

ANALYTIC STRUCTURE OF TWO-DIMENSIONAL QUANTUM FIELD THEORY

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There has been considerable progress in the past year in understanding the structure of two-dimensional quantum field theories. Some of this progress has come from an improved understanding of how algebraic geometry enters into the theory of determinants of differential operators on Riemann surfaces. In these notes I will sketch the connection briefly, en route to a description of some recent work done with L. Alvarez-Gaumé, J.-B. Bost, G. Moore, and C. Vafa.¹ Further details appear in Ref. 2. The motivation for these investigations is of course the fact that 2d field theory is an essential ingredient in the formulation of string theories, as discussed by many other speakers here. In particular the Fermi-Bose equivalence to be described below is a key tool used in many constructions.

It has been known for many years that quantum field theory in two dimensions has remarkable special features. For example, in many cases such theories are exactly solvable. The study of such solvable models led to the discovery that in some instances theories with commuting fields were exactly equivalent to other theories with anticommuting fields. In the following paragraphs I will outline some features of this equivalence and sketch how it can be demonstrated. A key feature of the development is that it applies to theories formulated on compact orientable

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surfaces of arbitrary genus, as it must in order to be of use for string theory.

A simple example of a theory with commuting (Bose) fields is the following. Consider a compact Riemann surface X and the "field space" \mathcal{X} of all real functions $\phi: X \rightarrow \mathbb{R}$ satisfying some sort of Sobolev condition, which we will not specify in detail. Define the functional $S_b[\phi] = \|\partial\phi\|^2 \equiv \int_X \partial\phi \bar{\partial}\phi$. S_b is called the action functional of a free massless boson; it depends on ϕ and also on the conformal structure chosen for X . It is possible to define a probability measure $[d\phi]$ on \mathcal{X} in such a way that

$$Z_b = \int_{\mathcal{X}} [d\phi] e^{-S_b[\phi]} \quad (1)$$

is well-defined. There are two complications, however. First, to define the measure $[d\phi]$ we must "regulate" it, i.e. suppress the contributions from rapidly-varying maps ϕ . We can only do this if we are given not just a conformal structure on X , but a *metric* g in the given conformal class. Thus $Z_b = Z_b[g]$. Secondly, as it stands Z_b includes an integral over constant maps ϕ , all of which have $S_b = 0$. To remedy this trivial infinity we must eliminate the zero mode of ∂ from \mathcal{X} and, it turns out, divide the measure by the metric norm of this zero mode, i.e. by $\|1\|^2 = \int_X \text{vol}_g$, the area of X . It turns out that we can define (1) by

$$Z_b[g] = \left(\frac{\det' \bar{\partial}^\dagger \bar{\partial}}{\|1\|^2} \right)^{-\frac{1}{2}}, \quad (2)$$

which is just the naive answer to (1) obtained by Gaussian integration, but made meaningful by removing the zero mode and defining the determinant via zeta-function regularization.

Equation (2) is called the partition function of the commuting field theory. It does not characterize the measure $[d\phi]$, since it is just one number for each g . Rather it is the simplest of a collection of moments of $[d\phi]$, which are called correlation functions. We will restrict attention to (2), but our conclusions generalize to the more complicated functions as well.

Next we can define an anticommuting (or Fermi) field theory. Again let X be a compact Riemann surface, but this time equipped with an even theta characteristic. This amounts to choosing a spin structure on X , or a holomorphic line bundle L such that $L \otimes L \cong K$, the canonical bundle on X . Analogous to (1) one can define

$$Z_f = \int [db] [dc] [d\bar{b}] [d\bar{c}] e^{-S_f} . \quad (3)$$

The functional measures in (3) are more formal than the one in (1). For the present purposes it suffices to define (3) by a formula analogous to (2)

$$Z_f[g; L] = \det \bar{\partial}_L^\dagger \bar{\partial}_L . \quad (4)$$

Here $\bar{\partial}_L$ is the Cauchy-Riemann operator on the holomorphic line bundle L . Again a metric is needed to define the "measures" in (3); it enters (4) via the adjoint. Since generically $H^0(X; L) = 0$, $\bar{\partial}_L$ has no zero mode and no prime or normalizing factor is needed in (4).

The expression (4) is the partition function of an anti-commuting field theory. It too admits generalizations to correlation functions which we will not discuss here.

Suppose that X is the Riemann sphere. Then the statement

of Bose-Fermi equivalence begins with the claim that

$$Z_b[g] = c Z_f[g] \quad (5)$$

where c is some constant independent of g . In fact (5) is true, but in a fairly uninteresting way. Both sides of (5) are unchanged if g is replaced by f^*g for some diffeomorphism f . Moreover, both sides change in the same simple way if we change g in its conformal class, i.e. if we replace g by $e^\sigma g$ for a function σ . (See e.g. Ref. 3.) Since up to diffeomorphism there is just one conformal structure on S^2 , (5) follows at once.

Equation (5) is bound to be more interesting for surfaces of genus $\gamma > 0$, since here there is a nontrivial space of conformal classes, the *moduli space* of genus γ . On such an X (5) cannot be literally true, however, since one side depends on a θ -characteristic while the other does not. Thus (5) should be replaced by

$$Z_b[g] = c \sum_L Z_f[g; L] \quad (5')$$

Moreover from earlier work involving the case where X is an infinite cylinder, we know that ϕ in (1) must be taken to be not a function to \mathbb{R} but rather a circle-valued function. The distinction matters when X is not simply-connected.

When X has genus $\gamma > 0$ we can classify maps ϕ into topological sectors as follows. Choose a canonical homology basis for X $\{a_i, b_i\}$ (see e.g. Refs. 4, 5) and let ϕ wind n_i, m_i times around S^1 as one walks around a_i, b_i respectively. Viewing S^1 as a group, we can realize any ϕ as $\phi_{n,m} \cdot \tilde{\phi}$ where $\tilde{\phi}$ has zero winding and $\omega_{n,m} \equiv \phi_{n,m}^{-1} d\phi_{n,m}$ is harmonic, by the Hodge theorem. Then (2) becomes

$$z_b[g] = \left(\sum_{n,m} e^{-\|\omega_{n,m}\|^2} \right) \left(\frac{\det' \bar{\partial}^\dagger \bar{\partial}}{\|1\|^2} \right)^{-\frac{1}{2}}. \quad (6)$$

The foregoing analysis was performed by the authors of Ref. 5, who then summed (6) to get

$$z_b[g] = \left(\sum_{\epsilon} |\vartheta[\epsilon](\tau)|^2 \right) \left(\frac{\det' \bar{\partial}^\dagger \bar{\partial}}{\det(\omega^i, \omega^j) \cdot \|1\|^2} \right)^{-\frac{1}{2}}. \quad (7)$$

Here ϑ is the Riemann theta function with characteristics and we sum over the half-points of the Jacobian of X ; this is because the physically appropriate prescription for (6) sums over half-integers n_i, m_i . τ is the period matrix of X in the given homology basis, and we have rewritten $\det \operatorname{Im} \tau$ as the determinant of the matrix of inner products of the Abelian differentials ω^i determined by the homology basis.⁴

In Ref. 5 Fermi-Bose equivalence and Quillen's theorem⁶ were used to deduce

$$\det \bar{\partial}_L^\dagger \bar{\partial}_L = c \left(\frac{\det' \bar{\partial}^\dagger \bar{\partial}}{\det(\omega^i, \omega^j) \cdot \|1\|^2} \right)^{-\frac{1}{2}} |\vartheta[\epsilon](\tau)|^2 \quad (8)$$

where on the right ϵ is the point in the Jacobian corresponding to the spin bundle L via the chosen homology basis and Riemann's theorem.^{4,5} We will sketch how (8) can be derived directly.

Both sides of (8) are functions of g with the same behavior under conformal rescaling. Thus their quotient F is a function on the moduli space \mathcal{M} of curves of genus γ (or more precisely on its cover $\hat{\mathcal{M}}$, curves with a θ -characteristic). We need two key facts about moduli space:

- (a) $\hat{\mathcal{M}}$ is a complex space, and $\bar{\partial}, \bar{\partial}_L$ vary holomorphically.
- (b) $\hat{\mathcal{M}}$ has only one holomorphic function in genus $\gamma > 2$;

indeed $\hat{\mathcal{M}}$ has only one pluriharmonic function. This follows from the existence of the Satake compactification of \mathcal{M} , which has codimension two.

Thus to show that F is a constant, and hence prove (8), one has only to show that

$$\delta_i \bar{\delta}_j \log F = 0 \quad (9)$$

where $\delta, \bar{\delta}$ are holomorphic and antiholomorphic derivatives on $\hat{\mathcal{M}}$.

To show (9), we note that the determinants in (8) are closely related to Quillen's construction of norms on determinant line bundles.⁶ Indeed over moduli space we have determinant bundles $D_0 = \text{DET } \bar{\partial}, D_{\frac{1}{2}} = \text{DET } \bar{\partial}_L$, and by the Riemann-Roch-Grothendieck theorem we have

$$D \equiv D_{\frac{1}{2}} \otimes D_{\frac{1}{2}} \otimes D_0 \cong \text{trivial}.$$

Moreover the trivializing section is $\sigma \equiv \sigma_{\frac{1}{2}} \otimes \sigma_{\frac{1}{2}} \otimes \sigma_0$, where $\sigma_{\frac{1}{2}}$ is the holomorphic section of $D_{\frac{1}{2}}$ vanishing whenever $H^0(X; L) \neq 0$, and

$$\sigma_0 = (\omega^1 \wedge \dots \wedge \omega^Y)^{-1} \otimes (1)^{-1} \cdot \theta[\varepsilon](\tau)^{-2}$$

(We are being careless about torsion since we are going to take the absolute square of σ .) σ never vanishes or blows up, since ε is chosen to make θ vanish whenever $\bar{\partial}_L$ has a zero mode.

The quotient of the two sides of (8) is then just $F = \|\sigma_{\frac{1}{2}} \otimes \sigma_{\frac{1}{2}} \otimes \sigma_0\|$, and so $\delta \bar{\delta} \log F$ is the Hermitian curvature of Quillen's norm on D . But remarkably, Quillen's norm has the property of being *flat* on any trivial combination of determinant bundles.⁷ Hence $\delta \bar{\delta} \log F = 0$, F is a constant, and (8) is established for genus $\gamma > 2$. (The cases $\gamma = 1, 2$ can be verified directly by other means.⁸)

Arguments like these based on the near-compactness of \mathcal{M} also let one prove Fermi-Bose equivalence for the correlation functions mentioned above. They have other applications to string theory as well, as discussed for instance in Ref. 2.

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