.

Table 7.1: Change in aggregate variables and factor prices

Variable	Moment	$\nu = 0.5$	$\nu = 1.0$
Agg. Output	mean	-1.10%	-2.79%
Agg. Capital	mean	-2.58%	-6.54%
Agg. Consumption	mean	0.10%	-0.08%
Agg. Labor	mean	-0.28%	0.11%
Risk-free return (bps)	mean	230	311
Equity return (bps)	mean	25	66
Wage rate (bps)	mean	-92	-210

7.2 Wealth tax

A wealth tax is a levy on the total value of personal/household assets. In this section, we conduct policy experiment on 2% flat-rate wealth tax on household's beginning-of-period wealth without exemption amount. The tax revenue from wealth tax can be thrown away by the government, since government consumption does not enter the utility function of household for our model, or used by the government to reduce outstanding government debt.

7.2.1 Transition path without tax revenue to reduce debt

Figure 7.7 and 7.8 report transition path for macroeconomic variables and factor prices. Introducing 2% wealth tax reduces aggregate capital stock by nearly 16% and boosts aggregate labor supply moderately. However, reduction in capital dominates the effect of labor supply and leads to decrease in GDP. Aggregate consumption has an initial increase by 6% and then goes down following the trend of aggregate capital. Risk-free rate has a moderate increase by 53 basis points. Equity return increases due to the decrease in capital stock.

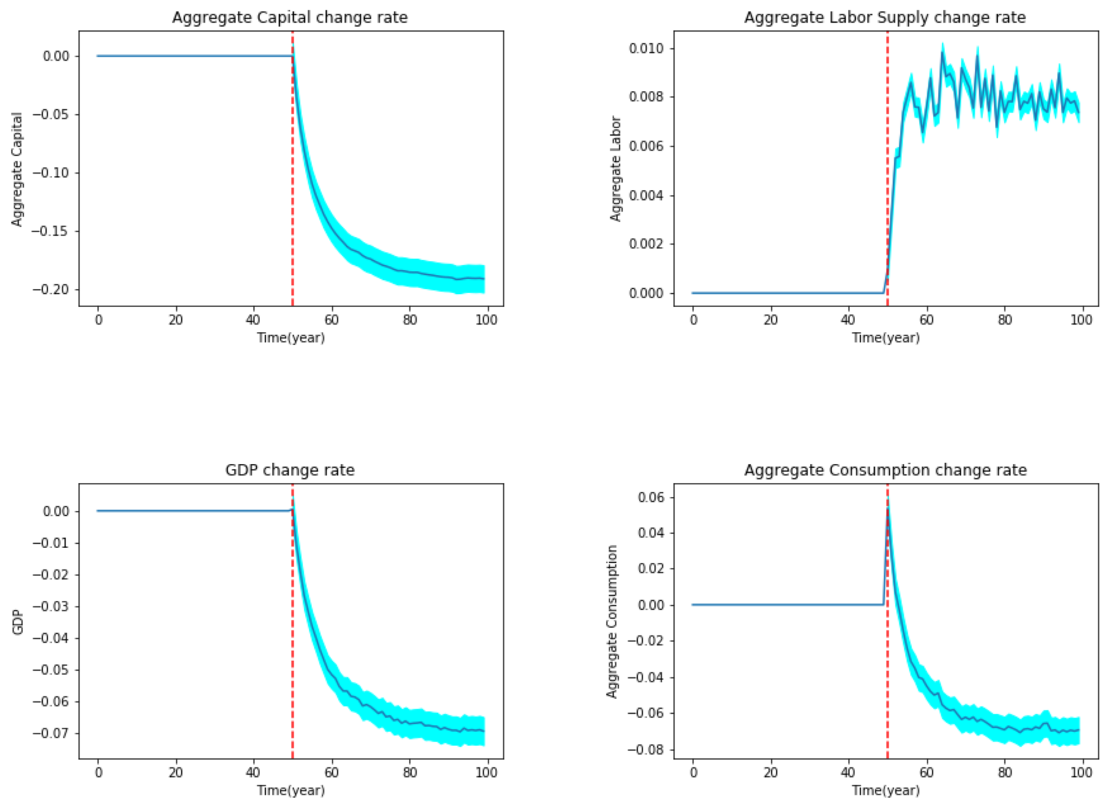


Figure 7.7: Transition path with 2% wealth tax

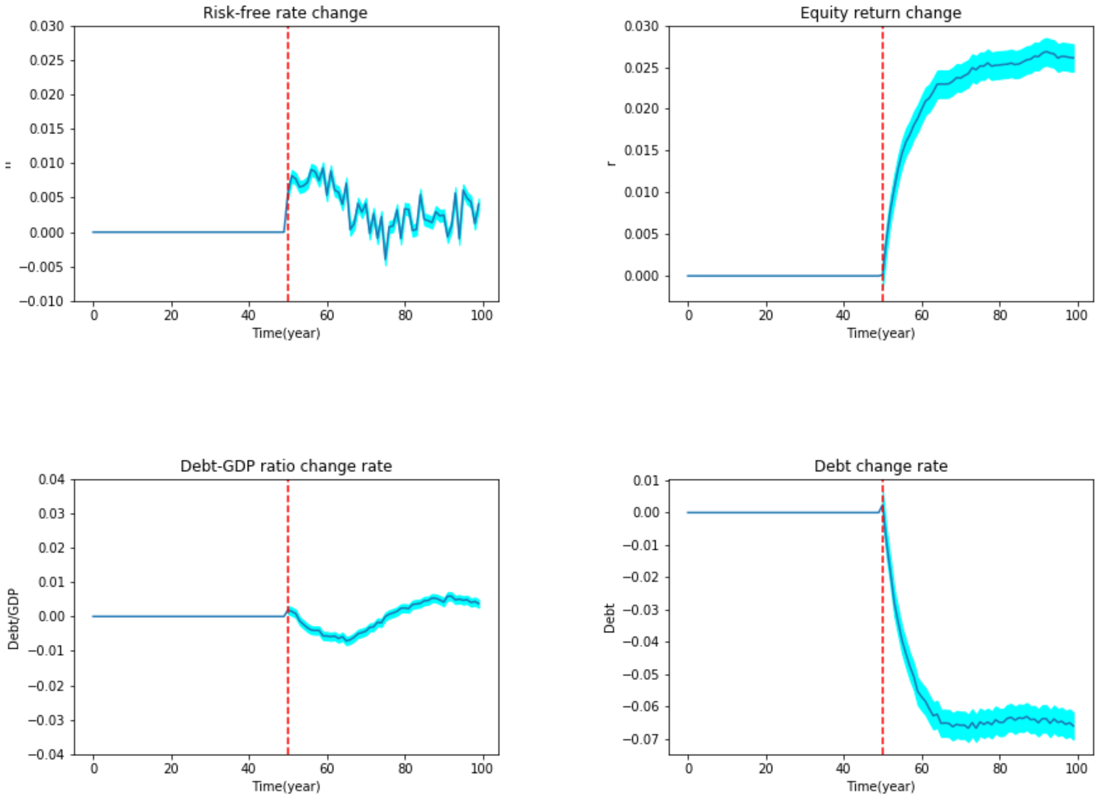


Figure 7.8: Transition path with 2% wealth tax

Table 7.2 reports the average change in aggregate variables and factor prices. Introducing wealth tax into the baseline economy results in sharp reduce in risky equity.

Table 7.2: Change in aggregate variables and factor prices

Variable	Moment	Wealth tax
Agg. Output	mean	-5.77%
Agg. Capital	mean	-16.32%
Agg. Consumption	mean	-5.37%
Agg. Labor	mean	0.76%
Risk-free return (bps)	mean	53
Equity return (bps)	mean	221

7.2.2 Transition path with tax revenue to reduce debt

In this part, we assume that the government collects tax revenue from wealth tax and uses it to reduce outstanding government debt. The dynamic debt-GDP rule government chooses becomes $((D - TW)/Y)_t = ((D - TW)/Y)_{t-1}/TFP_t$, where TW is tax revenue from wealth tax. Figure 7.9 and 7.10 report transition path for macroeconomic variables and factor prices. Table 7.3 reports the average change in aggregate variables and factor prices.

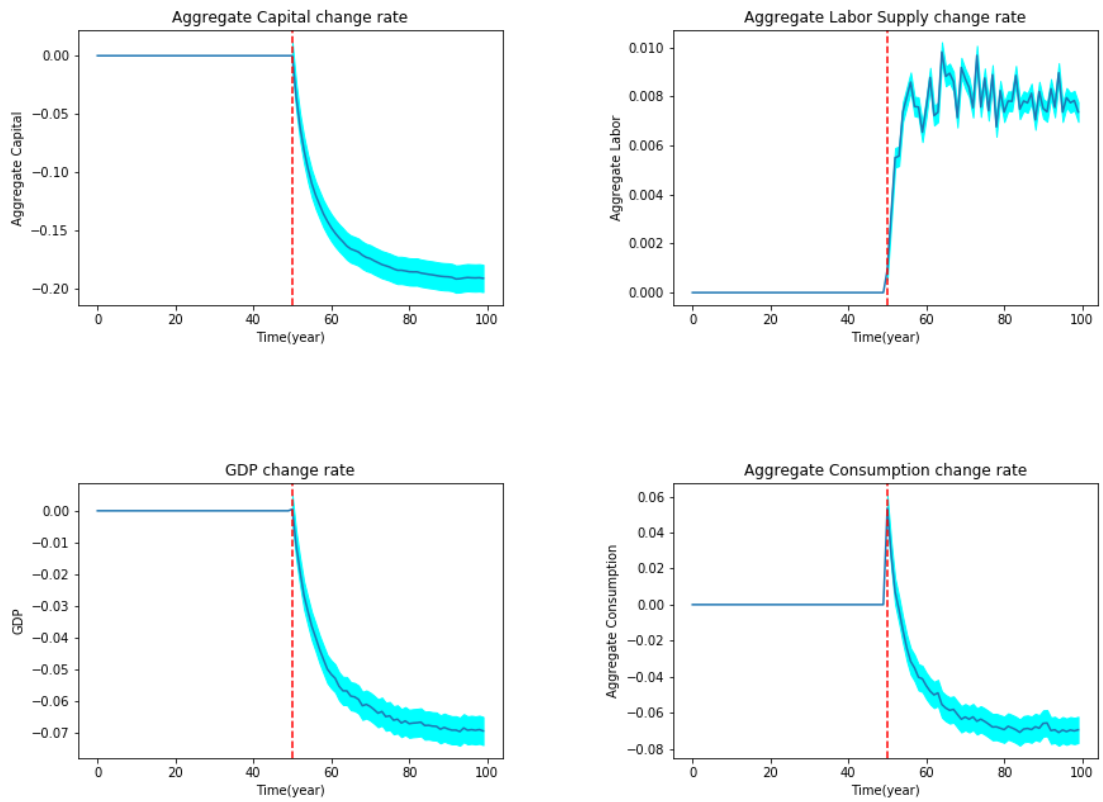


Figure 7.9: Transition path with 2% wealth tax

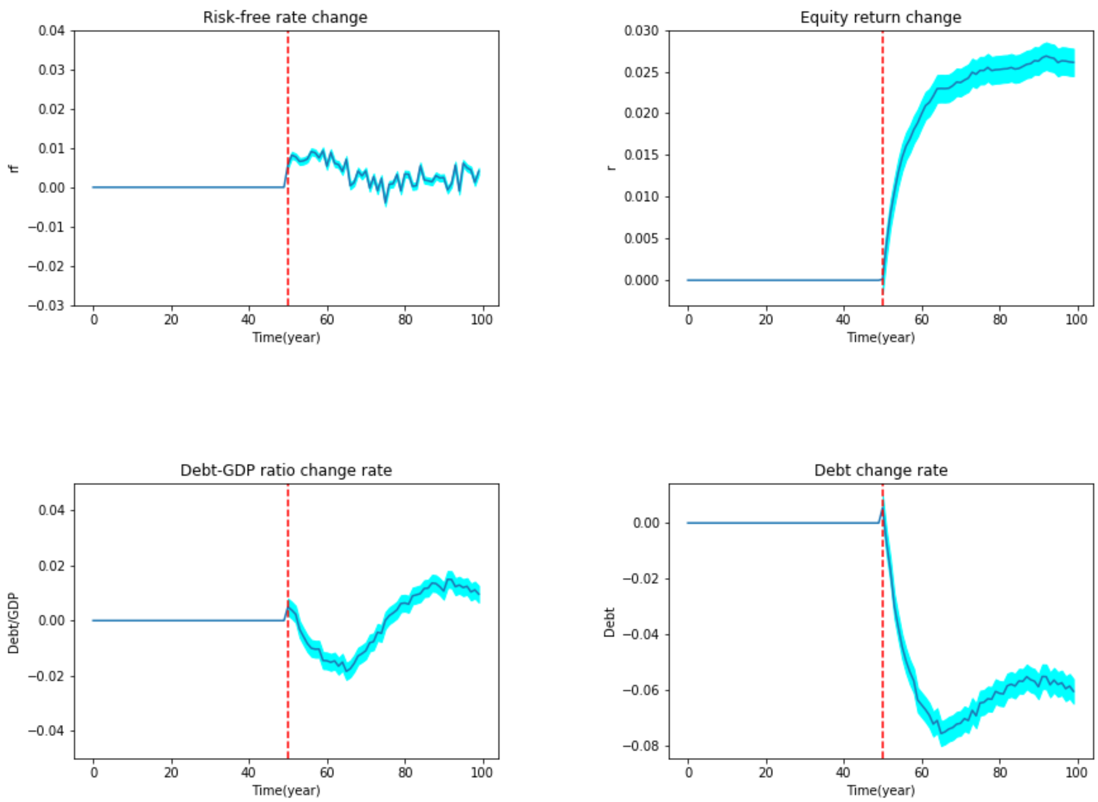


Figure 7.10: Transition path with 2% wealth tax

Table 7.3: Change in aggregate variables and factor prices

Variable	Moment	Wealth tax
Agg. Output (%)	mean	-5.83%
Agg. Capital (%)	mean	-16.35%
Agg. Consumption (%)	mean	-5.31%
Agg. Labor (%)	mean	0.72%
Risk-free return (bps)	mean	73
Equity return (bps)	mean	222

7.3 Capital income tax

A capital income tax is a levy on the return from risky equity investment. Suppose the capital income tax rate is τ_{cit}. The amount of capital income tax is $\tau_{\text{cit}}(r_t - r_{f,t})(1-d_t)a_t$. In this section, we conduct policy experiment on $\tau_{\text{cit}} = 25\%$ flat capital income tax on household's investment in risky equity. From the formula for capital income tax function, it is possible for capital income tax payable to be negative. In this case, household will receive refund from government.

7.3.1 Transition path without tax revenue to reduce debt

Table 7.4 reports average change in aggregate quantities and factor prices. Adding capital income tax into the economy leads to considerable decrease in aggregate output, capital and consumption.

Table 7.4: Change in aggregate variables and factor prices

Variable	Moment	Capital tax
Agg. Output (%)	mean	-3.01%
Agg. Capital (%)	mean	-8.10%
Agg. Consumption (%)	mean	-4.98%
Agg. Labor (%)	mean	-0.19%
Risk-free return (bps)	mean	51
Equity return (bps)	mean	85

Figure 7.11 and 7.12 plot transition path for aggregate quantities and factor prices. With capital income tax, aggregate amount of capital decreases and Aggregate amount of

labor also decreases, which leads to decrease in aggregate output. Risk-free rate increases by around 51 basis points and equity return on average increases 85 basis points.

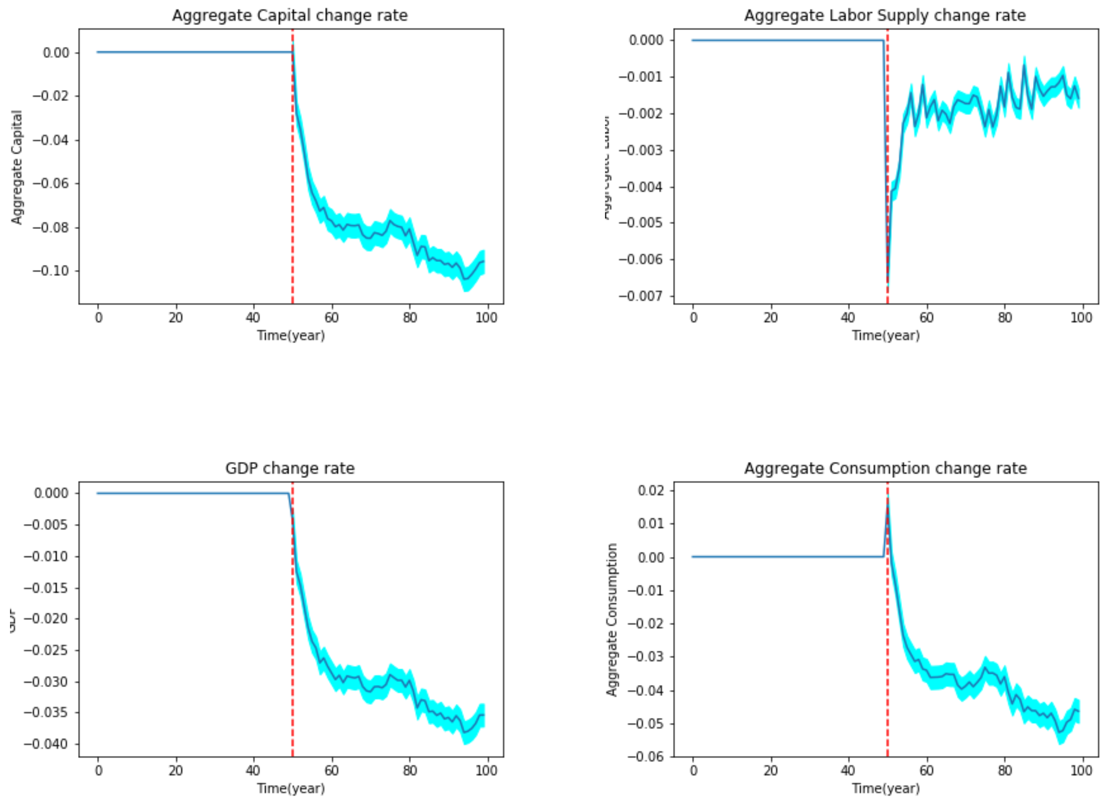


Figure 7.11: Transition path with capital income tax

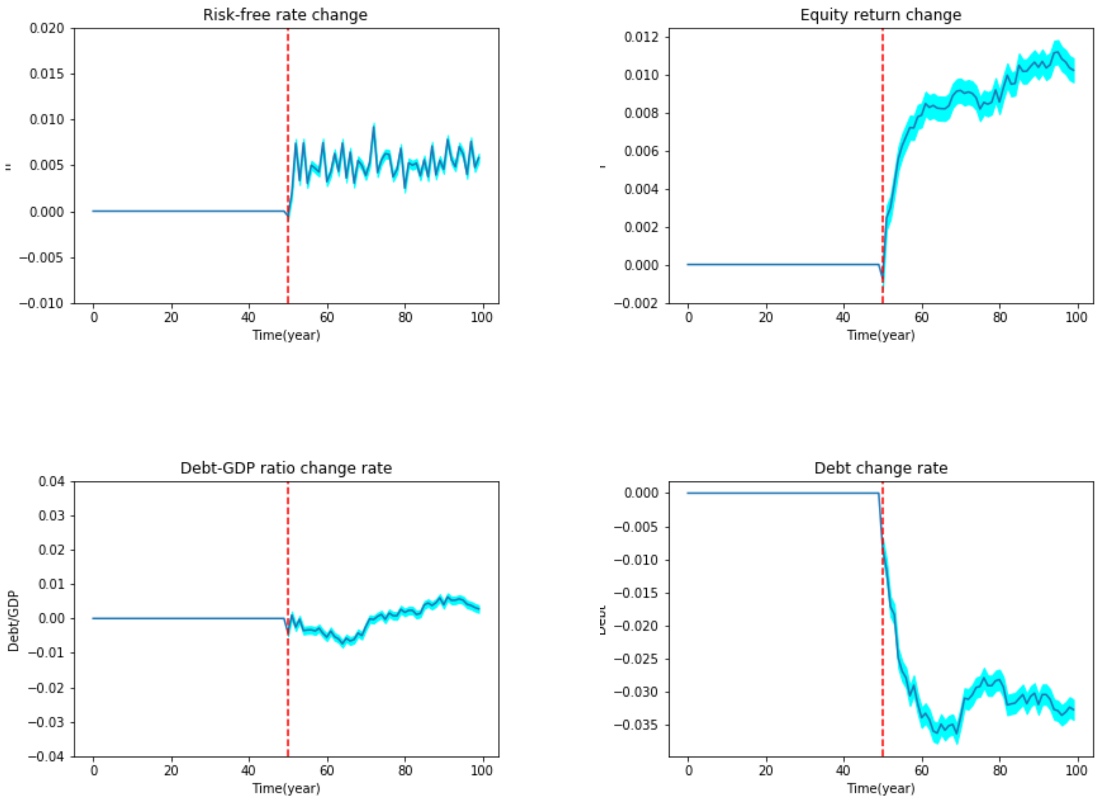


Figure 7.12: Transition path with capital income tax

7.3.2 Transition path with tax revenue to reduce debt

Transition path in this section is the case where the government uses revenue from capital income tax to reduce outstanding government debt in the economy. Table 7.5 reports average change for both aggregate variables and factor prices. Figure 7.13 and 7.14 plot transition path for aggregate quantities and factor prices.

Table 7.5: Change in aggregate variables and factor prices

Variable	Moment	Capital tax
Agg. Output (%)	mean	-3.67%
Agg. Capital (%)	mean	-9.70%
Agg. Consumption (%)	mean	-3.56%
Agg. Labor (%)	mean	-0.27%
Risk-free return (bps)	mean	95
Equity return (bps)	mean	106

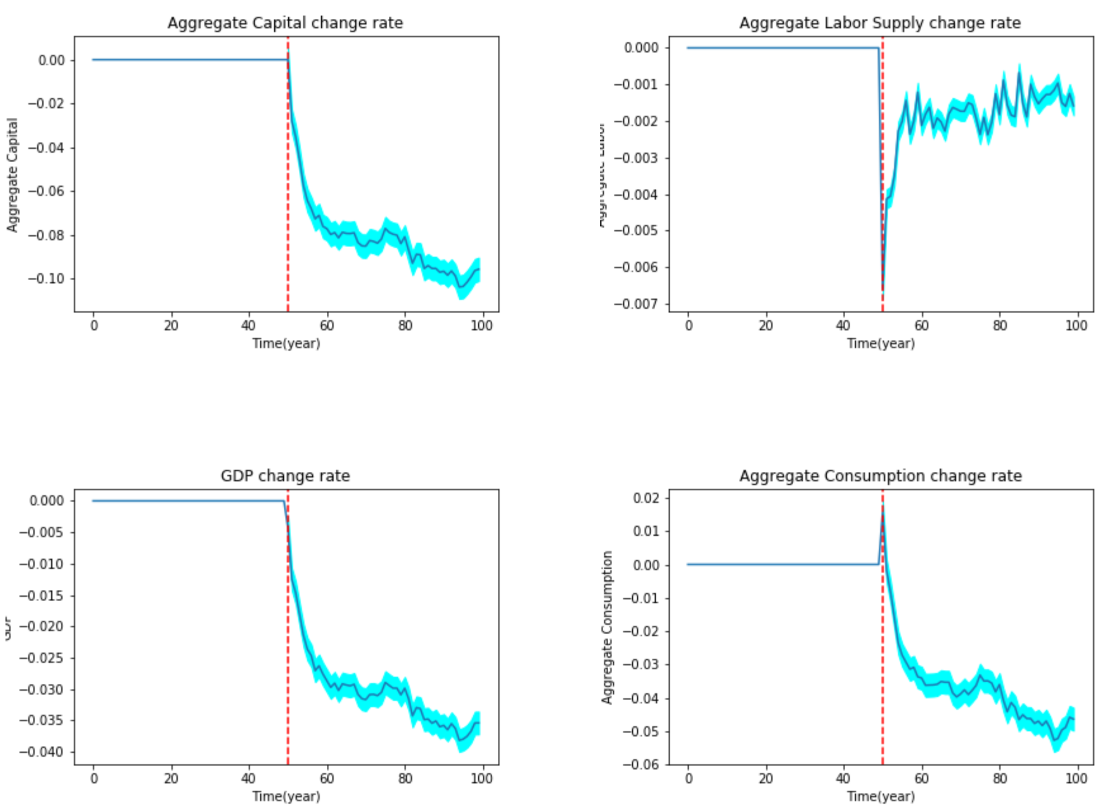


Figure 7.13: Transition path with capital income tax

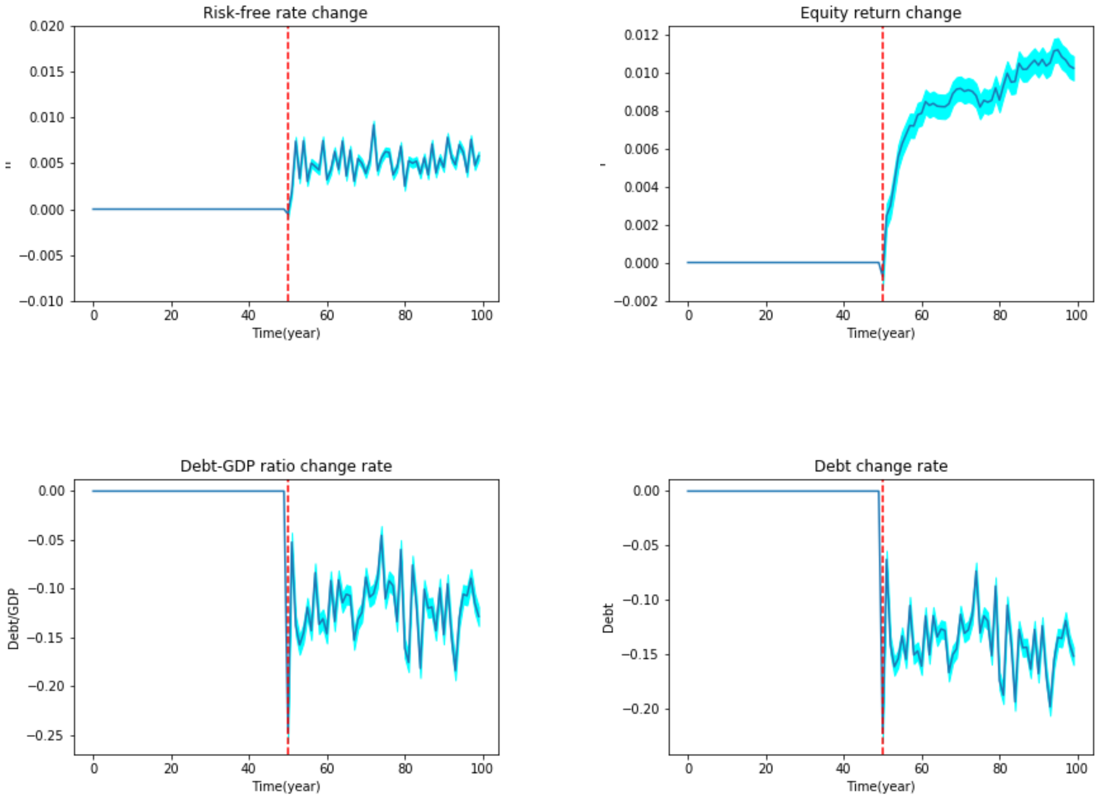


Figure 7.14: Transition path with capital income tax

Chapter 8

Discussion

In this part, we consider the robustness of our baseline model by changing some building blocks of the baseline model. First, a model with preference heterogeneity and Epstein-Zin preference, similar to Gomes and Michaelides (2007), is calibrated to match asset pricing moments and moments for macro variables. This could be a potential solution to reconcile artificial boundary for risk-free rate in the baseline model. Second, a model with rare shock following the setup and assumption in Barro (2006) is calibrated to match asset pricing moments and macro variables. Then, we focus on the influence of amount of minimum consumption to the macroeconomy and individual household behavior, especially wealth-income inequality. The fourth variation to the baseline model is on debt-GDP target, in which fixed, net positive debt-GDP target is applied to bond market. The fifth variation is to apply fixed debt-capital ratio target instead of debt-GDP target to understand the influence of different target for bond market to macroeconomy.

8.1 Preference heterogeneity

Participation barrier to equity market is always a prevalent topic in financial economics and asset pricing. In our baseline model, a small participation barrier is applied to match moment conditions in bond market and equity market. Beyond this setting, introducing preference heterogeneity is a potential solution to solve those asset pricing puzzles without assuming explicit participation barrier. Another intuition for preference heterogeneity is that: hitting boundary case in the baseline model is relevant with participation barrier.

As described above, we introduce preference heterogeneity along with Epstein-Zin preference. Type-A and Type-B agent each consist 50% of the population. For Type-A agent, the relative risk aversion and EIS are 1.5 and 0.666 (1.0/1.5) respectively. For Type-B agent, the relative risk aversion and EIS are 5.0 and 0.05 respectively. The consumption share parameter is 0.36 for both types of household. There is no positive participation barrier for either type of agent, which means that participation barrier is endogenous and generated by life-cycle property and risk preference. The time preference is 0.998 for both types of agent. The debt-GDP rule is the same with the baseline model.

Table 8.1 and Table 8.2 report asset pricing moments and aggregate variable moments implied by preference heterogeneity model. Participation ratio is higher than its counterpart in empirical data and this can be reconciled by introducing variable participation cost in Gomes and Michaelides (2007). Moreover, participation ratio is time-variant variable with much higher mean in recent year. Introducing preference heterogeneity and Epstein-Zin preference removes hitting boundary case in baseline model. The correlation between debt-GDP ratio and real interest rate is nearly zero since household with different preference responses differently to TFP shock.

Table 8.1: Asset pricing moments

Variable	Riskfree return		Equity return				
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt	Sharpe ratio
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585	0.3337
Baseline	1.36%	2.52%	5.44%	11.79%	-0.6405	2.8980	0.3112
EZ	1.34%	2.14%	5.34%	12.06%	-0.6267	2.8905	0.2918

Table 8.2: Aggregate variable moments

Variable	Moment	EZ	Baseline	Data
Log Agg. Output	Std. Dev.	6.32%	5.60%	5.41%
Log Agg. Cons.	Std. Dev.	6.48%	5.78%	3.52%
Debt-GDP,risk-free rate	Correlation	-0.0103	-0.1091	-0.2415
Debt-GDP,equity return	Correlation	0.0328	0.2835	0.2609
Consumption,equity return	Covariance	-0.0007	0.0003	0.0004
Participation ratio	Mean	0.7296	0.5820	0.5190
Consumption/Output	Mean	0.5570	0.5852	0.5950
SS Benefit/GDP	Mean	5.01%	5.00%	5.00%
Investment/Agg. Capital	Mean	11.21%	14.30%	14.50%
Working hour (unit)	Mean	0.3206	0.3006	0.3333

Table 8.3 and 8.4 report CBO regression analysis. For regression based on realized data, the coefficient for debt-GDP ratio is significantly negative. However, for the case based on conditional rational expectation, The coefficient is significantly positive due to the influence

of aggregate capital stock and GDP change. If we regress only on the first two variables, the coefficient for debt-GDP ratio is still negative.

Table 8.3: Regression based on realized simulation data

	A. $i_{t+5} = \beta_0 + \beta_1 D_{t+5} + \beta_2 K_t + \beta_3 \Delta Y_t$				
EZ	β_0	β_1	β_2	β_3	R^2
	0.0492	0.0863	-0.0226	-0.1121	0.226
	(11.396)	(8.589)	(-15.706)	(-7.437)	
	B. $i_{t+1} = \beta_0 + \beta_1 D_{t+1} + \beta_2 K_t + \beta_3 \Delta Y_t$				
EZ	β_0	β_1	β_2	β_3	R^2
	0.0778	0.0935	-0.0340	0.0605	0.463
	(21.128)	(11.260)	(-28.294)	(4.724)	

Table 8.4: Regression based on rational expectation

	A. $E_t[i_{t+5}] = \beta_0 + \beta_1 E_t[D_{t+5}] + \beta_2 K_t + \beta_3 \Delta Y_t$				
EZ	β_0	β_1	β_2	β_3	R^2
	0.0459	0.0367	-0.0164	0.0193	0.674
	(29.927)	(10.268)	(-31.170)	(3.550)	

8.2 Model with Rare Shock

Table 8.5 reports the main asset pricing moments implied by the rare shock model, along with their empirical U.S. counterparts as described in calibration part. Compared with the baseline model, the only difference is to add rare risk in TFP shock.

Table 8.5: Asset pricing moments

Variable	Riskfree return		Equity return			
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585
Rare shock	1.76%	2.49%	6.56%	11.64%	-0.6834	3.0328

8.2.1 Asset pricing implications

Table 8.6 reports moment conditions for macro variables indicated by model with rare shock with their empirical U.S. counterparts.

Table 8.6: Aggregate variable moments

Variable	Moment	Rare	Data
Log Agg. Output	Std. Dev.	6.52%	5.41%
Δ Agg. Consumption	Std. Dev.	5.32%	3.30%
Debt-GDP,risk-free rate	Correlation	-0.2501	-0.2415
Consumption,equity return	Covariance	0.0028	0.0004
Participation ratio	Mean	0.5916	0.5190
Consumption/Output	Mean	0.6286	0.5950
Investment/Agg. Capital	Mean	13.92%	14.50%
ΔAHW, Δw	Correlation	0.115	0.06

8.2.2 Life-cycle profile for optimal choice of household

Figure 8.1, 8.2 and 8.3 plot the life-cycle profile for consumption, labor choice, initial wealth, participation ratio and portfolio choice.

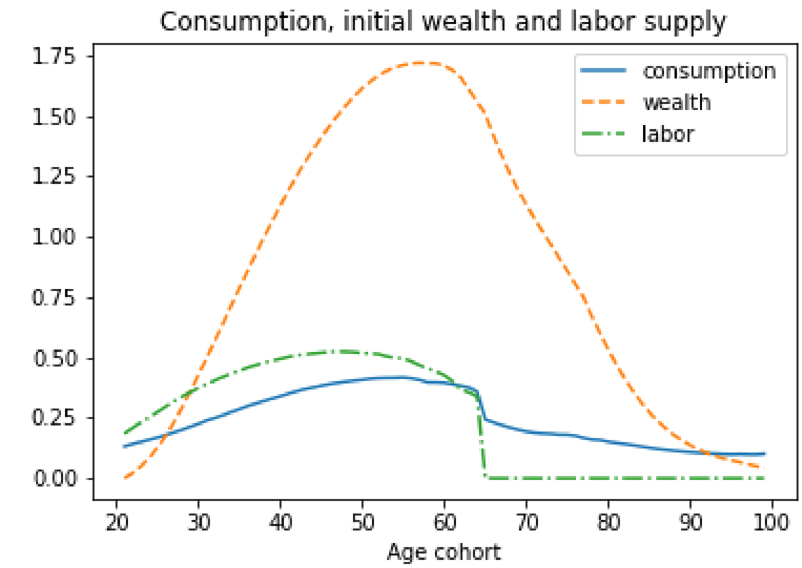


Figure 8.1: Consumption-Wealth-labor by age cohort

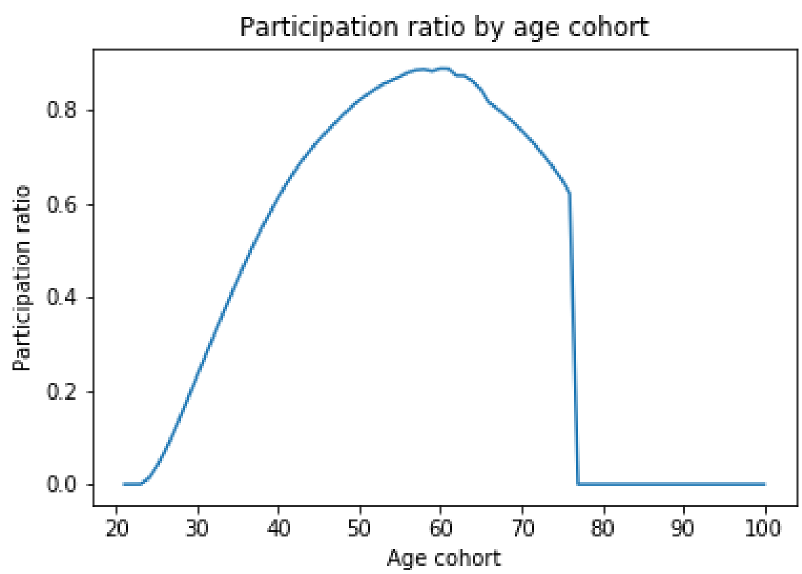


Figure 8.2: Participation ratio by age cohort

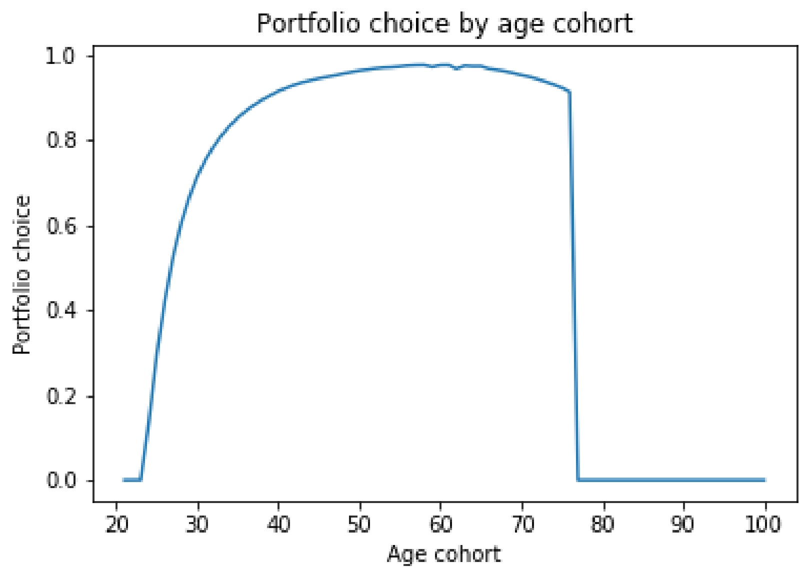


Figure 8.3: Portfolio choice by age cohort

8.3 Minimum consumption

In our baseline model, the government operates a minimum consumption program to ensure that poor household with zero total income will not have zero consumption. This minimum consumption program both has economic meaning and stabilizes numerical solution. This experiment is to increase minimum consumption amount to understand the role of minimum consumption in wealth, earning and income disparity. Compared with baseline model, we keep other parameters except the amount of minimum consumption the same and choose different amount of minimum consumption. In our baseline model, the minimum consumption is set at 0.001 model unit, which is around 140 in real dollar value. Beyond this, we calibrate another two model with minimum consumption at 0.01 and 0.03, which is around \$5000 per year. Similarly, we also calibrate a model with extremely low minimum consumption amount. Instead of placing a explicit lower bound for consumption in model unit, we place a lower bound for utility function, when consumption is zero, since utility function goes to minus infinity as consumption goes to zero. We choose an ad-hoc value for this lower bound as -10000.0.

Table 8.7 reports asset pricing moments indicated with model with different minimum consumption policy. Table 8.8 reports moment conditions for macroeconomic variables. From moment conditions, less minimum consumption forces household to save in terms of both equity and risk-free bond. Minimum consumption could act as safe asset to substitute risk-free bond for poor household. More minimum consumption also leads to less average hour of working, less participation to equity market.

Table 8.7: Asset pricing moments

Variable	Riskfree return		Equity return			
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585
Baseline	1.36%	2.52%	5.44%	11.79%	-0.6405	2.8980
10X	1.16%	3.06%	6.16%	11.77%	-0.6405	2.8997
30X	0.94%	3.16%	6.90%	11.77%	-0.6285	2.8960
No min	1.41%	1.46%	4.55%	11.79%	-0.6555	2.8998

Table 8.8: Macroeconomic variable moments

Variable	Moment	Baseline	10X	30X	No min
log GDP	Std. Dev.	5.60%	5.70%	5.65%	5.61
Δ Cons	Std. Dev.	5.94%	5.62%	5.43%	6.53
Debt/GDP ratio	Mean	0.22	0.20	0.17	0.24
Debt-GDP,r_f	Correlation	-0.1091	-0.1800	-0.1841	-0.4875
Cons,r	Covariance	0.0003	0.0004	0.0007	0.0006
Part. ratio	Mean	0.5820	0.5439	0.4900	0.6555
Cons/GDP	Mean	0.5852	0.5971	0.6162	0.5658
Working hour (unit)	Mean	0.3006	0.2856	0.2413	0.3022

We then compare consumption-saving-labor behavior of household under different minimum consumption policy, which is in Figure 8.4. With less amount of minimum consumption, household on average increase their saving in terms of a combination of risky equity and risk-free bond against uncertainty.

Figure 8.4: Average initial wealth by age cohort comparison

Table 8.9 reports distributions of labor earnings, total income and wealth by different minimum consumption policy. In general, more minimum consumption increases inequality in labor earning, income and wealth. When low-individual-productivity households have access to more minimum consumption, they tend to work less and just rely on minimum consumption.

Table 8.9: Distributions of Earnings and Wealth in the model economy

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1.0	Gini
				A. Distributions of Earning					
Baseline	0.000	0.026	0.108	0.227	0.212	0.139	0.171	0.117	0.6508
Data	0.000	0.016	0.106	0.216	0.178	0.132	0.185	0.167	0.6799
10X	0.000	0.024	0.106	0.225	0.214	0.140	0.173	0.19	0.6562
30X	0.000	0.001	0.103	0.231	0.221	0.144	0.177	0.122	0.6803
No minimum	0.000	0.026	0.109	0.229	0.211	0.138	0.170	0.117	0.6476
				B. Distributions of Income					
Baseline	0.017	0.070	0.167	0.277	0.191	0.109	0.112	0.059	0.5088
Data	0.026	0.061	0.101	0.166	0.132	0.100	0.169	0.244	0.5977
10X	0.015	0.057	0.157	0.279	0.198	0.114	0.118	0.062	0.5219
30X	0.008	0.039	0.143	0.285	0.209	0.122	0.128	0.066	0.5692
No minimum	0.022	0.097	0.179	0.268	0.177	0.100	0.103	0.054	0.4453
				C. Distributions of Wealth					
Baseline	0.000	0.062	0.178	0.302	0.197	0.108	0.103	0.050	0.5013
Data	-0.005	0.006	0.029	0.086	0.109	0.117	0.280	0.379	0.8595
10X	0.000	0.041	0.176	0.302	0.204	0.115	0.110	0.054	0.5345
30X	0.000	0.017	0.151	0.311	0.218	0.123	0.121	0.059	0.5847
No minimum	0.010	0.100	0.193	0.283	0.181	0.097	0.091	0.045	0.4365

8.4 Debt-GDP ratio target

In our baseline model, we apply a dynamic debt-GDP target to match dynamic property of bond market empirically. However, this is not the most prevalent assumption in past literature. Researchers almost always assume fixed ratio of debt-GDP target, net zero supply or net positive supply. For example Storesletten et al. (2007) assumes net zero supply; Gomes and Michaelides (2007) assumes net positive supply. If net zero supply, this means, in general equilibrium, representative household, if exists, invests all its asset in risky equity. However, from empirical data, federal debt is in net positive supply. To test the implication of fixed debt-GDP ratio, we also calibrate a model similar to our baseline model but assume fixed debt-GDP target, which matches the empirical mean of risk-free rate. Other implied moment conditions are also listed in Table 8.10 and 8.11 below.

Table 8.10: Asset pricing moments

Variable	Riskfree return		Equity return			
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585
Baseline	1.36%	2.52%	5.44%	11.79%	-0.6405	2.8980
D/Y=0.215	1.32%	2.92%	5.33%	11.74%	-0.6381	2.8708
D/Y=0.25	2.87%	1.91%	5.35%	11.72%	-0.6388	2.8714

Table 8.11: Aggregate variable moments

Aggregate variables moments				
Variable	Moment	D/Y=0.215	D/Y=0.25	Data
Log Agg. Output	Std. Dev.	5.69%	5.72%	5.41%
Δ Agg. Consumption	Std. Dev.	6.04%	6.07%	3.30%
Debt-GDP,risk-free rate	Correlation	-0.6381	-0.4422	-0.2415
Consumption,equity return	Covariance	0.0001	0.0001	0.0037
Participation ratio	Mean	0.5799	0.5850	0.519
Consumption/Output	Mean	0.5838	0.5866	0.595
Investment/Agg. Capital	Mean	13.70%	13.70	14.50%
ΔAHW, Δw	Correlation	0.1275	0.1257	0.06
Working hour (unit)	Mean	0.2968	0.2941	0.3333

Compared with our baseline model, the major difference is the correlation between risk-free rate and debt-GDP ratio. Hitting boundary phenomenon contributes to this result. Under fixed debt-GDP ratio setup, increasing debt-GDP ratio target will increase the level for risk-free return. The crowding-out effect with more supply of risk-free bond to equity market is nearly negligible in this case. In Table 8.12 and 8.13, we report regression result following the CBO setup described above. For regression analysis based on simulation data, the negative relationship between risk-free rate and debt-GDP ratio is statistically significant. Regression based on binomial model also shows this negative relationship.

Table 8.12: Regression based on realized simulation data

	A. $i_{t+5} = \beta_0 + \beta_1 D_{t+5} + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/Y=0.215	β_0	β_1	β_2	β_3	R^2
	0.5054	-2.0472	-0.0319	0.0421	0.446
	(27.509)	(-23.048)	(-8.200)	(3.104)	
D/Y=0.25	β_0	β_1	β_2	β_3	R^2
	0.8115	-2.9545	-0.0314	0.0302	0.285
	(15.917)	(-14.347)	(-11.001)	(3.132)	
	B. $i_{t+1} = \beta_0 + \beta_1 D_{t+1} + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/Y=0.215	β_0	β_1	β_2	β_3	R^2
	0.5126	-2.1587	-0.0197	-0.0189	0.430
	(27.504)	(-24.750)	(-5.169)	(-1.378)	
D/Y=0.25	β_0	β_1	β_2	β_3	R^2
	0.8792	-3.1697	-0.0413	0.0373	0.354
	(18.223)	(-16.405)	(-15.480)	(4.091)	

Table 8.13: Regression based on rational expectation

	$E_t[i_{t+5}] = \beta_0 + \beta_1 E_t[D_{t+5}] + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/Y=0.215	β_0	β_1	β_2	β_3	R^2
	0.0613	-0.0305	-0.0291	0.0258	0.736
	(22.405)	(-2.446)	(-51.441)	(12.093)	
D/Y=0.25	β_0	β_1	β_2	β_3	R^2
	0.0445	0.0400	-0.0221	0.0189	0.701
	(9.615)	(2.181)	(-47.938)	(11.975)	

8.5 Debt-Capital ratio target

In our baseline model and model with rare disaster shock, the government chooses dynamic rule on debt-GDP ratio to supply risk-free asset. The main concern is to match moment conditions on asset pricing, debt demand/supply and further correlation between several macroeconomic indicators. In this part, we propose another setup that pins down risk-free asset market by debt-capital ratio. Our primary concern is still to match level of risk-free asset to GDP as well as those moment conditions. The debt-capital policy is a fixed ratio chosen to match the mean return of risk-free asset.

Table 8.14 and Table 8.15 report asset pricing moments and aggregate variable moments implied by debt-capital ratio model.

Table 8.14: Asset pricing moments

Variable	Riskfree return		Equity return			
	Mean	Std.Dev.	Mean	Std.Dev.	Skew	Kurt
US data	1.31%	2.57%	5.17%	11.64%	-0.6203	3.0585
Baseline	1.36%	2.52%	5.44%	11.79%	-0.6405	2.8980
D/K=0.102	1.31%	2.81%	5.49%	11.79%	-0.6391	2.8980
D/K=0.30	4.54%	1.39%	5.95%	11.73%	-0.6356	2.8962

Table 8.15: Aggregate variable moments

Variable	Moment	D/K=0.30	D/K=0.102	Data
Log Agg. Output	Std. Dev.	6.14%	5.59%	5.41%
Δ Agg. Consumption	Std. Dev.	5.95%	5.81%	3.30%
Debt/GDP ratio	Mean	0.48	0.21	0.28
Debt-GDP,risk-free rate	Correlation	-0.2733	0.2013	-0.2415
Debt-GDP,equity return	Correlation	0.7312	-0.3185	0.2609
Consumption,equity return	Covariance	0.0000	0.0004	0.0037
Participation ratio	Mean	0.578	0.5854	0.519
Consumption/Output	Mean	0.6019	0.5859	0.595
SS Benefit/GDP	Mean	5.02%	5.00%	5.00%
Investment/Agg. Capital	Mean	14.25%	14.32%	14.50%
Working hour (unit)	Mean	0.2889	0.2963	0.3333

Table 8.16 and 8.17 report CBO regression analysis with debt-capital rule.

Table 8.16: Regression based on realized simulation data

	A. $i_{t+5} = \beta_0 + \beta_1 D_{t+5} + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/K=0.102	β_0	β_1	β_2	β_3	R^2
	0.1022	-0.0524	-0.0234	0.0206	0.166
	(24.263)	(-10.279)	(-8.675)	(2.569)	
D/K=0.30	β_0	β_1	β_2	β_3	R^2
	-0.0076	0.4114	-0.0464	0.0803	0.127
	(-0.814)	(9.608)	(-9.773)	(4.796)	
	B. $i_{t+1} = \beta_0 + \beta_1 D_{t+1} + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/K=0.102	β_0	β_1	β_2	β_3	R^2
	0.1407	-0.0457	-0.0546	0.0380	0.455
	(41.880)	(-11.065)	(-25.132)	(5.848)	
D/K=0.30	β_0	β_1	β_2	β_3	R^2
	-0.0413	0.2937	-0.0052	0.0160	0.041
	(-4.412)	(6.176)	(-0.987)	(0.909)	

Table 8.17: Regression based on rational expectation

	A. $E_t[i_{t+5}] = \beta_0 + \beta_1 E_t[D_{t+5}] + \beta_2 K_t + \beta_3 \Delta Y_t$				
D/K=0.102	β_0	β_1	β_2	β_3	R^2
	0.0560	-0.0008	-0.0144	0.0111	0.657
	(104.512)	(-1.225)	(-43.201)	(10.836)	
D/K=0.30	β_0	β_1	β_2	β_3	R^2
	0.0347	-0.0544	-0.0051	0.0163	0.486
	(43.993)	(-9.121)	(-9.713)	(11.268)	

Chapter 9

Concluding Remark

We present a DSGE model with heterogeneous-agent overlapping generation (OLG) household and incomplete market. The baseline model model can match moment conditions for risk-free asset, risky equity and macro variables in U.S economy. The equity risk premium is driven by financial friction and incomplete market with heterogeneous participants, instead of parameter selection or abnormal covariance between factor prices and macro variables. The answer to zero or even negative correlation between debt-GDP ratio and real interest rate is counter-cyclical debt-GDP policy. Beyond moment conditions, our baseline model can match labor earning and income inequality with empirical counterparts from SCF (2016). Wealth inequality could be potentially reconciled by introducing entrepreneur in the future. The heterogeneous-agent OLG model is the main workhorse model to analyze fiscal policy change and tax policy change. We test the influence to financial market and macroeconomy when introducing wealth tax, capital income tax and debt policy change. In general, flowing more debt or introducing wealth tax and capital income tax could have negative impact on macroeconomy. Robustness of our model is verified by modifying debt policy target, preference heterogeneity and minimum consumption plan.

From computational perspective, this paper presents a modified version of Krusell-Smith (1998) algorithm to solve the large-scale OLG model with feasible computational cost. Although Krusell Smith algorithm works pretty well in our model setup, Artificial Neural Network approach may replace Krusell Smith algorithm to approximate conditional expectation in some ill-mannered problem with R^2 below desired threshold. This improves the robustness of large-scale OLG model with approximation equilibrium. We also use high-performance computing technique, like MPI, OpenMP and CUDA to accelerate computation, which could be a general solution for large-scale heterogeneous-agent OLG model.

Appendix.A

Figures below is from SCF (2016). We divide households with age 1 to 80 into 16 bucket. Within each 5-year age bucket, we sort the households by their labor earning, total income and wealth into 8 sub-bucket: bottom 20%, 20%-40%, 40%-60%, 60%-80%, 80%-90%, 90%-95%, 95%-99%, 99%-100%. Earning, income and wealth amount are in real dollar.

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	-0.01233	0.002074	0.012051	0.034694	0.045787	0.038243	0.198943	0.680539
2	-0.02124	0.012139	0.053787	0.171733	0.149514	0.143283	0.348922	0.14186
3	-0.00012	0.006504	0.023928	0.062804	0.05666	0.046336	0.110828	0.693063
4	0.000146	0.003396	0.010978	0.029988	0.031846	0.031738	0.062404	0.829505
5	0.000952	0.010095	0.033241	0.095053	0.091559	0.087902	0.201131	0.480067
6	0.00139	0.010036	0.025965	0.069623	0.073701	0.066626	0.270738	0.481921
7	0.000689	0.006692	0.018314	0.050206	0.052914	0.065356	0.295371	0.510458
8	0.000965	0.008997	0.023038	0.076607	0.094827	0.09022	0.253992	0.451354
9	0.00163	0.008915	0.023749	0.075146	0.075203	0.080194	0.238659	0.496505
10	0.002073	0.008911	0.0218	0.055875	0.055605	0.05506	0.210813	0.589862
11	0.002253	0.012387	0.026328	0.064777	0.068296	0.059602	0.181801	0.584556
12	0.004347	0.010365	0.023917	0.064118	0.071605	0.068283	0.258569	0.498796
13	0.004664	0.014615	0.023523	0.075098	0.076459	0.062638	0.144285	0.598718
14	0.004918	0.020667	0.039256	0.112408	0.098171	0.06689	0.292792	0.364898
15	0.95115	0.000755	0.001538	0.002256	0.001859	0.002269	0.009302	0.030871
16								

Figure 1: Wealth share by wealth bucket and age bucket, SCF (2016)

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.037599	0.095877	0.158856	0.251286	0.17032	0.106999	0.131125	0.047938
2	0.058165	0.09749	0.148641	0.208521	0.134846	0.084308	0.098393	0.169635
3	0.026268	0.050948	0.080942	0.131463	0.083097	0.054769	0.11452	0.457992
4	0.046961	0.081147	0.138256	0.215332	0.154858	0.105513	0.160195	0.097737
5	0.031436	0.080058	0.130643	0.206389	0.157463	0.106999	0.177527	0.109485
6	0.034936	0.072581	0.114094	0.179275	0.133731	0.084237	0.175779	0.205367
7	0.002415	0.05333	0.09923	0.162634	0.12507	0.083984	0.276677	0.196659
8	0	0.037068	0.079016	0.148026	0.132804	0.10436	0.274835	0.22389
9	0	0.011133	0.067768	0.154402	0.135214	0.095504	0.244311	0.291669

Figure 2: Earning share by earning bucket and age bucket, SCF (2016)

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.057841	0.088246	0.142378	0.210457	0.142381	0.090517	0.128396	0.139784
2	0.063624	0.101531	0.142519	0.209197	0.13159	0.080616	0.097751	0.173172
3	0.034033	0.052522	0.078548	0.124322	0.078352	0.050897	0.110655	0.470671
4	0.042129	0.065424	0.099963	0.159299	0.117931	0.075184	0.167145	0.272925
5	0.039569	0.07003	0.106616	0.173385	0.127294	0.092693	0.183244	0.207169
6	0.027121	0.052797	0.081364	0.136979	0.101768	0.069044	0.216509	0.314417
7	0.017447	0.033177	0.054445	0.086706	0.069831	0.051449	0.249603	0.437342
8	0.016639	0.035931	0.057798	0.113836	0.098865	0.07616	0.2476	0.353171
9	0.017331	0.035492	0.064776	0.119166	0.090252	0.081406	0.234195	0.357381
10	0.016971	0.032109	0.04885	0.085836	0.065569	0.048694	0.180582	0.52139
11	0.028182	0.04918	0.076974	0.124798	0.100358	0.089182	0.142125	0.389201
12	0.028527	0.041948	0.061051	0.113541	0.116036	0.084011	0.155298	0.399588
13	0.030849	0.042428	0.068505	0.123208	0.080203	0.067123	0.174541	0.413143
14	0.031533	0.044958	0.072623	0.126918	0.12003	0.096276	0.15773	0.349934
15	0.026189	0.03601	0.045831	0.055653	0.053196	0.055516	0.12182	0.605785
16								

Figure 3: Income share by income bucket and age bucket, SCF (2016)

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	4105	48009	144604	352834	684771	1290084	4337089	49359462
1992	5903	47989	131952	310090	599049	1115033	3903880	48377055
1995	8143	52774	136438	308785	598214	1071225	3822460	68444235
1998	7259	58871	162824	403980	728728	1331822	5610448	78328988
2001	9213	67336	189029	514613	1009960	1783052	7946201	76153467
2004	8216	66770	196872	554802	1065493	1818012	8146400	1E+08
2007	8488	75173	229171	575418	1055386	2205671	9668267	1.13E+08
2010	4753	43882	152989	459485	1051997	2062101	7535855	92263560
2013	4434	39337	152191	441416	971326	1929815	8122527	1.08E+08
2016	4700	49100	169900	499275	1183805	2376810	10351400	1.37E+08

Figure 4: Wealth by wealth bucket in 2016 Dollar, SCF (2016)

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	13632	37001	58423	93477	126583	171374	353459	1951328
1992	12375	34375	60157	91095	125470	171877	343754	1718770
1995	13471	38490	60942	96224	126695	165184	360839	2084850
1998	17945	40377	64304	101690	134589	173471	355914	2990877
2001	19508	41803	69671	110080	151883	202046	480731	4876982
2004	16980	40490	65307	109716	151512	199840	417965	5159260
2007	13099	38107	64306	107177	142903	196491	506114	4620519
2010	9435	33696	57284	95473	140402	201055	545882	4245746
2013	8787	31383	54396	96240	141221	194572	460277	4121570
2016	10126	34430	60758	101264	151895	212654	506318	5336594

Figure 5: Earning by earning bucket in 2016 Dollar, SCF

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	19474	38949	60370	97372	138268	194743	455699	5209384
1992	18906	34375	56719	91095	130627	180471	386723	2870347
1995	19245	35282	57734	94620	128298	176410	384895	5147976
1998	20936	38881	62808	100194	139076	192912	518917	6011662
2001	22295	41803	69671	111474	161637	229915	700892	11147386
2004	23511	43103	67919	113634	164574	235106	611274	8424614
2007	23817	42871	69070	113131	163147	239362	790728	11325035
2010	22464	39312	64023	104459	157250	226889	664942	7003234
2013	20922	37659	61719	104608	159005	236415	711337	9215997
2016	23291	41518	67847	111390	176199	260248	848589	10979004

Figure 6: Income by income bucket in 2016 Dollar, SCF

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	-0.03435	0.012823	0.05277	0.130758	0.137233	0.128828	0.276179	0.29576
1992	-0.00209	0.014311	0.051961	0.125832	0.128324	0.121273	0.253234	0.307157
1995	-0.04316	0.016697	0.056125	0.127479	0.131483	0.121738	0.248106	0.341531
1998	-0.00887	0.01303	0.049969	0.120115	0.123842	0.112131	0.251662	0.338122
2001	-0.00185	0.011977	0.043475	0.114165	0.126051	0.120533	0.266636	0.319012
2004	-0.00173	0.010942	0.041624	0.114286	0.127884	0.114868	0.260338	0.331787
2007	-0.00189	0.010114	0.042088	0.106713	0.112671	0.1078	0.274546	0.347958
2010	-0.00885	0.006881	0.032473	0.099006	0.122914	0.12912	0.287239	0.331213
2013	-0.08806	0.006632	0.033326	0.101689	0.124343	0.125369	0.295832	0.400868
2016	-0.00514	0.00608	0.028715	0.08573	0.10861	0.117356	0.280084	0.378563

Figure 7: Wealth share by wealth bucket, SCF (2016)

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	0	0.025454	0.132658	0.246521	0.185203	0.127677	0.168213	0.114273
1992	0	0.026814	0.131921	0.249753	0.192502	0.130701	0.16681	0.101499
1995	0	0.027043	0.134211	0.247709	0.186051	0.125672	0.167073	0.112242
1998	0	0.037228	0.137749	0.246486	0.186182	0.123811	0.160657	0.107887
2001	0	0.039249	0.127279	0.227205	0.176056	0.119871	0.163522	0.147137
2004	0	0.034486	0.126596	0.231352	0.184381	0.125997	0.163009	0.134179
2007	0	0.026922	0.121346	0.228414	0.181607	0.125433	0.172587	0.143691
2010	0	0.023351	0.113424	0.221467	0.179075	0.129839	0.187932	0.144911
2013	0	0.019785	0.1117	0.223141	0.188383	0.135577	0.186087	0.135327
2016	0	0.015855	0.105531	0.21646	0.178303	0.132199	0.184854	0.166799

Figure 8: Earning share by earning bucket, SCF

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1989	0.030148	0.073556	0.129178	0.203465	0.151653	0.107116	0.155186	0.149698
1992	0.032469	0.07906	0.134306	0.213861	0.158823	0.111222	0.153809	0.11645
1995	0.029381	0.080272	0.137446	0.210595	0.157795	0.107447	0.148854	0.12821
1998	0.029176	0.075491	0.126298	0.200601	0.146693	0.100148	0.154359	0.167235
2001	0.027282	0.06715	0.111659	0.180183	0.13499	0.097153	0.162658	0.218925
2004	0.029053	0.070578	0.116645	0.186518	0.143107	0.103317	0.154999	0.195783
2007	0.028068	0.066228	0.109902	0.177864	0.134593	0.098215	0.169994	0.215136
2010	0.031461	0.07059	0.116392	0.184917	0.143897	0.1057	0.172767	0.174276
2013	0.029526	0.064531	0.106657	0.177598	0.14037	0.104457	0.172995	0.203866
2016	0.026442	0.060757	0.10148	0.166154	0.132206	0.099414	0.169192	0.244355

Figure 9: Income share by income bucket, SCF

Appendix.B

As described in calibration section, we divide household with age 1 to 80 into 16 bucket. Within each 5-year age bucket, we sort the household by its labor earning, total income and wealth into 8 sub-bucket: bottom 20%, 20%-40%, 40%-60%, 60%-80%, 80%-90%, 90%-95%, 95%-99%, 99%-100%. Earning, income and wealth amount are in model unit.

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0	0	0	0	0.133285	0.218499	0.418441	0.229775
2	0	0	0.04504	0.252475	0.25475	0.168114	0.183426	0.096196
3	0	0.040025	0.149952	0.287838	0.213464	0.125221	0.124399	0.059101
4	0.012239	0.092478	0.181322	0.285983	0.189076	0.101465	0.093998	0.043439
5	0.035786	0.121764	0.194416	0.275671	0.168387	0.087891	0.07972	0.036364
6	0.057432	0.140049	0.199494	0.26157	0.156451	0.079981	0.071919	0.033104
7	0.075756	0.149332	0.197431	0.255528	0.146942	0.0754	0.067838	0.031772
8	0.086681	0.151095	0.196092	0.247296	0.142336	0.074573	0.069003	0.032923
9	0.084806	0.145865	0.190109	0.237182	0.143941	0.079428	0.077389	0.04128
10	0.070797	0.143839	0.190484	0.244646	0.147251	0.080414	0.078253	0.044316
11	0.047504	0.138469	0.195172	0.256379	0.155214	0.083988	0.08125	0.042024
12	0.013782	0.110737	0.184342	0.268872	0.171868	0.097712	0.101936	0.050752
13	0	0.02351	0.145389	0.276024	0.211805	0.134311	0.142027	0.066935
14	0	0	0.000867	0.232455	0.259833	0.190909	0.213857	0.102079
15	0	0	0	0.007232	0.296729	0.256903	0.297807	0.141329
16	0	0	0	0	0.128176	0.332828	0.373058	0.165938

Figure 10: Wealth share by wealth bucket and age bucket

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.041196	0.086265	0.14861	0.258938	0.155153	0.111716	0.122625	0.075498
2	0.044095	0.088872	0.142238	0.247759	0.153324	0.122565	0.126002	0.075144
3	0.046702	0.083896	0.142556	0.248481	0.15389	0.122973	0.126384	0.075117
4	0.044383	0.081713	0.1418	0.249576	0.155323	0.124114	0.127532	0.075558
5	0.040196	0.0797	0.141249	0.249654	0.157527	0.125883	0.129312	0.076478
6	0.038746	0.078241	0.139848	0.245725	0.160072	0.127907	0.131427	0.078034
7	0.03731	0.076701	0.139517	0.235941	0.164446	0.131417	0.13471	0.079958
8	0.032239	0.076527	0.14074	0.226888	0.168509	0.134659	0.13814	0.082298
9	0.020317	0.067956	0.122747	0.230625	0.176155	0.132124	0.144047	0.106028

Figure 11: Earning share by earning bucket and age bucket

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.039412	0.082622	0.122536	0.23789	0.179653	0.114531	0.139291	0.084064
2	0.029659	0.060745	0.113408	0.242651	0.196729	0.126175	0.147294	0.083338
3	0.021541	0.069296	0.150541	0.26035	0.193021	0.117555	0.123416	0.06428
4	0.02831	0.096276	0.170711	0.266155	0.18063	0.103012	0.102764	0.052142
5	0.043096	0.116987	0.181154	0.262266	0.166572	0.093161	0.091121	0.045643
6	0.059538	0.13051	0.185187	0.254491	0.156515	0.086769	0.084496	0.042493
7	0.073655	0.138629	0.184921	0.248999	0.148577	0.083563	0.080806	0.040849
8	0.082085	0.141586	0.184856	0.240558	0.144741	0.083253	0.08127	0.041651
9	0.080443	0.136545	0.181	0.233839	0.146645	0.085228	0.087118	0.049182
10	0.069668	0.143889	0.190972	0.24546	0.147731	0.080656	0.078544	0.04308
11	0.045295	0.138489	0.195719	0.257368	0.15583	0.084334	0.08154	0.041425
12	0.016189	0.119584	0.196887	0.284972	0.181127	0.078991	0.081919	0.040331
13	0	0.023511	0.145394	0.276034	0.211813	0.134316	0.142028	0.066904
14	0	0	0.000867	0.232457	0.259835	0.190911	0.213859	0.102071
15	0	0	0	0.007232	0.296729	0.256903	0.297807	0.141328
16	0	0	0	0	0.128365	0.332828	0.372968	0.165839

Figure 12: Income share by income bucket and age bucket

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0	0	0	0	0.053347	0.185423	0.47692	3.867557
2	0	0	0.066403	0.346179	0.719975	1.058666	1.547949	12.11254
3	0	0.157631	0.533025	1.007837	1.574268	2.056273	2.715406	19.36634
4	0.084771	0.548474	1.063594	1.649458	2.294119	2.753572	3.391283	23.60095
5	0.289333	0.962168	1.507337	2.107722	2.717364	3.17621	3.835619	26.2908
6	0.525031	1.290976	1.789191	2.348592	2.93255	3.385584	4.058652	27.89957
7	0.748221	1.462052	1.914247	2.49193	2.98568	3.459481	4.155091	28.93324
8	0.86063	1.52298	1.964982	2.494979	2.966723	3.514855	4.309267	31.13957
9	0.832723	1.48216	1.940072	2.478763	3.10175	3.866229	4.988163	39.64858
10	0.577408	1.216874	1.628845	2.121049	2.773103	3.427432	4.457161	34.81345
11	0.297112	0.877059	1.231798	1.643332	2.156566	2.704307	3.555906	25.91131
12	0.061396	0.465897	0.773863	1.135708	1.580026	2.136492	3.070981	21.8125
13	0	0.050021	0.297624	0.584235	1.005102	1.541088	2.166661	14.7181
14	0	0	0	0.216712	0.575639	1.001481	1.485286	10.18487
15	0	0	0	0.001613	0.305174	0.659175	0.991077	6.90681
16	0	0	0	0	0.069559	0.4	0.613701	4.053

Figure 13: Wealth amount by wealth bucket and age bucket

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.034119	0.07587	0.130589	0.221759	0.280737	0.460011	0.661068	1.258752
2	0.071671	0.150598	0.241791	0.420459	0.549881	0.989509	1.356454	2.426444
3	0.099703	0.184467	0.318232	0.554762	0.726945	1.307049	1.791309	3.193524
4	0.106123	0.209717	0.358694	0.639581	0.841683	1.513183	2.073259	3.685246
5	0.103232	0.215101	0.379993	0.678736	0.905344	1.629115	2.231164	3.959726
6	0.099949	0.210715	0.379422	0.673653	0.928397	1.671242	2.288483	4.078488
7	0.089096	0.197159	0.357905	0.621762	0.916265	1.651143	2.244327	4.01819
8	0.069406	0.178841	0.334382	0.553395	0.875048	1.575557	2.137887	3.8469
9	0.028528	0.109852	0.223101	0.416112	0.648464	1.170166	1.592871	2.864564

Figure 14: Earning amount by earning bucket and age bucket

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1
1	0.039663	0.087815	0.129797	0.248463	0.395754	0.549007	0.917985	1.684891
2	0.086708	0.186756	0.351749	0.746805	1.275885	1.841936	2.868574	4.86889
3	0.120834	0.41458	0.878325	1.521684	2.382464	3.262461	4.563075	7.137498
4	0.242776	0.82873	1.465644	2.278019	3.257219	4.183508	5.560331	8.46563
5	0.454807	1.265716	1.944528	2.799851	3.76012	4.737272	6.178572	9.286852
6	0.700131	1.588781	2.234991	3.067402	3.979613	4.989487	6.465501	9.753886
7	0.911043	1.768561	2.341425	3.156702	3.979408	5.065099	6.510045	9.885296
8	0.999207	1.830224	2.377343	3.080982	3.907025	5.067673	6.569257	10.20055
9	0.911824	1.683025	2.256178	2.930296	3.841194	5.009837	6.705814	10.94099
10	0.5581	1.248677	1.678681	2.188884	2.85972	3.53671	4.622125	7.051552
11	0.26298	0.880548	1.259039	1.688953	2.209709	2.777853	3.662955	5.332488
12	0.054377	0.47867	0.802189	1.196646	1.7283	1.690919	2.439026	3.498093
13	0	0.035301	0.27863	0.568756	0.975719	1.490334	2.124576	2.92445
14	0	0	0	0.197019	0.553028	0.978121	1.453657	2.014198
15	0	0	0	0	0.293553	0.63027	0.970448	1.359342
16	0	0	0	0	0.058715	0.403949	0.602782	0.806623

Figure 15: Income amount by income bucket and age bucket

Appendix.C

In table below, we report Gini coefficient and wealth-income inequality implied by model with preference heterogeneity and Epstein-Zin preference.

Table 1: Distributions of Earnings and Wealth in the U.S. Economy

	0.2	0.4	0.6	0.8	0.9	0.95	0.99	1.0	Gini
	A. Distributions of Earning								
EZ	0.000	0.025	0.129	0.216	0.189	0.179	0.146	0.117	0.6452
Data	0.000	0.016	0.106	0.216	0.178	0.132	0.185	0.167	0.6799
	B. Distributions of Income								
EZ	0.023	0.063	0.144	0.256	0.188	0.130	0.138	0.059	0.5288
Data	0.026	0.061	0.101	0.166	0.132	0.100	0.169	0.244	0.5977
	C. Distributions of Wealth								
EZ	0.002	0.051	0.143	0.269	0.200	0.138	0.143	0.054	0.5643
Data	-0.005	0.006	0.029	0.086	0.109	0.117	0.280	0.379	0.8595

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