

On Fractional Paradigm and Intermediate Zones in Electromagnetism: II. Cylindrical and Spherical Observations[†]

Nader Engheta

University of Pennsylvania
Moore School of Electrical Engineering
Philadelphia, Pennsylvania 19104, U.S.A.
E-mail: engheta@ee.upenn.edu
URL: <http://www.ee.upenn.edu/~engheta/>

Key Words: Fractional kernels, Fractional Calculus, Fractional Paradigm, Intermediate zone, Electromagnetic Waves.

Abstract

Extending our previous work for the planar case [1], in this Letter we present fractionalization of the kernels of integral transforms that link the field quantities over two coaxial cylindrical surfaces of observation for the two-dimensional (2-D) monochromatic wave propagation, and over two concentric spherical surfaces of observation for the three-dimensional (3-D) wave propagation. With the proper radial normalizations, we show that the fractionalized kernels, with fractionalization parameter ν that here could attain complex values between zero and unity, can effectively be regarded as the kernels of the integral transforms that provide the radially normalized field quantities over the coaxial cylindrical surfaces (for 2-D case) and over the concentric spherical surfaces (for 3-D case) *between* the two original surfaces. Like in the planar case [1], here the fractionalized kernels can supply another way of interpreting the fields in the intermediate zones.

[†] Parts of the early results of this work were presented by the author at *the Progress in Electromagnetic Research Symposium (PIERS'98), Nantes, France, July 13-17, 1998.*

Introduction

In a previous Letter [1], we presented the concept of fractionalization of the kernels of integral transforms that relate a quantity of interest (e.g., potential or a Cartesian component of field) over an observation flat plane to the corresponding quantity over another flat plane parallel with the first one, and we showed that resulting fractionalized kernels can play the role of the kernels of integral transforms that “connect” the quantity on the first flat plane to the quantities on the *intermediate* parallel plane. That case study was part of our efforts in exploring the roles of fractionalization of operators in electromagnetics and developing and studying the area of fractional paradigm in electromagnetic theory [1-8]. In the present Letter we extend our previous work to the cases of cylindrical and spherical surfaces of observations. The geometry and analysis of the problem for the cylindrical case is given in the next section, and that of the spherical case is addressed in the subsequent section.

Geometry and Analysis: Cylindrical Case for 2-D Wave Propagation

We consider both Cartesian and cylindrical coordinate systems (x, y, z) and (r, φ, z) wherein $x = r \cos \varphi$ and $y = r \sin \varphi$. As in the planar case, here for the sake of simplicity of the analysis we again assume to have a monochromatic source independent of one coordinate (e.g., independent of z)¹, denoted as the volume current density $\mathbf{J}(x, y)$ in free space with its x - y transverse cross section limited to a finite region. The time dependence is again assumed to be $e^{-i\omega t}$. We take two coaxial cylindrical mathematical surfaces with radii r_o and r_1 where $r_1 \geq r_o$. The cylindrical surface of radius r_o contains the

¹ Note that in [1] the source and quantities of interest were assumed to be independent of the y coordinate, whereas here these are taken to be independent of the z coordinate.

source $\mathbf{J}(x, y)$ in its interior volume. The region outside this surface (i.e., $r \geq r_o$) is source free. The mathematical steps here follow the similar steps taken in the planar case, except the mathematical expansions here are done using Fourier series (in the φ variable of the 2-D cylindrical coordinate system (i.e., polar coordinate system)). Thus wherever possible, here we use similar notations for corresponding quantities in the planar case. We denote the scalar potential or a Cartesian component of fields in this problem by the symbol $\psi(r, \varphi)$. In the region outside the source, the function $\psi(r, \varphi)$ satisfies the standard Helmholtz equation $\nabla^2 \psi + k_o^2 \psi = 0$ where $k_o \equiv \omega \sqrt{\mu_o \varepsilon_o}$ and μ_o and ε_o are the permeability and permittivity of free space, respectively. This quantity on the two cylindrical surfaces mentioned above is shown as $\psi(r_o, \varphi)$ and $\psi(r_1, \varphi)$, respectively. Using the Fourier series expansion in the φ variable and from the fact that $\psi(r, \varphi)$ satisfies the Helmholtz equation, it can be shown that $\psi(r_o, \varphi)$ and $\psi(r_1, \varphi)$ may be expressed as

$$\psi(r_o, \varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \tilde{\psi}_n(r_o) e^{in\varphi} \quad (1a)$$

$$\psi(r_1, \varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \tilde{\psi}_n(r_1) e^{in\varphi} \quad (1b)$$

where the Fourier-series coefficients $\tilde{\psi}_n(r_o)$ and $\tilde{\psi}_n(r_1)$ can be explicitly written as

$$\tilde{\psi}_n(r_o) = a_n H_n^{(1)}(k_o r_o) \quad (2a)$$

$$\tilde{\psi}_n(r_1) = a_n H_n^{(1)}(k_o r_1) \quad (2b)$$

with $H_n^{(1)}(\cdot)$ being the n^{th} -order Hankel function of the first kind, and a_n 's being the constant coefficients determined from the knowledge of the source. One can “connect” the two functions $\psi(r_o, \varphi)$ and $\psi(r_1, \varphi)$ using the integral transform

$$\psi(r_1, \varphi) = \int_{-\pi}^{+\pi} \underline{K}(r_1, \varphi; r_o, \varphi') \psi(r_o, \varphi') d\varphi' \quad (3)$$

with the kernel $\underline{K}(r_1, \varphi; r_o, \varphi')$. Since the function $\psi(r, \varphi)$ drops as $r^{-1/2}$ in the far zone where $k_o r \rightarrow \infty$, and for the reason that will become evident later in this Letter, it is more suitable to rewrite the above integral transform between “radially normalized” functions $\sqrt{k_o r_o} \psi(r_o, \varphi)$ and $\sqrt{k_o r_1} \psi(r_1, \varphi)$. Furthermore, it can be easily recognized that the kernel is (azimuth) angle-shift-invariant, and thus Eq. (3) can be rewritten as

$$\sqrt{k_o r_1} \psi(r_1, \varphi) = \int_{-\pi}^{+\pi} K(r_1, \varphi - \varphi'; r_o) \sqrt{k_o r_o} \psi(r_o, \varphi') d\varphi'. \quad (4)^2$$

This kernel can also be expanded using the Fourier series expansion for the $\varphi - \varphi'$ variable as

$$K(r_1, \varphi - \varphi'; r_o) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \tilde{K}_n(r_1; r_o) e^{in(\varphi - \varphi')} \quad (5)$$

where $\tilde{K}_n(r_1; r_o)$ denotes the Fourier-series coefficients. From Eqs. (1), (2), (4), and (5), one concludes that

$$\sqrt{k_o r_1} \tilde{\psi}_n(r_1) = \sqrt{k_o r_o} \tilde{K}_n(r_1; r_o) \tilde{\psi}_n(r_o) \quad (6)$$

which in turn leads to the following expressions for $\tilde{K}_n(r_1; r_o)$ and $K(r_1, \varphi - \varphi'; r_o)$,

$$\tilde{K}_n(r_1, r_o) = \frac{\sqrt{r_1} H_n^{(1)}(k_o r_1)}{\sqrt{r_o} H_n^{(1)}(k_o r_o)}, \quad (7)$$

$$K(r_1, \varphi - \varphi'; r_o) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \frac{\sqrt{r_1} H_n^{(1)}(k_o r_1)}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} e^{in(\varphi - \varphi')}. \quad (8)$$

² Clearly the two kernels $K(r_1, \varphi; r_o, \varphi')$ and $\underline{K}(r_1, \varphi; r_o, \varphi')$ are simply related with each other through the multiplicative constant $\sqrt{r_o / r_1}$, i.e., $\underline{K}(r_1, \varphi; r_o, \varphi') = \sqrt{r_o / r_1} K(r_1, \varphi; r_o, \varphi')$.

Now we wish to fractionalize the operator $\underline{\underline{L}}$ expressed as the integral transform in Eq.

(4) that is

$$\underline{\underline{L}} = \int_{-\pi}^{+\pi} K(r_1, \varphi - \varphi'; r_o) \cdots d\varphi'. \quad (9)$$

The fractionalized operator, symbolically shown as $\underline{\underline{L}}^\nu$ can be written as

$$\underline{\underline{L}}^\nu = \int_{-\pi}^{+\pi} K_\nu(r_1, \varphi - \varphi'; r_o) \cdots d\varphi' \quad (10)$$

where $K_\nu(r_1, \varphi - \varphi'; r_o)$ denotes the “fractionalized” kernel, which we wish to determine, and ν is the fractionalization parameter. The steps for fractionalization of the kernel is given in [1] and is not repeated here. Following similar mathematical steps described in [1], we find that the Fourier-series coefficients of this fractionalized kernel, which are denoted as $(\tilde{K}_\nu)_n(r_1; r_o)$, can be written in terms of the ν^{th} power of the Fourier series coefficients of the original kernels as follows

$$(\tilde{K}_\nu)_n(r_1; r_o) = [\tilde{K}_n(r_1; r_o)]^\nu = \left[\frac{\sqrt{r_1} H_n^{(1)}(k_o r_1)}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} \right]^\nu. \quad (11)$$

From the above, one can then write the expression for the fractional kernel $K_\nu(r_1, \varphi - \varphi'; r_o)$ as

$$K_\nu(r_1, \varphi - \varphi'; r_o) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \left[\frac{\sqrt{r_1} H_n^{(1)}(k_o r_1)}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} \right]^\nu e^{in(\varphi - \varphi')}. \quad (12)$$

One can easily notice that when $\nu \rightarrow 1$, the fractional kernel $K_\nu(r_1, \varphi - \varphi'; r_o)$ becomes the original kernel $K(r_1, \varphi - \varphi'; r_o)$; and when $\nu \rightarrow 0$, the fractional kernel approaches the Dirac delta function $\delta(\varphi - \varphi')$, as expected. As in the planar case, here it can also be

shown that the integral transform (Eq. (10)) representing $\underline{\underline{L}}^\nu$ satisfies the additivity properties in fractionalization parameter ν , i.e., $\underline{\underline{L}}^{\nu_1} \circ \underline{\underline{L}}^{\nu_2} = \underline{\underline{L}}^{\nu_1+\nu_2}$.

When the fractionalization parameter ν is neither zero nor unity, would the fractional kernel $K_\nu(r_1, \varphi - \varphi'; r_o)$ represent the kernel of an integral transform that provides us with the radially-normalized quantity of interest on the intermediate cylindrical surface *between* the two original surfaces? To answer this question, let us utilize Eq. (4) to consider the integral transform that links $\sqrt{k_o r_o} \psi(r_o, \varphi)$ to the function $\sqrt{k_o r_{\text{int}}} \psi(r_{\text{int}}, \varphi)$ on the cylindrical surface of radius r_{int} where the subscript “int” indicates “intermediate”, so $r_o \leq r_{\text{int}} \leq r_1$. From Eqs. (4) and (8), one can write explicitly the kernel for such a transform as

$$K(r_{\text{int}}, \varphi - \varphi'; r_o) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \frac{\sqrt{r_{\text{int}}} H_n^{(1)}(k_o r_{\text{int}})}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} e^{in(\varphi - \varphi')} \quad (13)$$

If we equate the kernel for the intermediate radius (Eq. (13)) with the fractionalized kernel we obtained in Eq. (12), we will find the values of the fractionalization parameter ν in terms of all the other parameters involved. Specifically, from the following expression of equality of the two kernels,

$$\frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \frac{\sqrt{r_{\text{int}}} H_n^{(1)}(k_o r_{\text{int}})}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} e^{in(\varphi - \varphi')} = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \left[\frac{\sqrt{r_1} H_n^{(1)}(k_o r_1)}{\sqrt{r_o} H_n^{(1)}(k_o r_o)} \right]^\nu e^{in(\varphi - \varphi')} \quad (14)$$

we can find the values of fractionalization parameter ν as follows

$$\nu_n = \frac{\ln(\sqrt{r_{\text{int}}} H_n^{(1)}(k_o r_{\text{int}})) - \ln(\sqrt{r_o} H_n^{(1)}(k_o r_o))}{\ln(\sqrt{r_1} H_n^{(1)}(k_o r_1)) - \ln(\sqrt{r_o} H_n^{(1)}(k_o r_o))} \quad (15)$$

which shows that, unlike the planar case that we reported earlier [1], here for a given set of radii r_o , r_1 , and r_{int} we do not obtain a single value for fractionalization parameter ν .

Instead we have differing values ν_n for each Fourier component of these kernels. We notice that when $r_{\text{int}} \rightarrow r_1$, the parameters ν_n 's (for all n from $-\infty$ to $+\infty$) become unity; and when $r_{\text{int}} \rightarrow r_o$, all ν_n 's become zero, as expected. However, when $r_o < r_{\text{int}} < r_1$, the parameters ν_n 's are not necessarily the same value for all n , and in fact they can even attain complex values as can be implied from Eq. (15). So the fractionalized kernel given in Eq. (12) can be interpreted as the kernel of the integral transform linking the radially-normalized quantity of interest on the cylindrical surface r_o to the corresponding radially-normalized quantity on the *intermediate* coaxial cylindrical surface with radius r_{int} , if the fractionalization parameters ν_n 's are chosen according to Eq. (15). With these values of ν_n 's, one can then write

$$K_{\dots, \nu_{-2}, \nu_{-1}, \nu_0, \nu_1, \nu_2, \dots}(r_1, \varphi - \varphi'; r_o) = K(r_{\text{int}}, \varphi - \varphi'; r_o) \quad (16)$$

where the subscript of the fractional kernel is explicitly written as differing parameters ν_n 's. As these parameters vary from zero to unity according to Eq. (15), the radius of the intermediate cylindrical surface evolves from r_o to r_1 . If the original cylindrical surfaces r_o and r_1 are chosen at the near-zone radius and the far-zone radius, respectively, the fractional kernel with proper parameters ν_n 's would provide another way to interpret the fields in the intermediate zones in the cylindrical case.

If the original radii r_o and r_1 are chosen such that $k_o r_o \gg 1$ and $k_o r_1 \gg 1$, the expression for the Hankel functions may be simplified. It is well known [9, page 364] that for the large arguments, the Hankel functions can be asymptotically written as

$$H_n^{(1)}(k_o r) \underset{k_o r \rightarrow \infty}{\cong} \sqrt{\frac{2}{\pi k_o r}} e^{i(k_o r - \frac{1}{2}n\pi - \frac{1}{4}\pi)} \quad (17)$$

where $k_o r$ should also be large compared with the order n . Under these assumptions, Eq. (15) can be written as

$$\nu_n \cong \frac{r_{\text{int}} - r_o}{r_1 - r_o} \quad \text{when } k_o r_o \gg 1 \text{ and } k_o r_1 \gg 1, \text{ and also } k_o r_o \gg n \text{ and } k_o r_1 \gg n. \quad (18)$$

This indicates that when the above conditions are satisfied, the fractionalized parameters ν_n 's attain a single real value between zero and unity determined by the three radii r_o , r_1 , and r_{int} .³ The form of this expression is similar to the one we obtained for the planar case in which the fractionalization parameter was given as $\nu = \frac{z_\nu - z_o}{z_1 - z_o}$. [1] Now it has become clear why in writing the integral transform in Eq. (4), we used the radial normalization $\sqrt{k_o r}$. Like the case of planar observation, here under these conditions when the parameter ν is given at a non-integer real value between zero and unity, the radius of the intermediate cylindrical surface can be obtained using Eq. (18). Conversely, if again these conditions are satisfied and if the intermediate radius is specified, we can find the value of parameter ν .

To summarize this section, we have shown that by proper choice of fractionalization parameters the fractional kernel given in Eq. (12) can provide the kernel for the integral transform relating the radially-normalized function of interest on the cylindrical surface of radius r_o to the corresponding function on the intermediate coaxial

³ It must be noted that the index n takes all the integer values from $-\infty$ to $+\infty$. So when r_o and r_1 are specified and they satisfy the conditions $k_o r_o \gg 1$ and $k_o r_1 \gg 1$, there can always be the indexes n for which the conditions $k_o r_o \gg n$ and $k_o r_1 \gg n$ are no longer satisfied, and thus the expression given in Eq. (18) may not be applicable and one should then again use Eq. (15) for the value of ν_n . However, it must be noted that for a given source it is possible that for such large indexes the coefficients a_n 's in Eqs. (2a) and (2b) are negligibly small and therefore those Fourier-series terms with high indexes may not contribute significantly in this case. However, if they do become significant, then one should notice that Eq. (18) would be valid only for those indexes for which the conditions used in Eq. (17) are satisfied.

cylindrical surface with radius r_{int} where $r_o \leq r_{\text{int}} \leq r_1$. Some of the salient features of this fractional kernel are similar to its counterpart for the planar case.

In the following section, we address a similar fractionalization of the kernel for the spherical case.

Geometry and Analysis: Spherical Case for 3-D Wave Propagation

For this case, the monochromatic source is assumed to be in a finite region of three-dimensional space and is denoted as $\mathbf{J} = \mathbf{J}(x, y, z)$. The Cartesian and spherical coordinate systems (x, y, z) and (R, θ, φ) are used wherein $x = R \sin \theta \cos \varphi$, $y = R \sin \theta \sin \varphi$, and $z = R \cos \theta$. Following a similar approach, here we take two concentric spherical surfaces of radii R_o and R_1 , where $R_1 \geq R_o$. The source $\mathbf{J}(x, y, z)$ lies within the interior volume of the spherical surface of radius R_o . The potential function on these two surfaces are denoted by $\psi(R_o, \theta, \varphi)$ and $\psi(R_1, \theta, \varphi)$. Analogous to Eq. (4) in the previous section, here we use the following integral transform to relate the radially-normalized form of these two functions,

$$k_o R_1 \psi(R_1, \theta, \varphi) = \int_{-\pi}^{+\pi} \int_0^\pi K(R_1, \theta, \varphi; R_o, \theta', \varphi') k_o R_o \psi(R_o, \theta', \varphi') \sin \theta' d\theta' d\varphi' \quad (19)$$

where, using the standard spherical harmonics $Y_{lm}(\theta, \varphi)$ (see e.g., [10, page 107]) and knowing that $\psi(R, \theta, \varphi)$ satisfies the source-free Helmholtz equation in the region with $R \geq R_o$, one can show that the kernel $K(R_1, \theta, \varphi; R_o, \theta', \varphi')$ can be explicitly given as

$$K(R_1, \theta, \varphi; R_o, \theta', \varphi') = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \frac{R_1 h_l^{(1)}(k_o R_1)}{R_o h_l^{(1)}(k_o R_o)} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi'), \quad (20)$$

with $h_l^{(1)}(\cdot)$ being the l^{th} -order spherical Hankel function of the first kind, and $*$ indicates complex conjugation. In Eq. (19), the radial normalization $k_o R_o \psi(R_o, \theta, \varphi)$ and $k_o R_1 \psi(R_1, \theta, \varphi)$ are used following the similar reason that $\sqrt{k_o r}$ was used for the 2-D cylindrical case in the previous section. Using a similar approach for fractionalization of the kernel, the fractionalized kernel $K_\nu(R_1, \theta, \varphi; R_o, \theta, \varphi')$ can be explicitly written as

$$K_\nu(R_1, \theta, \varphi; R_o, \theta, \varphi') = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \left[\frac{R_1 h_l^{(1)}(k_o R_1)}{R_o h_l^{(1)}(k_o R_o)} \right]^\nu Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta, \varphi') \quad (21)$$

where ν is the fractionalization parameter. As in the 2-D cylindrical case, here again it can be easily seen that when $\nu \rightarrow 1$, the fractional kernel becomes the original kernel; and when $\nu \rightarrow 0$, the fractional kernel approaches

$$\sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta, \varphi') = \delta(\cos \theta - \cos \theta') \delta(\varphi - \varphi'), \text{ as anticipated.}$$

By equating the fractional kernel in Eq. (21) with the kernel of the integral transform that links the function $k_o R_o \psi(R_o, \theta, \varphi)$ with the function $k_o R_{\text{int}} \psi(R_{\text{int}}, \theta, \varphi)$ on the *intermediate* spherical surface of radius R_{int} where $R_o \leq R_{\text{int}} \leq R_1$, we obtain the values of the fractionalization parameter ν to be

$$\nu_l = \frac{\ln(R_{\text{int}} h_l^{(1)}(k_o R_{\text{int}})) - \ln(R_o h_l^{(1)}(k_o R_o))}{\ln(R_1 h_l^{(1)}(k_o R_1)) - \ln(R_o h_l^{(1)}(k_o R_o))} \quad (22)$$

As in the 2-D cylindrical case, here Eq. (22) also reveals that for any specific set of radii R_o , R_1 , and R_{int} one gets differing values for fractionalization parameter for each order l . One can easily observe that when $R_{\text{int}} \rightarrow R_1$, all the parameters ν_l 's become unity; and when $R_{\text{int}} \rightarrow R_o$, all ν_l 's become zero. When $R_o < R_{\text{int}} < R_1$, the parameters ν_l 's do not necessarily have the same value, and as in the 2-D cylindrical case, they can even attain

complex values. For these values of fractionalization parameters ν_l 's, the fractional kernel given in Eq. (21) can be equivalent with the kernel of the integral transform that relates the quantity of interest on the spherical surface R_o to the corresponding quantity on the *intermediate* concentric spherical surface with radius R_{int} . So for the 3-D spherical case, the counterpart of Eq. (16) can be given as follows

$$K_{\nu_o, \nu_1, \nu_2, \dots}(R_1, \theta, \varphi; R_o, \theta, \varphi') = K(R_{\text{int}}, \theta, \varphi; R_o, \theta, \varphi') \quad (23)$$

with the subscripts of the fractional kernel explicitly show the values of the fractionalization parameter ν_l 's obtained from Eq. (22) for each order l .

Finally, when the spherical Hankel functions can be approximated for the large argument, Eq. (22) can be simplified, leading to the following expression

$$\nu \cong \frac{R_{\text{int}} - R_o}{R_1 - R_o} \quad (24)$$

which is analogous to what we found for the 2-D cylindrical case. The implications of this result are similar to those discussed for the 2-D cylindrical case, and thus are not repeated here.

In summary, in this Letter we have extended the concept of the fractionalization of kernels, which we discussed in [1] for the planar case, to the cases of coaxial cylindrical surfaces for the 2-D wave propagation, and concentric spherical surfaces for the 3-D wave propagation. The salient features of these fractionalized kernels in the present cases are similar to those for the planar case, except here it turned out that the fractionalization parameters can in general attain differing values for each expansion term, and these values may even be complex. As the intermediate surface approaches either of the two originally selected surfaces, the fractionalization parameters approach either zero or unity, as anticipated. Under certain conditions on the radii of the two

original surfaces, the fractionalization parameters can be approximated and their form resembles that for the planar case.

Acknowledgements

This work is supported in part by the U.S. National Science Foundation Grant No. ECS-96-12634.

References

- [1] N. Engheta, "On Fractional Paradigm and Intermediate Zones in Electromagnetism: I. Planar Observation," accepted for publication in *Microwave and Optical Technology Letters*. Scheduled to appear in August 20, 1999.
- [2] N. Engheta, "Fractional Curl Operator in Electromagnetics," *Microwave and Optical Technology Letters*, Vol. 17, No. 2, pp. 86-91, February 5, 1998.
- [3] N. Engheta, "Fractional Paradigm in Electromagnetic Theory," a chapter in the book entitled *Frontiers of Electromagnetics*, D. H. Werner and R. Mittra (eds.), IEEE Press, to appear in 1999.
- [4] N. Engheta, "On Fractional Calculus and Fractional Multipoles in Electromagnetism," *IEEE Trans. Antennas & Propagation*, Vol. 44, No. 4, pp. 554-566, Apr. 1996. Erratum: Vol. 44, No. 9, p. 1307, September 1996.
- [5] N. Engheta, "Electrostatic "Fractional" Image Methods for Perfectly Conducting Wedges and Cones," *IEEE Trans. Antennas & Propagation*, Vol. 44, No. 12, pp. 1565-1574, Dec. 1996.
- [6] N. Engheta, "Use of Fractional Integration to Propose Some "Fractional" Solutions for the Scalar Helmholtz Equation," a chapter in *Progress in Electromagnetics Research (PIER)* monograph Series Vol. 12, Jin A. Kong (ed.), EMW Pub., Cambridge, MA, pp. 107-132, ch. 5, 1996.
- [7] N. Engheta, "A Note on Fractional Calculus and the Image Method for Dielectric Spheres," *J. of Electromagnetic Waves and Applications*, Vol. 9, No. 9, pp. 1179-1188, September 1995.

- [8] N. Engheta, "On the Role of Fractional Calculus in Electromagnetic Theory," in *IEEE Antennas and Propagation Magazine*, Vol. 39, No. 4, pp. 35-46, August 1997.
- [9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (ninth printing) Dover Publication, New York, 1970.
- [10] J. D. Jackson, *Classical Electrodynamics*, (3rd Edition) John Wiley & Sons, Inc., New York, 1999.