BROKEN TELEPHONE, AN ANALYSIS OF A REINFORCED PROCESS

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ABSTRACT

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We consider the following L player co-operative signaling game. Nature plays from the set $\{0, 0'\}$. Nature's play is observed by Player 1 who then plays from the set $\{1, 1'\}$. Player 1's play is observed by Player 2. Player 2 then plays from the set $\{2, 2'\}$. Player 2's play is observed by player 3. This continues until Player L observes Player L-1's play. Player L then guesses Nature's play. If he guesses correctly, then all players win. We consider an urn scheme for this where each player has two urns, labeled by the symbols they observe. Each urn has balls of two types, represented by the two symbols the player controlling the urn is allowed to play. At each stage each player plays by drawing from the appropriate urn, with replacement. After a win each player reinforces by adding a ball of the type they draw to the urn from which it was drawn. We attempt to show that this type of urn scheme achieves asymptotically optimal coordination. A lemma remains unproved but we have good numerical evidence for it's truth.

Contents

Chapter 1

Introduction

Since the fifth century BC, philosophers, such as Democratus the Atomist [Bar83], have debated how languages originally came into being. There are two types of theories about how language originally occurred [Ulb98]. The first, known as discontinuity theories, suggest that language in humans is very different from anything else found in nature and therefore must have appeared very suddenly [Cho72] as a technological innovation.

The other group of theories known as continuity theories which view language as evolving slowly over time in a fashion related to Darwinian evolution [Pin90].

A prevalent aspect of many theories of both types is that at least some words are intrinsically imbued with meaning. For an extreme example, it is not difficult to believe that “haha” might be a natural phrase for laughter.

The alternate viewpoint is that all language is purely by convention. It is hard to imagine a population without language explicitly and consciously agreeing on a vocabulary, without already having one.

A class of toy models called Lewis signaling games, after their inventor David Lewis, give us a test of the plausibility of continuity [Lew69] theories. Lewis signaling games involve two players: a sender and a receiver. Nature can be in any number of states and the sender can observe the state of nature. The sender then sends a signal to the receiver who acts in response to the signal. If the receiver acts in certain ways both players are rewarded.

In a Lewis signaling game, when the players are rewarded, they increase the probability of repeating the same action if the same situation should occur again. Lewis signaling games make no appeal to the players consciously deciding on which signals to send, or even to players being aware that they are in a signaling game at all.

This is a desirable property for our model to have as signaling and communication are commonplace among many simple organisms, which could not possibly be consciously communicating. For example, human cells communicate with each other frequently, bacteria exhibit signaling and all kinds of animals display cooperative hunting or warning signals.

There are several ways that the relevant probabilities can be adjusted in Lewis signaling games. One such example is Bush-Mosteller reinforcement [BM55]. We shall focus on a type of reinforcement called Skyrms reinforcement. We shall call these Lewis signaling games with Skyrms reinforcement, Skyrms games.

Of course communication is often not a simple case of Sender signals Receiver, and Receiver responds. Often there is back and forth communication between the agents. Often the agents have more than two signals available to them. Often more than two agents are present, arranged in any number of configurations. As long as all agents in a model are acting according to Skyrms reinforcement we shall still refer to such processes as Skyrms games.

While much is known about Skyrms games from simulations, very little has been proven rigorously. The recent paper [APSV08] deals with the simplest type of Skyrms game. It shows that efficient signaling occurs with probability one in this Skyrm game. In this game there are only two players, one of whom is always the Sender and the other is always the Receiver. Nature has only two symbols, each of which occurs with probability 0.5 independently of the past. The Sender has exactly two signals to play.

Here we attempt generalize this result to the case where there are $L \geq 2$ agents arranged in a fixed line. Each player still has two symbols that they can use and Nature still has two states which occur with probability 0.5 each, independently of the past. We attempt to show that in this more general model efficient signaling still occurs with probability one. There is a lemma which we could not prove however we will show that given this lemma efficent signalling will occur. We will give numerical evidence to support the idea that efficent signalling occurs and we we solve a toy model qualitatively similiar to the main model where the lemma fails.

Chapter 2

The Model

Motivation for the Model

An important point to remember about the paper of Argiento, Pemantle, Skyrms and Volkov [APSV08], is that it shows that urn models can produce efficient signaling, without any prior agreements between players. That is to say that the players do not need to sit down before the game starts and formally agree upon a language, or pieces of a language. They simply have to play according to the urn scheme and eventually a language will form.

One natural question that arises from the analysis of [APSV08] concerns the case of a game with more than two players. The simplest Skyrms game with L players is the case where all players are in a line, that is when Player 1 observes natures play and signals to Player 2 who in turn signals to Player 3 and so on until Player

L receives Player L-1's signal, at which point Player L guesses Nature's play.

In this paper we attempt to solve this generalization via a proof which closely follows the one discussed in [APSV08]. We fail to fully solve this because one lemma remains unproven. However we prove the resultholds if the lemma does and give some numericalevidence that it holds. In order to get these partial results we use the trick of factorizing individual coordinates of an appropriate vector field. This makes several computations much easier.

The Communication Game

We consider the following game where the players are Nature, Player 1, Player 2, Player 3, ..., Player L. Nature plays first by showing either 0 or 0' to Player 1, and no-one else. Player 1 sees Nature's play and responds to it by showing either 1 or 1' to Player 2, and no-one else. Player 2 then shows either 2 or 2' to Player 3 and so on until Player L-1 sends a symbol to Player L. Player L then guesses Nature's play. All the players win if Player L guesses correctly and all players lose if player n guesses incorrectly. This game is repeated ad infinitum.

If the players are allowed to confer beforehand they will simply decide on a language and win every time. Even if the players are only allowedto communicate enough to select a representative (Player k for some k) then every player except the representative,could simply pick a language arbitrarily and never deviate from the language they have chosen. The representative could then easily confirm thelanguage.

Below we study a protocol which does not require even this limited amount of pre-game communication between players.

The Urn Scheme

We assume that Nature's plays are i.i.d. and that Nature plays 0 half the time and 0' the other half of the time. Player 1 has two urns labeled 0 and 0'. Each of Player 1's urns contains balls labeled 1 and 1'. Player 1 draws, with replacement, a ball from the urn labeled by Nature's play and plays by sending the symbol corresponding to the ball he drew to Player 2. Effectively Player 1 shows the ball he drew to Player 2. Similarly Player 2 has urns labeled 1 and 1' containing balls labeled 2 and 2'. Player 2 draws, with replacement, a ball from the urn labeled by Player 1's play and plays by sending the symbol he draws to Player 3 and so on. The rest of the players' plays are also determined by similar urn schemes. Finally Player L guesses Nature's original play. If Player L guesses correctly everyone wins. As a result all players add a ball of the type they drew to the urn they drew it from.

Formal Construction of the Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a sufficiently rich source of randomness. We take it to be the probability space on which we have the family of i.i.d. uniform $(0,1)$ random variables $\{U_{n,j} : n \geq 1, j \in \{0, 1, ..., L\}\}$. We let $\mathcal{F}_t = \sigma(U_{k,j} : k \leq t, j \in \{0, 1, ..., L\})$ be the sigma field of information up until time t. For ease of notation, call Nature "Player 0". We define the random variables $V(n,k,k+1)$,$V(n,k,k+1')$,$V(n,k',k+1)$ and $V(n,k',k+1')$ inductively. Here $V(n,k,k+1)$ means the number of balls of type $k+1$ in urn k at time n, similiarly $V(n,k,k+1')$ means the number of balls of type $k+1'$ in urn k at time n, $V(n,k',k+1)$ means the number of balls of type $k+1$ in urn k' at time n and finally $V(n,k',k+1')$ means the number of balls of type $k+1'$ in urn k' at time n.

Define:

$V(1,k,k+1) = V(1,k,k+1') = V(1,k',k+1) = V(1,k',k+1') = 1$ for $k \in 0,1,2,...,n-2$. Also define $V(1,n-1,0) = V(1,n-1,0') = V(1,n-1',0) = V(1,n-1',0') = 1$. Intuitively this corresponds to declaring all the urns to have one ball of each relevant type, at the start.

Now, with the case of $n=1$ clearly specified we turn our attention to the induction step on n. Given the $4L$ values of $V(n-1,*,*)$ we can construct how Player 1's play at time n. Given Player 1's play at time n we can then construct Player 2's play at time n. When this is done we construct Player 3's play at time n and so on. To begin in ernest we construct Player k's play at time n, $M_{n,k}$ as follows. We start with Player 0 that is, with Nature. If $U_{n,0} < 0.5$ then $M_{t,0} = 0$ otherwise $M_{n,0} = 0'$. This reflects the fact that Nature plays 0 half the time and 0'the other half of the time.

For each $k \in 1, 2, 3, .., n-1$ we define $M_{n,k} = k$ if

$$U_{t,k} < \frac{V(t, M_{t-1,k-1}, k)}{V(t, M_{t-1,k-1}, k) + V(t, M_{t-1,k-1}, k')}$$

and as k' otherwise.

This corresponds to Player k choosing between sending symbol k or symbol k' to Player k+1, with weighted probabilities proportional to the number of balls of each type in the urn Player k controls labeled by Player $k-1$'s play at time n.

Finally if $M_{n,0} = M_{n,L}$ then for all k $V(n+1, M_{n,k}, M_{n,k+1}) = 1+V(n, M_{n,k}, M_{n,k+1})$ and $V(n+1, a, b) = V(n, a, b)$ for all other (a, b). That is if the Players collectively win then they each add a ball of the type they drew to the urn they drew it from, and they do nothing else when they win. However if $M_{n,0} \neq M_{n,L}$ then $V(t+1, a, b) = V(t, a, b)$ for all (a, b). That is when the players do not win they do nothing at all to any urn.

Chapter 3

The Main Theorem

Note that at time n (that is after the game has been repeated n times) that each player has the same total number of balls in urns belonging to them. Call this T_n. It makes sense to talk about $V(n,k,k+1)$ the number of balls of type $k+1$ in urn k. Similiarly we will talk about , $V(n,k,k+1')$, $V(n,k,k+1')$ and $V(n,k',k+1')$ the number of balls of type k, k' and k' in urns $k+1'$, $k+1$ and $k+1'$ respectively. For ease of notation we consider L equivalent to 0, L' equivalent to 0', $L+1$ equivalent to 1 and so on.

Define $x_{k,k+1}(n) = \frac{V_{(n,k,k+1)}}{T_n}$, $x_{k,k+1'}(n) = \frac{V_{(n,k,k+1')}}{T_n}$, $x_{k',k+1}(n) = \frac{V_{(n,k',k+1)}}{T_n}$ and $x_{k',k+1'}(n) = \frac{V_{(n,k',k+1')}}{T_n}$. It should be noted that the vector $B_k := (x_{k,k+1}, x_{k,k+1'}, x_{k',k+1}, x_{k',k+1'}) \in \Delta_3$ represents the proportion of Player k's ball of each kind (type and urn). Here Δ_3 is the 3-simplex.

We will thus call B_k Player k's view. Finally define the process

$$X_n := (B_0, B_1, B_2, ..., B_{L-1}). \tag{3.1}$$

Let $W_n := T_n - 4$ be the number of reinforcements. We now state the main conjecture of the thesis.

Conjecture 3.1. *With probability 1,* $\frac{W_n}{n} \to 1$*. Furthermore this occurs in one of* 2^{L-1} *specific ways, each of which is equally likely. These are the* 2^{L-1} *ways in which for each* k*, Player* k*'s view* B_k *tends to either* $(\frac{1}{2}, 0, 0, \frac{1}{2})$ *or* $(0, \frac{1}{2}, \frac{1}{2}, 0)$*, and for which the number of players whose views tend to* $(0, \frac{1}{2}, \frac{1}{2}, 0)$ *is even.*

The partial proof of the above result occupies most of the rest of this document.

To make our notation easier we make the following definitions.

$$\begin{aligned} s_k &:= x_{k,k+1} + x_{k,k+1'} &&= x_{k-1,k} + x_{k-1',k}, \\ s_{k'} &:= x_{k,k+1} + x_{k,k+1'} &&= x_{k-1,k} + x_{k-1',k} \end{aligned}$$

Notice that the s symbols defined above do not depend on which Players point of view we are looking at. For example $s_1 = x_{0,1} + x_{0',1} = x_{1,2} + x_{1,2'}$. This is because a win that involves (in the example) the 1 symbol must do the same thing to both $x_{0,1} + x_{0',1}$ and $x_{1,2} + x_{1,2'}$.

We also define,

$$Q_{k,k+1}(x) := x_{k,k+1}x_{k',k+1'} - x_{k,k+1'}x_{k',k+1},$$

$$Q_{k,k+1'}(x) := x_{k,k+1'}x_{k',k+1} - x_{k,k+1}x_{k',k+1'},$$

$$Q_{k',k+1}(x) := x_{k',k+1}x_{k,k+1'} - x_{k',k+1'}x_{k,k+1},$$

$$Q_{k',k+1'}(x) := x_{k',k+1'}x_{k,k+1} - x_{k,k+1'}x_{k',k+1}.$$

Clearly $Q_{k,k+1}(x) = Q_{k',k+1'}(x) = -Q_{k,k+1'}(x) = -Q_{k',k+1}(x)$. As it turns out these Q functions will make our notation a lot easier.

Also observe that

$$\begin{aligned} Q_{k,k+1}(x) &= x_{k,k+1}x_{k',k+1'} - x_{k,k+1'}x_{k',k+1} \\ &= x_{k,k+1}(1 - x_{k,k+1} - x_{k,k+1'} - x_{k',k+1}) - x_{k,k+1'}x_{k',k+1} \\ &= x_{k,k+1} - s_k s_{k+1} \end{aligned}$$

and similarly $Q_{k',k+1} = x_{k',k+1} - s_{k'}s_{k+1}$, $Q_{k,k+1'} = x_{k,k+1'} - s_k s_{k+1'}$ and $Q_{k',k+1'} = x_{k',k+1'} - s_{k'}s_{k+1'}$.

Define

$$R_{k,k+1}(x) := x_{k,k+1}(x_{k',k+1'} - 2Q_{k,k+1})$$

$$R_{k,k+1'}(x) := x_{k,k+1'}(x_{k',k+1} - 2Q_{k,k+1'})$$

$$R_{k',k+1}(x) := x_{k',k+1}(x_{k,k+1'} - 2Q_{k',k+1})$$

$$R_{k',k+1'}(x) := x_{k',k+1'}(x_{k,k+1} - 2Q_{k',k+1'})$$

As with the Q functions these R functions will make our notation much easier.

We define the sets $\Upsilon_k := \{k, k'\}$. For example $\Upsilon_0 := \{0, 0'\}$. We define $\Upsilon := \Upsilon_0 \times \Upsilon_1 \times ..._{L-1}$. Notice that this is the set of strings for which it is possible to reinforce along and every reinforcement must involve exactly one element υ of Υ.

We define P_υ as the probability of reinforcement via $\upsilon = (\upsilon_0, \upsilon_1, ..., \upsilon_{L-1})$ and observe:

$$P_\upsilon(x) := \frac{x_{\upsilon_0,\upsilon_1} x_{\upsilon_2,\upsilon_2} ... x_{\upsilon_{L-1},\upsilon_0}}{2 s_{\upsilon_0} s_{\upsilon_1} ... s_{\upsilon_{L-1}}}$$

Observe that $P(x)$ the probability of reinforcement via any string is

$$P(x) = \Sigma_{\upsilon \in \Upsilon} P_\upsilon(x)$$

For clarity, in the case $L = 3$

$$P(x) = \frac{x_{01}x_{12}x_{20}}{2s_0 s_1 s_2} + \frac{x_{01}x_{12'}x_{2'0}}{2s_0 s_1 s_{2'}} + \frac{x_{01'}x_{1'2}x_{20}}{2s_0 s_{1'} s_2} + \frac{x_{01'}x_{1'2'}x_{2'0}}{2s_0 s_{1'} s_{2'}} +$$
$$\frac{x_{0'1}x_{12}x_{20'}}{2s_{0'} s_1 s_2} + \frac{x_{0'1}x_{12'}x_{2'0'}}{2s_{0'} s_1 s_{2'}} + \frac{x_{0'1'}x_{1'2}x_{20'}}{2s_{0'} s_{1'} s_2} + \frac{x_{0'1'}x_{1'2'}x_{2'0'}}{2s_{0'} s_{1'} s_{2'}}$$

Define $D_L = s_0 s_{0'} s_1 s_{1'} ... s_{L-1} s_{L-1'}$. When it is obvious we will sometimes simply write D for D_L.

It is useful to observe $Q_{k,k+1}(x) = 0 \iff Q_{k,k+1'}(x) = 0 \iff Q_{k',k+1}(x) = 0 \iff Q_{k',k+1'}(x) = 0 \iff s_k s_{k+1} = x_{k,k+1} \iff s_k s_{k+1'} = x_{k,k+1'} \iff s_{k'} s_{k+1} = x_{k',k+1} \iff s_{k'} s_{k+1'} = x_{k',k+1'}$

If one (and therefore all) of these properties hold, then we will say that B_k has property I (I stands for ignore, this is precisely when a player ignores the information sent to him). We will say that B_k has property I_0 if least one of s_k,$s_{k'}$,s_{k+1} or $s_{k+1'}$ is 0. Notice that property I_0 implies property I.

Lemma 3.2. *If for* $B_k = (x_{k,k+1}, x_{k,k+1'}, x_{k',k+1}, x_{k',k+1'}) \in \Delta_3$ $R_{k,k+1}(x) = R_{k,k+1'}(x) = R_{k',k+1}(x) = R_{k',k+1'}(x) = 0$, *then exactly one of the following is true:*

1. $x = (1/2, 0, 0, 1/2)$.

2. $x = (0, 1/2, 1/2, 0)$.

3. x *has property* I_0.

Proof. If $x_{k,k+1} > 0$, and $x_{k,k+1'} > 0$, then as by assumption $R_{k,k+1} = R_{k,k+1'} = 0$,

$$x_{k',k+1'} - 2Q_{k,k+1} = 0$$

and

$$x_{k',k+1} - 2Q_{k,k+1'} = 0$$

Adding these together gives

$$x_{k',k+1} + x_{k',k+1'} = 0,$$

However $x_{k',k+1} + x_{k',k+1'} = 0$ implies property I_0. Similiarly if any of the pairs

$(x_{k',k+1}, x_{k',k+1'})$, $(x_{k,k+1}, x_{k,k+1'})$ or $(x_{k,k+1'}, x_{k',k'_1})$ consist of 2 positive values, property I_0 holds. This means that, the only way for B_k to have two positive coordinates (it clearly has property I_0 if it doesn't have two non-zero coordinates) is to have $x = (s, 0, 0, 1 - s)$ or $s = (0, s, 1 - s, 0)$.

Consider the case where $x = (s, 0, 0, 1 - s)$ (the other case is similar). Then,

$$R_{k,k+1}(x) = 0$$

$$\implies x_{k,k+1}(2(s_k s_{k+1} - x_{k,k+1}) + x_{k',k+1'}) = s(2(s^2 - s) + 1 - s) = 0,$$

which solves to $s = 1/2$. □

We define $\overline{Q} := Q_{0,1}Q_{1,2}...Q_{L-1,L}$ and $\overline{Q}_{k,k+1} := Q_{0,1}Q_{1,2}...Q_{L-1,L}$ where the $Q_{k,k+1}$ is omitted. When $Q_{k,k+1} \neq 0$ it follows that $\overline{Q}_{k,k+1} = \overline{Q}/Q_{k,k+1}$

Define also $\overline{Q}_{k',k+1'} = -\overline{Q}_{k,k+1'} = -\overline{Q}_{k',k+1} = \overline{Q}_{k,k+1}$

Lemma 3.3. *The following is true*

$$D(2P - 1) = Q_{0,1}Q_{1,2}...Q_{L-1,L} = \overline{Q}$$

Proof. It is routine (in maple) to check this for the case of three (or two) players. We take note that only the following properties are required

1. $s_k = x_{k-1,k} + x_{k-1',k} = x_{k,k+1} + x_{k,k+1'}$

2. $s_{k'} = x_{k-1,k'} + x_{k-1',k'} = x_{k',k+1} + x_{k',k+1'}$

3. $s_k + s_{k'} = 1 \ \forall k$

4. $P(L) = \sum_{v\in\Upsilon} \frac{\prod_i x_{v_i,v_{i+1}}}{\prod_i s_{v_i}}$

Here $P(L)$ is the probability of reinforcement for a game with L players.

It is important to notice that this is a purely algebraic fact, and doesn't directly depend on the process that generates these events.

Having checked this for $L = 3$, that is, having checked that,

$$D_3(2P(3) - 1) = Q_{0,1}Q_{1,2}Q_{2,0}$$

we use induction. Assuming true for $L = k$, that is, assuming, $D_k(2P(k) - 1) = Q_{0,1}Q_{1,2}Q_{2,3}...Q_{k-1,k}$, we shall use induction to show that this is true for $L = k+1$.

Define:

$$y_{k-1,0} = \frac{x_{k-1,k}x_{k,0}}{s_k} + \frac{x_{k-1,k'}x_{k',0}}{s'_k}$$

$$y_{k-1,0'} = \frac{x_{k-1,k}x_{k,0'}}{s_k} + \frac{x_{k-1,k'}x_{k',0'}}{s'_k}$$

$$y_{k-1',0} = \frac{x_{k-1',k}x_{k,0}}{s_k} + \frac{x_{k-1',k'}x_{k',0}}{s'_k}$$

$$y_{k-1',0'} = \frac{x_{k-1',k}x_{k,0'}}{s_k} + \frac{x_{k-1',k'}x_{k',0'}}{s'_k}$$

Observe that $s_{k-1} = y_{k-1,0} + y_{k-1,0'}$, $s_{k-1'} = y_{k-1',0} + y_{k-1',0'}$, $s_0 = y_{k-1,0} + y_{k-1',0}$, and $s_{0'} = y_{k-1,0'} + y_{k-1,0'}$. It is now easy to see that $P(k+1) = P'(k)$ where $P'(k)$ is defined the same way as $P(k)$ except writing the y symbols defined above for the x symbols with the same subscripts.

Hence

$$D_{k+1}(2P_{k+1}-1) = D_k s_{k+1} s_{k+1'}(2P'_k - 1)$$

$$= s_{k+1} s_{k+1'} Q_{01} Q_{12} ... Q_{k-2,k-1}(y_{k-1,0} y_{k-1',0'} - y_{k-1,0'} y_{k-1',0}).$$

Maple then tells us that $s_{k+1} s_{k+1'}(y_{k-1,0} y_{k-1',0'} - y_{k-1,0'} y_{k-1',0}) = Q_{k-1,k} Q_{k,0}$.

and we have

$$D_{k+1}(2P(k+1)-1) \tag{3.2}$$

Hence by the principle of induction,

$$D_L(2P-1) = Q_{0,1} Q_{1,2} Q_{2,3} ... Q_{L-1,0}, \forall L \geq 3 \in \mathbb{N} \tag{3.3}$$

It is worth noting that, while we have used $y_{k-1,0}$, we could have equally well used any $y_{j,j+1}$ to prove this result at the cost of some re-indexing.

□

For a symbol k, define,

$$P_k(x) := \frac{1}{2} \Sigma_{k \in v} P_v(x)$$

, which is the probability of a reinforcement that uses the symbol k.

Lemma 3.4. *The following identities hold for $k = 0, 1, 2, .., L-1$:*

1. $D(2P_k - s_k) = s_{k'} \overline{Q}$,
2. $D(2P_{k'} - s_{k'}) = s_k \overline{Q}$.

The proofs of these are almost identical to the proof of lemma 3.3. We will prove the first identity explicitly and note that the second follows by symmetry.

Proof. It is once again routine (in maple) to check this for the case of three (or two) players. Define $\overline{\Upsilon}_l = \Upsilon_0 \times \Upsilon_1 \times ...\Upsilon_{l-1} \times l \times \Upsilon_{l+1} \times ... \times \upsilon_{L-1}$ Again only the following properties are required

1. $s_l = x_{l-1,l} + x_{l-1',l} = x_{l,l+1} + x_{l,l+1'}$

2. $s_{l'} = x_{l-1,l'} + x_{l-1',l'} = x_{l',l+1} + x_{l',l+1'}$

3. $s_k + s_{k'} = 1 \ \forall k$

4. $P_k(L) = \sum_{\upsilon \in \overline{\Upsilon}_k} \frac{\prod_i x_{\upsilon_i,\upsilon_{i+1}}}{\prod_i s_{\upsilon_i}}$

Here $P_k(L)$ is the probability of reinforcement where the symbol k is used for a game with L players.

$s_{k^*} = x_{k-1,k^*} + x_{k-1',k^*} = x_{k^*,k+1} + x_{k^*,k+1'}$

$s_{k^*} = x_{k-1,k^*} + x_{k-1',k^*} = x_{k^*,k+1} + x_{k^*,k+1'}$ Once again note that this is a purely algebraic fact, and doesn't directly depend on the process that generates these events.

Having checked this for $L = 3$, that is, having checked that,

$$D_3(2P_l(3) - 1) = s_l Q_{0,1} Q_{1,2} Q_{2,0}$$

for $l = 0, 1, 2$ we use induction. Assuming true for $L = m$, that is, assuming, $D_k(2P_m(k) - 1) = Q_{0,1}Q_{1,2}Q_{2,3}...Q_{m-1,m}$, for $k = 0, 1, 2, .., m-1$ we shall use induction to show that this is true for each particular $k = 0, 1, 2, .., m$ when $L =$

$m+1$. For some $r \neq k$ define:

$$y_{r-1,r} = \frac{x_{r-1,r}x_{r,r+1}}{s_r} + \frac{x_{r-1,r'}x_{r',0}}{s'_r}$$

$$y_{r-1,r'} = \frac{x_{r-1,r}x_{r,r+1'}}{s_r} + \frac{x_{r-1,r'}x_{r',0'}}{s'_r}$$

$$y_{r-1',r} = \frac{x_{r-1',r}x_{r,r+1}}{s_r} + \frac{x_{r-1',r'}x_{r',0}}{s'_r}$$

$$y_{r-1',r'} = \frac{x_{r-1',r}x_{r,r+1'}}{s_r} + \frac{x_{r-1',r'}x_{r',0'}}{s'_r}$$

For $l < r$ define $y_{l,l+1} = xl, l+1$,$y_{l,l+1'} = xl, l+1'$,$y_{l',l+1} = xl', l+1$ and $y_{l',l+1'} = xl', l+1'$, when $l > r$ define $y_{l,l+1} = x_{l+1,l+2}$,$y_{l,l+1'} = x_{l+1,l+2'}$,$y_{l',l+1} = x_{l+1',l+2}$ and $y_{l',l+1'} = x_{l+1',l+2'}$. Observe that $s_{r-1} = y_{r-1,r} + y_{r-1,r'}$, $s_{r-1'} = y_{r-1',r} + y_{r-1',r'}$, $s_r = y_{r-1,r} + y_{r-1',r}$, and $s_{r'} = y_{r-1,r'} + y_{r-1,r'}$. It is now easy to see that $P_k(m+1) = P'_k(m)$ where $P'_k(m)$ is defined the same way as $P_k(m)$ except writing the y symbols defined above for the x symbols.

Hence

$$\begin{aligned}
D_{m+1}(2P_k(m+1) - s_k) &= D'_m(2P'_k(m) - s_k) \\
&= s_k Q_{01} Q_{12} ... Q_{k-2,k-1}(y_{k-1,0}y_{k-1',0'} - y_{k-1,0'}y_{k-1',0}) \\
&= s_k \overline{Q}.
\end{aligned}$$

That is when we treat Player $r-1$ and Player r as a single player (by using the y) transformation, $D(2P_k - s_k)$is unaffected and hence by induction is equal to $s_k\overline{Q}$. Hence by the principle of induction,

$$D_L(2P_k - s_K) = s_k Q_{0,1} Q_{1,2} Q_{2,3} ... Q_{L-1,0} = s_k \overline{Q}, \forall L \geq 3 \in \mathbb{N} \forall k \in \{0, 1, .., L-1\} \tag{3.4}$$

$\square$

We define $P_{k,k+1} = \frac{1}{2}\Sigma_{k,k+1\in\upsilon}P_\upsilon(x)$ The probability of reinforcement involving the both the symbols k and $k+1$. We define $P_{k,k+1'}$,$P_{k',k+1}$ and $P_{k',k+1'}$ similiarly.

Lemma 3.5. *The following four identities hold.*

$$D(2P_{k,k+1} - x_{k,k+1}) = x_{k,k+1}s_{k'}s_{k+1'}\overline{Q}_{k,k+1}$$

$$D(2P_{k,k+1'} - x_{k,k+1'}) = -x_{k,k+1'}s_{k'}s_{k+1}\overline{Q}_{k,k+1'}$$

$$D(2P_{k',k+1} - x_{k',k+1}) = -x_{k',k+1}s_k s_{k+1'}\overline{Q}_{k',k+1}$$

$$D(2P_{k',k+1'} - x_{k',k+1'}) = x_{k',k+1'}s_k s_{k+1}\overline{Q}_{k',k+1'}$$

The proofs of these are again almost identical to the proof of lemma 3.3 A brute force checking for the case of $L = 3$ (three players) and then an induction using the same substitution.

Proof. Once again we will only proof the first identity explicitly as the other three follow by symmetry. Again we begin with a routine (n Maple) checking for the case of three (or two) players. Again we require only the following algebraic conditions. Here $P_{k,k+1}(L)$ is the probability of reinforcement involving both k and $k+1$ for a game with L players.

Again this is a purely algebraic fact, and doesn't directly depend on the process that generates these events.

Having checked it for $L = 3$, that is, having checked that,

$$D_3(2P_{k,k+1}(3) - x_{k,k+1}) = x_{k,k+1}s_{k'}s_{k+1'}\overline{Q}_{k,k+1}$$

for $k \in [0, 1, 2]$ we use induction. Assuming true for $L = m$, that is, assuming, $D_m(2P_{k,k+1}(m) - x_{k,k+1}) = x_{k,k+1}s_{k'}s_{k+1'}\overline{Q}_{k,k+1}$, for $k \in [0, 1, .., m-1]$ we shall use induction to show that this is true for $L = m+1$. Once again we define $y_{k,k+1}$ as in the proof of 3.3 and 3.4

And recall that as before $s_{r-1} = y_{r-1,r} + y_{r-1,r'}$, $s_{r-1'} = y_{r-1',r} + y_{r-1',r'}$, $s_r = y_{r-1,r} + y_{r-1',r}$, and $s_{r'} = y_{r-1,r'} + y_{r-1,r'}$. It is now easy to see that $P_{k,k+1}(m+1) = P'_{k,k+1}(m)$ where $P'_{k,k+1}(m)$ is defined the same way as $P_{k,k+1}(m)$ except writing the y symbols defined above for the x symbols with the same subscripts.

Hence

$$
\begin{aligned}
D_{k+1}(2P_{k+1} - 1) &= D_k s_{k+1} s_{k+1'}(2P'_k - 1) \\
&= s_{k+1}s_{k+1'}Q_{01}Q_{12}...Q_{k-2,k-1}(y_{k-1,0}y_{k-1',0'} - y_{k-1,0'}y_{k-1',0}).
\end{aligned}
$$

Hence by the principle of induction,

$$D_L(2P_{k,k+1} - x_{k,k+1}) = x_{k,k+1}s_{k'}s_{k+1'}\overline{Q}_{k,k+1}, \forall L \geq 3 \in \mathbb{N} and all$$

k $\in o, 1, 2, ..., L-1$ $\square$

Chapter 4

Relation to stochastic approximation and an ODE

A common version of the stochastic approximation process is one that satisfies

$$X_{n+1} - X_n = \gamma_t(F(X_n) + \xi_n) \tag{4.1}$$

where $\{\gamma_n\}$ are constants such that $\Sigma_n \gamma_n = \infty$ and $\Sigma_n \gamma_n^2 < \infty$, and where ξ_n are bounded and $\mathbb{E}(\xi_n | \mathcal{F}_n) = 0$. There is no precise definition of an urn model, but the normalized content in an urn model is typically a stochastic approximation process with $\gamma_n = 1/n$. One sees this by computing $\mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n)$ and seeing that when scaled by $1/n$, it converges to a vector function F.

We define $\psi_n := V_{n+1} - V_n$

To analyze our particular chain V_n or the scaled chain X_n, note that

$$X_{n+1} - X_n = \frac{V_{n+1}}{1+T_n} - \frac{V_n}{1+T_n} + \frac{V_n}{1+T_n} - \frac{V_n}{T_n} = \frac{1}{1+T_n}(\psi_n - X_n) \tag{4.2}$$

if $|\psi_n| = \sqrt{L}$ and 0 otherwise.

Taking expectations gives

$$\mathbb{E}(X_{n+1} - X_n)|\mathcal{F}_n) = \frac{1}{1+T_n} F(X_n) \tag{4.3}$$

where

$$F(X) := \mathbb{E}[1_{|\psi|>0}(\psi - X_n)|X_n = x]$$

Letting $\xi_n = (1+T_n)(X_{t+n} - X_n - F(X_n))$ be a noise term we see that (4.1) is a variant of (4.3) with non-deterministic γ_n

For processes obeying (4.1) or (4.3) the heuristic is that the trajectories of the process should approximate trajectories of the corresponding differential equation $X' = F(X)$. Let $Z(F)$ denote the set of zeros of the vector field F. The heuristic says that if there are no cycles in the vector field F, then the process should converge to the set $Z(F)$. A sufficient condition for the nonexistence of cycles is the existence of a Lyapunov function, namely a function $\mathcal{L}$ such that $\nabla \mathcal{L} \cdot F \geq 0$ with equality only where F vanishes. When $Z(F)$ is large enough to contain a curve, there is a question unsettled by the heuristic, as to whether the process can continue to move around in $Z(F)$. There is however a non-convergence heuristic saying the process should not converge to an unstable equilibrium.

Lemma 4.1. *The component of F associated to the pair of symbols* $(k, k+1)$ *is given by* $\frac{R_{k,k+1}\overline{Q}_{k,k+1}}{2D}$*, similiarly the components of F associated to* $(k, k+1')$*,* $(k', k+1)$ *and*

$(k', k+1')$ *are given by* $\frac{R_{k,k+1'}\overline{Q}_{k,k+1'}}{2D}$, $\frac{R_{k',k+1}\overline{Q}_{k',k+1}}{2D}$ *and* $\frac{R_{k',k+1'}\overline{Q}_{k',k+1'}}{2D}$ *respectively.*

For purposes of illustration we show the vector fields for $L = 3$ and $L = 4$ below. They are written as column vectors so as to fit better.

$$F_3(x) = \begin{pmatrix} \frac{R_{01}Q_{12}Q_{20}}{2D} \\ \frac{R_{01'}Q_{1'2}Q_{20}}{2D} \\ \frac{R_{0'1}Q_{12}Q_{20'}}{2D} \\ \frac{R_{0'1'}Q_{1'2}Q_{20'}}{2D} \\ \frac{R_{12}Q_{20}Q_{01}}{2D} \\ \frac{R_{12'}Q_{2'0}Q_{01}}{2D} \\ \frac{R_{1'2}Q_{20}Q_{01'}}{2D} \\ \frac{R_{1'2'}Q_{2'0}Q_{01'}}{2D} \\ \frac{R_{20}Q_{01}Q_{12}}{2D} \\ \frac{R_{20'}Q_{0'1}Q_{12}}{2D} \\ \frac{R_{2'0}Q_{01}Q_{12'}}{2D} \\ \frac{R_{2'0'}Q_{0'1}Q_{12'}}{2D} \end{pmatrix}, \quad F_4(x) = \begin{pmatrix} \frac{R_{01}Q_{12}Q_{23}Q_{30}}{2D} \\ \frac{R_{01'}Q_{1'2}Q_{23}Q_{30}}{2D} \\ \frac{R_{0'1}Q_{12}Q_{23}Q_{30'}}{2D} \\ \frac{R_{0'1'}Q_{1'2}Q_{23}Q_{30'}}{2D} \\ \frac{R_{12}Q_{23}Q_{30}Q_{01}}{2D} \\ \frac{R_{12'}Q_{2'3}Q_{30}Q_{01}}{2D} \\ \frac{R_{1'2}Q_{23}Q_{30}Q_{01'}}{2D} \\ \frac{R_{1'2'}Q_{2'3}Q_{30}Q_{01'}}{2D} \\ \frac{R_{23}Q_{30}Q_{01}Q_{12}}{2D} \\ \frac{R_{23'}Q_{3'0}Q_{01}Q_{12}}{2D} \\ \frac{R_{2'3}Q_{30}Q_{01}Q_{12'}}{2D} \\ \frac{R_{2'3'}Q_{3'0}Q_{01}Q_{12'}}{2D} \\ \frac{R_{30}Q_{01}Q_{12}Q_{23}}{2D} \\ \frac{R_{30'}Q_{0'1}Q_{12}Q_{23}}{2D} \\ \frac{R_{3'0}Q_{01}Q_{12}Q_{23'}}{2D} \\ \frac{R_{3'0'}Q_{0'1}Q_{12}Q_{23'}}{2D} \end{pmatrix}$$

The coordinate associated to the pair of symbols $(k, k+1)$ is by equation (4.3) $P_{k,k+1}(x) - x_{k,k+1}P(x)$. Similiarly the coordinates associated to the pairs of symbols

$(k, k+1')$,$(k', k+1)$ and $(k', k+1')$ are given by $P_{k,k+1'}(x) - x_{k,k+1'}P(x)$, $P_{k',k+1}(x) - x_{k',k+1}P(x)$ and $P_{k',k+1'}(x) - x_{k',k+1'}P(x)$ respectively. To prove the lemma we apply lemmas 3.3 and 3.5. It is now clear that the zero set $Z(F)$ of F consists of two types of points. We shall call, the first type of zero point, "language points"'. These are points where every player's view $B_k = (1/2, 0, 0, 1/2)$ or $(0, 1/2, 1/2, 0)$. We shall call the second type of point "babble points", these are points where at least two players views have property I.

It is worth noting that a single players view having property I_0 produces a language point because if Player k's view B_k has property I_0 then either Player $k-1$ or Player $k+1$ also has property I_0

Lemma 4.2. *We show $\nabla(\overline{Q} \cdot F) \geq 0$ and equality occurs only when $\overline{Q} = 0$ or at language points.*

It should be noted that $F = 0 \Rightarrow \overline{Q} = 0$ or x is a language point.

Proof.

$$\begin{aligned}
\nabla(\overline{Q}) \cdot F =& \Sigma_{k=1}^{n} \overline{Q}_{k,k+1} x_{k',k+1'} \frac{\overline{Q}_{k,k+1} R_{k,k+1}}{2D} - \overline{Q}_{k,k+1} x_{k',k+1} \frac{\overline{Q}_{k,k+1'} R_{k,k+1'}}{2D} \\
&- \overline{Q}_{k,k+1} x_{k,k+1'} \frac{\overline{Q}_{k',k+1} R_{k',k+1}}{2D} \\
&+ \overline{Q}_{k,k+1} x_{k,k+1} \frac{\overline{Q}_{k',k+1'} R_{k',k+1'}}{2D} \\
=& \Sigma_{k=1}^{n} \frac{\overline{Q}_{k,k+1}^2}{2D} (x_{k',k+1'} R_{k,k+1} + x_{k',k+1} R_{k,k+1'} + x_{k,k+1'} R_{k',k+1} + x_{k,k+1} R_{k',k+1'}) \\
=& \Sigma_{k=1}^{n} \frac{\overline{Q}_{k,k+1}^2}{2D} [x_{k',k+1'}(x_{k,k+1} - 2Q_{k,k+1})^2 + x_{k',k+1}(x_{k,k+1'} - 2Q_{k,k+1})^2 \\
&+ x_{k,k+1'}(x_{k',k+1} - 2Q_{k,k+1})^2 + x_{k,k+1}(x_{k',k+1'} - 2Q_{k,k+1})^2]
\end{aligned}$$

This last step is seen by defining a random varible Y with the following distribution.

$$Y = \begin{cases} x_{k,k+1} & \text{w.p.} x_{k',k+1'} \\ \\ -x_{k,k+1'} & \text{w.p.} x_{k',k+1} \\ \\ -x_{k',k+1} & \text{w.p.} x_{k,k+1'} \\ \\ x_{k',k+1'} & \text{w.p.} x_{k,k+1} \end{cases}$$

and observing that both of the last 2 steps are equal to the variance of Y, which is certianly positive. $\square$

Chapter 5

Probabilistic Analysis

Lemma 5.1. *We begin by showing that for each L there exists $\epsilon_L > 0$ with probability* 1,

$$\epsilon_L \leq liminf \frac{T_n}{n} \leq limsup \frac{T_n}{n} \leq 1$$

Proof. The upper bound is trivial as $T_n \leq n + 4$. We claim that for the case of L players there exists $\epsilon_L > 0$ such that for all obtainable values of X_n, $P(X_L) > \epsilon_L$ independent of n. Notice that there are 2^L ways in which it is possible to reinforce. That is, there are 2^L paths along which reinforcement can occur. At any time step n there must be a (perhaps not unique) path ζ which has been reinforced along most often. Which, means that, at least $1/2^L$ of all reinforcements have occurred along ζ.

This means that path ζ must be followed with probability at least $1/2 * (1/2^L)^L$. The reason for this is that at each of the L steps involving two players along ζ the probability of continuing to follow ζ is at least $1/2^L$, while nature plays the required symbol with probability 1/2.

Hence reinforcement happens with probability at least $\frac{1}{2}(\frac{1}{2^L})^L = \frac{1}{2^{L^2+1}}$, and combining this with the conditional Borel-Cantelli lemma [[Dur04], Theorem I.6]$\liminf \frac{T_n}{n} \leq \frac{1}{2^{L^2+1}}$ □

With this preliminary result out of the way, the remainder of the proof of Theorem 3.1, may be broken into three pieces namely Propositions 5.2, 5.3 and 5.4 below. We need to define a few sets in order to state these propositions. These sets are defined below.

$$Z(F) := \{x \in \Delta_3^n | F(x) = 0\}$$

$$Z(\overline{Q}) := \{x \in Z(F) | \overline{Q} = 0\}$$

$$dS_k := \{x \in B_k | s_k = 0 \cup s_k = 1\}$$

$$dS := \cup dS_k$$

Notice that $dS \subseteq ZZ_0(\overline{Q})$.

Proposition 5.2. *(Lypunov function implies convergence) The function $\mathcal{L}(x) = \overline{Q}$ converges almost surely to* 0 *or to* $1/4^L$

Proposition 5.3. *(no convergence to boundary, from good side) If $\overline{Q}$ is eventually greater than* 0 *then limit* $\lim_{n\to\infty} X_n$ *exists with probability 1. Furthermore,* $\mathbb{P}($

$limX_{n \to \infty} \in dS) = 0$

Proposition 5.4. *(saddle implies no convergence) The probability that* $lim_{n \to \infty} X_n$ *exists is 1. Furthermore,* $\mathbb{P}(lim_{n \to \infty} X_n) \in ZZ_0(\overline{Q}) = 0$.

These three results together imply Theorem 3.1. The first is shown via martingale methods that $\{X_n\}$ cannot continue to cross regions where F vanishes. The second relies on a comparison to a polya urn. The third could not be proved entirely however some partial results where obtained. In particular we will see that the result holds when $\overline{Q} > 0$ eventually using a proof mimicking [Pem90] and it's generalizations such as [Ben99]. We can also fill in the gaps assuming a condition that we have some numerical evidence for.

We begin by proving proposition 5.2

Proof. Let $Y_n := \mathcal{L}(X_n) = \overline{Q}_n$. We decompose Y_n into a martingale and a predictable process $Y_n = M_n + A_n$ where $A_{n+1} - A_n = \mathbb{E}(Y_{n+1} - Y_n | \mathcal{F}_)$. By Lemma 5.1 the increments in Y_n are $O(1/n)$ almost surely, hence the martingale M_n is in L^2 and hence almost surely convergent. We use the Taylor expansion

$$L(x + y) = L(x) + y \cdot \nabla L(x) + R_x(y) \tag{5.1}$$

with $R_x(y) = O(|y|^2)$ uniformly in x. Then

$$A_{n+1} - A_n = \mathbb{E}[L(X_{n+1}) - L(X_n)|\mathcal{F}_n] =$$

$$\mathbb{E}[\nabla(L)(X_n) \cdot (X_{n+1} - X_n) + R_{X_t}(X_{n+1} - X_n)|\mathcal{F}_n] =$$

$$\frac{1}{1+T_n}(\nabla(L \cdot F)(X_n)) + \mathbb{E}[R_{X_n}(X_{n+1} - X_n)|\mathcal{F}_n]$$

As $R_{X_n}(X_{n+1} - X_n) = O(T_n^{-2}) = O(n^{-2})$ it is summable , this gives

$$A_n = \eta + \Sigma_{k=1}^{n} \frac{1}{1+T_n}(\nabla\mathcal{L} \cdot F)(X_n)$$

for some almost surely convergent η. We now argue that if X_n is found infinitely often away from the critical values of Y_n then the drift would cause Y_n would blow up. Observe first that as $\{Y_n\}$ and $\{M_n\}$ are bounded it follows that $\{A_n\}$ is also bounded. For $\epsilon \in (0, \frac{0.5}{4^{L-1}})$, let Δ_ϵ denote $Y^{-1}[\epsilon, 1/4^L - \epsilon]$. On δ_ϵ the function $\nabla\mathcal{L} \cdot F$ which is always non-negative, is bounded below by some constant c_ϵ. Let δ be the distance from Δ_ϵ to the complement of $\Delta_{\epsilon/2}$. Suppose $X_t \in \Delta_\epsilon$ and $X_{t+k} \notin \Delta_{\epsilon/2}$. Then since $|\phi_n|$ and $|X_n|$ are at most $\sqrt{L}$ from equation 4.2 we see that

$$\begin{aligned}
\delta \leq & \Sigma_{j=n}^{n+k+1} |X_{j+1} - X_j| \\
\leq & \Sigma_{j=n}^{n+k+1} \frac{2\sqrt{L}}{1+T_j} \\
\leq & \frac{c_\epsilon \sqrt{L}}{\epsilon}[A_{n+k} - A_n - (\eta(n+k) - \eta(n))]
\end{aligned}$$

It follows that if $X_n \in \Delta_\epsilon$ infinitely often then A_n increases without bound. A contradiction therefore for every $\epsilon > 0$, X_n is eventually outside of Δ_ϵ $\square$

We now turn our attention to the proof of 5.3, we will first need the following lemma.

Lemma 5.5. *Suppose an urn has balls of two colours, white and black. Suppose that the number of balls increases by precisely 1 at each time step. Denote the number of white balls at time n by W_n and the number of black balls at time n by B_n. Let $X_n := W_n/(W_n + B_n)$ denote the fraction of white balls at time n, and let $\mathcal{F}_n$ denote the σ-field of information up to time n. Suppose further that there is some $p \in (0,1)$ such that that the fraction of white balls is always attracted towards p in the following sense.*

$$(\mathbb{P}(X_{n+1} > X_n | \mathcal{F}_n) - X_n) \cdot (p - X_n) \geq 0 \tag{5.2}$$

Then the limiting fraction $lim_{n\to\infty} X_n$ almost surely exists and is strictly between zero and one.

Lemma 5.5 appears as Lemma 3.9 in [APSV08]. The proof is given there and reproduced below.

Proof. Let $\tau_N := \inf\{k \geq N | X_k \leq p\}$ be the first time after N that the fraction fo white balls fals below p. The process $\{X_{k\wedge\tau_N} | k \geq N\}$ is a bounded supermartingale, and hence converges almost surely. Let $\{(W_k', B_k') : k \geq N\}$ be a Polya urn process coupled to $\{(W_k, B_k)\}$ as follows. Let $(W_n', B_n') = (W_n, B_n)$. We will verify inductively that $X_k \leq X_k' := W_k'/(W_k' + B_k')$ for all $k < \tau_N$. If $k < \tau_n$ and $W_{k+1} - W_k = 1$ then $W_{k+1}' = W_k' + 1$. If $k < \tau_N$ and $W_{k+1} = W_k$

then let Y_{k+1} be a bernoulli random varible independent of everything else with $\mathbb{P}(Y_{k+1} = 0|\mathcal{F}_k) = (1-X'_k)/(1-X_k)$ which is non-negetive. Let $W'_{k+1} := W'_k + Y_{k+1}$. This construction guareentees that $X'_{k+1} \geq X_{k+1}$, completeing the induction, and it is easy to see that $\mathbb{P}(W_{k+1} > W'_k) = X'_k$, so that $\{X'_k | N \leq \tau_N\}$ is a Polya Process. Complete the definition by letting $\{X'_k\}$ evole independently as a Polya urn process once $k \geq \tau_N$. It is well known that X'_k converges almost surely and that the conditional law of $X'_\infty := \lim_{k\to\infty} X'_k$ given $\mathcal{F}_N$is a beta distribution, $\beta(W_N, B_N)$. For future use we remark that that beta distribution satisfies the estimate

$$\mathbb{P}(|\beta(xn,(1-x)n) - x| > \delta) \quad \leq \quad c_1 e^{-c_2 n\delta} \tag{5.3}$$

uniformly for x in a compact subinterval of $(0,1)$. Since the beta distribution ha no atom at 1, we see that $\lim_{k\to\infty} X_k$ is strictly less than 1 on the event $\{\tau_N = \infty\}$. An entirely analogous argument with τ_N repalces by $\sigma_N := \inf\{k \geq N | X_k \geq p\}$ shows that $\lim_{k\to\infty}$ is strictly greater than 0 on the event $\{\sigma_N = \infty\}$. Taking the union over N shows that $\lim_{k\to\infty} X_k$ exists on the event $\{(X_k - p)(X_{k+1} - p) < 0$ finitely often $\}$ and is strictly between zero and one. The proof of the lemma will therefore be done once we show that $X_k \to p$ on the event that $X_k - p$ changes sign infinitely often.

Let $G(N,\epsilon)$ denote the event that $X_{N-1} < p < X_N$ and there exists $k \in [N, \tau_N]$ such that $X_k > p + \epsilon$. Let $H(N,\epsilon)$ denote the event that $X_{N-1} > p > X_N$ and there exists $k \in [N, \sigma_N]$ such that $X_k < p - \epsilon$. It suffices to show that for ever $\epsilon > 0$, the sums $\sum_{N=1}^\infty \mathbb{P}(G(N,\epsilon))$ and $\sum_{N=1}^\infty \mathbb{P}(H(N,\epsilon))$ are finite; for then by Borel-Cantelli these occur finitely often; implying $p-\epsilon \le \liminf X_k \le \limsup X_k \le p+\epsilon$ on the event that $X_k - p$ changes sign infinity often; since ϵ is arbitary this suffices. Recall the Polya urn coupled to $\{X_k | N \le k \le \tau_N\}$. On the event $G(N,\epsilon)$ either $X'_\infty \ge p+\epsilon/2$ or $X'_\infty - X_\rho \le -\epsilon/2$ where $\rho \ge k$ is the least $m \ge N$ such that $X'_m \ge p+\epsilon$. The conditional distribution of $X'_\infty - X_\rho$ given $\mathcal{F}_\rho$ is $\beta(W'_\rho, B'_\rho)$. Hence

$$\mathbb{P}(G(N,\epsilon) \le \tag{5.4}$$

$$\mathbb{E}1_{X_{N-1}<p<X_N}\mathbb{P}(\beta(W_N,B_N) \ge p+\frac{\epsilon}{2}) + \mathbb{E}1_{\rho<\infty}\mathbb{P}(\beta(W'_\rho, B'_\rho) \le p-\epsilon/2) \tag{5.5}$$

Combining this with the estimate 5.3 establishes summability of $\mathbb{P}(G(N,\epsilon))$. An entirely analogous argument establishes the smmability of $\mathbb{P}(H(N,\epsilon))$, finishing the proof of the lemma. $\square$

We now turn our attention to the proof of proposition 5.3

Proof. We consider $B_k := (x_{k,k+1}, x_{k,k+1'}, x_{k',k+1}, x_{k',k+1'})$. That is the balls and urns that Player k controls. We color balls of type $k+1$ white and balls of type $k+1'$ black. That is we consider the process s_k as a function of t. It turns out that $s_k(t)$ satisfies 5.2 with $p = 1/2$, provided we rescale time by ignoring the times when we fail to reinforce. Assuming this for the moment, we obtain that $\lim_{t\to\infty} s_k$ exists and is niether 0 nor 1. It follows trivially that $\lim_{t\to\infty} s_{k'}$ exists and is neither 0 nor 1.

We now need only verify that the process described above does indeed satisfy (5.2) Now, substituting in the results of Lemma 3.3 and Lemma 3.4 we see that for our time-rescaled process.

$$\mathbb{P}(s_k(n+1) > s_k(n)|\mathcal{F}_t) = P_k/P = s'_k + \frac{s_k - s_{k'}}{2P}$$

This gives us.

$$\begin{aligned}(\mathbb{P}(s_k(n+1) > s_k(n)|\mathcal{F}_n) - s_k(n)) &= \\ (s'_k + \frac{s_k - s_{k'}}{2P} - X_n) &= \\ (s'_k + \frac{s_k - s_{k'}}{2P} - s_k)&\end{aligned}$$

Which, is positive at the same times that $s_k - 1/2$ is. So the hypothesis of lemma 5.5 are fulfilled and we are done. □

We now turn our attention to a discussion of proposition 5.4 and its partial proofs. These will depend upon the unproven technical condition

$$\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}] \quad \geq \quad n^{-1}\frac{-\overline{Q}}{2D} \tag{5.6}$$

being eventually true when $\overline{Q} < 0$.

We will split this into two parts namely 5.6 and 5.7. In the first we will show that assuming the technical condition 5.6 that when $\overline{Q}_n < 0$ that there is almost surely some $m > n$ such that $\overline{Q}_m > 0$. In the second part we will show that if $\overline{Q}_n > 0$ for some sufficently large n then with probability at least $a > 0$ we converge to a language.

Proposition 5.6. *(gets to $\overline{Q}=0$) If $\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}]/(\frac{-\overline{Q}}{D})$ is eventually greater than kn^{-1} for some $k>1/2$ when $\overline{Q}<0$ then with probability 1 $\overline{Q}_m>0$ for some m.*

Proposition 5.7. *(escapes $\overline{Q}=0$) There exists a constant $c>0$ such that if $\overline{Q}>0$ then $\mathbb{P}(\overline{Q}\to 1/4^L)>c$.*

We now turn our attention to the proof of proposition 5.6

Proof. We assume that there exists some k and N such that $n\geq N\Rightarrow n\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}]\geq \frac{-k\overline{Q}}{D}$ for $k>1/2$ or $\frac{\overline{Q}}{D}>0$. Intuitively this assumption means that if X_n is behind the barrier then it is eventually approaching the barrier quickly relative to it's distance from the barrier.

We will show that X_n crosses the barrier assuming this condition, more formally that $\overline{Q}$ eventually becomes positive (at least temporarily) and combining this with proposition 5.7 we obtain proposition 5.4. To begin we define for $m\geq N$ a process $W_m:=\frac{\overline{Q}_m}{D_m}-\sum_{n=N}^m\eta_n$ where η_n is choosen so that $\mathbb{E}_n[\Delta W_n]=kn^{-1}$. By assumption $\eta_n>0$. Write $W_{n+1}:=W_n+A_n+Y_n$ where $A_n:=\mathbb{E}_n[W_{n+1}-W_n]$ is $\mathcal{F}_n$-measurable and $Y_n:=W_{n+1}-W_n-A_n$, which is to say that $Y_n:=\Delta(\overline{Q}/D)_n-\mathbb{E}_n[(\overline{Q}/D)_n]$ Hence $\mathbb{E}_n[Y_n]=0$ and in a neighbourhood $\mathcal{N}$ of the set $\{\overline{Q}=0\}$, we have the bound, $\mathrm{E}_n[Y_n^2]>bn^{-1}$ for some b depending on $\mathcal{N}$. In fact for a specific b we may use $\mathcal{N}$ as the region on which $\mathbb{E}_n[Y_n^2]>bn^{-1}$

Define $Z_{n,m}:=\sum_{i=n}^{m-1}Y_i$ yields for each fixed n a martingale $\{Z_{n,m},\mathcal{F}_m\}$. We note the L^2-bound $\mathbb{E}[Z_{n,\infty}^2]\leq\sum_{i=n}^{\infty}\frac{1}{i+1}\leq 1/n$. Further when X_n is in the region $\mathcal{N}$ and τ is any stopping time bounded by the exit time of $\mathcal{N}$ we have,

$\mathrm{E}_n[Z_{n,\infty}^2] \geq \mathbb{E}_n[Z_{n,\tau}^2] \geq \mathbb{P}_n(\tau = \infty)b/(n+1)$

Lemma 5.8. *There are constants a,c and a neighbourhood of* $\overline{Q} = 0$, $\mathcal{N}$ *such that* $\mathbb{P}_n(Z_{n,\infty} > cn^{-1/2} > a$ *or* $\frac{\overline{Q}}{D}_{n+j} \notin \mathcal{N} | \mathcal{F}_n) > a$.

Proof. For $k > 0$, let $\tau \leq \infty$ be the first time W_j exits $\mathcal{N}$ or $Z_{n,j}$ exits $(-kn^{-1/2}, kn^{-1/2})$. Then we have $\mathbb{P}_n(\tau = \infty)b/(n+1) \leq \mathbb{E}_n[Z_{n,\tau}^2]\mathbb{E}[(W_\tau - W_n)^2] \leq k/n$. Which gives immediately $\mathbb{P}_n(\tau = \infty|\mathcal{F}_n) \leq k^2(n+1)/b^2 n$ and choosing k small enough makes this at most $1/3$. Let $q := \mathbb{P}_n(\tau < \infty, X_\tau \notin \mathcal{N})$ so that the conditional probability of $Z_{n,j}$ exiting $(-kn^{1/2}, kn^{1/2})$ given $\mathcal{F}_n$ is at least $2/3 - q$. Any martingale $\mathcal{M}$ started at zero that exits an interval $(-L, L)$ with probability at least r and has increments bounded by $L/2$ satisfies $\mathbb{P}(\sup \mathcal{M} \geq L/2) \geq (3r-1)/4$;stopping $\mathcal{M}$ upon exiting $(-L, -L/2)$ and lettign $s = \mathbb{P}(\sup \mathcal{M} > L/2)$ gives $0 = E[\mathcal{M}] \leq sL + (r-s)(-L) + (1-r)(L/2)) = 2L(s - (3r-1)/4))$. Thus $Z_{n,j} \geq k/2\sqrt{n}$ for some j with probability at least $(1-3q)/4$. Now for any j, condition on the event $Z_{n,\infty} < k/4\sqrt{n}$ can be bounded away from 1 using the following 1-sided Tschebysheff estimate 5.9:

Lemma 5.9. *If* $\mathcal{M}$ *is a mean zero random varible and* $L < 0$, *then* $P(\mathcal{M} < L) \leq \mathbb{E}[\mathcal{M}^2]/(\mathbb{E}[\mathcal{M}^2] + L^2)$.

Proof. Write ω for prob$(\mathcal{M} \leq L)$. From

$$0 = \mathbb{E}[M] = \omega\mathbb{E}[\mathcal{M}|\mathcal{M} \leq L] + (1-\omega)\mathbb{E}[\mathcal{M}|\mathcal{M} \geq L]]$$

and $\mathbb{E}[\mathcal{M}|\mathcal{M} \leq L] \leq L$, it is immediate that

$$\mathbb{E}[\mathcal{M}|M > L] \geq -L\frac{\omega}{1-\omega}$$

Then

$$\begin{aligned}
\mathbb{E}[\mathcal{M}^2] &= \omega\mathbb{E}[\mathcal{M}^2|\mathcal{M} \leq L] + (1-\omega)\mathbb{E}[\mathcal{M}^2|\mathcal{M} > L] \\
&\geq \omega L^2 + (1-\omega)(\mathbb{E}[\mathcal{M}|\mathcal{M} > L])^2 \\
&\geq \omega L^2 + (1-\omega)L^2(\omega^2/(1-\omega)^2)
\end{aligned}$$

from which the desired conclusion follows.

□

We now continute with the proof of 5.8, by applying 5.9 to the process $Z_{n,i}$ stopped at the entrance time σ of the interval $(-\infty, -k/4\sqrt{n})$ to get

$$\begin{aligned}
\mathbb{P}_n(Z_{n,\infty} \leq k/4\sqrt{n}) &\leq \mathbb{P}_n(Z_{n,\tau}k/4\sqrt{n}) \\
&\leq \mathbb{E}[Z_{n,\tau}^2]/(\mathbb{E}[Z_{n,\tau}^2] + k^2/16n) \\
&\leq \mathbb{E}[Z_{n,\infty}^2]/(\mathbb{E}[Z_{n,\infty}^2] + k^2/16n) \\
&16/(k^2+16)
\end{aligned}$$

Combining this with the previous result shows that $\mathbb{P}_n(Z_{n,\infty} > k/4\sqrt{n}) \geq \frac{(1-3q)k^2}{64+4k^2}$, recall that q is the conditional probability of the process W_n exiting $\mathcal{N}$ given $\mathcal{F}_n$, so that the probability we're trying to bound is at least $k^2/(64+7k^2)$ thus the lemma is proved with $c = k/4$ and $a = k^2/(64+7k^2)$ $\square$

To complete the proof of 5.6 it remains to show that the probability is zero that W_n eventually resides in the interval $(-\epsilon, 0)$ If the probability where non-zero then for any δ there would be an event β in some $\mathcal{F}_M$ for which $\mathbb{P}(X_{M+j} \in (-\epsilon, 0) \forall j \geq 0) > 1 - \delta$. In fact, conditioning on W_M, β may be taken to determine X_M. For what follows condition on $\mathcal{F}_M$ and on $W_M \in (-\epsilon, 0)$. Also choose M large enough that for any $n > M$, $n^{-k/2k_1} < cn^{-1/2}$ where c is choosen as in Lemma 5.8, and choose ϵ small enough that $(-\epsilon, 0)\mathcal{N}$ to which Lemma 5.8 applies.

Begin by setting up constants and stopping times. Pick $1/2 < k_1 < k < 3/4$. For $n \geq M$ define

$V_n = (k/k_1)ln(n) + 2ln(-W_n)$ for $W_n < 0$ and $-\infty$ otherwise.

By assumption on W_n, $V_n > -\infty$. Let τ be the least $n \geq M$ such that $W_n \notin (-\epsilon, 0)$ or $V_n < 0$.Observe that if $V_n > 0$ then $1/n < (-W_n)^{2k_1/k} \leq (-W_n)^{4/3}$, so $|W_{n+1} - W_n|$ is small compared to $-W_n$, so $V_{\tau \wedge n}$ can never reach ∞ and is in fact bounded below by $\min(-1, V_M)$. Now for $n < \tau$ calculate

$$\mathbb{E}_n[-W_n] \leq ln\mathbb{E}_n[-W_n]$$

$$= \ln(-W_n - A_n)$$

$$\leq \ln(-W_n)(1 - k/(n+1))$$

$$\ln(-W_n + ln(1 - k/(n+1));$$

so

$$\mathbb{E}_n[V_{n+1}] \leq V_n + (k/k_1)(ln(n+1) - ln(n)) + 2ln(1 - k/(n+1))$$

$$= V_n + (k/k_1)(n^{-1} + O(n^{-2})) - 2k(n^{-1} + O(n^{-2})$$

$$V_n - (2 - 1/k_1)k + O(1)n^{-1}$$

$$< V_n - Cn^{-1}$$

for large n and some $C > 0$. So $V_{n \wedge \tau}$ is a supermartingale for large n bounded below by $\min(1, V_M)$, and hence converges almost surely. Clearly it cannot converge without stopping, since the increments of the expectation sum to $-\infty$, therefore the stopping time is reached almost surely.

In other words conditional upon any event in $\mathcal{F}_M$, the probability is 1 that for some $n > M$, either W_n will leave $(-\epsilon, 0)$ or $(k/k_1)\ln(n) < -2ln(-W_n)$. Let $\sigma \leq \infty$ be the least $n > M$ for which $(k/k_1)\ln(n) < -2\ln(-W_n)$. We have just shown that the conditional probability of some W_n leaving $(-\epsilon, 0)$ is one. On the other hand, the conditional probability of of some W_{n+j} leaving $(-\epsilon, 0)$ given $\sigma = n$ is at least a by lemma 5.8 since $W_{n+j} \notin \mathcal{N}$ trivially implies $W_{n+j} \notin (-\epsilon, 0)$, while $Z_{n,\infty} > cN^{1/2}$

implies $Z_{n,n+j} > cn^{1/2}?n^{-k/2k_1} > -W_n$ for some j which implies $W_{n+j} > 0$ and hence $\overline{Q}(X_{n+j}) > 0$

□

We now turn our attention to the proof of proposition 5.7

Proof. The idea of this proof first appeared in [Pem88], it has also been discussed in [Pem90] , [Ben99] and [APSV08]. However there are slightly different hypothesis there,in particular the vector field there points away from the still-set there and here points away on one side only.

For any process $\{Y_n\}$ we define $\Delta Y_n := Y_{n_1} - Y_n$. We let $\mathcal{N} \subset \mathbb{R}^d$ be any closed set, let $\{X_n : n \geq 0\}$ be a process adapted to a fitration $\{\mathcal{F}_n\}$ and let $\sigma := inf\{k : X_k \nsubseteq \mathcal{N}\}$ be the time the process takes to exit $\mathcal{N}$. Let $\mathbb{P}_n$ and $\mathbb{E}_n$ denote conditional probability and expectation with respect to $\mathcal{F}_n$

We will impose several hypothesis, (5.7), (5.8), (5.9), on a process $\{X_n\}$ and associated functions $Q_{k,k+1}$then check that our process $\{X_n\}$ defined in (3.1) and the $Q_{k,k+1}$ we've been working with satisfy these conditions on appropriate neighbourhoods. We require

$$\mathbb{E}_n|X_n|^2 \leq c_1 n^{-2} \tag{5.7}$$

for some $c_1 > 0$, which also implies $\mathbb{E}_n|X_n| \leq \sqrt{c_1} n^{-1}$. Let $Q_{k,k+1}$ be a twice differentiable functions on a neighbourhood $\mathcal{N}'$ of $\mathcal{N}$. We require

$$sgn(Q_{k,k+1})[\nabla Q_{k,k+1}(X_n) \cdot \mathbb{E}_n \Delta X_n] \geq -c_2 n^{-2} \tag{5.8}$$

for $k \in [0,1,2,...,L-1]$ whenever $X_n \in \mathcal{N}'$. Let c_3 be an upper bound for the matrix of second partial derivatives of Q on $\mathcal{N}'$

$$\mathbb{E}_n(\Delta Q_{k,k+1}(X_n))^2 \geq c_4 n^{-2} \tag{5.9}$$

for each $k \in [0,1,2,..,L-1]$ when $n < \sigma$. The relation between these assumptions and our process $\{X_n\}$ defined in (3.1) is as follows.

Lemma 5.10. *For our process X_n and the function $\mathcal{L} := Q_{k,k+1}$ (5.7) and (5.9) are true on all of Δ_3^L while 5.8 is true when either $\overline{Q} > -c_5 n^{-1}$ for some c_5 depending on c_2*

Proof. (5.7) holds because $|X_n|$ is bounded above by $\sqrt{L}n^{-1}$. To see (5.9) observe that $|\nabla Q_{k,k+1}| \geq \epsilon > 0$ on any closed set disjoint from dS and also that $P_{k,k+1}$, $P_{k,k+1'}$,$P_{k',k+1}$ and $P_{k',k+1'}$ are bounded from below and the lower bound on the second moment of $\Delta Q_{k,k+1}$,tht is (5.9), follows. Lastly we see that (5.8) holds when $\overline{Q} > -c_5 n^{-2}$ for some c_5 depending on $\mathcal{N}$ and c_2 by recalling that $\nabla Q_{k,k+1} \cdot F = \frac{\overline{Q}_{k,k+1}}{2D}[M_{k,k+1} - 4Q_{k,k+1}^2]$. □

We now continue with the proof of 5.7. Define $Q'(x) := Q_{k,k+1}$ such that $|Q_{k,k+1}|$ is minimized and $\tau := \inf m \geq N : |Q'(X_m)| > \epsilon m^{-1/2}$

To begin in earnest we set $\epsilon = \sqrt{\frac{c_4}{2}}$ and fix $\mathrm{N}_0 \geq \frac{16(c_2+c_1c_3)}{c_4^2}$.

Let $\tau := inf\{m \geq N_0 : |\mathcal{L}(X_m)| \geq \epsilon m^{-1/2}\}$. Suppose that $N_0 \leq n \leq \sigma \wedge \tau$. From the Taylor estimate $|\mathcal{L}(x+y) - \mathcal{L}(x)\nabla\mathcal{L}(x) \cdot y| \leq C|y|^2$ where C is an upper bound on the Hessian Determinant for $\mathcal{L}$ on the ball of radius $|y|$ about x, we see that

$$
\begin{aligned}
\mathbb{E}_n\Delta\mathcal{L}(X_n)^2) =& \mathbb{E}_n 2\mathcal{L}\Delta(\mathcal{L}(X_n)^2) + \mathbb{E}_n\Delta(\mathcal{L}(X_n))^2 \\
\geq& 2\mathcal{L}\nabla\mathcal{L}\cdot\mathbb{E}_n\Delta X_n - 2c_3\mathcal{L}\mathbb{E}_n|\Delta X_n|^2 + \mathbb{E}_n|\mathcal{L}(X_n)|^2 \\
\geq& \mathcal{L}(X_n)(c_2 + c_1c_3) + c_4]n^{-2}
\end{aligned}
$$

The proof is now completed by establishing the two lemmas; lemma 5.11 and lemma 5.12

Lemma 5.11. *(Leaves neighbourhood infinitely often) If ϵ is taken to equal $c_4/2$ in the definition of τ , then $\mathbb{P}_n(\tau\wedge\sigma\leq\infty)\geq 1/2$.*

Proof.

For any $m\geq n$ it is clear that $|Q'(X_{n\wedge\sigma\wedge\tau})|\leq\epsilon n^{-1/2}$. Thus

$$
\begin{aligned}
\epsilon n^{-1} \geq& \mathbb{E}_n|Q'^2(X_{n\wedge\sigma\wedge\tau})| \\
\geq& \mathbb{E}_n|Q'^2(X_{n\wedge\sigma\wedge\tau})| - \mathbb{E}_n|Q'^2(X_n)| \\
=& \sum_{k=n}^{m-1}\mathbb{E}_n[\Delta Q'(X_k)^2]1_{\sigma\wedge\tau<k} \\
\geq& \sum_{k=n}^{m-1} c_4k^{-2}\mathbb{P}_t\sigma\wedge\tau > k) \\
\geq& \frac{c_4}{2}(n^{-1} - m^{-1}\mathbb{P}_t(\sigma\wedge\tau=\infty)
\end{aligned}
$$

Letting $m\to\infty$ we conclude that $\epsilon<\frac{c_4}{2}$ implies $\mathbb{P}(\tau\wedge\sigma=\infty)\leq\frac{1}{2}$

□

Lemma 5.12. *There is an N_0 and a $c_6 > 0$ such that for all $n \geq N_0$, $\mathbb{P}_t(\sigma < \infty$ or $\forall k$ and $\forall m \geq n$, $|Q_{k,k+1}| \geq \frac{c_4}{2})n^{-1/2} \geq c_6$ whenever $|Q_{k,k+1}| \geq (c_4/2)n^{-1/2}$, $\forall k$*

Proof. Let $\widetilde{Q} = \phi(Q'(x)) = Q'(x) + Q'^2(x)$. First we establish that there is a $\lambda > 0$ such that $\widetilde{Q}_{k,k+1}(x)$ is a submartingale when $\overline{Q} \geq 0$ and $n \geq N_0$.

$$\begin{aligned} \mathbb{E}_n \Delta \widetilde{Q}(X_n) = \\ \mathbb{E}_n \Delta Q'(X_n) + \lambda \mathbb{E}_n (Q'(X_n)^2) = \\ \geq \nabla Q'(X_n) \cdot \mathbb{E}_n \Delta X_n - c_3 \mathbb{E}_n |\Delta X_n|^2 + \lambda \frac{c_4}{2} n^{-2} \end{aligned}$$

Next let $M_n + A_n$, denote the Doob decomposition of $\{\widetilde{Q}(X_n)\}$; in other words, M_n is a martingale and A_n is predictable and increasing. An upper bound on $|\widetilde{Q}_{k,k+1}(X_n))|$ is $c_8 := 1 + 2\lambda$. From the definition of $Q_{k,k+1}$, we see that $|Q_{k,k+1}| \leq 1$. It follows from these two facts that

$$\frac{\widetilde{Q}(x+y) - \widetilde{Q}(x)}{|y|} \leq 1 + 2\lambda$$

It is now easy to estimate

$$\begin{aligned} \mathbb{E}_n(\Delta M_n)^2 \leq & \mathbb{E}_n(\Delta \widetilde{Q})^2 \\ \leq & (sup \frac{|\widetilde{Q}(x+y) - \widetilde{Q}(x)|}{|y|}) \mathbb{E}_n |\Delta X_n|^2 \\ \leq & c_1 c_7 n^{-2} sup \frac{d\widetilde{Q}}{dQ} \end{aligned}$$

We conclude that there is a constant $c_6 > 0$ such that $\mathbb{E}_n(\Delta M_n)^2 \leq c_6 n^{-2}$ and consequently $\mathbb{E}_n(M_{n+m} - M_n)^2 \leq c_6 n^{-1}$ for all $m \geq 0$ on the event $\overline{Q}(X_t) \geq 0\}$.

For any $a, nV > 0$ and any martingale M_k satisfying $M_n \geq a$ and $sup_m \mathbb{E}_n(M_{n+m} - M_n)^2 \leq V$, there holds an inequality

$$\mathbb{P}(\inf_m M_{m+n} \leq \frac{a}{2}) \leq \frac{4V}{4V + a^2}$$

To see this, let $\tau = \inf\{k \geq n : M_k \leq a/2\}$ and let $p := \mathbb{P}_n(\tau \leq \infty)$. Then

$$V \geq p(\frac{a}{2})^2 + (1-p)\mathbb{E}_n(M_\infty - M_n | \tau = \infty)^2 \geq p(\frac{a}{2})^2 + (1-p)(\frac{p(a/2)}{1-p})^2$$

which is equivalent to $p \leq 4V/(4V + a^2)$.

It follows that

$$\mathbb{P}_n(inf_{k \geq n} M_k \leq \frac{c_4}{4} n^{-1/2}) \leq c_5 := \frac{4c_6}{4c_6 + (1/4)c_4^2}$$

But $M_k \leq \widetilde{Q}(X_k)$ for $k \geq n$, so $Q(X_k) \leq (c_4/5)n^{-1/2}$. Thus the conclusion of the lemma is established. $\square$

Chapter 6

Numerical Evidence

We wish to establish some numerical evidence for the unproven hypothesis 5.6. We begin by computing $\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}]$. We begin with:

$$\begin{aligned}
\mathbb{E}_n[\Delta Q_{k,k+1}] &= \frac{1}{n}(x_{k',k+1'}F_{k,k+1} - x_{k',k+1}F_{k,k+1'} - x_{k,k+1'}F_{k',k+1} + x_{k,k+1}F_{k',k+1'}) + O(n^{-2}) \\
&= \frac{1}{n}(\frac{\overline{Q}_{k,k+1}}{2D}x_{k',k+1'}R_{k,k+1} + x_{k',k+1}R_{k',k+1} + x_{k,k+1'}R_{k,k+1'} + x_{k,k+1}R_{k',k+1'}) + O(n^{-2}) \\
&= \frac{1}{n}(\frac{\overline{Q}_{k,k+1}}{2D}[M_{k,k+1} - 4Q^2_{k,k+1}]) + O(n^{-2})
\end{aligned}$$

Where $M_{k,k+1} := x_{k,k+1}x_{k',k+1'}(x_{k,k+1} + x_{k',k+1'}) + x_{k,k+1'}x_{k',k+1}(x_{k,k+1'} + x_{k',k+1})$

From this we calculate

$$\begin{aligned}
\mathbb{E}_n[\Delta\overline{Q}] &= \sum_{k=1}^{L-1}\overline{Q}_{k,k+1}\mathbb{E}_n[\Delta Q_{k,k+1}] + O(n^{-2}) \\
&= \frac{1}{n}\sum_{k=0}^{L-1}\frac{\overline{Q}^2_{k,k+1}}{2D}[M_{k,k+1} - 4Q^2_{k,k+1}] + O(n^{-2}).
\end{aligned}$$

The next step is to compute $\mathbb{E}_n[\Delta s_k]$

$$
\begin{aligned}
\mathbb{E}_n[\Delta s_k] &= \mathbb{E}_n[\Delta x_{k,k+1} + x_{k,k+1'}] \\
&= \mathbb{E}_n[\Delta x_{k,k+1}] + \mathbb{E}_n[\Delta x_{k,k+1'}] \\
&= \frac{1}{n}\frac{1-2s_k}{2}\frac{\overline{Q}}{D} + O(n^{-2})
\end{aligned}
$$

We can use this to compute

$$
\begin{aligned}
\mathbb{E}_n[\Delta\frac{1}{s_k(1-s_k)}] &= -\frac{1-2s_k}{s_k^2(1-s_k)^2}\mathbb{E}[\Delta s_k] + O(n^{-2}) \\
&= -\frac{1}{n}\left(\frac{(1-2s_k)}{s_k(1-s_k)}\right)^2\frac{\overline{Q}}{2D} + O(n^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_n[\Delta\frac{1}{D}] &= \sum_{k=0}^{L-1}\frac{1}{D_k}\mathbb{E}_n[\Delta\frac{1}{s_k s_{k'}}] + O(n^{-2}) \\
&= \sum_{k=0}^{L-1}\frac{1}{D_k}\left(-\frac{1}{n}\left(\frac{(1-2s_k)}{s_k(1-s_k)}\right)^2\frac{\overline{Q}}{2D}\right) + O(n^{-2}) \\
&= n^{-1}\left(\sum_{k=0}^{L-1} -\frac{(1-2s_k)^2}{2s_k(1-s_k)}\frac{\overline{Q}}{D^2}\right) + O(n^{-2})
\end{aligned}
$$

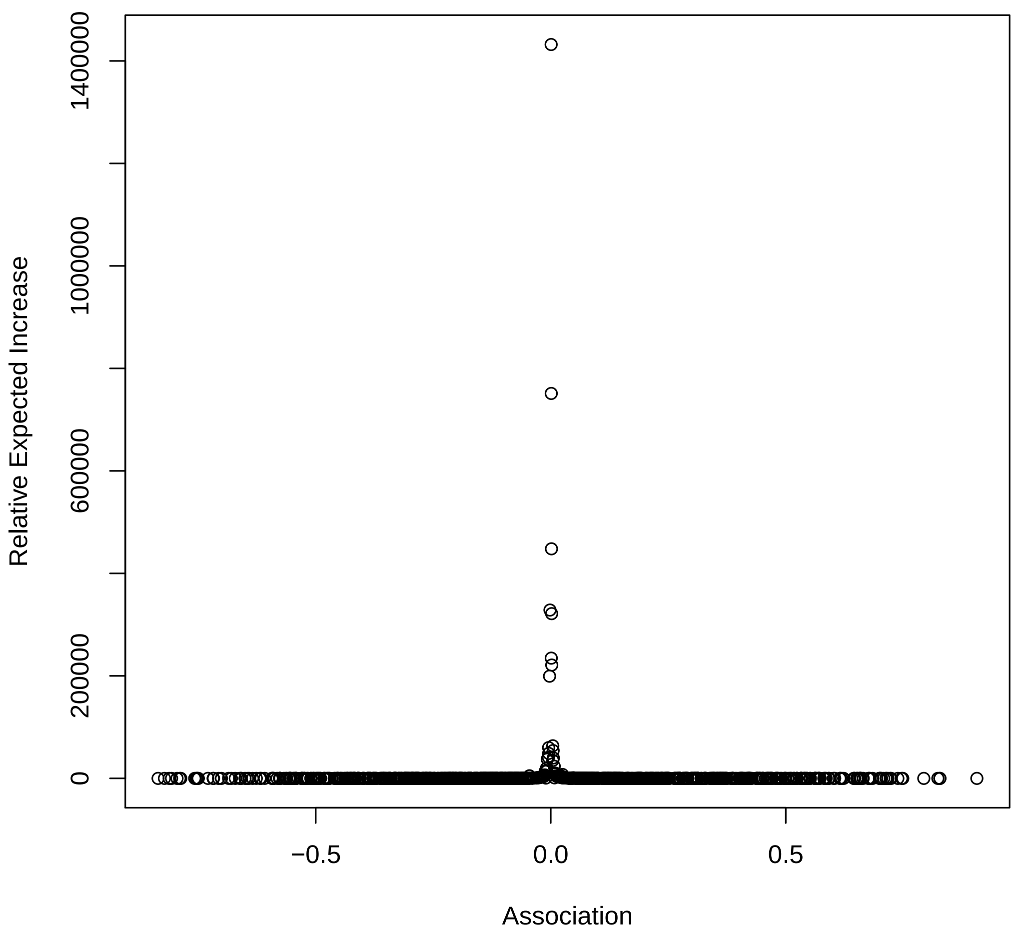
Combining these we may finally compute

$$\begin{aligned}
\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}] &= \mathbb{E}[\Delta\overline{Q}]\frac{1}{D} + \overline{Q}\mathbb{E}[\Delta\frac{1}{D}] + O(n^{-2}) \\
&= \frac{1}{n}\left(\frac{\overline{Q}}{D}\right)^2 \left(\sum_{k=0}^{L-1} \frac{M_{k,k+1}}{2Q_{k,k+1}^2} - \frac{(1-2s_k)^2}{2s_k(1-s_k)} - 2\right) + O(n^{-2}) \\
&= \frac{1}{n}\left(\frac{\overline{Q}}{D}\right)^2 \left(\sum_{k=0}^{L-1} \frac{M_{k,k+1}}{2Q_{k,k+1}^2} - \frac{1}{2s_k(1-s_k)}\right) + O(n^{-2}) \\
&= \frac{1}{n}\frac{1}{2}\left(\frac{\overline{Q}}{D}\right)^2 \left(\sum_{k=0}^{L-1} \left(\frac{s_k(1-s_k)}{Q_{k,k+1}}\right)^2 \left(\frac{x_{k,k+1}x_{k,k+1'}}{s_k^3} + \frac{x_{k',k+1}x_{k',k+1'}}{s_{k'}^3}\right)\right)
\end{aligned}$$

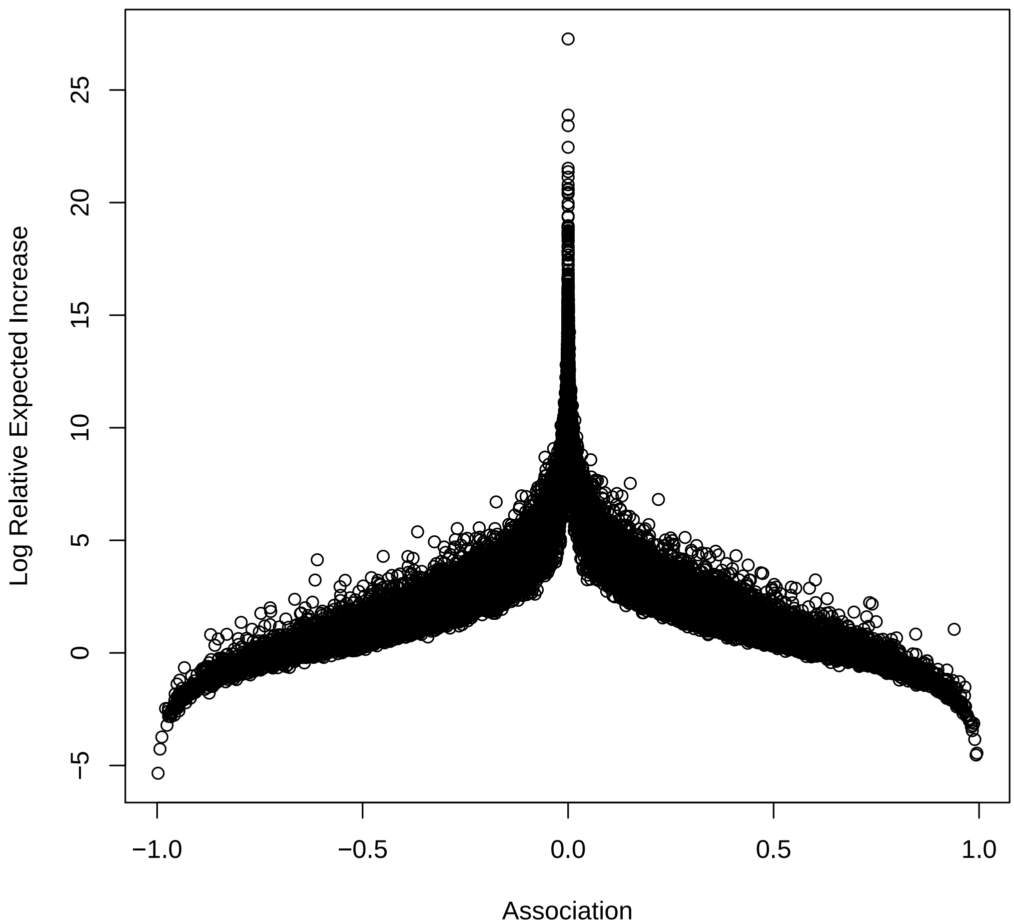
It is now clear that $\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}] > 0$. In order to be guareenteed to push through the barrier we would like that when $\frac{\overline{Q}}{D} < 0$ then we eventually have $n\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}] > \frac{-\overline{Q}}{2D}$. That is we would have condition 5.6 and would be able to prove the main conjecture. It follows from the above computation of $\mathbb{E}_n[\Delta\overline{Q}/D]$ that condition 5.6 is equivalent to requiring that eventually $\frac{-\overline{Q}}{D}\left(\frac{M_{k,k+1}}{2Q_{k,k+1}^2} - \frac{1}{2s_k s_{k'}}\right) > 1$.

Consider the set of values $G := \{\frac{|Q_{k,k+1}|}{s_k s)k'}|k \in \{0, 1, ..., L-1\}\}$, intuitively $\mathbb{E}_n[\Delta\frac{\overline{Q}}{D}]$ will be large relative to $\frac{\overline{Q}}{D}$, when some element of G is much smaller than all other other elements of G. The intuition here is that, when each player makes an association between the symbols they recieve and the symbols they send out, in such a way that the they are globally inefficent (i.e. $\overline{Q} < 0$). Then each player starts to realize that his symbol-association doesn't work with the other players associations, and starts to adjust towards the other possible association of pairs, at the same time all other players do. If one player has a particularly weak association then he will change first and everyone else will begin to reaffirm there original association. We postulate that if a player has an even slightly weaker association than his fellows

then his association will become weaken more quickly than that of his fellows. This will eventually make his association small enough to quickly change. We observe:
$\mathbb{E}_n[\Delta \frac{Q_{k,k+1}}{s_k s_{k'}}] = n^{-1}\left(\frac{\overline{Q}}{2D}\right)\left(\frac{Q_{k,k+1}}{s_k s_{k'}}\right)\left(\frac{M_{k,k+1}}{Q^2_{k,k+1}} - \frac{1}{s_k(1-s_k)}\right) + O(n^{-2})$
That is $\mathbb{E}_n[\Delta \frac{Q_{k,k+1}}{s_k s_{k'}}]/\frac{Q_{k,k+1}}{s_k s_{k'}} \propto \frac{M_{k,k+1}}{Q^2_{k,k+1}} - \frac{1}{s_k(1-s_k)}$. Hence if it is true that $\frac{M_{k,k+1}}{Q^2_{k,k+1}} - \frac{1}{s_k(1-s_k)}$ is large when, $\frac{|Q_{k,k+1}|}{s_k s_{k'}}$ is small, then we, might reasonably be hopeful that our technical condition is eventually satisfied with probability 1. We plugged a thousand points into Δ_3 the 3 simplex.



The graph above shows that when association ($\frac{Q_{k,k+1}}{s_k s_{k'}}$) is very small, that relative expected increase $\frac{M_{k,k+1}}{Q^2_{k,k+1}} - \frac{1}{s_k(1-s_k)}$ is very large. This effect is true to the point where we can't observe much away from the origin. To combat this we take the log of relative expected increase. This time we use a hundered thousand points. The graph is shown below.



There is a clear trend for smaller associtions to be associaed with faster relative increases. This is our first piece of numerical evidence for the eventual establishment of the technical condition.

Further in pursuit of showing that the technical condition eventually holds we compute:

$$\mathbb{E}_n[\Delta M_{k,k+1}] = \frac{1}{n}\left[(2x_{k,k+1}x_{k',k+1'} + x^2_{k',k+1'})F_{k,k+1} + (2x_{k,k+1'}x_{k',k+1} + x^2_{k',k+1})F_{k,k+1'}\right.$$
$$\left. +(2x_{k',k+1}x_{k,k+1'} + x^2_{k,k+1'})F_{k',k+1} + (2x_{k',k+1'}x_{k,k+1} + x^2_{k,k+1})F_{k',k+1'}\right] + O(n^{-2}) =$$
$$\frac{1}{n}\frac{\overline{Q}_{k,k+1}}{2D}[O_{k,k+1} - 6Q_{k,k+1}M_{k,k+1}] + O(n^{-2})$$

Where $O_{k,k+1} := x_{k,k+1}x_{k',k+1'}(x^2_{k,k+1}+4x_{k,k+1}x_{k',k+1'}+x^2_{k,k+1})-x_{k,k+1'}x_{k',k+1}(x^2_{k,k+1'}+4x_{k,k+1'}x_{k',k+1} + x^2_{k',k+1})$ From this we compute

$$\begin{aligned}\mathbb{E}_n[\Delta\frac{M_{k,k+1}}{2Q^2_{k,k+1}}] &= \frac{\mathbb{E}_n[\Delta M_{k,k+1}]}{2Q^2_{k,k+1}} + \mathbb{E}_n[\Delta Q^{-2}_{k,k+1}/2]M_{k,k+1} \\ &= \frac{\mathbb{E}_n[\Delta M_{k,k+1}]}{2Q^2_{k,k+1}} - \frac{M_{k,k+1}\mathbb{E}_n[\Delta Q_{k,k+1}]}{Q^3_{k,k+1}} \\ &= \frac{\overline{Q}}{D}\frac{O_{k,k+1}Q_{k,k+1} + 2Q^2_{k,k+1}M_{k,k+1} - 2M^2_{k,k+1}}{4Q^4_{k,k+1}}\end{aligned}$$

Which immediately gives.

$$\mathbb{E}_n[\Delta\frac{M_{k,k+1}}{2Q^2_{k,k+1}} - \frac{1}{2s_ks_{k'}}] =$$
$$n^{-1}\left[\frac{\overline{Q}}{4D}\left(\frac{(O_{k,k+1}Q_{k,k+1} + 2Q^2_{k,k+1}M_{k,k+1} - 2M^2_{k,k+1})}{Q^4_{k,k+1}} + \frac{(1-2s_k)^2}{s^2_ks^2_{k'}}\right)\right] + O(n^{-2})$$

From which it follows that

$$\mathbb{E}_n[\Delta\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q^2_{k,k+1}} - \frac{1}{2s_ks_{k'}}] =$$
$$\sum_{k=0}^{L-1}\mathbb{E}_n[\Delta\frac{M_{k,k+1}}{2Q^2_{k,k+1}} - \frac{1}{2s_ks_{k'}}] =$$
$$\frac{1}{n}\left(\frac{\overline{Q}}{4D}\sum_{k=0}^{L-1}\left(\frac{(O_{k,k+1}Q_{k,k+1} + 2Q^2_{k,k+1}M_{k,k+1} - 2M^2_{k,k+1})}{Q^4_{k,k+1}} + \frac{(1-2s_k)^2}{s^2_ks^2_{k'}}\right)\right) + O(n^{-2})$$

Which now allows us to compute.

$$\mathbb{E}_n[\Delta\left(\frac{-\overline{Q}}{D}\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{1}{2s_ks_{k'}}\right)\right)] =$$

$$\left(\frac{-\overline{Q}}{D}\right)\mathbb{E}_n[\Delta\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{1}{2s_ks_{k'}}\right)]+\sum_{k=0}^{L-1}\left(\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{1}{2s_ks_{k'}}\right)\mathbb{E}_n[\Delta\frac{-\overline{Q}}{D}] =$$

$$\frac{1}{n}\left(\frac{\overline{Q}}{2D}\right)^2\left[(\sum_{k=0}^{L-1}\frac{2M_{k,k+1}^2-2Q_{k,k+1}^2M_{k,k+1}-O_{k,k+1}Q_{k,k+1}}{Q_{k,k+1}^4})-\frac{(1-2s_k)^2}{s_k^2s_{k'}^2}\right]$$

$$-\frac{1}{n}\left(\frac{\overline{Q}}{2D}\right)^2\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{Q_{k,k+1}^2}-\frac{1}{s_ks_{k'}}\right)^2+O(n^{-2})$$

We would like to have that

$$\sum_{k=0}^{L-1}\left(\frac{2M_{k,k+1}^2-2Q_{k,k+1}^2M_{k,k+1}-O_{k,k+1}Q_{k,k+1}}{Q_{k,k+1}^4}-\frac{(1-2s_k)^2}{s_k^2s_{k'}^2}\right)+(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{Q_{k,k+1}}-\frac{1}{s_ks_{k'}})^2$$

is eventually positive. This would suggest that $(\mathbb{E}_n[\Delta\overline{Q}/D])/(-\overline{Q}/D)$ will eventually grow beyond $1/2$, in turn this would imply that $\overline{Q}$, will eventually grow become positive.

It is unfortunately false in general that $\sum_{k=0}^{L-1}\left(\frac{2M_{k,k+1}^2-2Q_{k,k+1}^2M_{k,k+1}-O_{k,k+1}Q_{k,k+1}}{Q_{k,k+1}^4}-\frac{(1-2s_k)^2}{s_k^2s_{k'}^2}\right)+$ $(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{Q_{k,k+1}}-\frac{1}{s_ks_{k'}})^2>0$

We generated eight million random data points and conditioned on $\overline{Q}<0$ and $\frac{\mathbb{E}_n[\Delta\overline{Q}/D]}{\overline{Q}/D}<0.5$, which left 3861854 points.

The lowest ten values of $\mathbb{E}_n[\Delta\frac{-\overline{Q}}{D}\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{(1-2s_k)^2}{2s_k(1-s_k)}-2\right)]$ were -0.008864631, -0.008686591, -0.008309375, -0.007317739, -0.006820120, -0.006609380, -0.006514855, -0.005941558, -0.005706664 and -0.005616857.

This is evidence that $\mathbb{E}_n[\Delta\frac{-\overline{Q}}{D}\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{(1-2s_k)^2}{2s_k(1-s_k)}-2\right)]$ can't go below -0.01 or so.

We'd like it to be eventually positive, so we take the two lowest values and simulate the process. Results are below in table form. The lowest two values of $\mathbb{E}_n[\Delta\frac{-\overline{Q}}{D}\left(\sum_{k=0}^{L-1}\frac{M_{k,k+1}}{2Q_{k,k+1}^2}-\frac{(1-2s_k)^2}{2s_k(1-s_k)}-2\right)]$ are $-0.008864631$and -0.008686591.

We rescale these so that the total number of balls controlled by each player is ten-thousand. Each starting position was run twice for one hundered million iterations and results are given below, in table format.

The tables require some explanation. The first 5 coloums labeled "Association 0" through "Association 4" represent the quantaties $\frac{Q_{0,1}}{s_0s_{0'}}$, $\frac{Q_{1,2}}{s_1s_{1'}}$, $\frac{Q_{2,3}}{s_2s_{2'}}$,$\frac{Q_{3,4}}{s_3s_{3'}}$ and $\frac{Q_{4,0}}{s_4s_{4'}}$. The sixth coloumb, labeled "QD" is the product of these first five coloums $\frac{\overline{Q}}{D}$. The seventh coloum labeled "Ψ_1" is $n\frac{\mathbb{E}_n[\Delta\overline{Q}/D]}{-\overline{Q}/D}$. That is to say:

$$\Psi_1 = \frac{1}{2}\left(\frac{\overline{Q}}{D}\right)^2\left(\sum_{k=0}^{4}\left(\frac{s_k(1-s_k)}{Q_{k,k+1}}\right)^2\left(\frac{x_{k,k+1}x_{k,k+1'}}{s_k^3}+\frac{x_{k',k+1}x_{k',k+1'}}{s_{k'}^3}\right)\right)$$

.

Finally the eigth coloum labeled Ψ_2 represents $n\mathbb{E}_n[\Delta\Psi_1]$, which can from the above computations be seen to be

$$\Psi_2 := \left(\frac{\overline{Q}}{2D}\right)^2\left[(\sum_{k=0}^{4}\frac{2M_{k,k+1}^2-2Q_{k,k+1}^2M_{k,k+1}-O_{k,k+1}Q_{k,k+1}}{Q_{k,k+1}^4})-\frac{(1-2s_k)^2}{s_k^2s_{k'}^2}\right]$$
$$-\left(\frac{\overline{Q}}{2D}\right)^2\left(\sum_{k=0}^{4}\frac{M_{k,k+1}}{Q_{k,k+1}^2}-\frac{1}{s_ks_{k'}}\right)^2 \qquad [$$

We now look at the generated data:

Table 6.1: First Run, First Start Position

Time	Association 0	Association 1	Assoction 2	Association 3	Association 4	QD	Ψ_1	Ψ_2
0	0.307002311	-0.627261850	-0.292135736	-0.414475738	0.403155103	-0.009400403	0.146153306	-0.008864631
10^3	0.30319569	-0.62551187	-0.29256314	-0.41701594	0.40016523	-0.00925913	0.14526124	-0.00875835
10^4	0.296356156	-0.622129734	-0.278603887	-0.407715910	0.395595416	-0.008284971	0.138799383	-0.007901130
10^5	0.27083046	-0.61568680	-0.23790957	-0.38853954	0.36816608	-0.00567477	0.11940264	-0.00548992
10^6	0.234982186	-0.613310712	-0.197630297	-0.365192950	0.344175992	-0.003579909	0.101026652	-0.003502302
10^7	0.205338382	-0.606399970	-0.162370473	-0.346879606	0.327038835	-0.002293583	0.087337766	-0.002163870
10^8	0.183700053	-0.601403550	-0.133523531	-0.335938675	0.316047571	-0.001566194	0.079502276	-0.001282927

Table 6.2: Second Run, First Start Position

Time	Association 0	Association 1	Assoction 2	Association 3	Association 4	QD	Ψ_1	Ψ_2
0	0.307002311	-0.627261850	-0.292135736	-0.414475738	0.403155103	-0.009400403	0.146153306	-0.008864631
10^3	0.308953871	-0.630748997	-0.293272408	-0.412601797	0.401685846	-0.009471943	0.146542209	-0.008956542
10^4	0.301619559	-0.624177120	-0.283681611	-0.405304759	0.392522875	-0.008496601	0.139302933	-0.008082359
10^5	0.274503706	-0.611500394	-0.254885810	-0.379981053	0.370073599	-0.006016455	0.119522299	-0.005851083
10^6	0.231090310	-0.602133100	-0.218681505	-0.362268893	0.354543937	-0.003908296	0.101894232	-0.003897594
10^7	0.196600814	-0.594074282	-0.183185089	-0.345292869	0.337365406	-0.002492323	0.087048484	-0.002515115
10^8	0.170847228	-0.588516584	-0.156275634	-0.334093587	0.325809610	-0.001710369	0.077549568	-0.001727108

Table 6.3: First Run, Second Start Position

Time	Association 0	Association 1	Assoction 2	Association 3	Association 4	QD	Ψ_1	Ψ_2
0	-0.529087808	-0.365582062	0.497942997	0.415501464	-0.292231270	-0.011694765	0.147333923	-0.008686591
10^3	-0.525628218	-0.363191563	0.496117131	0.417435102	-0.288315118	-0.011398692	0.146332741	-0.008467282
10^4	-0.516513250	-0.359463277	0.492035126	0.415951288	-0.273914946	-0.010408552	0.142164862	-0.007680255
10^5	-0.492212535	-0.331669805	0.482450847	0.392123271	-0.227525842	-0.007026920	0.125634383	-0.004823847
10^6	-0.469117511	-0.295791218	0.469333252	0.373111207	-0.172723918	-0.004197000	0.114019858	-0.001713395
10^7	-0.454273850	-0.271889408	0.454340676	0.358344215	-0.127435283	-0.002562605	0.111974035	0.001450349
10^8	-0.443966053	-0.254538883	0.445961901	0.349322280	-0.084023780	-0.001479211	0.130902828	0.008179117

Table 6.4: Second Run, Second Start Position

Time	Association 0	Association 1	Assoction 2	Association 3	Association 4	QD	QD'	QD"
0	-0.529087808	-0.365582062	0.497942997	0.415501464	-0.292231270	-0.011694765	0.147333923	-0.008686591
10^3	-0.528922689	-0.364145954	0.496153907	0.415230697	-0.289391783	-0.011483116	0.146355321	-0.008518849
10^4	-0.514437109	-0.333756552	0.476234237	0.386808081	-0.238670806	-0.007270255	0.124057576	-0.005295034
10^5	-0.495455203	-0.333756552	0.476234237	0.386808081	-0.238670806	-0.007270255	0.124057576	-0.005295034
10^6	-0.473215302	-0.299778368	0.455030156	0.363101811	-0.189540447	-0.004442522	0.107220155	-0.002718292
10^6	-0.459144452	-0.271963636	0.440853639	0.346871751	-0.143133486	-0.002733158	0.101001476	-0.000331242
10^7	-0.449360393	-0.253136062	0.431215790	0.336576169	-0.103484404	-0.001708448	0.106198720	0.002871559

In all four runs, each association became progressively weaker, as expected. Also as expected the weakest associations fall fastest. In the two runs from the first starting point Ψ_2 remains negetive but became far far less negetive. It seems likely that it will eventually become positive (although it would take a very very long time), once this happens we expect Ψ_1 to grow and eventually exceed one half. For the pair of runs started at the second start position we again notice that the weakest associations fall fastest relative to there current positions. Here Ψ_2 does in fact become positive which leads to an increase in Ψ_1. It looks as though Ψ_1 will eventually grow large, and in particlar grow beyond 1/2, hence satisfying the technical condition and forcing $\overline{Q}$ to be eventualy positive.

All in all this is strong numerical evidence that the process eventually escapes from behind the boundary.

Chapter 7

Related Toy Model

We consider the following urn model. Two urns contain black and white balls, initally each urn contains one ball of each colour. At each discrete time step a ball is added to each urn. We let X_n represent the proportion of white balls in urn one and Y_n, the proportion of white balls in urn two.

The probability of adding a white ball to urn one at time n is:

$$\min(X_n + (0.5 - X_n)^2/Y_n, 1)$$

and the probability of adding a white ball to urn two at time n is:

$$\max(\min(Y_n - (0.5 - X_n)/Y_n, 1), 0)$$

Further the probability of adding a white ball to urn one is independent of the probability of adding a white ball to urn two at any time step. Also the probability of adding a white ball to either urn depends only on the proportion of white balls in the two urns, not on the order in which those balls were added.

As it turns out

$$\mathbb{E}[(X_{n+1},Y_{n+1})-(X_n,Y_n)|\mathcal{F}_n]=\frac{1}{n+1}F(x,y)$$

For the vector field $F(x,y)=(\min(\frac{(0.5-x)^2}{y},1-y),\max(\min(\frac{x-0.5}{y},1-y),-y))$

The geometry on the vector field here closely resembles the vector field discussed in the previous chapters of my thesis in several ways. The most obvious is that in both vector fields we have connected sets where the vector field is 0 ($x=1/2$ and $\overline{Q}=0$), we'll call these "still" sets. In the case of the two-urn model presented here the still set divides the underlying set ($[0,1]\times[0,1]$) into two regions. On one of these regions, the vector field flows towards the still set and in the other it flows away from the still set. In the case of the vector field in the thesis the still set partitions the region into 2^L regions on 2^{L-1} of which the vector field flows towards the still set and on 2^{L-1} of which the vector field flows away.

Call regions where the vector field flows towards the still set "bad", and regions where the vector field flows away from the still set "good". Each good region in both examples has exactly one point where the vector field has a stable equilibrium, to which the process will tend unless by chance it gets pushed back behind the boundary into a bad region.

In both models when in a bad region tending towards the boundary the question of interest is weather or not the flow from the vector field is enough to push it through the field. It is known that in the 1-dimensional analog of this model (the touchpoint paper), that there is a chance we do not break through the still set. However in both the 2-dimensional urn scheme and in the model presented in my thesis it seems plausible that when in a bad region near the still set, that we'll be eventually forced into a region where the vector field becomes large. When this happens it's possible that the vector field will be large enough to force a particle over the boundary.

Theorem 7.1. *For any $0 < c_1 < c_2 < 1$, we have that it is possible for X_n to tend to $1/2$ from the left and for Y_n to eventually reside in $[c_1, c_2]$.*

We write $X_{n+1} = X_n + A_n + S_n$ where $A_n := \mathbb{E}_n[X_{n+1} - X_n]$ and S_n is a mean zero random variable given $\mathcal{F}_n$. For each fixed n this defines a martingale $\{Z_{n,m}, \mathcal{F}_n\}$ where $Z_{n,m} := \Sigma_{i=n}^{m-1} S_i$. This martingale has the L^2-bound $\mathbb{E}[Z_{n,\infty}^2] \leq \Sigma_{i=n}^{\infty}(1+i)^{-2} \leq 1/n$

Addtionally pick l, l_1 and $\gamma > 1$ such that $l < l_1 < \gamma l_1 < 1/2$ and such that in some left neighbourhood $\mathcal{N}$ of $x = 0.5$ and $y \in [c_1, c_2]$, $\mathbb{E}_n[\Delta x] \leq \frac{l(0.5-x)}{n}$.

The function $g(s) := se^{(1-s)/(2\gamma l_1)}$ has value 1 at $s = 1$ and derivative $g'(1) = 1 - 1/2\gamma k_1 < 0$ so here is an $s \in (0,1)$ such that $g(s) > 1$. Fix an r such that $g(r^3) > 1$

Now define

$$T(k) := e^{k(1-r^3)/(\gamma l_1)}$$

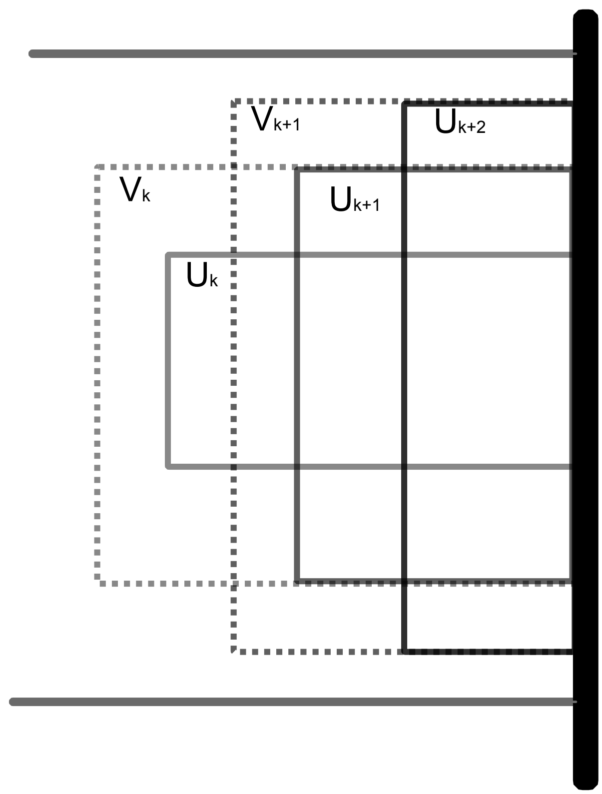
,so

$$g(r^3)^k = r^{3k} T(k)^{1/2}$$

We now define two regions $U_k \subset V_k$ by $U_k = [0.5-r^{3k}, 0.5-r^{3(k+1)}] \times [c_1+r^k, c_2-r^k]$

and $V_k = [0.5 - \gamma r^{3k}, 0.5 - r^{3(k+1)}]/[c_1 + r^{(k+1)}, c_2 - r^{(k+1)}]$.

We define $\tau_M := \inf\{j > T(m) | (X_j, Y_j) \in (0, 0.5 - r^{3M}) \times [c_1 + r^M, c_2 - r^M]\}$ and$\tau_{k+1} := \inf\{n \geq \tau_k | (X_n, Y_n) \notin V_n\}$ for all $n \geq M$



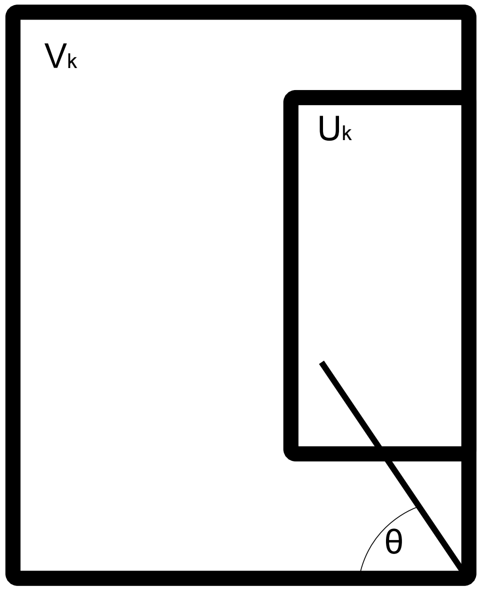
We assume that $(X_{\tau_M}, Y_{\tau_M}) \in U_M$ and $\tau_M \geq T(M)$ for some large enough M. Let β_l,β_b, β_t and β_r be the events that (X_n, Y_n) first leaves V_n on the left, bottum, top or right respectively. Let $\beta = \beta_l \cup \beta_b \cup \beta_t$. We calculate an upper bound on $P(\beta)$ using the relation $\mathbb{P}(\beta) \leq \mathbb{P}(\beta_l) + \mathbb{P}(\beta_b) + \mathbb{P}(\beta_t)$ and individual bounds on these 3 quantities.

We begin by finding a bound on $\mathbb{P}(\beta_l|\tau_k > T(k))$.

$$\begin{aligned}
\mathbb{P}(\beta_l|\tau_k > T(k)) &= \mathbb{P}(\inf_{j>\tau_k} X_j < 0.5 - \gamma r^{3n}|\tau_k > T(k)) \\
&\leq \mathbb{P}(\inf_{j>\tau_k} Z_{\tau_k,j} < -(\gamma-1)r^{3k}|\tau_k > T(k)) \\
&\leq \mathbb{E}[Z^2_{\tau_k,\infty}|\tau_k > T(k)]/((\gamma-1)r^{3k})^2 \\
&\leq e^{-\frac{k(1-r)}{l_1\gamma}}(\gamma-1)^{-2}r^{-6k} \\
&= (\gamma-1)^{-2}[g(r^3)]^{-2k}
\end{aligned}$$

Lemma 7.2. $\mathbb{P}(\beta_b) < r^k$

Proof. We define the constants $Y_0 := c_1 + r^{k+1}$ and $X_0 := 0.5 - r^{3(k+1)}$ We start by observing that $Z_n := \arctan\left(\frac{Y_n - Y_0}{X_0 - X_n}\right)$ is a submartingale on $W_k := V_k \cap \{(y - y_0) > r^{2k}(x_0 - x)\}$. Observe that $U_k \subset W_k$.



$$
\begin{aligned}
\mathbb{E}[\Delta Z_n] &= \mathbb{E}[\Delta \arctan\left(\frac{Y_n - Y_0)}{X_0 - X_n}\right)] \\
&= n^{-1}\left(\frac{d(\arctan(\frac{y-y_0}{x_0-x}))}{dx}\right)\cdot\frac{(x-0.5)^2}{y} + n^{-1}\left(\frac{d(\arctan(\frac{y-y_0}{x_0-x}))}{dy}\right)\cdot\frac{x-0.5}{y} + O(n^{-2}) \\
&= n^{-1}\left(\frac{(y_0-y)/(x_0-x)^2}{1+[(y-y_0)/(x_0-x)]^2}\right)\cdot\frac{(x-0.5)^2}{y} \\
&+ n^{-1}\left(\frac{1/(x_0-x)}{1+[(y-y_0)/(x_0-x)]^2}\right)\cdot\frac{x-0.5}{y} + O(n^{-2}) \\
&= n^{-1}\left(\frac{1/(x_0-x)^2}{1+[(y-y_0)/(x_0-x)]^2}\right)\cdot\frac{x-0.5}{y}\cdot[(y-y_0)\cdot(x-0.5)+(x_0-x)] + O(n^{-2})
\end{aligned}
$$

For large n this is positive when $(y - y_0) \cdot (x - 0.5) + (x_0 - x) < 0$ as it is on W_k.

$$\begin{aligned}
\arctan\left(\frac{Y_{\tau_n} - (c_1 + r^{(k+1)})}{(0.5 - r^{3(k+1)}) - X_{\tau_k})}\right) &\geq \\
\arctan\left(\frac{(c_1 + r^k) - (c_1 + r^{(k+1)})}{(0.5 - r^{3(k+1)}) - (0.5 - r^{3k}))}\right) &\geq \\
\arctan\left(\frac{(c_1 + r^k) - (c_1 + r^{k+1})}{(0.5 - r^{3(k+1)}) - (0.5 - r^{3k})}\right) &\geq \\
\arctan\left(\frac{r^k(1 - r)}{r^{3n}(1 - r^3)}\right) &= \\
\arctan\left(\frac{1}{r^{2k}(1 + r + r^2)}\right) &\geq \\
\arctan\left(r^{-n}\right) &= \\
\operatorname{arccot}\left(r^n\right) &\geq \\
\pi/2 - r^k &
\end{aligned}$$

Hence $\mathbb{P}(\beta_b) < r^k$ $\square$

We can use an analogous argument to show that $\mathbb{P}(\beta_t) < r^k$. Finally we compute an upper bound on the probabiliy that (X_n, Y_n) leaves V_k from the right but does so too early. To begin more formally we note that if β_r holds, then

$$\sum_{T(k)<j<T(k+1)} A_j = \sum_{T(k)<j<T(k+1)} \frac{(0.5-X_j)^2}{y(j+1)}$$
$$< \sum \frac{r^{6k}}{jc_1}$$
$$\leq [ln(\lceil T(k+1)\rceil) - ln(\lceil T(k)\rceil)]r^{6n}/c_1$$
$$\leq [(1-r^3)/(\gamma l_1 c_1) + 1/T(k)]r^{6n}$$

But then if β_r holds and $\tau_{k+1} = L \leq T(k+1)$, it must be the case that

$$Z_{\tau_k,L} = X_L - X_{\tau_k} - \sum \lim_{j=\tau_k}^{L-1} A_j$$
$$\geq X_l - X_{\tau_n} - \sum \lim_{j=T(k)}^{T(k+1)} A_j$$
$$= r^{3k} - r^{3k+3} - \zeta_k - [(1-r^3)/(\gamma l_1 c_1) + 1/T(k)]r^{6k}]$$
$$= r^{3k}(1-r^3) - \tilde{\zeta}_k$$

Now the ζ_k denotes the fact that X may overshoot $0.5 - r^{3n}$. While the $\tilde{\zeta}_k :=$ $\zeta_k + [(1-r^3)/(\gamma l_1 c_1) + 1/T(k)]r^{6k}]$, which vanishes assymptotically.

Noticing that $\mathbb{P}(\beta \cup \tau_{k+1} < T(k+1) | \tau_k > T(k))$ is summable over k completes the proof of 7.1. $\square$

We will now show that it is possible to get away from a half. We begin by showing that if $X_n \geq 0.5$ then with macroscopic probability X_n is eventually far away from the $x = 0.5$ line. We then show that with positive probability X_n never returns much closer to 0.5 than this.

Lemma 7.3. *If $X_n \geq 0.5$ then with probability greater than $1/4$, there exists m s.t. $X_m \geq 0.5 + c_4 m^{-1/2}$.*

Proof. To show this we consider a related process (U_n, V_n). Which has vector field.

$$G(u, v) = (sgn(u - 0.5)\frac{(u - 0.5)^2}{v}, sgn(u - 0.5)\frac{u - 0.5}{v})$$

To the right of the line $x = 1/2$ ($u = 1/2$) this new process behaves exactly like (X_n, Y_n). To the left of this barrier it has had it's direction reversed (i.e. it has been multiplied by negetive 1). We shall show that for some ϵ and sufficently large N that if $U_N \in (0.5 - \epsilon N^{-0.5}, 0.5 + \epsilon N^{-0.5})$ then with probability at least $1/2$ that $U_n - 0.5$ eventually leaves $(\epsilon n^{-0.5}, \epsilon n^{-0.5})$.

By the symmetry of G, we have that probability of leaving on the right side ($U_N > 0.5$) must be greater than $1/4$. Intuitively F is more right skewed than G (i.e. urn X is always more likely to ad a white ball than urn U),using the obvious coupling shows that if G crosses on the right so does F.

To begin in ernest. Observe that for some $c > 0$, that $\mathbb{E}_n[\Delta(X_n - 0.5)^2] \leq cn^{-2}$

Set $\epsilon^2 = c/2$ Let $\tau = inf\{k \geq N_0 : |U_k - 0.5| \geq \epsilon k^{-1/2}\}$

For any $m > n$ we have $|U_{m\wedge\tau} - 0.5| \leq \epsilon n^{-1/2}$, thus

$$\begin{aligned}
\epsilon^2 m^{-1} \geq & (U_{m\wedge\tau} - 0.5)^2 \\
\geq & (U_{m\wedge\tau} - 0.5)^2 - (U_n - 0.5)^2 \\
\geq & \Sigma_{k=n}^{m-1} \Delta (U_n - 0.5)^2 \mathbf{1_{k<\tau}} \\
\geq & \Sigma_{k=n}^{m-1} c k^{-2} \mathbb{P}(\tau < k) \\
\geq & c/2(n^{-1} - m^{-1}) \mathbb{P}_n(\tau < \infty)
\end{aligned}$$

Sending m to infiniy shows that $\mathbb{P}_n(\tau = \infty) \leq 1/2$.Hence with probability at least $1/4$ $U_n > 0.5 + \epsilon n^{-0.5}$) for some n. Coupling (X_n, Y_n) and (U_n, V_n) in the obvious way shows that X_n also crosses this boundary line. □

Lastly we show that.

Lemma 7.4. *There is an N_0 and a c_1 such that for all $n > N_0$.*

$$\mathbb{P}_n(\forall m > n,\ |X_{n+m} - 0.5| > \frac{2c}{5}) > c_1$$

whenever $X_n \geq 0.5 + cn^{-1/2}$

Proof. Let $M_n + A_n$ denote the Doob decomposiiton of X_n. Then $\mathbb{E}_n[(\Delta M_n)^2] \leq \mathbb{E}_n[(\Delta X_n)^2] \leq c_2 n^{-2}$ for some $c_2 > 0$ Hence $\mathbb{E}_n(M_{n+m} - M_n)^2 < c_2 n^{-1}$ for all $m > 0$ on the event $X_n > 0.5$. For any $a, n, V > 0$ and any martingale M_k satisfying $M_n > a$ and $sup_m \mathbb{E}_n(M_{n+m} - M_n)^2 \leq V$, there holds an inequality

$$\mathbb{P}(inf_m M_{n+m} \leq a/2) < \frac{4V}{4V + a^2}$$

. Setting $a = cn^{-1/2}$ and $V = c_2 N^{-1}$ It follows that $\mathbb{P}_n(inf_{k \geq n M_k \leq c/2n^{-0.5}}) \leq c_1 :=$

$\frac{c_2}{c_2+c^2}$

and we are done. $\square$

Chapter 8

Discussion and Conclusion

Above we solved the simplest Skyrms game with L players. There are still many other unsolved problems relating to Skyrms games. These are listed and discussed in turn. Following the seminal work of [APSV08] for the $L = 2$ case, [HT] solved the $L = 2$ case when Nature and Player 2 have M_2 useable and player 1 has M_1 usable symbols. They give a complete list of possible equilibria for this model. The most notable feature of the solution is that here it is possible for the equilibria to occur at points of ineffcient signalling.

Another unsolved problem is the question of what happens when nature's plays are not i.i.d. fair coin flips. The most natural deviation from this is for nature's plays to be i.i.d. unfair coin flips. For the case of two players simulations suggest that a language does not always occur. It is entirely unknown how likely languages are to develop as a function of p, the probability that nature sends a 0, in this case.

Another interesting question is the case where instead of players being in series they are in paralell. The simplest case of this is as follows. Nature plays from the set 1,2,3,4 in an i.i.d. uniform manner and Player 1 observes weather the symbol is in $\{1, 3\}$ or $\{2, 4\}$ and Player 2 observes weather the symbol is in $\{1, 2\}$ or $\{3, 4\}$. Player 1 and Player 2 then both signal Player 3 there choice of a or b. Player 3 then guesses Nature's play. Here player 3 has 4 urns each with 4 types of ball and players 1 and 2 have 2 urns each with 2 types of ball. Simulations and analysis of the mean vector fields suggest that effcient signalling occurs here with probability 1. A third open question is what happens when Players can create new symbols. In the 2 player case simulations suggest that effceint signalling evolves. In this case it seems a language always develops. Finally it is possible to combine these is essentially any combination. For example the case where the senders are in parallel and nature does not play evenly could be analysed.

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