EMBEDDINGS AMONG TORUSES AND MESHES

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Abstract – Toruses and meshes include graphs of many varieties of topologies, with lines, rings, and hypercubes being special cases. Given a d-dimensional torus or mesh G and a c-dimensional torus or mesh H of the *same size*, we study the problem of embedding G in H to minimize the dilation cost. For *increasing dimension* cases ($d < c$) in which the shapes of G and H satisfy the condition of *expansion*, the dilation costs of our embeddings are either 1 or 2, depending on the types of graphs of G and H. These embeddings are optimal except when G is a torus of even size and H is a mesh. For *lowering dimension* cases ($d > c$) in which the shapes of G and H satisfy the condition of *reduction*, the dilation costs of our embeddings depend on the shapes of G and H. These embeddings, however, are not optimal in general. For the special cases in which G and H are *square*, the embedding results above can always be used to construct embeddings of G in H: these embeddings are all optimal for increasing dimension cases in which the dimension of H is divisible by the dimension of G, and all optimal to within a constant for fixed values of d and c for lowering dimension cases. Our main analysis technique is based on a generalization of Gray code for radix-2 (binary) numbering system to similar sequences for mixed-radix numbering systems.

Key words and phrases - Torus, mesh, hypercube, ring, interconnection network, embedding, dilation cost, Hamiltonian circuit.

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Contents

1 Introduction

An embedding of a graph G in a graph H is an injection (one-to-one mapping) of the nodes in G to the nodes in H. The graph embedding problem can be stated as follows: given a pair of graphs G and H, and a set of constraints and optimization measures, find an embedding of G in H that satisfies these constraints and optimizes these measures. Many variations of the graph embedding problem have been studied in the literature [AR82, BMS87, CS86, DEL78, DJ86, Ell88, Fit74, Har66, HJ87, HMR83, HMR73, KA88, LC76, LW87, MS88, RS78, Ros75, Ros78, Ros79, S87, Wu85]. These variations differ mainly in the relative sizes of G and H, the constraints imposed on the embeddings, and the optimization measures used in the embeddings. Many important problems in parallel processing can be formulated as the graph embedding problem. They include the problem of matching the communication requirements of a task with the communication support of a parallel system (by interpreting G as the task graph and H as the interconnection network) and the problem of evaluating the relative performance of a pair of interconnection networks (by interpreting G and H as interconnection networks).

This paper studies embeddings among toruses and meshes of various dimensions. A d-dimensional torus is a graph in which each node has two neighbors in each of the d dimensions. A d-dimensional mesh is a graph in which each node, except those at the boundaries, has two neighbors in each of the d dimensions, while a boundary node in any dimension has only one neighbor in that dimension. (The terms *array* and *grid* have also been used for *mesh* in the literature.) Toruses and meshes are two families of graphs that are important in parallel processing. These two families include lines, rings, and hypercubes. Many of these graphs arise naturally as task graphs in parallel processing, particularly in the application areas of image processing, robotics, and scientific computation [Fox83, HKS*83, RK82, BB82]. Furthermore, because of their regularity and simplicity, many of these graphs have also been used widely as the topologies of large-scale interconnection networks [LM87, Oru84, KWA82, PV79].

The most commonly used optimization measure in graph embeddings is *dilation cost*. The dilation cost of an embedding of G in H is the maximum distance in H between the images of any two adjacent nodes in G [HMR83]. This cost gives a measure of the *proximity* in H of the adjacent nodes in G under an embedding. In this paper, we study embeddings for which G and H are of the *same* size, using dilation cost as the optimization measure. Based on the dimension of G, we divide the embeddings among toruses and meshes into two classes: (i) basic embeddings, those for which the dimension of G is 1, that is, G is either a ring or a line; and (ii) generalized embeddings, those for which the dimension of G is greater than 1. Based on the dimensions of G and H, we further divide generalized embeddings into two classes: (i) generalized embeddings for increasing dimension, those for which the dimension of G is lower than the dimension of H; and (ii) generalized embeddings for lowering dimension, those for which the dimension of G is higher than the dimension of H. We study only those cases in generalized embeddings that satisfy some particular conditions: the condition of *expansion* for increasing dimension cases and the condition of *reduction* for lowering dimension cases.

All of our generalized embeddings are constructed from several optimal, basic embeddings, which are derived by generalizing the concept of Gray code for the radix-2 (binary) numbering system to similar sequences for mixed-radix numbering systems. Given a torus, we take the lengths of its dimensions as the radices of a mixed-radix numbering system, and for each pair of numbers in such a numbering system, we define a distance measure between them in the same way as the distance between a pair of nodes in the torus is defined. The problem of finding an embedding of a line (a ring) in a torus with minimum dilation cost is then transformed into the problem of finding a sequence (a cyclic sequence) of all numbers in a mixed-radix numbering system to minimize the *spread*, which is the maximum distance between any pair of consecutive numbers in the sequence. For the problem of embedding a line or a ring in a mesh, a similar transformation is made by using a different distance measure for a mixed-radix numbering system (one corresponding to the distance measure of a mesh instead of a torus).

All of our basic embeddings are optimal. With two exceptions, our embeddings have unit dilation cost. Our embeddings have an optimal dilation cost of 2 when (i) G is a ring of odd size and H is a mesh, and (ii) G is a ring and H is a line.

For increasing dimension cases in which the shapes of G and H satisfy the condition of expansion, our embeddings have dilation costs of either 1 or 2, depending on the types of graphs of G and H. Except for the case in which G is a torus of even size and H is a mesh, these embeddings are all optimal. For lowering dimension cases in which the shapes of G and H satisfy the condition of reduction, the dilation costs of our embeddings depend on the shapes of G and H. These embeddings, however, are not optimal in general.

For the special cases in which both G and H are square, we can always construct an embedding of G in H using our results for generalized embeddings. For increasing dimension cases in which the dimension of G is divisible by the dimension of H, our embeddings have a dilation cost of 2 if G is a torus of odd size and H is a mesh, and have unit dilation cost otherwise. These embeddings are all optimal. For lowering dimension cases, our embeddings have dilation cost $2\ell^{(d-c)/c}$ if G is a torus and H is a mesh, and $\ell^{(d-c)/c}$ otherwise, where ℓ is the length of the dimensions of G, d the dimension of G, and c the dimension of H. For fixed values of d and c, these embeddings are all optimal to within a constant.

A few special cases of the embedding problem studied in this paper have been solved optimally in the literature: embedding a mesh (of size some power of 2) in a hypercube [CS86], embedding a 2-dimensional square torus in a ring [MN86], embedding a 2-dimensional square mesh in a line [Fit74], embedding a 3-dimensional square mesh in a line [Fit74], and embedding a hypercube in a line [Har66]. We compare in detail the dilation costs of our embeddings with the dilation costs of these optimal embeddings in Sections 4 and 5. In addition to having minimum dilation cost, the embeddings of meshes in hypercubes given in [CS86] also satisfy other proximity properties, and they are derived using binary reflected Gray codes. Our basic embeddings and generalized embeddings for increasing dimension are derived using a generalization of the technique used in [CS86].

Other closely related results in the literature include the following: embeddings of 2-dimensional square meshes in lines to minimize *average proximity* [DEL78], embeddings of finite arrays (meshes), prism arrays, and orthant arrays in lines to minimize proximity in various local and global senses [Ros75], embeddings of 2-dimensional rectangular meshes in 2-dimensional square meshes to minimize the dilation costs while satisfying constraints on expansion costs [AR82, Ell88], embeddings of meshes in hypercubes with various expansion costs and dilation costs [S87, HJ87, BMS87], and simulations between rectangular meshes [KA88]. (In a *simulation* of G in H, a constant number of nodes in G can be mapped into a single node in H; thus, a simulation is not an injection but a many-to-one mapping.) With the exception of [KA88], in which the *costs* are expressed in terms of big O notation (thus, ignoring any constant factor), the *costs* in the papers cited above and in this paper are all exact.

2 Preliminaries

Unless stated otherwise, variables denote positive integers, logarithms refer to base 2, graphs are unweighted and undirected. Given an integer $n \geq 1$, we use $[n]$ to denote the set $\{0, 1, \ldots, n-1\}$, and $[n]^+$ to denote the set $\{1, 2, \ldots, n\}$. Given a list $(x_1, x_2, \ldots, x_p)$ and a list $(y_1, y_2, \ldots, y_q)$, we use $(x_1, x_2, \ldots, x_p) \diamond (y_1, y_2, \ldots, y_q)$ to denote the concatenation of the two lists: $(x_1, x_2, \ldots, x_p) \diamond (y_1, y_2, \ldots, y_q) = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$. Given two functions f and g, we use $f \circ g$ to denote the composition of f and g: $(f \circ g)(x) = f(g(x))$ for all x in the domain of g. Given a positive integer k, a list $(i_1, i_2, \ldots, i_k)$, and a permutation $\pi : [k]^+ \rightarrow [k]^+$, we use $\pi((i_1, i_2, \ldots, i_k))$ to denote $(i_{\pi(1)}, \ldots, i_{\pi(k)})$. Given a rational number x, we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

A *graph* $G = (V_G, E_G)$ is a pair consisting of a set V_G of nodes and a set E_G of edges. The size of G is $|V_G|$.

Definition 1 An *embedding* f of a graph $G = (V_G, E_G)$ in a graph $H = (V_H, E_H)$ is an injection $f : V_G \rightarrow V_H$. The *dilation cost* of f is $\max_{(i,j) \in E_G}\{\text{distance between nodes } f(i) \text{ and } f(j) \text{ in } H\}$. □

Definition 2 Let d be a positive integer, and $l_1, l_2, \ldots, l_d$ be integers greater than 1. An $(l_1, l_2, \ldots, l_d)$-*torus* is a connected graph with $\prod_{i \in [d]^+} l_i$ nodes. The nodes are all lists $(i_1, i_2, \ldots, i_d)$, where for all $j \in [d]^+$, $i_j \in [l_j]$. For each node $A = (i_1, i_2, \ldots, i_d)$ and each $j \in [d]^+$, A has in the j-th dimension a *left neighbor* $(i_1, i_2, \ldots, i_{j-1}, (i_j - 1) \bmod l_j, i_{j+1}, \ldots, i_d)$ and a *right neighbor* $(i_1, i_2, \ldots, i_{j-1}, (i_j + 1) \bmod l_j, i_{j+1}, \ldots, i_d)$. □

Given an $(l_1, l_2, \ldots, l_d)$-torus, $(l_1, l_2, \ldots, l_d)$ is the *shape* of the torus; d is the *dimension* of the torus; and for all $j \in [d]^+$, l_j is the *length* of the j-th dimension of the torus. If $l_1 = l_2 = \cdots = l_d$, we say that the torus is *square*. A torus of dimension 1 is a *ring*. For convenience in presentation,

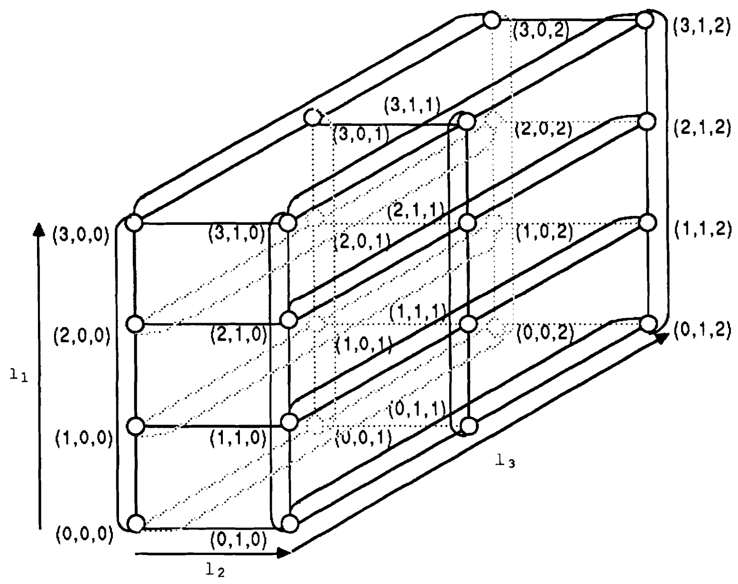


Figure 1: A (4, 2, 3)-torus

given a ring of size n, instead of using the lists (0), (1), ..., $(n-1)$ to denote its nodes, we often use the integers 0, 1, ..., $n-1$. An example of a $(4,2,3)$-torus is given in Figure 1.

Definition 3 Let d be a positive integer, and $l_1, l_2, \ldots, l_d$ be integers greater than 1. An $(l_1, l_2, \ldots, l_d)$-*mesh* is a connected graph with $\prod_{i\in[d]^+} l_i$ nodes. The nodes are all lists $(i_1, i_2, \ldots, i_d)$, where for all $j \in [d]^+$, $i_j \in [l_j]$. For each node $A = (i_1, i_2, \ldots, i_d)$ and each $j \in [d]^+$, if $i_j \notin \{0,\ l_j - 1\}$, then A has in the j-th dimension a *left neighbor* $(i_1, i_2, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_d)$ and a *right neighbor* $(i_1, i_2, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_d)$. If $i_j = 0$, then A has no left neighbor in the j-th dimension, and if $i_j = l_j - 1$, then A has no right neighbor in the j-th dimension. $\Box$

The terms *shape*, *dimension*, *length* of a dimension, and *square* for meshes are defined in the same way as for toruses. A mesh of dimension 1 is a *line*. Given a line of size n, we often use the integers 0, 1, ..., $n-1$ to denote its nodes. An example of a $(4,2,3)$-mesh is given in Figure 2.

Given a torus or a mesh G, the *type* of G refers to whether G is a torus or a mesh. Two graphs are of the *same type* if they are both toruses or both meshes.

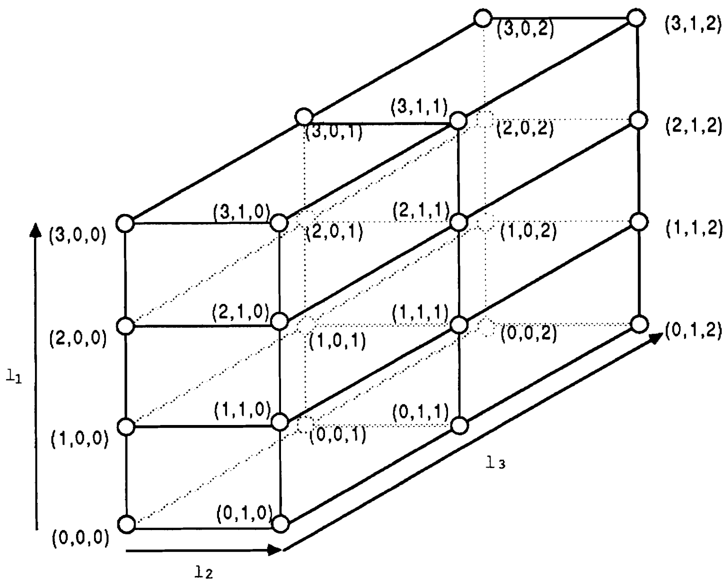


Figure 2: A (4, 2, 3)-mesh

Definition 4 Let $n = 2^d$, for some positive integer d. A *hypercube* of size n is a connected graph in which the nodes are all lists $(i_1, i_2, \ldots, i_d)$, where for all $i \in [d]^+$, $i_j \in \{0, 1\}$. A pair of nodes A and B are *neighbors* if the lists A and B differ in exactly one position. □

A graph G is a hypercube *if and only if* G is both a torus and a mesh: a hypercube of size n is both a $(\log n)$-dimensional torus and a $(\log n)$-dimensional mesh in which the length of each dimension is 2.

For every pair of nodes v and v' in a connected graph G, the *distance* between v and v' in G is the length of the shortest paths between v and v' in G. The following two lemmas follow directly from the definitions of toruses and meshes.

Lemma 5 *Let G be an $(l_1, l_2, \ldots, l_d)$-torus, and $A = (i_1, i_2, \ldots, i_d)$ and $B = (i'_1, i'_2, \ldots, i'_d)$ be a pair of nodes in G. The distance between A and B in G, denoted by $\delta_t(A, B)$, is $\sum_{k=1}^{d} \min\{|i_k - i'_k|, l_k - |i_k - i'_k|\}$.* □

Lemma 6 *Let G be an $(l_1, l_2, \ldots, l_d)$-mesh, and $A = (i_1, i_2, \ldots, i_d)$ and $B = (i'_1, i'_2, \ldots, i'_d)$ be a pair of nodes in G. The distance between A and B in G, denoted by $\delta_m(A, B)$, is $\sum_{k=1}^{d} |i_k - i'_k|$.* □

In the torus given in Figure 1, the distance between the nodes $(0,0,1)$ and $(3,0,0)$ is 2, and in the mesh given in Figure 2, the distance between the nodes $(0,0,1)$ and $(3,0,0)$ is 4.

Definition 7 Let d be a positive integer, and $l_1, l_2, \ldots, l_d$ be integers greater than 1. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$, and $n = \prod_{i=1}^{d} l_i$. For all $i \in [d+1]$, let $w_i = \prod_{j=i+1}^{d} l_j$. For all $x \in [n]$, the *radix-$\mathcal{L}$ representation* of x is the d-tuple $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)$ such that for all $j \in [d]^+$, $\hat{x}_j = \lfloor x/w_j \rfloor \bmod l_j$. $\mathcal{L}$ is a *radix-base*, and w_0, w_1, ..., w_d are the *weights* in the radix-$\mathcal{L}$ representation. The set of all radix-$\mathcal{L}$ numbers, denoted by $\Omega_{\mathcal{L}}$, is the set of radix-$\mathcal{L}$ representation of x, for all $x \in [n]$. $\Omega_{\mathcal{L}}$ is a *mixed-radix numbering system.* Let $u_{\mathcal{L}} : [n] \to \Omega_{\mathcal{L}}$ denote the bijection given above that maps each integer in $[n]$ to its radix-$\mathcal{L}$ representation in $\Omega_{\mathcal{L}}$. Let $u_{\mathcal{L}}^{-1} : \Omega_{\mathcal{L}} \to [n]$ denote the inverse of $u_{\mathcal{L}}$. For every number $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d) \in \Omega_{\mathcal{L}}$, $u_{\mathcal{L}}^{-1}((\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)) = \sum_{k=1}^{d} \hat{x}_k w_k$. $\Box$

Every integer in $[n]$ has unique radix-$\mathcal{L}$ representation [TM75]. Note that the weight w_0 is not used in the definition of radix-$\mathcal{L}$ representation of numbers. This weight is included only for the simplification of our later definitions and analyses. Again, for convenience in presentation, when $d = 1$, instead of using the list (l_1) to denote a radix-base $\mathcal{L}$, and the lists (0), (1), ..., $(l_1 - 1)$ to denote the numbers in $\Omega_{\mathcal{L}}$, we often use the integer l_1, and 0, 1, ..., $l_1 - 1$, respectively. An example of the radix-$(4,2,3)$ numbering system is given in Figure 9 on page 26. In this example, $l_1 = 4$, $l_2 = 2$, $l_3 = 3$, $w_1 = 6$, $w_2 = 3$, and $w_3 = 1$.

Given a radix-base $\mathcal{L} = (l_1, l_2, \ldots, l_d)$, we can view the radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$ as either the nodes in an $(l_1, l_2, \ldots, l_d)$-torus or the nodes in an $(l_1, l_2, \ldots, l_d)$-mesh using the obvious bijections. We can thus define the *δ_t-distance* and the *δ_m-distance* between a pair of radix-$\mathcal{L}$ numbers as the distances between the corresponding pair of nodes in a torus and in a mesh, respectively. By the definitions of δ_m-distance and δ_t-distance, the δ_m-distance between any two numbers in $\Omega_{\mathcal{L}}$ is always greater than or equal to their δ_t-distance.

Definition 8 Let n be a positive integer, $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ a radix-base, and $f : [n] \to \Omega_{\mathcal{L}}$ a bijection. Such a function f is often treated as an *acyclic sequence,* namely, $f(0)$, $f(1)$, ..., $f(n-1)$. For all $i \in [n-1]$, $f(i)$ and $f(i+1)$ are *successive* elements in the acyclic sequence f. If the first and the last elements, $f(0)$ and $f(n-1)$, are also taken to be successive, then f is called a *cyclic sequence.* The *δ_m-spread* of the acyclic sequence f is the maximum of the

i	0	1	2	3	4	5	6	7	8
$f(i)$	(0,0)	(0,1)	(0,2)	(2,2)	(2,1)	(2,0)	(1,0)	(1,1)	(1,2)

(a)

i	0	1	2	3	4	5	6	7	8
$\delta_m(f(i), f((i+1) \bmod 9))$	1	1	2	1	1	1	1	1	3
$\delta_t(f(i), f((i+1) \bmod 9))$	1	1	1	1	1	1	1	1	2

(b)

Figure 3: A function f with $n = 9$ and $\mathcal{L} = (3, 3)$

δ_m-distances among all pairs of successive elements in f, and the *δ_t-spread* of the acyclic sequence f is the maximum of the δ_t-distances among all pairs of successive elements in f. The *δ_m-spread* and *δ_t-spread* of the cyclic sequence f are defined similarly. □

In the definition above, a function f can be treated as either an acyclic sequence or a cyclic sequence, depending on the way that successive elements are defined. Furthermore, whether f is viewed as cyclic or acyclic, we can always define a δ_m-distance and a δ_t-distance between pairs of elements of f. In the remainder of this paper, we simply call an acyclic sequence a *sequence*. Figure 3(a) gives an example of a function $f : [9] \to \Omega_{(3,3)}$, and Figure 3(b) shows the δ_m-distance and δ_t-distance between the pair $f(i)$ and $f((i+1) \bmod 9)$, for all $i \in [9]$. In this example, if we view f as an acyclic sequence, then the δ_m-spread of f is 2, and the δ_t-spread of f is 1. If we view f as a cyclic sequence, then the δ_m-spread of f is 3, and the δ_t-spread of f is 2.

As will be discussed in detail in the next section, given an embedding f of G in H, we often view f as an acyclic sequence if G is a line, and as a cyclic sequence if G is a ring. We use δ_m-distance measure on f if H is a mesh, and δ_t-distance measure if H is a torus.

For the special case in which $n = 2^d$ and $\mathcal{L}$ is a list of d elements each equal to 2, if the function $f : [n] \to \Omega_{\mathcal{L}}$ has unit δ_t-spread (which is the same as the δ_m-spread in this case), then the sequence f is called a *Gray code* [RJD77].

3 Basic embeddings

In this section, we consider the embeddings of either a line or a ring in a mesh or a torus. The major results of this section are the following:

(i) A line can always be embedded in a mesh or a torus with unit dilation cost.

(ii) A ring can always be embedded in a torus with unit dilation cost.

(iii) A ring can be embedded in a mesh with unit dilation cost if the ring is of even size and the mesh has dimension greater than 1, and with an optimal dilation cost of 2 otherwise.

3.1 Embedding a line in a mesh or a torus

Let G be a line of size n, and H be an $(l_1, l_2, \ldots, l_d)$-mesh or an $(l_1, l_2, \ldots, l_d)$-torus such that $n = \prod_{i=1}^{d} l_i$. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$. The problem of embedding G in H can be considered in terms of the radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$: the nodes in G are all numbers in $[n]$; the nodes in H are all radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$; and an embedding f of G in H is a bijection from $[n]$ to $\Omega_{\mathcal{L}}$. Since the neighbors in G correspond to the pairs of successive numbers in the sequence $0,\ 1,\ ,\ldots,\ n-1$, the dilation cost of an embedding f is the δ_m-spread of the sequence f if H is a mesh, and the δ_t-spread if H is a torus. The problem of finding an embedding of G in H with minimum dilation cost thus corresponds to the problem of finding a sequence of all numbers in $\Omega_{\mathcal{L}}$ with minimum δ_m-spread if H is a mesh, and finding one with minimum δ_t-spread if H is a torus.

Since the δ_t-spread of a sequence is never greater than its δ_m-spread, to prove that a line can be embedded in a mesh or a torus with unit dilation cost, it suffices to prove that we can construct a sequence of all numbers in $\Omega_{\mathcal{L}}$ with unit δ_m-spread.

Let P be the sequence of numbers $0, 1, \ldots, n-1$ in their radix-$\mathcal{L}$ representations. In the following, we first show that the δ_m-spread of P is at least 2 for all $d > 1$, and then construct another sequence P' from P with unit δ_m-spread.

In the sequence P, every element a is of the form $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_d)$, where $\hat{a}_i \in [l_i]$, for all $i \in [d]^+$. Every element a in P thus consists of d components. The sequence P can be viewed as consisting of d separate sequences of natural numbers, namely p_1, p_2, $\ldots$, p_d, all of length n,

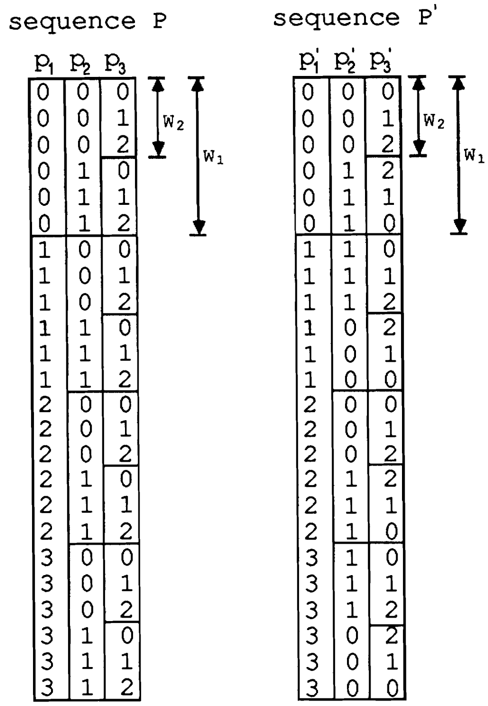


Figure 4: Sequences P and P' for $\mathcal{L} = (4,\ 2,\ 3)$

one for each of the d components of the elements in P. Let w_0, w_1, $\ldots$, w_d be the weights in the radix-$\mathcal{L}$ representation. From the properties of the radix-$\mathcal{L}$ representation of numbers, for all $i \in [d]^+$, the sequence p_i can be partitioned into n/w_{i-1} segments, each with w_{i-1} elements and of the form $\underbrace{0\cdots0}_{w_i}\underbrace{1\cdots1}_{w_i}\underbrace{(l_i-1)\cdots(l_i-1)}_{w_i}$. We number these segments from 0 to $n/w_{i-1}-1$ successively. For every pair of successive elements in p_i, for all $i \in [d]^+$, if they belong to the same segment in p_i, then their difference is at most 1; otherwise, their difference is $l_i - 1$. The sequence P has thus a δ_m-spread greater than 1 for all $d > 1$. An example of the sequence P for $\mathcal{L} = (4,\ 2,\ 3)$ and $n = 24$ is shown in Figure 4.

We next construct a sequence P' with unit δ_m-spread from P. The sequence P' can also be viewed as consisting of d sequences, p_1', p_2', $\ldots$, p_d'. For all $i \in [d]^+$, p_i' is constructed from p_i by reversing all of the odd-numbered segments of p_i, which produces segments of the form

$\underbrace{(l_i-1)\cdots(l_i-1)}_{w_i}\underbrace{1\cdots1}_{w_i}\underbrace{0\cdots0}_{w_i}$, and by leaving all of the even-numbered segments unchanged. As will be proved below, for every pair of successive elements in p_i', if they belong to the same segment, their difference is at most 1; otherwise, their difference is 0. The sequence P' has unit δ_m-spread. An example of P' for $\mathcal{L} = (4,\ 2,\ 3)$ and $n = 24$ is shown in Figure 4.

We now define a function $f_{\mathcal{L}} : [n] \to \Omega_{\mathcal{L}}$. Lemma 10 shows that the sequence $f_{\mathcal{L}}$ is a sequence of all numbers in $\Omega_{\mathcal{L}}$, and Lemma 11 and Lemma 12 show respectively that the sequence $f_{\mathcal{L}}$ has unit δ_m-spread and unit δ_t-spread. The sequence $f_{\mathcal{L}}$ is P'.

Definition 9 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. Let $w_0,\ w_1,\ \ldots,\ w_d$ be the weights in the radix-$\mathcal{L}$ representation. For all $x \in [n]$, let $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)$ be the radix-$\mathcal{L}$ representation of x. The function $f_{\mathcal{L}} : [n] \to \Omega_{\mathcal{L}}$ is defined as follows: for all $x \in [n]$, $f_{\mathcal{L}}(x) = (x_1, x_2, \ldots, x_d)$, where for all $i \in [d]^+$,

$$x_i = \begin{cases} \hat{x}_i, & \text{if } \lfloor x/w_{i-1} \rfloor \text{ is even;} \\ l_i - \hat{x}_i - 1, & \text{if } \lfloor x/w_{i-1} \rfloor \text{ is odd.} \end{cases}$$

□

In the definition above, for all $i \in [d]^+$, $\lfloor x/w_{i-1} \rfloor$ determines the segment in the sequence p_i to which $\hat{x}_i$ belongs. An example of the function $f_{\mathcal{L}}$ is given in Figure 9 on page 26.

We say that two numbers have the same *parity* if they are both even or both odd.

Lemma 10 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. The function $f_{\mathcal{L}}$ is bijective.* □

Proof. Since $|\Omega_{\mathcal{L}}| = n$, to show that $f_{\mathcal{L}}$ is bijective, it is sufficient to show that $f_{\mathcal{L}}$ is injective. Let x and y be an arbitrary pair of distinct integers in $[n]$. We want to show that $f_{\mathcal{L}}(x) \neq f_{\mathcal{L}}(y)$. Let $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)$ and $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_d)$ be the radix-$\mathcal{L}$ representations of x and y. Let $f_{\mathcal{L}}(x) = (x_1, x_2, \ldots, x_d)$, and $f_{\mathcal{L}}(y) = (y_1, y_2, \ldots, y_d)$. Since every integer in $[n]$ has a unique radix-$\mathcal{L}$ representation, there is at least one index $i \in [d]^+$ such that $\hat{x}_i \neq \hat{y}_i$. Let $k \in [d]^+$ be the smallest index such that $\hat{x}_k \neq \hat{y}_k$. We first show that $\lfloor x/w_{k-1} \rfloor$ and $\lfloor y/w_{k-1} \rfloor$ have the same parity. There are two cases:

Case 1. $k = 1$.

Since $w_0 = n$, $\lfloor x/w_0 \rfloor = \lfloor y/w_0 \rfloor = 0$. Thus, $\lfloor x/w_0 \rfloor$ and $\lfloor y/w_0 \rfloor$ have the same parity.

Case 2. $k > 1$.

Assume for contradiction that $\lfloor x/w_{k-1} \rfloor$ and $\lfloor y/w_{k-1} \rfloor$ have different parities. This implies that $\lfloor x/w_{k-1} \rfloor \neq \lfloor y/w_{k-1} \rfloor$. Since $\hat{x}_{k-1} = \hat{y}_{k-1}$, we also have $\lfloor x/w_{k-1} \rfloor \bmod l_{k-1} = \lfloor y/w_{k-1} \rfloor \bmod l_{k-1}$. It follows that $|\,\lfloor x/w_{k-1} \rfloor - \lfloor y/w_{k-1} \rfloor\,| = c\, l_{k-1}$, for some positive integer c. By the definition of radix-base, $l_{k-1} > 1$, and hence, $|\,\lfloor x/w_{k-1} \rfloor - \lfloor y/w_{k-1} \rfloor\,| > 1$. This implies that $|x - y| > w_{k-1}$. On the other hand, since k is the smallest index such that $\hat{x}_k \neq \hat{y}_k$, we have

$$|x - y| \leq \sum_{j=k}^{d} |\ \hat{x}_j w_j - \hat{y}_j w_j\ | \leq \ \sum_{j=k}^{d} (l_j - 1)\ w_j.$$

Since by definition, for all $j \in [d+1]$, $w_j = \prod_{j=i+1}^{d} l_j$, we have for all $j \in [d]^+$, $l_j w_j = w_{j-1}$. Thus,

$$|x - y| \leq \sum_{j=k-1}^{d-1} w_j - \sum_{j=k}^{d} w_j = w_{k-1} - w_d < w_{k-1},$$

which is a contradiction. Therefore, $\lfloor x/w_{k-1} \rfloor$ and $\lfloor y/w_{k-1} \rfloor$ have the same parity.

If $\lfloor x/w_{k-1} \rfloor$ and $\lfloor y/w_{k-1} \rfloor$ are both even, then we have $x_k = \hat{x}_k$ and $y_k = \hat{y}_k$. If they are both odd, then we have $x_k = l_k - \hat{x}_k - 1$ and $y_k = l_k - \hat{y}_k - 1$. In either case, the fact that $\hat{x}_k \neq \hat{y}_k$ implies that $x_k \neq y_k$. Thus, $f_{\mathcal{L}}(x) \neq f_{\mathcal{L}}(y)$. The function $f_{\mathcal{L}}$ is therefore bijective. $\Box$

Lemma 11 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. For all $x \in [n-1]$, $\delta_m(f_{\mathcal{L}}(x), f_{\mathcal{L}}(x+1)) = 1$.* $\Box$

Proof. Let x be an arbitrary number in $[n-1]$, and let $y = x + 1$. Let $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_d)$ and $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_d)$ be the radix-$\mathcal{L}$ representations of x and y. Let $f_{\mathcal{L}}(x) = (x_1, x_2, \ldots, x_d)$, and $f_{\mathcal{L}}(y) = (y_1, y_2, \ldots, y_d)$. We want to show that $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ differ by 1 in exactly one position.

First we look at the relationship between the values of $\hat{x}_i$ and $\hat{y}_i$ for all $i \in [d]^+$. Since $x < n - 1$, by the properties of the radix-$\mathcal{L}$ representation of numbers, there exists exactly one index $k \in [d]^+$ such that $\hat{x}_k < l_k - 1$ and for all $i \in \{k+1,\ \ldots,\ d\}$, $\hat{x}_i = l_i - 1$. Since $y = x + 1$, for all $i \in \{k+1,\ \ldots,\ d\}$, $\hat{y}_i = 0$; $\hat{y}_k = \hat{x}_k + 1$; and for all $i \in \{1,\ \ldots,\ k-1\}$, $\hat{y}_i = \hat{x}_i$.

We now look at the relationship between x_i and y_i, for all $i \in [d]^+$. There are three cases:

Case 1. $i \in \{k+1, \ldots, d\}$.

First we show that $\lfloor x/w_{i-1} \rfloor$ and $\lfloor y/w_{i-1} \rfloor$ have different parities. Since $\hat{x}_{i-1} \neq \hat{y}_{i-1}$, we have $\lfloor x/w_{i-1} \rfloor \bmod l_{i-1} \neq \lfloor y/w_{i-1} \rfloor \bmod l_{i-1}$, and hence, $\lfloor x/w_{i-1} \rfloor \neq \lfloor y/w_{i-1} \rfloor$. Furthermore, since x and y differ only by 1, $\lfloor x/w_{i-1} \rfloor = \lfloor y/w_{i-1} \rfloor + 1$. Therefore, $\lfloor x/w_{i-1} \rfloor$ and $\lfloor y/w_{i-1} \rfloor$ have different parities. Since $\hat{x}_i = l_i - 1$ and $\hat{y}_i = 0$, we have $x_i = y_i$.

Case 2. $i \in [k-1]^+$.

First we show that $\lfloor x/w_{i-1} \rfloor$ and $\lfloor y/w_{i-1} \rfloor$ have the same parity. If $i = 1$, then since $w_0 = n$, we have $\lfloor x/w_0 \rfloor = \lfloor y/w_0 \rfloor = 0$. Therefore, $\lfloor x/w_{i-1} \rfloor$ and $\lfloor y/w_{i-1} \rfloor$ have the same parity. If $i \in \{2, 3, \ldots, k-1\}$, then since $\hat{x}_{i-1} = \hat{y}_{i-1}$, we have $\lfloor x/w_{i-1} \rfloor \bmod l_{i-1} = \lfloor y/w_{i-1} \rfloor \bmod l_{i-1}$. Furthermore, since $l_{i-1} > 1$ and x and y differ only by 1, we have $\lfloor x/w_{i-1} \rfloor = \lfloor y/w_{i-1} \rfloor$. Therefore, $\lfloor x/w_{i-1} \rfloor$ and $\lfloor y/w_{i-1} \rfloor$ also have the same parity. Since $\hat{x}_i = \hat{y}_i$, we have $x_i = y_i$.

Case 3. $i = k$.

Using a proof as the one in Case 2, we can show that $\lfloor x/w_{k-1} \rfloor$ and $\lfloor y/w_{k-1} \rfloor$ have the same parity. Since $\hat{y}_k = \hat{x}_k + 1$, we have $|y_k - x_k| = 1$.

Since $\delta_m(f_{\mathcal{L}}(x), f_{\mathcal{L}}(x+1)) = \sum_{i=1}^{d} |x_i - y_i|$, we have $\delta_m(f_{\mathcal{L}}(x), f_{\mathcal{L}}(x+1)) = 1$. □

Lemma 12 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. For all $x \in [n-1]$, $\delta_t(f_{\mathcal{L}}(x), f_{\mathcal{L}}(x+1)) = 1$.* □

Proof. Since for any two numbers in $\Omega_{\mathcal{L}}$ their δ_m-distance is never less than their δ_t-distance, the claim follows from Lemma 11. □

Theorem 13 *Let G be a line, and H be either an $(l_1, l_2, \ldots, l_d)$-torus or an $(l_1, l_2, \ldots, l_d)$-mesh such that G and H are of the same size. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$. The line G can be embedded in H with unit dilation cost. The function $f_{\mathcal{L}}$ gives such an optimal embedding.* □

Proof. The theorem follows from Lemmas 10, 11, and 12 by interpreting the numbers in $[n]$ as the nodes in G, and the radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$ as the nodes in H. □

An example of embedding a line in a mesh using the function $f_{\mathcal{L}}$ is given in Figure 10 on page 27.

3.2 Embedding a ring in a mesh or a torus

Let G be a ring of size n, and H be either an $(l_1, l_2, \ldots, l_d)$-mesh or an $(l_1, l_2, \ldots, l_d)$-torus such that $n = \prod_{i=1}^{d} l_i$. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$. As with the problem of embedding a line in a mesh, we can consider this problem in terms of the radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$. The neighbors in a ring of size n correspond to the pairs of successive numbers in the *cyclic* sequence $0,\ 1,\ ,\ldots,\ n-1$. The problem of finding an embedding of G in H with minimum dilation cost thus corresponds to the problem of finding a cyclic sequence of all radix-$\mathcal{L}$ numbers in $\Omega_{\mathcal{L}}$ with minimum δ_m-spread if H is a mesh and finding one with minimum δ_t-spread if H is a torus.

In this section, we first show that the δ_m-spread of the cyclic sequence $f_{\mathcal{L}}$ is at least $l_1 - 1$. We then construct from $f_{\mathcal{L}}$ another cyclic sequence $g_{\mathcal{L}}$ with a δ_m-spread of 2. The function $g_{\mathcal{L}}$ provides an embedding of a ring in a mesh with a dilation cost of 2. We also prove that a ring of odd size cannot be embedded in a mesh of the same size with unit dilation cost. The embedding function $g_{\mathcal{L}}$ is therefore optimal for all rings and meshes of odd sizes. Finally, we construct a cyclic sequence $h_{\mathcal{L}}$ that has unit δ_m-spread if $\mathcal{L}$ consists of at least two components, and with the first component being an even number. The function $h_{\mathcal{L}}$ can be used to construct an embedding of a ring of even size in a higher-dimensional mesh with unit dilation cost. Furthermore, the cyclic sequence $h_{\mathcal{L}}$ has unit δ_t-spread. Thus, the function $h_{\mathcal{L}}$ also provides an optimal embedding of a ring in a torus with unit dilation cost.

3.2.1 Embedding a ring in a mesh

The embedding function $g_{\mathcal{L}}$

Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. Let $f_{\mathcal{L}}(n-1) = (n_1,\ n_2,\ \ldots,\ n_d)$. The radix-$\mathcal{L}$ representation of $n-1$ is $(l_1 - 1,\ l_2 - 1,\ \ldots,\ l_d - 1)$. Since $w_0 = n$, we have $\lfloor (n-1)/w_0 \rfloor = 0$, which is even. It follows from the definition of $f_{\mathcal{L}}$ that $n_1 = l_1 - 1$. Hence, the δ_m-distance between $f_{\mathcal{L}}(0)$ and $f_{\mathcal{L}}(n-1)$ is at least $l_1 - 1$. The cyclic sequence $f_{\mathcal{L}}$ therefore has a δ_m-spread of at least $l_1 - 1$.

A cyclic sequence with a δ_m-spread of 2 can be constructed from $f_{\mathcal{L}}$ in the following way. We number all the elements in $f_{\mathcal{L}}$ successively from 0 to $n-1$. Let R' and R'' be the following two sequences: R' consists of all even-numbered elements in $f_{\mathcal{L}}$ in the same order as they appear in $f_{\mathcal{L}}$, and R'' consists of all odd-numbered elements in $f_{\mathcal{L}}$ in the reverse order. Since the sequence $f_{\mathcal{L}}$ has unit δ_m-spread, both R' and R'' have a δ_m-spread of 2. The cyclic sequence $R'R''$, the concatenation of R' and R'', has a δ_m-spread of 2: the first element in R' and the last element in R'' correspond to the first two elements in $f_{\mathcal{L}}$; the last element in R' and the first element in R'' correspond to the last two elements in $f_{\mathcal{L}}$; and the sequence $f_{\mathcal{L}}$ has a unit δ_m-spread.

We first define the function $t_n : [n] \to [n]$. This function defines a cyclic sequence of all numbers in $[n]$ with a δ_m-spread of 2. We then define the function $g_{\mathcal{L}} : [n] \to \Omega_{\mathcal{L}}$ using $f_{\mathcal{L}}$ and t_n. The sequence $g_{\mathcal{L}}$ is $R'R''$.

Definition 14 Let n be any positive integer. The function $t_n : [n] \to [n]$ is defined as follows: for all $x \in [n]$,

if n is even, then

$$t_n(x) = \begin{cases} 2x, & \text{if } x < n/2; \\ n - 2(x - n/2) - 1, & \text{otherwise;} \end{cases}$$

if n is odd, then

$$t_n(x) = \begin{cases} 2x, & \text{if } x < (n+1)/2; \\ n - 2(x - (n+1)/2) - 2, & \text{otherwise.} \end{cases}$$

□

Definition 15 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. The function $g_{\mathcal{L}} : [n] \to \Omega_{\mathcal{L}}$ is defined as follows: for all $x \in [n]$,

$$g_{\mathcal{L}}(x) = f_{\mathcal{L}}(t_n(x)).$$

□

An example of the function $g_{\mathcal{L}}$ for $\mathcal{L} = (4, 2, 3)$ is given in Figure 9 on page 26. It is clear that the function $g_{\mathcal{L}}$ is bijective. The next lemma follows directly from the definition of $g_{\mathcal{L}}$ and the properties of $f_{\mathcal{L}}$.

Lemma 16 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. For all $x \in [n]$, $\delta_m(g_{\mathcal{L}}(x), g_{\mathcal{L}}((x+1) \bmod n)) \leq 2$.* □

Theorem 17 *Let G be a ring, and H be an $(l_1, l_2, \ldots, l_d)$-mesh such that G and H are of the same size. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$. The ring G can always be embedded in H with a dilation cost of 2. The function $g_{\mathcal{L}}$ gives such an embedding. Furthermore, such an embedding is optimal if H is a line of size greater than 2 or has odd size.* □

Proof. We need only prove that a ring cannot be embedded in either a line of size greater than 2 or a mesh of odd size with unit dilation cost. The other part of the theorem follows from Lemma 16.

For the case in which H is a line of size greater than 2, it suffices to notice that since each of the two boundary nodes of H has only one neighbor, a ring cannot be embedded in the line with unit dilation cost. For the case in which H is of odd size and of dimension greater than 1, we prove the theorem by showing that there is no Hamiltonian circuit in such a mesh.

Assume for contradiction that a Hamiltonian circuit exists in an $(l_1, l_2, \ldots, l_d)$-mesh of odd size. Since the mesh has an odd number of nodes, the circuit also has an odd number of edges. By specifying a direction in the circuit, we can view all of the edges in the circuit as directed. Each node in the mesh is a list of d components, $(i_1, i_2, \ldots, i_d)$, where $i_j \in [l_j]$, for all $j \in [d]^+$. Since each edge (u, v) in the circuit connects a pair of neighboring nodes in the mesh, u and v differ in exactly one component by 1, that is, v can be obtained from u by either increasing or decreasing exactly one component of u by 1. Furthermore, for each edge (u, v) in the circuit, if v can be obtained from u by increasing the k-th component of u from a to $a+1$, where $k \in [d]^+$ and a, $a+1 \in [l_k]$, then there must exist an edge (s, t) in the circuit such that t can be obtained from s by decreasing the k-th component of s from $a+1$ to a; otherwise, if we traverse the circuit starting from the node u, we will not be able to return to u in the circuit. For a similar reason, the reverse of the above condition is also true: if v can be obtained from u by decreasing the k-th component of u from $a+1$ to a, then there must exist an edge (s, t) in the circuit such that t can be obtained from s by increasing the k-th component of s from a to $a+1$. It follows that every edge in the circuit has a unique *mate*. Therefore, the number of edges in the circuit is even. This contradicts the assumption that H is of odd size. □

An example of an embedding of a ring of size 24 in a $(4, 2, 3)$-mesh using the function $g_{\mathcal{L}}$ is given in Figure 10 on page 27.

The proof of the following corollary is contained in the proof of the theorem above.

Corollary 18 *There is no Hamiltonian circuit in any mesh of odd size.* □

For the special case in which the mesh is of even size and of dimension at least 2, a ring can always be embedded in it with unit dilation cost. In the following, we first construct an embedding function $r_{\mathcal{L}}$ for the simple case in which the dimension of the mesh is exactly 2, and then construct a function $h_{\mathcal{L}}$ for the case in which the dimension of the mesh is at least 2.

The embedding function $r_{\mathcal{L}}$

The following lemma gives a property of $f_{\mathcal{L}}$ that will be used in the construction of the function $r_{\mathcal{L}}$.

Lemma 19 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. If l_1 is even, then $f_{\mathcal{L}}(n-1) = (l_1 - 1,\ 0, \ldots,\ 0)$.* □

Proof. By definition, the radix-$\mathcal{L}$ representation of $n-1$ is $(l_1 - 1, l_2 - 1, \ldots, l_d - 1)$. Since $w_0 = n$, we have $\lfloor (n-1)/w_0 \rfloor = 0$, which is even. We want to show that if l_1 is even, then for all $i \in \{2, \ldots, d\}$, $\lfloor (n-1)/w_{i-1} \rfloor$ is odd. These results together with the definition of the function $f_{\mathcal{L}}$ will then imply the lemma.

Since $n = \prod_{k=1}^{d} l_k$, and, by definition, for all $i \in \{2, \ldots, d\}$, $w_{i-1} = \prod_{j=i}^{d} l_j$, we can write $\lfloor (n-1)/w_{i-1} \rfloor$ as $\lfloor \prod_{j=1}^{i-1} l_j - (1/w_{i-1}) \rfloor$. Furthermore, since $0 < (1/w_{i-1}) \leq 1$, we have $\lfloor (n-1)/w_{i-1} \rfloor = \prod_{j=1}^{i-1} l_j - 1$. Therefore, for all $i \in \{2, \ldots, d\}$, $\lfloor (n-1)/w_{i-1} \rfloor$ is odd if l_1 is even. □

Let G be a ring, and H be an $(l_1,\ l_2)$-mesh such that l_1 is even, and G and H are of the same size. Let $\mathcal{L} = (l_1,\ l_2)$. We assume the following coordinates: the origin of the mesh H, $(0, 0)$, is at the lower left corner, the first dimension increases vertically upward, and the second dimension increases horizontally to the right. If we use the function $f_{\mathcal{L}}$ to embed the ring in

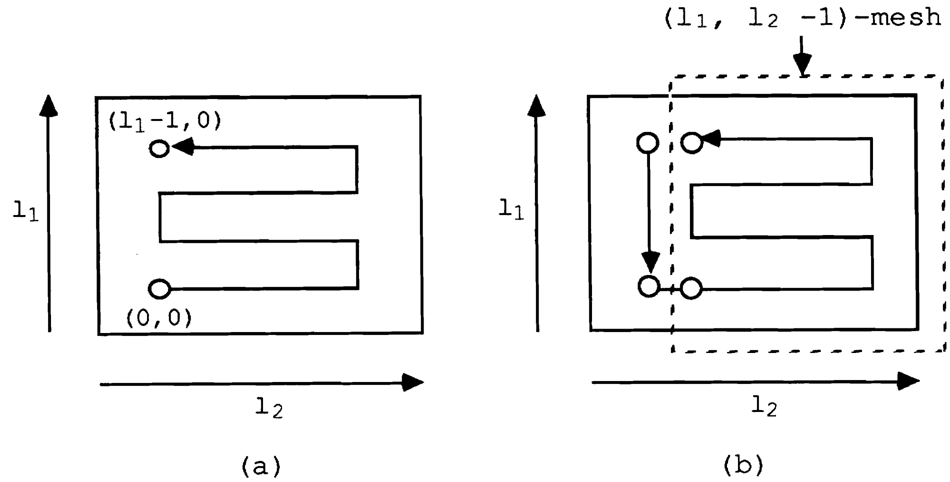


Figure 5: Embedding a ring in an (l_1, l_2)-mesh with $l_1 = 4$ and $l_2 > 2$

the mesh, then by Lemma 19, both the first and the last nodes from the ring are embedded in the first column of the mesh, with node 0 at the bottom (node (0, 0) in the mesh) and node $n-1$ at the top (node $(l_1 - 1,\ 0)$ in the mesh) (see Figure 5(a)). The δ_m-distance between $f_{\mathcal{L}}(0)$ and $f_{\mathcal{L}}(n-1)$ is thus $l_1 - 1$. For the case in which $l_2 > 2$, the following simple modification of $f_{\mathcal{L}}$ gives an embedding of G in H with unit dilation cost. We first embed the nodes from the ring successively in the first column of the mesh, from top to bottom, and then by treating the remaining nodes in the mesh as an $(l_1, l_2 - 1)$-mesh, we embed the remaining nodes from the ring using the function $f_{(l_1, l_2-1)}$. (See Figure 5(b).) In this embedding, all neighboring nodes in the ring are embedded in neighboring nodes in the mesh.

For the case in which $l_2 = 2$, the function $f_{(l_1, l_2-1)}$ is not defined because every component in a radix-base must be greater than 1. For this case, we simply embed the nodes from the ring successively in the first column of the mesh, from top to bottom, and then embed the remaining nodes from the ring in the second column of the mesh, from bottom to top. This embedding also has unit dilation cost.

We next define the function $r_{\mathcal{L}} : [n] \rightarrow \Omega_{\mathcal{L}}$. This function $r_{\mathcal{L}}$ gives the embedding above.

Definition 20 Let $\mathcal{L} = (l_1,\ l_2)$ be a radix-base, and let $n = l_1 l_2$. The function $r_{\mathcal{L}} : [n] \rightarrow \Omega_{\mathcal{L}}$ is defined as follows: for all $x \in [n]$, if $l_2 > 2$, then

$$r_{\mathcal{L}}(x) = \begin{cases} (l_1 - 1 - x, 0), & \text{if } x < l_1; \\ (x_1, x_2 + 1) \text{ where } (x_1, x_2) = f_{(l_1, l_2 - 1)}(x - l_1), & \text{if } x \geq l_1; \end{cases}$$

if $l_2 = 2$, then

$$r_{\mathcal{L}}(x) = \begin{cases} (l_1 - 1 - x, 0), & \text{if } x < l_1; \\ (x - l_1, 1), & \text{if } x \geq l_1. \end{cases}$$

□

The next lemma follows directly from the definition of $r_{\mathcal{L}}$ and the properties of the function $f_{\mathcal{L}}$.

Lemma 21 *Let $\mathcal{L} = (l_1,\ l_2)$ be a radix-base for which l_1 is even, and let $n = l_1 l_2$. For all $x \in [n]$, $\delta_m(r_{\mathcal{L}}(x), r_{\mathcal{L}}((x+1) \bmod n)) = 1$.* □

The embedding function $h_{\mathcal{L}}$

We next consider the case of embedding a ring of even size in a mesh of dimension at least 3. Given a mesh of even size, first we assume that the length of its first dimension is even.

Let $d \geq 3$, let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base for which l_1 is even, and let $n = \prod_{i=1}^{d} l_i$. Let $\mathcal{L}' = (l_1,\ l_2)$, $\mathcal{L}'' = (l_3,\ l_4, \ldots,\ l_d)$, and $m = \prod_{i=3}^{d} l_i$. We now construct a cyclic sequence of the numbers in $\Omega_{\mathcal{L}}$ with unit δ_m-spread. This sequence is defined in terms of $r_{\mathcal{L}'}$ and $f_{\mathcal{L}''}$. We first define m sequences q_0, q_1, …, q_{m-1}, each of which has length $l_1 l_2$. For all $i \in [m]$, let q_i be the sequence $r_{\mathcal{L}'}(0) \diamond f_{\mathcal{L}''}(i)$, $r_{\mathcal{L}'}(1) \diamond f_{\mathcal{L}''}(i)$, …, $r_{\mathcal{L}'}(l_1 l_2 - 1) \diamond f_{\mathcal{L}''}(i)$. ($\diamond$ is the operator for concatenating two lists, as defined in Section 2, page 4.) Since the function $r_{\mathcal{L}'} : [l_1 l_2] \rightarrow \Omega_{\mathcal{L}'}$ and the function $f_{\mathcal{L}''} : [m] \rightarrow \Omega_{\mathcal{L}''}$ are both bijective, each of these sequences consists of $l_1 l_2$ distinct numbers in $\Omega_{\mathcal{L}}$. Next we construct two disjoint segments from each of these sequences: for all $i \in [m]$, the segment q_i' consists of the first $l_1 l_2 - 1$ elements of q_i, with these elements in the same order as they appear in q_i if i is even and in the reverse order if i is odd; and the segment q_i'' consists of the last element in q_i. Let $Q' = q_0'\ q_1'\ \cdots\ q_{m-1}'$, $Q'' = q_{m-1}''\ q_{m-2}''\ \cdots\ q_0''$, and $Q = Q'Q''$. An example of Q, Q', and Q'' is given in Figure 6 for even m. The sequence Q

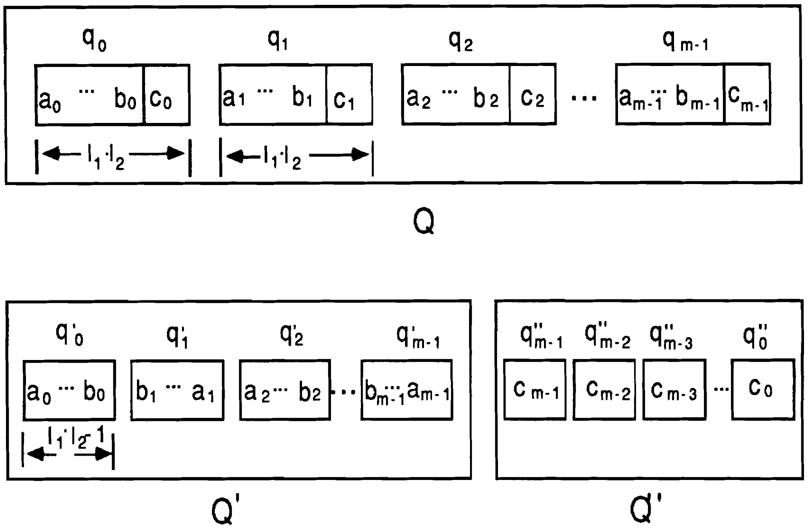


Figure 6: Q, Q' and Q'' for even m

consists of all numbers in $\Omega_{\mathcal{L}}$, and each element in Q is a list of d components. We now show that the cyclic sequence Q has unit δ_m-spread by establishing the following claims.

Claim 1. The sequence Q' has unit δ_m-spread.

For every pair of successive elements in Q', if they belong to the same segment q'_i, for some $i \in [m]$, then they have the same rightmost $d-2$ components, which are the components of $f_{\mathcal{L}''}(i)$, and their leftmost two components correspond to successive elements in the sequence $r_{\mathcal{L}'}$. Therefore the δ_m-distance between them is 1. If they belong to different segments, then they have the same leftmost two components, which are either the components of $r_{\mathcal{L}'}(0)$ or the components of $r_{\mathcal{L}'}(l_1 l_2 - 2)$, and their rightmost $d-2$ components correspond to successive elements in the sequence $f_{\mathcal{L}''}$. Therefore the δ_m-distance between them is also 1. The sequence Q' thus has unit δ_m-spread.

Claim 2. The sequence Q'' has unit δ_m-spread.

All elements in Q'' have the same leftmost two components, which are the components of $r_{\mathcal{L}'}(l_1 l_2 - 1)$. Furthermore, for every pair of successive elements in Q'', their rightmost $d-2$ components correspond to successive elements, in reverse order, in $f_{\mathcal{L}''}$. The sequence Q'' therefore has unit δ_m-spread.

Claim 3. The cyclic sequence Q has unit δ_m-spread.

Let y' and z' be the first and last elements of Q', and y'' and z'' be the first and last elements of Q''. We show that the δ_m-distance between z' and y'' and the δ_m-distance between y' and z'' are both 1. Both z' and y'' come from the sequence q_{m-1}, with y'' being the last element in q_{m-1}, and depending on whether m is even or odd, z' being either the first or the second to last element in q_{m-1}. Since l_1 is even, by Lemma 21, the cyclic sequence $r_{\mathcal{L}'}$ has unit δ_m-spread. The δ_m-distance between z' and y'' is therefore 1. For the pair y' and z'' , since they both come from the sequence q_0, with y' being the first element and z'' being the last element, again since the cyclic sequence $r_{\mathcal{L}'}$ has unit δ_m-spread, the δ_m-distance between y' and z'' is also 1. Using claims 1 and 2, we conclude that the cyclic sequence Q has unit δ_m-spread.

We next define the function $h_{\mathcal{L}} : [n] \rightarrow \Omega_{\mathcal{L}}$. When $d \geq 3$ and l_1 is an even number, the sequence $h_{\mathcal{L}}$ is $Q'Q''$. To simplify our presentation, we also define the function $h_{\mathcal{L}}$ for the special cases $d = 1$ and $d = 2$. For $d = 2$, we define $h_{\mathcal{L}}$ to be $r_{\mathcal{L}}$. For $d = 1$, we define $h_{\mathcal{L}}$ to be the identity function. (The function $h_{\mathcal{L}}$ with $d = 1$ appears only in the embedding of a ring in a torus, which will be discussed in the next subsection, but not in the embedding of a ring in a mesh.)

Definition 22 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. The function $h_{\mathcal{L}} : [n] \rightarrow \Omega_{\mathcal{L}}$ is defined as follows: for all $x \in [n]$,

if $d \geq 3$, then let $\mathcal{L}' = (l_1,\ l_2)$, $\mathcal{L}'' = (l_3,\ l_4, \ldots,\ l_d)$, $m = \prod_{i=3}^{d} l_i$, $a = \lfloor x/(l_1 l_2 - 1) \rfloor$, $b = x \bmod (l_1 l_2 - 1)$, and

$$h_{\mathcal{L}}(x) = \begin{cases} r_{\mathcal{L}'}(b) \diamond f_{\mathcal{L}''}(a), & \text{if } x < m(l_1 l_2 - 1) \text{ and } a \text{ is even;} \\ r_{\mathcal{L}'}(l_1 l_2 - b - 2) \diamond f_{\mathcal{L}''}(a), & \text{if } x < m(l_1 l_2 - 1) \text{ and } a \text{ is odd;} \\ r_{\mathcal{L}'}(l_1 l_2 - 1) \diamond f_{\mathcal{L}''}(n - x - 1), & \text{otherwise;} \end{cases}$$

if $d = 2$, then $\quad h_{\mathcal{L}}(x) = r_{\mathcal{L}}(x)$; $\quad$ and

if $d = 1$, then $\quad h_{\mathcal{L}}(x) = x$. $\quad \square$

In the definition above, $l_1 l_2 - 1$ corresponds to the length of each segment in Q', $m(l_1 l_2 - 1)$ corresponds to the length of the sequence Q', a determines a particular segment in Q', and b determines a particular element inside the segment. An example of the function $h_{\mathcal{L}}$ for $\mathcal{L} = (4, 2, 3)$ is given in Figure 9 on page 26.

The function $h_{\mathcal{L}}$ is clearly bijective. The following lemma follows from the definition of $h_{\mathcal{L}}$ and the properties of $r_{\mathcal{L}'}$ and $f_{\mathcal{L}''}$.

Lemma 23 *Let $d > 1$, let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. If l_1 is even, then for all $x \in [n]$, $\delta_m(h_{\mathcal{L}}(x), h_{\mathcal{L}}((x+1) \bmod n)) = 1$.* □

We can view the function $h_{\mathcal{L}}$ as embedding a ring in an $(l_1, l_2, \ldots, l_d)$-mesh for which $d \geq 2$ and l_1 is even in the following way. Let $m = \prod_{i=3}^{d} l_i$. We first divide the $(l_1, l_2, \ldots, l_d)$-mesh into m (l_1, l_2)-meshes, which we simply call *planes*. All nodes in each plane have the same rightmost $(d-2)$ components. The values of these components are used to order the planes from 0 to $m-1$ according to the sequence $f_{\mathcal{L}''}(0)$, $f_{\mathcal{L}''}(1)$, ..., $f_{\mathcal{L}''}(m-1)$. We refer to the nodes in each plane only by their leftmost two components. The embedding function $h_{\mathcal{L}}$ *marches* through these planes in two passes: first a forward pass from plane 0 to plane $m-1$, and then a backward pass from plane $m-1$ to plane 0. In the forward pass, $h_{\mathcal{L}}$ fills up $l_1 l_2 - 1$ nodes in each plane according to the sequence $r_{\mathcal{L}'}(0)$, $r_{\mathcal{L}'}(1)$, ..., $r_{\mathcal{L}'}(l_1 l_2 - 2)$ for even-numbered planes, and according to the sequence $r_{\mathcal{L}'}(l_1 l_2 - 2)$, $r_{\mathcal{L}'}(l_1 l_2 - 3)$, ..., $r_{\mathcal{L}'}(0)$ for odd-numbered planes. In the backward pass, $h_{\mathcal{L}}$ fills up the last node $r_{\mathcal{L}'}(l_1 l_2 - 1)$ in each plane. (See Figure 7.) An example of an embedding of a ring of size 24 in a $(4, 2, 3)$-mesh using the function $h_{\mathcal{L}}$ is given in Figure 10 on page 27.

Given a ring G of even size and an $\mathcal{L}$-mesh H of the same size and of dimension greater than 1, the function $h_{\mathcal{L}}(x)$ gives a unit dilation cost embedding of G in H only if the first component of $\mathcal{L}$ is an even number. If this condition is not satisfied, we can define an $\mathcal{L}^*$-mesh H^* such that $\mathcal{L}^* = (l_1^*, l_2^*, \ldots, l_d^*)$, l_1^* is even, and $\pi(\mathcal{L}^*) = \mathcal{L}$, for some permutation $\pi : [d]^+ \rightarrow [d]^+$. (The application of a permutation to a list is defined in Section 2 on page 4.) Since H is of even size, $\mathcal{L}^*$ must exist. The ring G can be embedded in H by first embedding G in H^* using $h_{\mathcal{L}^*}$ and then embedding H^* in H using π. For any pair of neighboring nodes A and B in H^*, $\pi(A)$ and $\pi(B)$ remain neighbors in H because π is only a permutation of the lists A and B. Hence, the function $\pi \circ h_{\mathcal{L}^*}$ gives a unit dilation cost embedding of the ring G in the mesh H. ($\circ$ is the function composition operator defined in Section 2 on page 4.)

Theorem 24 *Let G be a ring of even size, and H be an $\mathcal{L}$-mesh of the same size and of dimension d, for $d \geq 2$. Let $\mathcal{L}^*$ be a list such that $\pi(\mathcal{L}^*) = \mathcal{L}$ for some permutation $\pi : [d]^+ \rightarrow [d]^+$, and the*

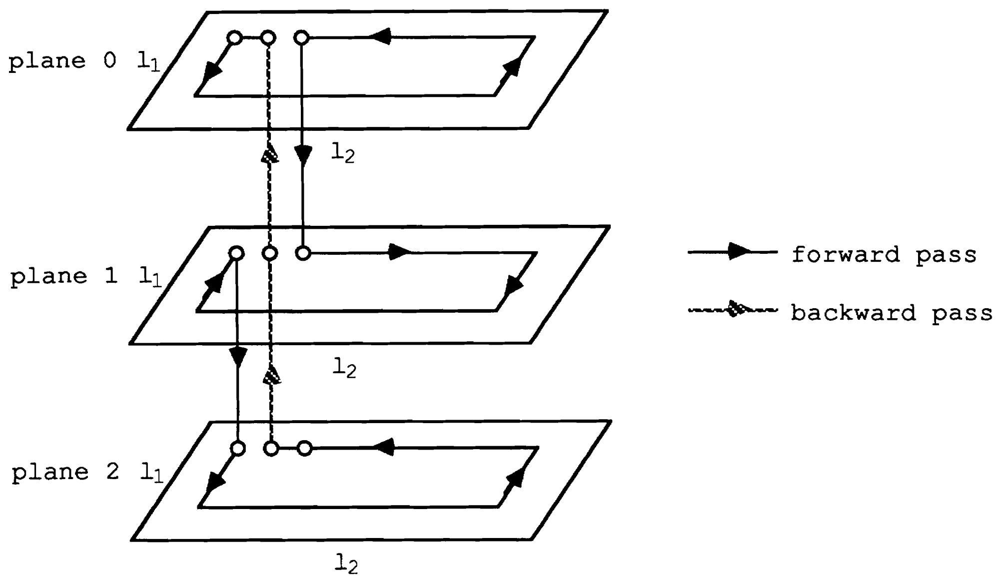


Figure 7: Embedding scheme of $h_{\mathcal{L}}$ with $\mathcal{L} = (l_1, l_2, l_3)$ and $l_3 = 3$

first component of $\mathcal{L}^*$ is even. The ring G can be embedded in H with unit dilation cost. The function $\pi \circ h_{\mathcal{L}^*}$ gives such an optimal embedding. □

The next corollary follows from Theorem 24.

Corollary 25 *Every mesh of even size and of dimension greater than 1 has a Hamiltonian circuit.* □

3.2.2 Embedding a ring in a torus

By Lemma 19, if l_1 is even, then $f_{\mathcal{L}}(n) = (l_1 - 1, 0, \ldots, 0)$. In this case, while the δ_m-distance between $f_{\mathcal{L}}(0) = (0, 0, \ldots, 0)$ and $f_{\mathcal{L}}(n-1) = (l_1 - 1, 0, \ldots, 0)$ is $l_1 - 1$, the δ_t-distance between them is 1. On the other hand, if l_1 is odd, then $\lfloor (n-1)/w_1 \rfloor$ (which was shown to be $l_1 - 1$ in the proof of Lemma 19) is even. It follows that the sublist corresponding to the leftmost two components of $f_{\mathcal{L}}(n-1)$ is $(l_1 - 1, l_2 - 1)$, and thus the δ_t-distance between $f_{\mathcal{L}}(0)$ and $f_{\mathcal{L}}(n-1)$ is greater than 1.

Let G be a ring, and H be an $\mathcal{L}$-torus of the same size and of dimension d. If the size of G and H is even, we can define an $\mathcal{L}^*$-torus H^* such that the first component of $\mathcal{L}^*$ is an even

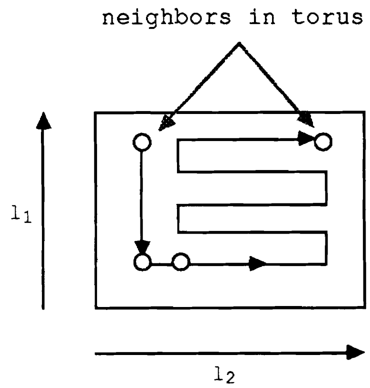


Figure 8: The function $r_{\mathcal{L}}$ for odd l_1

number, and $\pi(\mathcal{L}^*) = \mathcal{L}$ for some permutation $\pi : [d]^* \to [d]^+$. The ring can be embedded in H^* using $f_{\mathcal{L}^*}$, and H^* can be embedded in H using π, both with unit dilation cost. The function $\pi \circ f_{\mathcal{L}^*}$ thus gives a unit dilation cost embedding of G in H. On the other hand, if the size of G and H is odd, then all the components in $\mathcal{L}$ are odd numbers. In this case, we cannot construct a unit dilation cost embedding of G in H in this way because the intermediate graph H^* does not exist.

We now show that the embedding function $h_{\mathcal{L}}$ always embeds a ring in an $\mathcal{L}$-torus of the same size with unit dilation cost, whether their size is even or odd.

Let $\mathcal{L} = (l_1, l_2)$ be a radix-base. While the cyclic sequence $r_{\mathcal{L}}$ has unit δ_m-spread only when l_1 is even, this cyclic sequence always has unit δ_t-spread. When l_1 is odd, $r_{\mathcal{L}}(n-1) = (l_1 - 1, l_2 - 1)$, which is the top node in the last column of a torus. (See Figure 8.) Since this node and $r_{\mathcal{L}}(0)$, which is the top node in the first column, are neighbors in a torus, $\delta_t(r_{\mathcal{L}}(0), r_{\mathcal{L}}(n-1)) = 1$. This property is summarized in the following lemma.

Lemma 26 *Let* $\mathcal{L} = (l_1, l_2)$ *be a radix-base, and let* $n = l_1 l_2$. *For all* $x \in [n]$, $\delta_t(r_{\mathcal{L}}(x), r_{\mathcal{L}}((x+1) \bmod n)) = 1$. □

Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $\mathcal{L}' = (l_1, l_2)$. For the case in which $d \geq 2$, since the cyclic sequence $r_{\mathcal{L}'}$ in Definition 22 always has unit δ_t-spread, whether l_1 is odd or even, the cyclic sequence $h_{\mathcal{L}}$ has unit δ_t-spread. For the case in which $d = 1$, the cyclic sequence $h_{\mathcal{L}}$ is 0, 1,

..., $n-1$, which also has unit δ_t-spread. The function $h_{\mathcal{L}}$ therefore always provides an optimal, unit dilation cost embedding of a ring in an $\mathcal{L}$-torus. We summarize these results in Lemma 27 and Theorem 28.

Lemma 27 *Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base, and let $n = \prod_{i=1}^{d} l_i$. For all $x \in [n]$, $\delta_t(h_{\mathcal{L}}(x), h_{\mathcal{L}}((x+1) \bmod n)) = 1$.* □

Theorem 28 *Let G be a ring, and H be an $\mathcal{L}$-torus of the same size. The ring G can be embedded in H with unit dilation cost. The function $h_{\mathcal{L}}$ gives such an optimal embedding.* □

The next corollary follows from the theorem above.

Corollary 29 *Every torus has a Hamiltonian circuit.* □

4 Generalized embeddings

In this section, we study embeddings for which the dimensions of the two graphs are greater than 1. We analyze only the cases in which the shapes of the two graphs satisfy certain conditions: the condition of *expansion* for increasing dimension cases (G has lower dimension than H) and the condition of *reduction* for lowering dimension cases (G has higher dimension than H). The embedding functions for these cases are defined in terms of the basic embedding functions $f_{\mathcal{L}}$, $g_{\mathcal{L}}$, and $h_{\mathcal{L}}$.

Except when G is a torus of even size and H is a mesh, our embeddings for increasing dimension are all optimal. For the exception above, our embeddings can always achieve a dilation cost of 2, and when a certain condition on the shapes of G and H is satisfied, unit dilation cost is also achievable.

The dilation costs of our embeddings for lowering dimension depend on the shapes of G and H. They are not optimal in general.

x	radix-$\mathcal{L}$ rep. of x	$f_{\mathcal{L}}(x)$	$g_{\mathcal{L}}(x)$	$h_{\mathcal{L}}(x)$
0	(0,0,0)	(0,0,0)	(0,0,0)	(3,0,0)
1	(0,0,1)	(0,0,1)	(0,0,2)	(2,0,0)
2	(0,0,2)	(0,0,2)	(0,1,1)	(1,0,0)
3	(0,1,0)	(0,1,2)	(1,1,0)	(0,0,0)
4	(0,1,1)	(0,1,1)	(1,1,2)	(0,1,0)
5	(0,1,2)	(0,1,0)	(1,0,1)	(1,1,0)
6	(1,0,0)	(1,1,0)	(2,0,0)	(2,1,0)
7	(1,0,1)	(1,1,1)	(2,0,2)	(2,1,1)
8	(1,0,2)	(1,1,2)	(2,1,1)	(1,1,1)
9	(1,1,0)	(1,0,2)	(3,1,0)	(0,1,1)
10	(1,1,1)	(1,0,1)	(3,1,2)	(0,0,1)
11	(1,1,2)	(1,0,0)	(3,0,1)	(1,0,1)
12	(2,0,0)	(2,0,0)	(3,0,0)	(2,0,1)
13	(2,0,1)	(2,0,1)	(3,0,2)	(3,0,1)
14	(2,0,2)	(2,0,2)	(3,1,1)	(3,0,2)
15	(2,1,0)	(2,1,2)	(2,1,0)	(2,0,2)
16	(2,1,1)	(2,1,1)	(2,1,2)	(1,0,2)
17	(2,1,2)	(2,1,0)	(2,0,1)	(0,0,2)
18	(3,0,0)	(3,1,0)	(1,0,0)	(0,1,2)
19	(3,0,1)	(3,1,1)	(1,0,2)	(1,1,2)
20	(3,0,2)	(3,1,2)	(1,1,1)	(2,1,2)
21	(3,1,0)	(3,0,2)	(0,1,0)	(3,1,2)
22	(3,1,1)	(3,0,1)	(0,1,2)	(3,1,1)
23	(3,1,2)	(3,0,0)	(0,0,1)	(3,1,0)

Figure 9: Embedding functions $f_{\mathcal{L}}$, $g_{\mathcal{L}}$, and $h_{\mathcal{L}}$ for $n = 24$ and $\mathcal{L} = (4,2,3)$

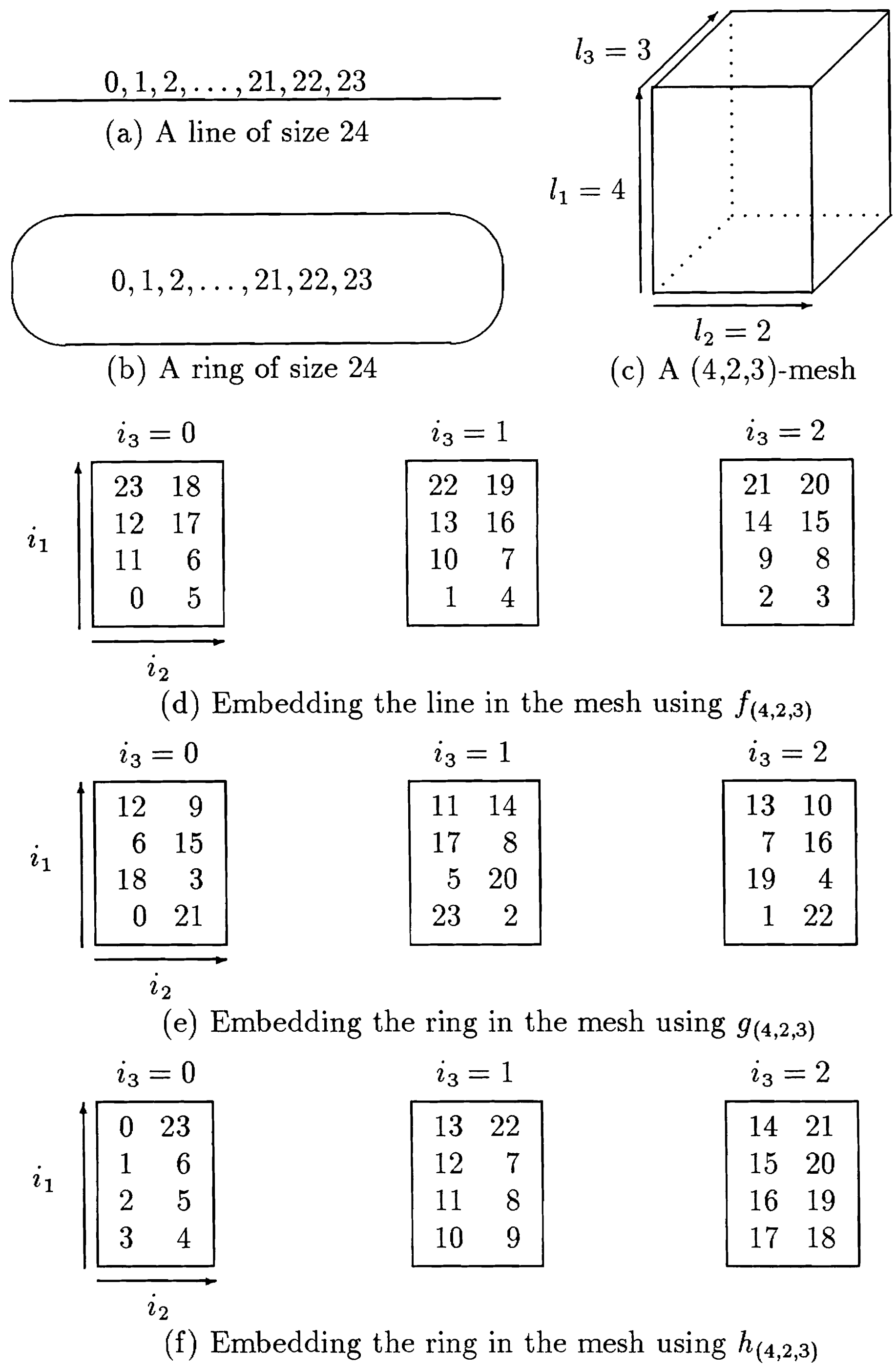


Figure 10: Embedding a line or a ring of size 24 in a $(4, 2, 3)$-mesh

4.1 Embeddings for increasing dimension

Given a list $\mathcal{A} = (a_1, a_2, \ldots, a_k)$, we use $\prod \mathcal{A}$ to denote the product $a_1 a_2 \cdots a_k$.

Definition 30 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be lists of positive integers for which $d < c$. The list $\mathcal{M}$ is an *expansion* of the list $\mathcal{L}$ if there exist d lists of integers $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_d$ such that (i) for all $i \in [d]^+$, $\prod \mathcal{V}_i = l_i$; and (ii) the list $\mathcal{M}$ is a permutation of the list $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \diamond \cdots \diamond \mathcal{V}_d$. We call $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d)$ an *expansion factor* of $\mathcal{L}$ into $\mathcal{M}$. □

For example, the list $\mathcal{M} = (2, 4, 3, 8, 5, 4)$ is an expansion of the list $\mathcal{L} = (6, 8, 80)$ because we can have $\mathcal{V}_1 = (2, 3)$, $\mathcal{V}_2 = (8)$, and $\mathcal{V}_3 = (4, 5, 4)$. The list $\mathcal{V} = ((2, 3), (8), (4, 5, 4))$ is an expansion factor of $\mathcal{L}$ into $\mathcal{M}$. Expansion factors may not be unique: the list $((3, 2), (8), (5, 4, 4))$ is also an expansion factor of $\mathcal{L}$ into $\mathcal{M}$.

Let G be a torus or a mesh of shape $\mathcal{L}$, and let H be a torus or a mesh of shape $\mathcal{M}$ such that $\mathcal{M}$ is an expansion of $\mathcal{L}$ with an expansion factor $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d)$. Let $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \cdots \diamond \mathcal{V}_d$, and let H' be a graph of shape $\tilde{\mathcal{V}}$ and of the same type as H. (*type* of a graph is defined in Section 2 on page 5.) We now construct an embedding of G in H in two steps: $G \to H' \to H$.

Let $\pi : [c]^+ \to [c]^+$ be a permutation such that $\pi(\tilde{\mathcal{V}}) = \mathcal{M}$. By the definition of expansion, such a permutation always exists. Since H' has shape $\tilde{\mathcal{V}}$ and H has shape $\mathcal{M}$, H' can be embedded in H with unit dilation cost using the permutation π. Next we construct an embedding of G in H'.

We first consider the case in which G and H' are meshes. We map each node $(i_1, i_2, \ldots, i_d)$ in G to the node $f_{\mathcal{V}_1}(i_1) \diamond f_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond f_{\mathcal{V}_d}(i_d)$ in H'. Since the functions $f_{\mathcal{V}_1} : [l_1] \to \Omega_{\mathcal{V}_1}$, $f_{\mathcal{V}_2} : [l_2] \to \Omega_{\mathcal{V}_2}$, ..., $f_{\mathcal{V}_d} : [l_d] \to \Omega_{\mathcal{V}_d}$ are all bijective, this mapping is an embedding of G in H'. For every pair of neighboring nodes $(i_1, i_2, \ldots, i_d)$ and $(i'_1, i'_2, \ldots, i'_d)$ in G, by definition, there exists exactly one index $k \in [d]^+$ such that $|i_k - i'_k| = 1$ and $i_j = i'_j$, for all $j \in [d]^+$ such that $j \neq k$. Since the sequences $f_{\mathcal{V}_1}, f_{\mathcal{V}_2}, \ldots, f_{\mathcal{V}_d}$ all have unit δ_m-spread, we have $\delta_m(f_{\mathcal{V}_k}(i_k), f_{\mathcal{V}_k}(i'_k)) = 1$, and $\delta_m(f_{\mathcal{V}_j}(i_j), f_{\mathcal{V}_j}(i'_j)) = 0$, for all $j \in [d]^+$ such that $j \neq k$. The nodes $f_{\mathcal{V}_1}(i_1) \diamond f_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond f_{\mathcal{V}_d}(i_d)$ and $f_{\mathcal{V}_1}(i'_1) \diamond f_{\mathcal{V}_2}(i'_2) \diamond \cdots \diamond f_{\mathcal{V}_d}(i'_d)$ thus have unit δ_m-distance in H', and hence must be neighbors in H'. This embedding therefore has unit dilation cost. Furthermore, since the sequences $f_{\mathcal{V}_1}$,

$f_{\mathcal{V}_2}$, …, $f_{\mathcal{V}_d}$ all have unit δ_t-spread. This embedding also has unit dilation cost when G is a mesh and H' is a torus.

When G is a torus and H' is a mesh, we can define a similar embedding by replacing the functions $f_{\mathcal{V}_1}$, $f_{\mathcal{V}_2}$, …, $f_{\mathcal{V}_d}$ with the functions $g_{\mathcal{V}_1}$, $g_{\mathcal{V}_2}$, …, $g_{\mathcal{V}_d}$. Since the cyclic sequences $g_{\mathcal{V}_1}$, $g_{\mathcal{V}_2}$, …, $g_{\mathcal{V}_d}$ all have a δ_m-spread of 2, by a similar argument, we can show that the embedding has a dilation cost of 2.

For the remaining case in which G and H' are toruses, we can construct a similar embedding by replacing $f_{\mathcal{V}_1}$, $f_{\mathcal{V}_2}$, …, $f_{\mathcal{V}_d}$ with $h_{\mathcal{V}_1}$, $h_{\mathcal{V}_2}$, …, $h_{\mathcal{V}_d}$. Since the cyclic sequences $h_{\mathcal{V}_1}$, $h_{\mathcal{V}_2}$, …, $h_{\mathcal{V}_d}$ all have unit δ_t-spread, the embedding also has unit dilation cost.

The sequence of embeddings $G \to H' \to H$ described above gives an embedding of G in H with a dilation cost of 2 if G is a torus and H is a mesh, and with unit dilation cost otherwise.

As will be proved in Theorem 32, when G is a torus and H is a mesh, a dilation cost of 2 is optimal for all G of odd size. On the other hand, if each dimension of G has even length and there is at least one expansion factor of $\mathcal{L}$ into $\mathcal{M}$ such that each list in the factor has at least two components, then we can choose an expansion factor $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d)$ of $\mathcal{L}$ into $\mathcal{M}$ such that for all $i \in [d]^+$, $\mathcal{V}_i$ has length at least 2, and its first component is an even number. If we use such an expansion factor $\mathcal{V}$ to define the shape of H', then by Lemma 23, G can be embedded in H' with unit dilation cost by mapping each node $(i_1, i_2, \ldots, i_d)$ in G to the node $h_{\mathcal{V}_1}(i_1) \diamond h_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond h_{\mathcal{V}_d}(i_d)$ in H'. Such an embedding sequence $G \to H' \to H$ gives a unit dilation cost embedding of G in H.

For example, if $\mathcal{L} = (6, 12)$ and $\mathcal{M} = (6, 3, 2, 2)$, then both $((6), (3, 2, 2))$ and $((2, 3), (6, 2))$ are expansion factors of $\mathcal{L}$ into $\mathcal{M}$. If we choose the expansion factor $((2, 3), (6, 2))$ to define the shape of H', then we get a unit dilation cost embedding of a $(6, 12)$-torus G in a $(6, 3, 2, 2)$-mesh H. On the other hand, if we choose $((6), (3, 2, 2))$ to define the shape of H', then we get an embedding with a dilation cost of 2.

We formalize the above results in the following definition and theorems.

Definition 31 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be radix-bases such that $\mathcal{M}$ is an expansion of $\mathcal{L}$ with an expansion factor $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d)$. Let $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \diamond \cdots \diamond \mathcal{V}_d$. The functions $\mathcal{F}_{\mathcal{V}}\colon \Omega_{\mathcal{L}} \to \Omega_{\tilde{\mathcal{V}}}$, $\mathcal{G}_{\mathcal{V}}\colon \Omega_{\mathcal{L}} \to \Omega_{\tilde{\mathcal{V}}}$ and $\mathcal{H}_{\mathcal{V}}\colon \Omega_{\mathcal{L}} \to \Omega_{\tilde{\mathcal{V}}}$ are defined as follows: for all

$(i_1,\ i_2,\ \ldots,\ i_d) \in \Omega_{\mathcal{L}}$,

$$\mathcal{F}_{\mathcal{V}}((i_1,\ i_2,\ \ldots,\ i_d)) = f_{\mathcal{V}_1}(i_1) \diamond f_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond\ f_{\mathcal{V}_d}(i_d),$$

$$\mathcal{G}_{\mathcal{V}}((i_1,\ i_2,\ \ldots,\ i_d)) = g_{\mathcal{V}_1}(i_1) \diamond g_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond\ g_{\mathcal{V}_d}(i_d),$$

$$\mathcal{H}_{\mathcal{V}}((i_1,\ i_2,\ \ldots,\ i_d)) = h_{\mathcal{V}_1}(i_1) \diamond h_{\mathcal{V}_2}(i_2) \diamond \cdots \diamond\ h_{\mathcal{V}_d}(i_d).$$

Furthermore, let $\pi : [c]^+ \rightarrow [c]^+$ be a permutation such that $\pi(\tilde{\mathcal{V}}) = \mathcal{M}$. Then we have the functions $\pi \circ \mathcal{F}_{\mathcal{V}} : \Omega_{\mathcal{L}} \rightarrow \Omega_{\mathcal{M}}$, $\pi \circ \mathcal{G}_{\mathcal{V}} : \Omega_{\mathcal{L}} \rightarrow \Omega_{\mathcal{M}}$, and $\pi \circ \mathcal{H}_{\mathcal{V}} : \Omega_{\mathcal{L}} \rightarrow \Omega_{\mathcal{M}}$. □

Examples of the functions $\mathcal{F}_{\mathcal{V}}$, $\mathcal{G}_{\mathcal{V}}$, and $\mathcal{H}_{\mathcal{V}}$ for $\mathcal{L} = (4,6)$, $\mathcal{M} = (2,2,2,3)$, and $\mathcal{V} = ((2,2),(2,3))$ are given in Figure 11. In this example, we have $\mathcal{M} = \mathcal{V}_1 \diamond \mathcal{V}_2$.

Theorem 32 *Let G be an $(l_1,l_2,\ldots,l_d)$-torus or an $(l_1,l_2,\ldots,l_d)$-mesh, and let H be an $(m_1,m_2,\ldots,m_c)$-torus or an $(m_1,m_2,\ldots,m_c)$-mesh. Assume that $(m_1,m_2,\ldots,m_c)$ is an expansion of $(l_1,l_2,\ldots,l_d)$ with an expansion factor $\mathcal{V} = (\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_d)$. Let $\pi : [c]^+ \rightarrow [c]^+$ be a permutation such that $\pi(\mathcal{V}_1 \diamond \mathcal{V}_2 \cdots \diamond \mathcal{V}_d) = (m_1,m_2,\ldots,m_c)$. Then*

(i) *If G is a mesh, then G can be embedded in H with unit dilation cost. The function $\pi \circ \mathcal{F}_{\mathcal{V}}$ gives such an optimal embedding.*

(ii) *If G and H are toruses, then G can be embedded in H with unit dilation cost. The function $\pi \circ \mathcal{H}_{\mathcal{V}}$ gives such an optimal embedding.*

(iii) *If G is a torus and H is a mesh, then G can be embedded in H with a dilation cost of 2. The function $\pi \circ \mathcal{G}_{\mathcal{V}}$ gives such an embedding. Furthermore, such an embedding is optimal for all G of odd size. If G is of even size, and for all $i \in [d]^+$, $\mathcal{V}_i$ consists of at least two components such that the first component is an even number, then G can be embedded in H with unit dilation cost. The function $\pi \circ \mathcal{H}_{\mathcal{V}}$ gives such an optimal embedding.*

□

Proof. We prove only the claim in (iii) that $\mathcal{G}_{\mathcal{V}}$ is optimal for all toruses of odd sizes. We prove this by showing that such a torus cannot be embedded in a mesh with unit dilation cost. The other parts of the theorem follow from the definitions of $\mathcal{F}_{\mathcal{V}}$, $\mathcal{G}_{\mathcal{V}}$, and $\mathcal{H}_{\mathcal{V}}$.

(i_1, i_2)	$\Omega_{(2,2)} \diamond \Omega_{(2,3)}$	$\mathcal{F}_{\mathcal{V}} = f_{(2,2)} \diamond f_{(2,3)}$	$\mathcal{G}_{\mathcal{V}} = g_{(2,2)} \diamond g_{(2,3)}$	$\mathcal{H}_{\mathcal{V}} = h_{(2,2)} \diamond h_{(2,3)}$
(0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,1,0)
(0,1)	(0,0,0,1)	(0,0,0,1)	(0,0,0,2)	(0,0,0,0)
(0,2)	(0,0,0,2)	(0,0,0,2)	(0,0,1,1)	(0,0,0,1)
(0,3)	(0,0,1,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,2)
(0,4)	(0,0,1,1)	(0,0,1,1)	(0,0,1,2)	(0,0,1,2)
(0,5)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,1,1)
(1,0)	(0,1,0,0)	(0,1,1,0)	(1,1,0,0)	(1,0,1,0)
(1,1)	(0,1,0,1)	(0,1,1,1)	(1,1,0,2)	(1,0,0,0)
(1,2)	(0,1,0,2)	(0,1,1,2)	(1,1,1,1)	(1,0,0,1)
(1,3)	(0,1,1,0)	(0,1,0,2)	(1,1,1,0)	(1,0,0,2)
(1,4)	(0,1,1,1)	(0,1,0,1)	(1,1,1,2)	(1,0,1,2)
(1,5)	(0,1,1,2)	(0,1,0,0)	(1,1,0,1)	(1,0,1,1)
(2,0)	(1,0,0,0)	(1,1,0,0)	(1,0,0,0)	(1,1,1,0)
(2,1)	(1,0,0,1)	(1,1,0,1)	(1,0,0,2)	(1,1,0,0)
(2,2)	(1,0,0,2)	(1,1,0,2)	(1,0,1,1)	(1,1,0,1)
(2,3)	(1,0,1,0)	(1,1,1,2)	(1,0,1,0)	(1,1,0,2)
(2,4)	(1,0,1,1)	(1,1,1,1)	(1,0,1,2)	(1,1,1,2)
(2,5)	(1,0,1,2)	(1,1,1,0)	(1,0,0,1)	(1,1,1,1)
(3,0)	(1,1,0,0)	(1,0,1,0)	(0,1,0,0)	(0,1,1,0)
(3,1)	(1,1,0,1)	(1,0,1,1)	(0,1,0,2)	(0,1,0,0)
(3,2)	(1,1,0,2)	(1,0,1,2)	(0,1,1,1)	(0,1,0,1)
(3,3)	(1,1,1,0)	(1,0,0,2)	(0,1,1,0)	(0,1,0,2)
(3,4)	(1,1,1,1)	(1,0,0,1)	(0,1,1,2)	(0,1,1,2)
(3,5)	(1,1,1,2)	(1,0,0,0)	(0,1,0,1)	(0,1,1,1)

Figure 11: Embedding functions $\mathcal{F}_{\mathcal{V}}$, $\mathcal{G}_{\mathcal{V}}$, $\mathcal{H}_{\mathcal{V}}$ for $\mathcal{L} = (4,6)$, $\mathcal{M} = (2,2,2,3)$, and $\mathcal{V} = ((2,2),(2,3))$

Assume for contradiction that a torus G of odd size can be embedded in a mesh H with unit dilation cost. Let p be such an embedding. Since G is a torus, by Corollary 29, there exists at least one Hamiltonian circuit $v_0 - v_1 - \cdots - v_{n-1} - v_n\ (= v_0)$ in G. By the definition of a Hamiltonian circuit, for all $i \in [n]$, v_i and v_{i+1} are neighbors in G. Since the embedding p has unit dilation cost, $p(v_i)$ and $p(v_{i+1})$ must also be neighbors in H. This implies that the path $p(v_0) - p(v_1) - \cdots - p(v_{n-1}) - p(v_n)\ (= p(v_0))$ is a Hamiltonian circuit in H, contradicting the fact that no mesh of odd size has a Hamiltonian circuit (Corollary 18). □

The embeddings for increasing dimension given in this subsection can be applied only if the shapes of the two graphs satisfy the condition of expansion. The next theorem states that if H is a hypercube, then the shapes of G and H always satisfy the condition of expansion.

Theorem 33 *Let G be a torus or a mesh of size some power of 2, and let H be a hypercube of the same size. Then the shape of H is an expansion of the shape of G.* □

Proof. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be the shape of G, and $\mathcal{M}$ be the shape of H. Since G is of size some power of 2, for all $k \in [d]^+$, $l_k = 2^{q_k}$, for some positive integer q_k. Since G and H are of the same size, $2^{q_1}2^{q_2}\cdots 2^{q_d}$ is the size of H. The list $\mathcal{M}$ is thus an expansion of the list $\mathcal{L}$ with an expansion factor

$$((\underbrace{2,2,\ldots,2}_{q_1}),(\underbrace{2,2,\ldots,2}_{q_2}),\ldots,(\underbrace{2,2,\ldots,2}_{q_d})).$$

□

By viewing a hypercube as a special case of a torus, the next corollary follows directly from Theorems 32 and 33. This corollary was proved in [CS86].

Corollary 34 *A torus or a mesh of size some power of 2 can be embedded in a hypercube of the same size with unit dilation cost.* □

4.2 Embeddings for lowering dimension

Our embeddings for lowering dimension are defined using two types of embeddings: embeddings for increasing dimension (from preceding subsection) and embeddings among toruses and meshes of the same *shape.*

Given a torus or a mesh G and a torus or a mesh H of the same shape $(l_1, l_2, \ldots, l_d)$, G can be embedded in H with unit dilation cost using the identity function, except for the case in which G is a torus, H is a mesh, and they are not hypercube. In this exceptional case, G clearly cannot be embedded in H with unit dilation cost because each boundary node in H has degree less than that of any node in G. An optimal embedding of G in H with a dilation cost of 2 can be constructed by embedding each node $(i_1, i_2, \ldots, i_d)$ of G in the node $(t_{l_1}(i_1), t_{l_2}(i_2), \ldots, t_{l_d}(i_d))$ of H. Since for all $i \in [d]^+$, the function $t_{l_i} : [l_i] \to [l_i]$ defines a cyclic sequence of all numbers in $[l_i]$ with a δ_m-spread of 2 (Definition 14), every two neighboring nodes in G are mapped to nodes in H at a distance no greater than 2. This embedding thus has a dilation cost of 2. The following definition and lemma summarize these results.

Definition 35 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ be a radix-base. The function $\mathcal{T}_{\mathcal{L}} : \Omega_{\mathcal{L}} \to \Omega_{\mathcal{L}}$ is defined as follows: for all $(x_1, x_2, \ldots, x_d) \in \Omega_{\mathcal{L}}$,

$$\mathcal{T}_{\mathcal{L}}((x_1, x_2, \ldots, x_d)) = (t_{l_1}(x_1), t_{l_2}(x_2), \ldots, t_{l_d}(x_d)).$$

□

Lemma 36 *Let G be a torus or a mesh, and let H also be a torus or a mesh of the same shape $\mathcal{L}$. If G is a torus, H is a mesh, and G and H are not hypercube, then G can be embedded in H with an optimal dilation cost of 2 using the embedding function $\mathcal{T}_{\mathcal{L}}$. Otherwise, G can be embedded in H with unit dilation cost using the identity function.* □

For lowering dimension, we consider only those cases in which the shapes of G and H satisfy the condition of reduction. We define two types of reduction: (i) simple reduction and (ii) general reduction.

4.2.1 Simple reduction

Definition 37 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be lists of positive integers for which $d > c$. The list $\mathcal{M}$ is a *simple reduction* of the list $\mathcal{L}$ with a reduction factor $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_c)$ if $\mathcal{L}$ is an expansion of $\mathcal{M}$ with an expansion factor $\mathcal{V}$. □

Let $\mathcal{L}$ be a radix-base. We next define a function that will be used to construct our embeddings. This function is defined in terms of the function $u_{\mathcal{L}}^{-1}$, which maps each mixed-radix number in $\Omega_{\mathcal{L}}$ to the corresponding natural number in $|\Omega_{\mathcal{L}}|$, defined on page 7, Section 2.

Definition 38 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be radix-bases such that $\mathcal{M}$ is a simple reduction of $\mathcal{L}$ with a reduction factor $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_c)$. Let $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \cdots \diamond \mathcal{V}_c$. For all $k \in [c]^+$, let $u_{\mathcal{V}_k}^{-1} : \Omega_{\mathcal{V}_k} \rightarrow [m_k]$. The function $\mathcal{U}_{\mathcal{V}} : \Omega_{\tilde{\mathcal{V}}} \rightarrow \Omega_{\mathcal{M}}$ is defined as follows: for all $(i_1,\ i_2,\ \ldots,\ i_d) \in \Omega_{\tilde{\mathcal{V}}}$,

$$\mathcal{U}_{\mathcal{V}}((i_1,\ i_2,\ \ldots,\ i_d)) = u_{\mathcal{V}_1}^{-1}(I_1) \diamond u_{\mathcal{V}_2}^{-1}(I_2) \diamond \cdots \diamond\ u_{\mathcal{V}_c}^{-1}(I_c),$$

where $I_1,\ I_2,\ \ldots,\ I_c$ are partitions of $(i_1, i_2, \ldots, i_d)$ such that for all $k \in [c]^+$, $|I_k| = |\mathcal{V}_k|$, and $I_1 \diamond I_2 \diamond \cdots \diamond I_c = (i_1, i_2, \ldots, i_d)$. Furthermore, let $\pi : [d]^+ \rightarrow [d]^+$ be a permutation such that $\pi(\mathcal{L}) = (\tilde{\mathcal{V}})$. Then we have the function $\mathcal{U}_{\mathcal{V}} \circ \pi : \Omega_{\mathcal{L}} \rightarrow \Omega_{\mathcal{M}}$. $\Box$

Let G be a torus or a mesh with shape $\mathcal{L}$, and let H be a torus or a mesh with shape $\mathcal{M}$ such that $\mathcal{M}$ is a simple reduction of $\mathcal{L}$. Let $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_c)$ be a reduction factor of $\mathcal{L}$ into $\mathcal{M}$ such that for all $i \in [c]^+$, the components in the list $\mathcal{V}_i$ are in non-increasing order. Let v_i denote the index in $[d]^+$ such that l_{v_i} is the first component in $\mathcal{V}_i$. Let $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \cdots \diamond \mathcal{V}_c$, and let G' be a graph with shape $\tilde{\mathcal{V}}$ and of the same type of graph as G. Let $\pi : [d]^+ \rightarrow [d]^+$ be a permutation such that $\pi(\mathcal{L}) = \tilde{\mathcal{V}}$. The graph G can be embedded in G' using the permutation π with unit dilation cost. We next construct an embedding of G' in H.

Let $A = I_1 \diamond I_2 \diamond \cdots \diamond I_k \diamond \cdots \diamond I_c$ and $B = I_1' \diamond I_2' \diamond \cdots \diamond I_k' \diamond \cdots \diamond I_c'$ be an arbitrary pair of neighboring nodes in G', where for all $i \in [c]^+$, $|I_i| = |I_i'| = |\mathcal{V}_i|$. Let $q = |\mathcal{V}_k|$, and $(l_1', l_2', \ldots, l_q') = \mathcal{V}_k$, where $k \in [c]^+$. Without loss of generality, assume that A and B differ at the r-th position in I_k, for some $r \in [q]^+$. Let i_r and i_r' denote respectively the components of A and B at this position.

We first consider the case in which G' and H are meshes. We use the function $\mathcal{U}_{\mathcal{V}}$ to embed G' in H. The distance between the images of A and B in H is $\delta_m(\mathcal{U}_{\mathcal{V}}(A), \mathcal{U}_{\mathcal{V}}(B)) = |u_{\mathcal{V}_k}^{-1}(I_k) - u_{\mathcal{V}_k}^{-1}(I_k)| = |i_r - i_r'| \prod_{j=r+1}^{q} l_j'$. Since G' is a mesh, $|i_r - i_r'| = 1$, and since $m_k = \prod_{j=1}^{q} l_j'$, we have $\delta_m(\mathcal{U}_{\mathcal{V}}(A), \mathcal{U}_{\mathcal{V}}(B)) = m_k / \prod_{j=1}^{r} l_q' \leq m_k / l_1'$. Therefore, the function $\mathcal{U}_{\mathcal{V}}$ gives an embedding of G' in H with a dilation cost of $\max_{1 \leq i \leq c}\{m_i / l_{v_i}\}$.

For the cases in which either (i) G' is a mesh and H is a torus or (ii) G' and H are toruses, we use the same embedding function $\mathcal{U}_{\mathcal{V}}$ to embed G' in H. The distance between the images of A and B in H is $\delta_t(\mathcal{U}_{\mathcal{V}}(A),\mathcal{U}_{\mathcal{V}}(B)) = \min\{|i_r - i'_r| \prod_{j=r+1}^{q} l'_j, m_k - |i_r - i'_r| \prod_{j=r+1}^{q} l'_j\}$. For case (i), $|i_r - i'_r|$ is 1 and for case (ii) $|i_r - i'_r|$ is either 1 or $l'_r - 1$. In either case, using the fact that for all $j \in [q]^+$, $l'_j \geq 2$, we can show that the embedding also has a dilation cost of $\max_{1 \leq i \leq c}\{m_i/l_{v_i}\}$.

For the remaining case in which G' is a torus and H is a mesh, using the embedding function $\mathcal{T}_{\tilde{\mathcal{V}}}$, we first embed G' in an intermediate mesh G'' that has the same shape as G'. Such an embedding has a dilation cost of 2. We then embed the mesh G'' in the mesh H using the function $\mathcal{U}_{\mathcal{V}}$. This sequence gives an embedding of G' in H with a dilation cost of $2\max_{1 \leq i \leq c}\{m_i/l_{v_i}\}$.

Theorem 39 *Let G be a torus or a mesh of shape $\mathcal{L} = (l_1, l_2, \ldots, l_d)$, and let H be a torus or a mesh of shape $\mathcal{M} = (m_1, m_2, \ldots, m_c)$. Assume that $\mathcal{M}$ is a simple reduction of $\mathcal{L}$. Let $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_c)$ be a reduction factor of $\mathcal{L}$ into $\mathcal{M}$ such that for all $i \in [c]^+$, the components in the list $\mathcal{V}_i$ are in non-increasing order. Let v_i denote the index in $[d]^+$ such that l_{v_i} is the first component in $\mathcal{V}_i$. Let $\tilde{\mathcal{V}} = \mathcal{V}_1 \diamond \mathcal{V}_2 \diamond \cdots \diamond \mathcal{V}_c$. Let $\pi : [d]^+ \to [d]^+$ be a permutation such that $\pi(\mathcal{L}) = \tilde{\mathcal{V}}$. If G is a torus and H is a mesh, then G can be embedded in H with a dilation cost of $2\max_{1 \leq i \leq c}\{m_i/l_{v_i}\}$, and the function $\mathcal{U}_{\mathcal{V}} \circ \mathcal{T}_{\tilde{\mathcal{V}}} \circ \pi$ gives such an embedding; otherwise, G can be embedded in H with a dilation cost of $\max_{1 \leq i \leq c}\{m_i/l_{v_i}\}$, and the function $\mathcal{U}_{\mathcal{V}} \circ \pi$ gives such an embedding.* $\square$

By the definition of simple reduction and Theorem 33, given a hypercube G and a torus or a mesh H of the same size, the shape of H is always a simple reduction of the shape of G. The next corollary thus follows from Theorem 39 by treating hypercube as a special case of a mesh.

Corollary 40 *A hypercube can be embedded in an $(m_1, m_2, \ldots, m_c)$-torus or an $(m_1, m_2, \ldots, m_c)$-mesh of the same size with a dilation cost of $\max\{m_1, m_2, \ldots, m_c\}/2$.* $\square$

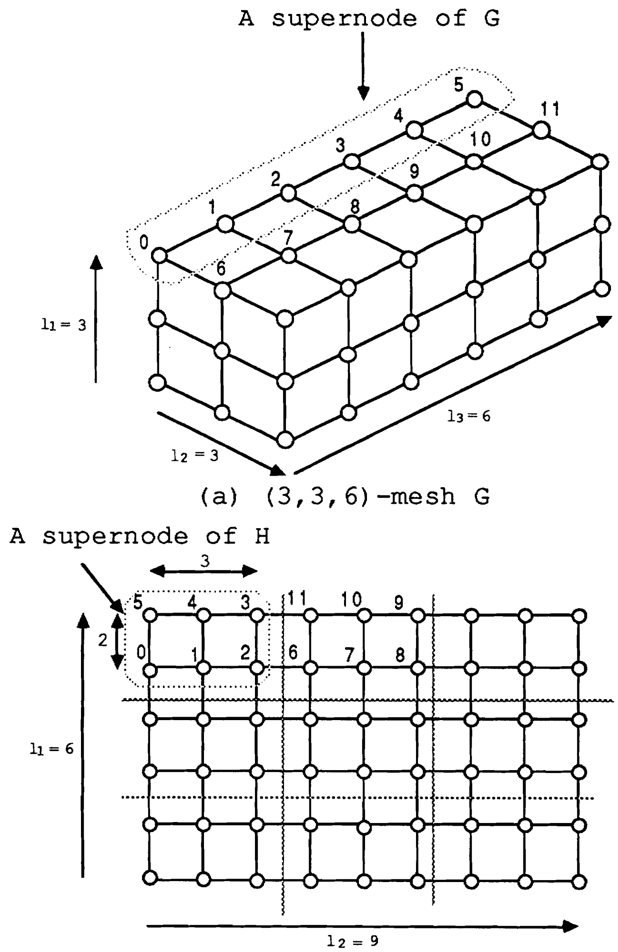
4.2.2 General reduction

We first illustrate through a simple example the embeddings to be constructed under general reduction. Let G be a $(3,3,6)$-mesh, and H be a $(6,9)$-mesh. We can view G as a $(3,3)$-mesh of *supernodes*, each of which is a line of length 6, and view H as a $(3,3)$-mesh of *supernodes*, each of which is a $(2,3)$-mesh. (See Figure 12.) With respect to supernodes, G and H have the same shape: a $(3,3)$-mesh. With the identity function, neighboring supernodes of G can be embedded in neighboring supernodes of H. Since the supernodes of G are lines of length 6, and the supernodes of H are $(2,3)$-meshes, the nodes belonging to a single supernode of G can be embedded in the nodes belonging to the corresponding supernode of H by using the embedding function $f_{(2,3)}$. This embedding of G in H is achieved by embedding nine separate lines of length 6 in nine separate $(2,3)$-meshes, with neighboring lines embedded in neighboring meshes. Such an embedding gives a dilation cost of 3.

In general, given a torus or a mesh G and a torus or a mesh H whose shape is a *general reduction* (to be defined below) of the shape of G, G and H can be viewed as graphs of some supernodes such that (i) with respect to supernodes, G and H have the same shape; and (ii) the shape of the supernodes of H is an expansion of the shape of the supernodes of G. An embedding of G in H can be achieved as follows: first establish a one-to-one correspondence between the supernodes of G and the supernodes of H, and then by using the embedding functions for increasing dimension defined in the preceding subsection, embed the nodes belonging to a single supernode of G in the nodes belonging to the corresponding supernode of H.

We now define the relation *general reduction* between two lists of different lengths for which the shorter list is longer than half of the longer list. Given a list $\mathcal{A} = (a_1, a_2, \ldots, a_k)$ and a list $\mathcal{B} = (b_1, b_2, \ldots, b_k)$, we use $\mathcal{A} \times \mathcal{B}$ to denote the list $(a_1b_1, a_2b_2, \ldots, a_kb_k)$ and $\mathcal{A} + \mathcal{B}$ to denote the list $(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$. We use [ ] for grouping.

Definition 41 Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be lists of positive integers for which $c < d < 2c$. The list $\mathcal{M}$ is a *general reduction* of the list $\mathcal{L}$ if (i) there exist a list $\mathcal{L}'$ of length c and a list $\mathcal{L}''$ of length $d-c$ such that $\mathcal{L}$ is a permutation of the list $\mathcal{L}' \diamond \mathcal{L}''$; (ii) there exist $d-c$ lists $\mathcal{S}_1$, $\mathcal{S}_2$, …, $\mathcal{S}_{d-c}$, the components of each of which are integers all greater than 1, such that the list $\mathcal{L}''$ is $(\prod \mathcal{S}_1, \prod \mathcal{S}_2, \ldots, \prod \mathcal{S}_{d-c})$ and the list $\tilde{\mathcal{S}} = \mathcal{S}_1 \diamond \mathcal{S}_2 \ldots \diamond \mathcal{S}_{d-c}$ has length b, where $d - c < b \leq c$; and (iii) $\mathcal{M}$ is a permutation of the list $[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'$, where $\mathcal{I} = (\underbrace{1, 1, \ldots, 1}_{c-b})$. We call $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d-c})$ a *reduction factor* of $\mathcal{L}$ into $\mathcal{M}$, $\mathcal{L}'$ the *multiplicant sublist*, and



(a) (3,3,6)-mesh G

(b) (6,9)-mesh H

Figure 12: Supernode view

$\mathcal{L}''$ the *multiplier sublist.* □

For example, the list $\mathcal{M} = (4,3,5,28,10,18)$ is a general reduction of the list $\mathcal{L} = (2,3,2,10,6,21,5,4)$ because we can choose $\mathcal{L}' = (2,2,6,4,3,5)$, $\mathcal{L}'' = (10,21)$, $\mathcal{S}_1 = (5,2)$, and $\mathcal{S}_2 = (3,7)$. The list $[\tilde{\mathcal{S}} \diamond (1,1)] \times \mathcal{L}' = (10,4,18,28,3,5)$ is a permutation of $\mathcal{M}$. The list $\mathcal{S} = ((5,2),(3,7))$ is a reduction factor of $\mathcal{L}$ into $\mathcal{M}$. Reduction factors may not be unique: the list $((2,5),(3,7))$ is also a reduction factor of $\mathcal{L}$ into $\mathcal{M}$.

By the definition, if $\mathcal{M}$ is a general reduction of $\mathcal{L}$ with a reduction factor $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d-c})$, then the list $\tilde{\mathcal{S}} = \mathcal{S}_1 \diamond \mathcal{S}_2 \diamond \cdots \diamond \mathcal{S}_{d-c}$ is an expansion of $\mathcal{L}''$ with an expansion factor $\mathcal{S}$.

Note that if $\mathcal{M}$ is a simple reduction of $\mathcal{L}$, then each component in $\mathcal{M}$ is the product of one or more components of $\mathcal{L}$. On the other hand, if $\mathcal{M}$ is a general reduction of $\mathcal{L}$, then each component in $\mathcal{M}$ is either (i) a component in the multiplicant sublist $\mathcal{L}'$ or (ii) the product of a component in $\mathcal{L}'$ and a factor of one of the components in the multiplier sublist $\mathcal{L}''$.

Let G be a torus or a mesh of shape $\mathcal{L} = (l_1, l_2, \ldots, l_d)$, and let H be a torus or a mesh of shape $\mathcal{M} = (m_1, m_2, \ldots, m_c)$. Assume that $\mathcal{M}$ is a general reduction of $\mathcal{L}$ with a reduction factor $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d-c})$. Let $\tilde{\mathcal{S}} = (s_1, s_2, \ldots, s_b) = \mathcal{S}_1 \diamond \mathcal{S}_2 \diamond \cdots \diamond \mathcal{S}_{d-c}$, where $d - c < b \le c$, and let $\mathcal{I} = (\underbrace{1, \ldots, 1}_{c-b})$. Let G' be a graph of shape $\mathcal{L}' \diamond \mathcal{L}''$ and of the same type as G, and let H' be a graph of shape $[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'$ and of the same type as H. We now construct an embedding of G in H in three steps: $G \to G' \to H' \to H$. Let $\alpha : [d]^+ \to [d]^+$ be a permutation such that $\alpha(\mathcal{L}) = \mathcal{L}' \diamond \mathcal{L}''$, and let $\beta : [c]^+ \to [c]^+$ be a permutation such that $\beta([\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}') = \mathcal{M}$. By the definition of general reduction, such permutations always exist. The graph G can be embedded in G' with unit dilation cost using the permutation α, and H' can be embedded in H with unit dilation cost using the permutation β. Next we construct an embedding of G' in H'.

The graph G' has shape $\mathcal{L}' \diamond \mathcal{L}'' = (l_{\alpha(1)}, l_{\alpha(2)}, \ldots, l_{\alpha(c)}) \diamond (l_{\alpha(c+1)}, l_{\alpha(c+2)}, \ldots l_{\alpha(d)})$. If G' is a mesh, we can think of G' as an $\mathcal{L}'$-mesh of *supernodes* with each supernode being an $\mathcal{L}''$-mesh, that is, the supernode $(i_1, i_2, \ldots, i_c)$ consists of all nodes $(i_1, i_2, \ldots, i_c) \diamond (*, *, \ldots, *)$ in G', where for all $j \in [c]^+$, $i_j \in [l_{\alpha(j)}]$, and $(*, *, \ldots, *)$ denotes all lists in $\Omega_{\mathcal{L}''}$. For example, if we view the $(3,3,6)$-mesh given in Figure 12(a) as a $(3,3)$-mesh of supernodes, then the supernode $(2,0)$

consists of the nodes $(2,0,0)$, $(2,0,1)$, $(2,0,2)$, $(2,0,3)$, $(2,0,4)$, and $(2,0,5)$. These nodes are labeled 0, 1, 2, 3, 4, and 5 in the figure. Similarly, if G' is a torus, we can think of G' as an $\mathcal{L}'$-torus of supernodes with each supernode being an $\mathcal{L}''$-torus.

The graph H' has shape $[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}' = (s_1 l_{\alpha(1)}, s_2 l_{\alpha(2)}, \ldots, s_b l_{\alpha(b)}, l_{\alpha(b+1)} \ldots, l_{\alpha(c)})$. If H' is a mesh, we can think of H' as an $\mathcal{L}'$-mesh of supernodes with each supernode being an $\tilde{\mathcal{S}}$-mesh, that is, the supernode $(i_1, i_2, \ldots, i_c)$ consists of all nodes $[(s_1 i_1, s_2 i_2, \ldots, s_b i_b) + (*, *, \ldots, *)] \diamond (i_{b+1}, i_{b+2}, \ldots, i_c)$ in H', where for all $j \in [c]^+$, $i_j \in [l_{\alpha(j)}]$, and $(*, *, \ldots, *)$ denotes all lists in $\Omega_{\tilde{\mathcal{S}}}$. For example, if we view the $(6,9)$-mesh in Figure 12(b) as a $(3,3)$-mesh of supernodes, then the supernode $(2,0)$ consists of the nodes $(4,0)$, $(4,1)$, $(4,2)$, $(5,0)$, $(5,1)$, and $(5,2)$. These nodes are labeled 0, 1, 2, 5, 4, and 3 in the figure. If H' is a torus, we can also think of H' as an $\mathcal{L}'$-torus of supernodes. Each supernode in H' is now an $\tilde{\mathcal{S}}$-mesh instead of an $\tilde{\mathcal{S}}$-torus. Notice that we cannot divide a torus into toruses of the same dimension and of smaller sizes because the neighborship required at the boundary nodes of the smaller toruses cannot be satisfied.

In summary, the supernodes of G' are formed by partitioning the shape of G' into two parts, with one part forming the shape of the supernodes, and the other the shape of the graph consisting of these supernodes. On the other hand, the supernodes of H' are formed by factoring the length of each dimension of H' into one or two factors, with one factor forming the length of a dimension of the graph consisting of the supernodes, and the other factor, if present, forming the length of a dimension of the supernodes. The dimensions of the supernodes of G' and the graph consisting of these supernodes are both lower than the dimension of G'. On the other hand, the dimension of the supernodes of H' may be lower than the dimension of H', while the dimension of the graph consisting of these supernodes is always the same as the dimension of H'. With respect to supernodes, G' and H' have the same shape $\mathcal{L}'$. The shape of the supernodes of H' ($\tilde{\mathcal{S}}$) is an expansion of the shape of the supernodes of G' ($\mathcal{L}''$) with an expansion factor of $\mathcal{S}$.

We consider the following four cases for constructing an embedding of G' in H'.

Case 1. G' and H' are meshes.

In this case, G' and H' are $\mathcal{L}'$-meshes of supernodes. Neighboring supernodes in G' can be mapped to neighboring supernodes in H' using the identity function. The $\mathcal{L}''$-meshes (supernodes of G') can then be embedded in the $\tilde{\mathcal{S}}$-meshes (supernodes of H') using the embedding function $\mathcal{F}_{\mathcal{S}} : \Omega_{\mathcal{L}''} \rightarrow \Omega_{\tilde{\mathcal{S}}}$ defined in the preceding subsection. Hence, we map each node $(i_1, i_2, \ldots, i_d)$ in G' to the node

$$\mathcal{F}'_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) = [(s_1 i_1, s_2 i_2, \ldots, s_b i_b) + \mathcal{F}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \diamond (i_{b+1}, i_{b+2}, \ldots, i_c)$$

in H'. We call $(s_1 i_1, s_2 i_2, \ldots, s_b i_b)$ and $(i_{b+1}, \ldots, i_c)$ the *base*, and $\mathcal{F}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))$ the *offset*.

Let $\mathcal{F}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d)) = (e_1, e_2, \ldots, e_b)$. We can write

$$\mathcal{F}'_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) \text{ as } (s_1 i_1 + e_1, s_2 i_2 + e_2, \ldots, s_b i_b + e_b, i_{b+1}, \ldots, i_c).$$

Since $\mathcal{F}_{\mathcal{S}}: \Omega_{\mathcal{L}''} \rightarrow \Omega_{\tilde{\mathcal{S}}}$ is bijective, and for all $i \in [b]^+$, $0 \leq e_i < s_i$, the function $\mathcal{F}'_{\mathcal{S}}: \Omega_{\mathcal{L}' \diamond \mathcal{L}''} \rightarrow \Omega_{[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'}$ is injective. Since $|\Omega_{\mathcal{L}' \diamond \mathcal{L}''}| = |\Omega_{[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'}|$, $\mathcal{F}'_{\mathcal{S}}$ is bijective. Therefore, the function $\mathcal{F}'_{\mathcal{S}}$ is an embedding of G' in H'.

This embedding has a dilation cost of $\max\{s_1, s_2, \ldots, s_b\}$. Let $A = (i_1, i_2, \ldots, i_d)$ and $B = (i'_1, i'_2, \ldots, i'_d)$ be an arbitrary pair of neighboring nodes in G', and let $k = [d]^+$ be the index at which $i_k \neq i'_k$. Let $A' = \mathcal{F}'_{\mathcal{S}}(A)$ and $B' = \mathcal{F}'_{\mathcal{S}}(B)$. If $k \in [c]^+$, then A' and B' have the same offset but different bases. Since H' is a mesh, the distance between A' and B' is $|s_k i_k - s_k i'_k|$ if $k \in [b]^+$, and $|i_k - i'_k|$ if $k \in \{b+1, \ldots, c\}$. Since G' is also a mesh, we have $|i_k - i'_k| = 1$. Therefore, the distance between A' and B' in H' is s_k if $k \in [b]^+$, and 1 if $k \in \{b+1, \ldots, c\}$. If $k \in \{c+1, \ldots, d\}$, then A' and B' have the same base but different offsets. Since the function $\mathcal{F}_{\mathcal{S}}$ embeds an $\mathcal{L}''$-mesh in an $\tilde{\mathcal{S}}$-mesh with unit dilation cost, the distance between A' and B' in H' is 1.

Case 2. G' is a mesh and H' is a torus.

We use the embedding function $\mathcal{F}'_{\mathcal{S}}$ from Case 1 but modifying the analysis slightly. We change the distance measure between A' and B' from δ_m-distance to δ_t-distance, and use the relation that for all $k \in [b]^+$, $m_k = s_k l_k$ and $l_k > 1$. In this way, we can show that this embedding also gives a dilation cost of $\max\{s_1, s_2, \ldots, s_b\}$.

Case 3. G' and H' are toruses.

Since G' and H' are both $\mathcal{L}'$-toruses of supernodes, neighboring supernodes in G' can be mapped to neighboring supernodes in H' using the identity function. The $\mathcal{L}''$-toruses (supernodes of G') can then be embedded in the $\tilde{\mathcal{S}}$-meshes (supernodes of H') using the function $\mathcal{G}_{\mathcal{S}} : \Omega_{\mathcal{L}''} \to \Omega_{\tilde{\mathcal{S}}}$ defined in the preceding subsection. Hence, we map each node $(i_1, i_2, \ldots, i_d)$ in G' to the node

$$\mathcal{G}'_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) = [(s_1 i_1, s_2 i_2, \ldots, s_b i_b) + \mathcal{G}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \diamond (i_{b+1}, i_{b+2}, \ldots, i_c)$$

in H'. This mapping is also bijective, and is therefore an embedding of G' in H'.

This embedding also has a dilation cost of $\max\{s_1, s_2, \ldots, s_b\}$. Let A, B, and k be defined as in Case 1; and let $A' = \mathcal{G}'_{\mathcal{S}}(A)$, and $B' = \mathcal{G}'_{\mathcal{S}}(B)$. Since G' is a torus, $|i_k - i'_k|$ is either 1 or $l_k - 1$. If $k \in [b]^+$, then the distance between A' and B' is $\min\{|s_k i_k - s_k i'_k|,\ m_k - |s_k i_k - s_k i'_k|\}$; since $m_k = s_k l_k$, this distance is s_k. If $k \in \{b+1, \ldots, c\}$, the distance between A' and B' is $\min\{|i_k - i'_k|,\ m_k - |i_k - i'_k|\}$; since $m_k = l_k$, this distance is 1. If $k \in \{c+1, \ldots, d\}$, then the distance between A' and B' in H' is at most 2 because the function $\mathcal{G}_{\mathcal{S}}$ embeds an $\mathcal{L}''$-torus in an $\tilde{\mathcal{S}}$-mesh with a dilation cost of 2. Finally, since for all $i \in [d]^+$, $l_i > 1$, we have $\max\{s_1, s_2, \ldots, s_b\} \geq 2$. Therefore, the embedding has a dilation cost of $\max\{s_1, s_2, \ldots, s_b\}$.

Case 4. G' is a torus and H' is a mesh.

By Lemma 36, neighboring supernodes of G' can be mapped to some supernodes in H' at a distance no greater than 2 by embedding each supernode $(i_1, i_2, \ldots, i_c)$ in G' in the supernode $(t_{l_{\alpha(1)}}(i_1), t_{l_{\alpha(2)}}(i_2), \ldots, t_{l_{\alpha(c)}}(i_c))$ in H'. The $\mathcal{L}''$-toruses in G' are then embedded in the $\tilde{\mathcal{S}}$-meshes using the function $\mathcal{G}_{\mathcal{S}}$. Hence, we can map each node $(i_1, i_2, \ldots, i_c)$ in G' to the node

$$\begin{aligned}\mathcal{G}''_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) &= [(s_1 t_{l_{\alpha(1)}}(i_1), s_2 t_{l_{\alpha(2)}}(i_2), \ldots, s_b t_{l_{\alpha(b)}}(i_b)) + \mathcal{G}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \\ &\quad \diamond (t_{l_{\alpha(b+1)}}(i_{b+1}), t_{l_{\alpha(b+2)}}(i_{b+2}) \ldots, t_{l_{\alpha(c)}}(i_c))\end{aligned}$$

in H'. This mapping is also bijective, and is therefore an embedding of G' in H'.

Let A, B, and k be defined as in Case 1, and let $A' = \mathcal{G}''_{\mathcal{S}}(A)$ and $B' = \mathcal{G}''_{\mathcal{S}}(B)$. The distance between A' and B' is $|s_k t_{l_k}(i_k) - s_k t_{l_k}(i'_k)|$ if $k \in [b]^+$, and $|t_{l_k}(i_k) - t_{l_k}(i'_k)|$ if $k \in \{b+1, \ldots, c\}$. Since for all $j \in [c]^+$, the cyclic sequence $t_{l_{\alpha(j)}}$ has a δ_m-spread of 2 if $l_{\alpha(j)} > 2$, and 1 otherwise, this distance is at most $2s_k$ if $k \in [b]^+$, and at most 2 if $k \in \{b+1, \ldots, c\}$. If $k \in \{c+1, \ldots, d\}$, then as in Case 3, the distance between A' and B' in H' is at most 2. Hence, the embedding has a dilation cost at most $2\max\{s_1, s_2, \ldots, s_b\}$.

In summary, the sequence of embeddings $G \to G' \to H' \to H$ defined above has a dilation cost at most $2\max\{s_1, s_2, \ldots, s_b\}$ if G is a torus, H is a mesh, and a dilation cost of $\max\{s_1, s_2, \ldots, s_b\}$ otherwise.

We formalize the above results in the following definition and theorem.

Definition 42 Let d and c be positive integers such that $c < d < 2c$. Let $\mathcal{L} = (l_1, l_2, \ldots, l_d)$ and $\mathcal{M} = (m_1, m_2, \ldots, m_c)$ be radix-bases. Assume that $\mathcal{M}$ is a general reduction of $\mathcal{L}$ with a reduction factor $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d-c})$, multiplicant sublist $\mathcal{L}'$, and multiplier sublist $\mathcal{L}''$. Let $\alpha : [d]^+ \to [d]^+$ be a permutation such that $\alpha(\mathcal{L}) = \mathcal{L}' \diamond \mathcal{L}''$. Let $\tilde{\mathcal{S}} = (s_1, s_2, \ldots, s_b) = \mathcal{S}_1 \diamond \mathcal{S}_2 \diamond \cdots \diamond \mathcal{S}_{d-c}$, and let $\mathcal{I} = (\underbrace{1, 1, \ldots, 1}_{c-b})$. Let $\mathcal{F}_{\mathcal{S}} : \Omega_{\mathcal{L}''} \to \Omega_{\tilde{\mathcal{S}}}$ and $\mathcal{G}_{\mathcal{S}} : \Omega_{\mathcal{L}''} \to \Omega_{\tilde{\mathcal{S}}}$. The functions $\mathcal{F}'_{\mathcal{S}} : \Omega_{\mathcal{L}' \diamond \mathcal{L}''} \to \Omega_{[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'}$, $\mathcal{G}'_{\mathcal{S}} : \Omega_{\mathcal{L}' \diamond \mathcal{L}''} \to \Omega_{[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'}$, and $\mathcal{G}''_{\mathcal{S}} : \Omega_{\mathcal{L}' \diamond \mathcal{L}''} \to \Omega_{[\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}'}$ are defined as follows: for all $(i_1, i_2, \ldots, i_d) \in \Omega_{\mathcal{L}' \diamond \mathcal{L}''}$,

$$\mathcal{F}'_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) = [(s_1 i_1, s_2 i_2, \ldots, s_b i_b) + \mathcal{F}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \diamond (i_{b+1}, i_{b+2} \ldots, i_c),$$

$$\mathcal{G}'_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) = [(s_1 i_1, s_2 i_2, \ldots, s_b i_b) + \mathcal{G}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \diamond (i_{b+1}, i_{b+2} \ldots, i_c),$$

$$\begin{aligned} \mathcal{G}''_{\mathcal{S}}((i_1, i_2, \ldots, i_d)) \;&=\; [(s_1 t_{l_{\alpha(1)}}(i_1), s_2 t_{l_{\alpha(2)}}(i_2), \ldots, s_b t_{l_{\alpha(b)}}(i_b)) + \mathcal{G}_{\mathcal{S}}((i_{c+1}, i_{c+2}, \ldots, i_d))] \\ &\quad \diamond (t_{l_{\alpha(b+1)}}(i_{b+1}), t_{l_{\alpha(b+2)}}(i_{b+2}) \ldots, t_{l_{\alpha(c)}}(i_c)). \end{aligned}$$

Furthermore, let $\beta : [c]^+ \to [c]^+$ be a permutation such that $\beta([\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}') = \mathcal{M}$. Then we have the functions $\beta \circ \mathcal{F}'_{\mathcal{S}} \circ \alpha : \Omega_{\mathcal{L}} \to \Omega_{\mathcal{M}}$, $\beta \circ \mathcal{G}'_{\mathcal{S}} \circ \alpha : \Omega_{\mathcal{L}} \to \Omega_{\mathcal{M}}$, and $\beta \circ \mathcal{G}''_{\mathcal{S}} \circ \alpha : \Omega_{\mathcal{L}} \to \Omega_{\mathcal{M}}$. □

Theorem 43 *Let d and c be positive integers such that $c < d < 2c$. Let G be a torus or a mesh of shape $\mathcal{L} = (l_1, l_2, \ldots, l_d)$, and let H be a torus or a mesh of shape $\mathcal{M} = (m_1, m_2, \ldots, m_c)$. Assume that $\mathcal{M}$ is a general reduction of $\mathcal{L}$ with a reduction factor $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{d-c})$, multiplicant sublist $\mathcal{L}'$, and multiplier sublist $\mathcal{L}''$. Let $\tilde{\mathcal{S}} = (s_1, s_2, \ldots, s_b) = \mathcal{S}_1 \diamond \mathcal{S}_2 \diamond \cdots \diamond \mathcal{S}_{d-c}$, and let $\mathcal{I} = (\underbrace{1, 1, \ldots, 1}_{c-b})$. Let $\alpha : [d]^+ \to [d]^+$ be a permutation such that $\alpha(\mathcal{L}) = \mathcal{L}' \diamond \mathcal{L}''$, and let $\beta : [c]^+ \to [c]^+$ be a permutation such that $\beta([\tilde{\mathcal{S}} \diamond \mathcal{I}] \times \mathcal{L}') = \mathcal{M}$. Then*

(i) *If G is a mesh, then G can be embedded in H with a dilation cost of* $\max\{s_1, s_2, \ldots, s_b\}$. *The function $\beta \circ \mathcal{F}'_{\mathcal{S}} \circ \alpha$ gives such an embedding.*

(ii) *If G and H are toruses, then G can be embedded in H with a dilation cost of* $\max\{s_1, s_2, \ldots, s_b\}$. *The function* $\beta \circ \mathcal{G}'_S \circ \alpha$ *gives such an embedding.*

(iii) *If G is a torus and H is a mesh, then G can be embedded in H with a dilation cost at most* $2\max\{s_1, s_2, \ldots, s_b\}$. *The function* $\beta \circ \mathcal{G}''_S \circ \alpha$ *gives such an embedding.*

□

The condition of general reduction requires that the dimension of H must be higher than half of the dimension of G. If this condition is not satisfied, an embedding of G in H can still be constructed using the results in this subsection provided that there exists a sequence of intermediate graphs in which every pair of successive graphs have shapes satisfying the condition of general reduction.

As will be shown in Section 5, if G and H are square, then one of the following two conditions must be true: (i) their shapes satisfy the condition of simple reduction, and (ii) the sequence of intermediate graphs described above exists.

5 Generalized embeddings among square toruses and square meshes

The results for generalized embeddings developed in the preceding section can be applied only if the shapes of G and H satisfy either the condition of expansion (for increasing dimension cases) or the condition of reduction (for lowering dimension cases). In this section, we study the cases in which G and H are square. For these cases, we can always construct an embedding of G in H through a sequence of one or more embedding steps using the embedding functions defined in Section 4.

Let d be the dimension of G, c be the dimension of H, a be the greatest common denominator of d and c, and ℓ be the length of the dimensions of G. The major results of this section are the following:

For the case of lowering dimension ($c < d$), G can be embedded in H with a dilation cost of $2\ell^{(d-c)/c}$ if G is a torus and H is a mesh, and with a dilation cost of $\ell^{(d-c)/c}$ otherwise. For fixed values of d and c, these dilation costs are optimal to within a constant.

For the case of increasing dimension ($d < c$), if c is divisible by d, then G can be embedded in H with an optimal dilation cost of 2 if G is a torus of odd size and H is a mesh, and with unit dilation cost otherwise. If c is not divisible by d, then G can be embedded in H with a dilation cost of $2\ell^{(d-a)/c}$ if G is a torus of odd size and H is a mesh, and with a dilation cost of $\ell^{(d-a)/c}$ otherwise; these dilation costs, however, may not be optimal.

A lower bound on dilation cost for lowering dimension

In [Ros75], Rosenberg studied the problem of embedding finite arrays (meshes), prism arrays, and orthant arrays in lines to minimize proximity in various local and global senses. Let t be an embedding of a d-dimensional mesh G in a line. For any positive integer k, *the diameter of preservation* σ_k is the smallest positive integer i such that for every node v in G, and for every pair of nodes u and w in G whose distances from v are no greater than k, $\delta_m(t(u), t(w)) < i$. Rosenberg proved that $\sigma_k > bk\mu^{d-1}$, where μ is the length of the shortest dimension of G, and b depends only on d and is a constant with respect to μ.

Let G be a d-dimensional torus or a d-dimensional mesh, and H be a c-dimensional torus or a c-dimensional mesh such that $c < d$ and G and H are of the same size. In the following, using a straightforward modification of Rosenberg's proof for the lower bound on the diameter of preservation [Ros75], we show that the dilation cost of any embedding of G in H is bounded from below by $b\mu^{(d-c)/c}$, where μ is the length of the shortest dimension of G, and b is a constant with respect to μ and depends only on d and c. This lower bound on dilation cost will be used to prove the optimality properties of our embeddings among square toruses and square meshes in the lowering dimension case.

Given a d-dimensional mesh G, a node v in G, and a positive integer k, let $Q(v, k)$ denote the set of nodes in G whose distances from v are no greater than k.

Lemma 44 *[Ros75] Let G be a d-dimensional mesh. Let μ be the length of the shortest dimension of G. For any positive integer k such that $k < \mu$, $\max_{v \in G}|Q(v, k)| \geq \binom{k+d}{d} > bk^d$, where $b > 0$ is a constant with respect to k, and depends only on d.* □

Lemma 45 *Let G be a d-dimensional mesh, and H be a c-dimensional mesh such that $c < d$ and G and H are of the same size. Let t be an embedding of G in H with a dilation cost of ρ. Then for any node v in G and any positive integer k, $|Q(v,k)| \leq (2k\rho + 1)^c$.* □

Proof. Let p_1, p_2, ..., p_c be nonnegative integers. A set of lists is said to *lie within* a c-dimensional interval $[p_1, p_2, \ldots, p_c]$ if the lists are all of the form $(i_1 + e_1, i_2 + e_2, \ldots, i_c + e_c)$, where for all $j \in [c]^+$, i_j is some fixed integer and $e_j \in [p_j]$. For v an arbitrary node in G and k an arbitrary positive integer, let $t(Q(v,k))$ be the set of images of all the nodes in $Q(v,k)$ under the embedding t. We first show by induction on k that $t(Q(v,k))$ lies within a c-dimensional interval $[2k\rho+1, 2k\rho+1, \ldots, 2k\rho+1]$.

Induction basis: $k = 1$.

Let $q = |Q(v,1)|$. Let $(a_1^1, a_2^1, \ldots, a_c^1)$, $(a_1^2, a_2^2, \ldots, a_c^2)$, ..., $(a_1^q, a_2^q, \ldots, a_c^q)$ denote the nodes in $t(Q(v,1))$. For all $j \in [c]^+$, let $\alpha_j = \min\{a_j^1, a_j^2, \ldots, a_j^q\}$, and let $\beta_j = \max\{a_j^1, a_j^2, \ldots, a_j^q\}$. Since for all $u, w \in Q(v,1)$, $\delta_m(t(u), t(w)) \leq \delta_m(t(u), t(v)) + \delta_m(t(v), t(w)) \leq 2\rho$, we have for all $j \in [c]^+$, $|\alpha_j - \beta_j| \leq 2\rho$. Therefore, $t(Q(v,1))$ must lie within a c-dimensional interval $[2\rho+1, 2\rho+1, \ldots, 2\rho+1]$.

Induction hypothesis: Assume that for all $k \leq k'$, $t(Q(v,k'))$ lies within a c-dimensional interval $[2k'\rho+1, 2k'\rho+1, \ldots, 2k'\rho+1]$.

Induction step: $k = k' + 1$.

Since every node u in $Q(v, k'+1)$ must either belong to $Q(v,k')$ or be a neighbor of some node w in $Q(v,k')$, the smallest c-dimensional interval containing $t(Q(v,k'+1))$ contains at most 2ρ elements more in each of the c dimensions than the corresponding interval for $t(Q(v,k'))$. Therefore, by our induction hypothesis, $t(Q(v,k'+1))$ must lie within a c-dimensional interval $[2\rho+2k'\rho+1, 2\rho+2k'\rho+1, \ldots, 2\rho+2k'\rho+1] = [2(k'+1)\rho+1, 2(k'+1)\rho+1, \ldots, 2(k'+1)\rho+1]$.

For any positive integer k, the maximum number of lists that can lie within a c-dimensional interval $[2k\rho+1, 2k\rho+1, \ldots, 2k\rho+1]$ is $(2k\rho+1)^c$. Since t is bijective, we have $|Q(v,k)| \leq (2k\rho+1)^c$. □

Lemma 46 *Let G and H be meshes of the same size. Let G' be a torus of the same shape as G, and H' be a torus of the same shape as H. Assume that the dilation cost of any embedding* *of G in H is bounded from below by x. Then the dilation cost of any embedding of G' in H, G in H', or G' in H' is bounded from below by bx, for some constant b.* □

Proof. Let ζ be the dilation cost of an arbitrary embedding of the torus G' in the torus H'. By Lemma 36, the mesh G can be embedded in the torus G' with unit dilation cost, and the torus H' can be embedded in the mesh H with a dilation cost of 2. Since the sequence of embeddings $G \xrightarrow{1} G' \xrightarrow{\zeta} H' \xrightarrow{2} H$ provides an embedding of G in H with a dilation cost of 2ζ, we have $\zeta \geq x/2$.

Similarly, let λ be the dilation cost of an arbitrary embedding of G' in H, and γ the dilation cost of an arbitrary embedding of G in H'. Since the sequence $G \xrightarrow{1} G' \xrightarrow{\lambda} H$ and the sequence $G \xrightarrow{\gamma} H' \xrightarrow{2} H$ also provide embeddings of G in H with dilation costs of λ and 2γ respectively, we have $\lambda \geq x$ and $\gamma \geq x/2$. □

Theorem 47 *Let G be a d-dimensional torus or a d-dimensional mesh, and let H be a c-dimensional torus or a c-dimensional mesh such that $c < d$ and G and H are of the same size. Let μ be the length of the shortest dimension of G. Then the dilation cost of any embedding of G in H is bounded from below by $b\mu^{(d-c)/c}$, for some positive number b that is a constant with respect to μ and depends only on d and c.* □

Proof. We first assume that G and H are meshes. Let ρ be the dilation cost of an arbitrary embedding of G in H. By Lemmas 44 and 45, for any positive integer k such that $k < \mu$, $(2k\rho + 1)^c > bk^d$, for some positive number b that depends only on d. We thus have $\rho > (\frac{b^{1/c}}{2})k^{(d-c)/c} - \frac{1}{2k} \geq (\frac{b^{1/c}}{2})k^{(d-c)/c}$. By letting $k = \mu - 1$, we have $\rho \geq (\frac{b^{1/c}}{2})(\mu - 1)^{(d-c)/c}$. Since $\mu \geq 2$, $\mu - 1 \geq \frac{\mu}{2}$. Therefore $\rho \geq b'\mu^{(d-c)/c}$, for some b' that is a constant with respect to μ and depends only on d and c. The other cases follow from Lemma 46. □

Embeddings for lowering dimension

Theorem 48 *Let G be a square torus or a square mesh of dimension d, and H be a square torus or a square mesh of dimension c such that $c < d$ and G and H are of the same size. Let ℓ be the length of the dimensions of G. Assume that d is divisible by c. Then the shapes of G and*

H always satisfy the condition of simple reduction. Furthermore, G can be embedded in H with a dilation cost of $2\ell^{(d-c)/c}$ if G is a torus and H is a mesh, and with a dilation cost of $\ell^{(d-c)/c}$ otherwise; for fixed values of d and c, such dilation costs are optimal to within a constant. □

Proof. Let $b = d/c$. Since d is divisible by c, b is an integer. Let m be the length of the dimensions of H. Since G and H are of the same size, we have $m^c = \ell^d$, and $m = \ell^b$. Hence, H is a simple reduction of G with a reduction factor $((\underbrace{\ell,\ldots,\ell}_{b}),\ldots,(\underbrace{\ell,\ldots,\ell}_{b}))$. Therefore, by Theorem 39, G can be embedded in H with a dilation cost of $2m/\ell = 2\ell^{(d-c)/c}$ if G is a torus and H is a mesh, and with a dilation cost of $\ell^{(d-c)/c}$ otherwise.

By Theorem 47, the optimal dilation cost of embedding G in H is bounded from below by $b\ell^{(d-c)/c}$, for some positive number $b > 0$ that is a constant with respect to ℓ and depends only on d and c. Since the dilation costs of our embeddings are either $2\ell^{(d-c)/c}$ or $\ell^{(d-c)/c}$, they are optimal to within a constant for fixed values of d and c. □

The next corollary follows directly from Theorem 48. This corollary also follows as a special case of Corollary 40.

Corollary 49 *A hypercube can be embedded in a square torus or a square mesh of the same size with a dilation cost of $m/2$, for m the length of the dimensions of the given torus or mesh.* □

The following lemma states a property of integers that will be used in Theorem 51 to construct our embeddings for lowering dimension cases in which d is not divisible by c. This lemma in turn uses the following properties of integers [Bun72]:

(∗) Any positive integer $N > 1$ can be written uniquely in a *standard form* $N = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ such that for all $i \in [r]^+$, b_i is a positive integer and each p_i is a prime with $1 < p_1 < \cdots < p_r$.

Lemma 50 *Let x be any integer greater than 1, and let u and v be any integers that are relatively prime. Assume that $x^{u/v}$ is an integer. Then $x^{1/v}$ is also an integer.* □

Proof. Let $y = x^{u/v}$. By assumption, y is an integer. Furthermore, since x is an integer greater than 1, y must also be an integer greater than 1. By property (∗) of integers, x can be written in its unique standard form $p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$ in which $r, b_1, b_2, \ldots, b_r$ are positive integers and p_1, p_2, $\ldots$, p_r are distinct primes with $p_1 < p_2 < \cdots < p_r$. Similarly, y can be written in its unique standard form $q_1^{c_1}q_2^{c_2}\cdots q_s^{c_s}$ in which $s, c_1, c_2, \ldots, c_s$ are positive integers and q_1, q_2, $\ldots$, q_s are distinct primes with $q_1 < q_2 < \cdots < q_s$.

Since $y^v = x^u$, we have $q_1^{vc_1}q_2^{vc_2}\cdots q_s^{vc_s} = p_1^{ub_1}p_2^{ub_2}p_r^{ub_r}$. Since $q_1, q_2, \ldots, q_s$ are distinct primes with $q_1 < q_2 < \ldots < q_s$ and $p_1, p_2, \ldots, p_r$ are also distinct primes with $p_1 < p_2 < \ldots < p_r$, we have $r = s$ and for all $i \in [r]^+$, $q_i = p_i$ and $vc_i = ub_i$. Hence, for all $i \in [r]^+$, we have $ub_i/v = c_i$. Since c_i is an integer, and u and v are relatively prime, b_i must be divisible by v. It follows that $p_1^{b_1/v}p_2^{b_2/v}\cdots p_r^{b_r/v}$, which is $x^{1/v}$, must be an integer. □

Theorem 51 *Let G be a square torus or a square mesh of dimension d, and H a square torus or a square mesh of dimension c such that $c < d$ and G and H are of the same size. Let ℓ be the length of the dimensions of G. Assume that d is not divisible by c. Then there always exists a sequence of intermediate graphs in which the shapes of every pair of successive graphs satisfy the condition of general reduction. Furthermore, G can be embedded in H with a dilation cost of $2\ell^{(d-c)/c}$ if G is a torus and H is a mesh, and with a dilation cost of $\ell^{(d-c)/c}$ otherwise. For fixed values of d and c, these dilation costs are optimal to within a constant.* □

Proof. We first treat the case in which G and H are meshes. Let m be the length of the dimensions of H. Since G and H are of the same size, we have $m^c = \ell^d$, and $m = \ell^{d/c}$. Since m is an integer, $\ell^{d/c}$ must also be an integer.

We first consider the simple case in which d and c are relatively prime. By the definition of meshes, $\ell > 1$, and hence by Lemma 50, $\ell^{1/c}$ is an integer. Let I_0, I_1, $\ldots$, I_{d-c} be meshes such that for all $k \in [d-c+1]$, I_k has dimension $d-k$ and shape

$$(\underbrace{\ell^{(c+k)/c}, \ldots, \ell^{(c+k)/c}}_{c}, \underbrace{\ell, \ldots, \ell}_{d-c-k}).$$

We have $I_0 = G$; $I_{d-c} = H$; I_0, I_1, $\ldots$, I_{d-c} all have the same size ℓ^d; and, except for I_0 and I_{d-c}, none of the meshes I_1, I_2, $\ldots$, I_{d-c-1} is square. For all $k \in [d-c]$, the dimension of I_k is greater than the dimension of I_{k+1} by 1, and the shape of I_{k+1} is a general reduction of the shape of I_k

with a reduction factor

$$((\underbrace{\ell^{1/c},\ldots,\ell^{1/c}}_{c})).$$

By Theorem 43, the mesh I_k can be embedded in I_{k+1} with a dilation cost of $\ell^{1/c}$. The sequence of embeddings $G = I_0 \to I_1 \to \cdots \to I_{d-c-1} \to I_{d-c} = H$ has a total of $d-c$ steps, with a dilation cost of $\ell^{1/c}$ in each step. This embedding of G in H therefore has a dilation cost of $\ell^{(d-c)/c}$.

Next we consider the case in which d and c are not relatively prime. Let a be the greatest common denominator of d and c, and let $u = d/a$ and $v = c/a$. Since d is not divisible by c, u and v are integers and relatively prime. We can write $\ell^{d/c}$ as $\ell^{u/v}$. Since $\ell^{u/v}$ is an integer and u and v are relatively prime, by Lemma 50, $\ell^{1/v}$ is an integer.

As in the preceding case, we can define a sequence of embeddings from G to H. This sequence consists of $u-v$ embedding steps, in each step of which the dimensions of the two corresponding graphs differ by a. Let I_0, I_1, …, I_{u-v} be meshes such that for all $k \in [u-v+1]$, I_k has dimension $a(u-k)$ and shape

$$\mathcal{L}_k = (\underbrace{\ell^{(v+k)/v},\ldots,\ell^{(v+k)/v}}_{av}, \underbrace{\ell,\ldots,\ell}_{a(u-v-k)}).$$

We have $I_0 = G$; $I_{u-v} = H$; I_0, I_1, …, I_{u-v} all have the same size $\ell^{au} = \ell^d$; and, except for I_0 and I_{u-v}, none of the meshes I_1, I_2, …, I_{u-v-1} is square.

For all $k \in [u-v]$, let $\mathcal{L}'_k$ be a list of length $a(u-k-1)$, and $\mathcal{L}''_k$ be a list of length a such that

$$\mathcal{L}'_k = (\underbrace{\ell^{(v+k)/v},\ldots,\ell^{(v+k)/v}}_{av}, \underbrace{\ell,\ldots,\ell}_{a(u-v-k-1)}) \quad \text{and} \quad \mathcal{L}''_k = (\underbrace{\ell,\ldots,\ell}_{a}).$$

$\mathcal{L}'_k \diamond \mathcal{L}''_k$ is a permutation of $\mathcal{L}_k$. Let

$$\mathcal{R}_k = (\underbrace{\ell^{1/v},\ldots,\ell^{1/v}}_{v}) \quad \text{and} \quad \mathcal{R}'_k = \underbrace{\mathcal{R}_k \diamond \cdots \diamond \mathcal{R}_k}_{a}.$$

The list $\mathcal{R}'_k$ has length av. We have

$$\mathcal{L}''_k = (\underbrace{\prod \mathcal{R}_k,\ldots,\prod \mathcal{R}_k}_{a}).$$

The list $[\mathcal{R}'_k \diamond (\underbrace{1, \ldots, 1}_{a(u-v-k-1)})] \times \mathcal{L}'_k$ is $\mathcal{L}_{k+1}$. Therefore, the list $\mathcal{L}_{k+1}$ is a general reduction of the list $\mathcal{L}_k$ with a reduction factor of

$$\underbrace{(\mathcal{R}_k, \ldots, \mathcal{R}_k)}_{a}.$$

By Theorem 43, the mesh I_k can be embedded in the mesh I_{k+1} with a dilation cost of $\ell^{1/v}$.

In the sequence of embeddings $G = I_0 \to I_1 \to \cdots \to I_{u-v-1} \to I_{u-v} = H$, each embedding step has a dilation cost of $\ell^{1/v}$. Since there are a total of $u - v$ steps, this embedding of G in H has a dilation cost of $\ell^{(u-v)/v} = \ell^{(d-c)/c}$.

We next consider the case in which G is a torus and H is a mesh. For all $i \in [u-v]$, let I_i be a torus, and let I_{u-v} be a mesh. For all $k \in [u-v+1]$, the shape of I_k is defined as in the preceding case. Again by Theorem 43, for all $k \in [u-v-1]$, the torus I_k can be embedded in the torus I_{k+1} with a dilation cost of $\ell^{1/v}$, and the torus I_{u-v-1} can be embedded in the mesh I_{u-v} with a dilation cost of $2\ell^{1/v}$. Therefore, the sequence of embeddings $G = I_0 \to I_1 \to \cdots \to I_{u-v} = H$ has a total dilation cost of $2\ell^{(d-c)/c}$. The proofs of the dilation costs for the other cases of G and H are similar and thus omitted.

The optimality condition of these dilation costs follows from Theorem 47. □

Notice that in Theorems 48 and 51 and Corollary 49, the ratio of our dilation cost to the optimal dilation cost is bounded from above by $1/b$, for some positive number b that depends only on d and c. For fixed values of d and c, this upper bound on the ratio is a constant. Since in Theorems 48 and 51, an instance of G and H depends on d, c, and ℓ (or equivalently, on b, c, and m, since $\ell^d = m^c$), we can fix the values of d and c without fixing an instance of G and H. Therefore, in Theorems 48 and 51, for all problem instances in which d and c are fixed but ℓ is any integer greater than 1, the ratio of our dilation cost to the optimal dilation cost is bounded from above by a constant. On the other hand, in Corollary 49, in which case G is a hypercube, an instance of G and H depends only on d and c. Fixing d and c fixes such an instance. Therefore, in this case, the upper bound $1/b$ on the ratio of our dilation cost to the optimal dilation cost varies with each problem instance.

A few special cases of embeddings among toruses and meshes of the same size for lowering dimension have been solved optimally in the literature: optimal embedding of an (ℓ, ℓ, ℓ)-mesh in a line of the same size with a dilation cost of $\lfloor 3\ell^2/4 + \ell/2 \rfloor$ [Fit74], optimal embedding of an (ℓ, ℓ)-mesh in a line of the same size with a dilation cost of ℓ [Fit74], optimal embedding of an (ℓ, ℓ)-torus in a ring of the same size with a dilation cost of ℓ [MN86], and optimal embedding of a hypercube of size 2^d in a line of the same size with a dilation cost of $\sum_{k=0}^{d-1} \binom{k}{\lfloor k/2 \rfloor}$ [Har66].

For the cases of embedding an (ℓ, ℓ)-mesh in a line and embedding an (ℓ, ℓ)-torus in a ring, our embeddings also give a dilation cost of ℓ. Thus, both are truly optimal. For the case of embedding an (ℓ, ℓ, ℓ)-mesh in a line, our embedding gives a dilation cost of ℓ^2. Thus, it is optimal to within a constant 4/3.

For the case of embedding a hypercube of size 2^d in a line, our embedding gives a dilation cost of 2^{d-1}. The optimal dilation cost $\sum_{k=0}^{d-1} \binom{k}{\lfloor k/2 \rfloor}$ can be written as $\varepsilon_{d-1} 2^{d-1}$, where $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$, and for all $d \geq 3$, $\varepsilon_{d-1} > \varepsilon_d$. (See Appendix.) Hence, our embedding is truly optimal for $1 \leq d \leq 3$. However, for all $d > 3$, the ratio of our dilation cost to the optimal dilation cost, which is $1/\varepsilon_{d-1}$, is strictly greater than 1. Furthermore, for all $d > 3$, this ratio is an increasing function of d, and hence, as we have discussed earlier, cannot be bounded from above by a constant.

Embeddings for increasing dimension

Theorem 52 *Let G be a square torus or a square mesh of dimension d, and let H be a square torus or a square mesh of dimension c such that $d < c$ and G and H are of the same size. Assume that c is divisible by d. Then G can be embedded in H with an optimal dilation cost of 2 if G is a torus of odd size and H is a mesh, and with unit dilation cost otherwise.* □

Proof. Let $a = c/d$. By the assumption of the theorem, a is an integer. Let ℓ be the length of the dimensions of G, and m be the length of the dimensions of H. Let $\mathcal{L}$ be the shape of G, and $\mathcal{M}$ be the shape of H. We have

$$\mathcal{L} = (\underbrace{\ell, \ldots, \ell}_{d}) \quad \text{and} \quad \mathcal{M} = (\underbrace{m, \ldots, m}_{c})$$

Since G and H are of the same size, we have $\ell^d = m^c$, and $\ell = m^a$. Let

$$\mathcal{R} = (\underbrace{m, \ldots, m}_{a}).$$

Since $\prod \mathcal{R} = \ell$, and

$$\mathcal{M} = \underbrace{\mathcal{R} \diamond \cdots \diamond \mathcal{R}}_{d},$$

the list $\mathcal{M}$ is an expansion of the list $\mathcal{L}$, with an expansion factor of

$$\underbrace{(\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R})}_{d}.$$

Assume that G is a torus of even size and H is a mesh of the same size. Since $d < c$, we have $a \geq 2$. Hence, the list $\mathcal{R}$ consists of at least two components. Furthermore, since the size of H is even, m must also be even, and hence, all of the components of $\mathcal{R}$ are even. Therefore, by Theorem 32, G can be embedded in H with unit dilation cost. The other cases of G and H also follow from Theorem 32. □

Theorem 53 *Let G be a square torus or a square mesh of dimension d, and let H be a square torus or a square mesh of dimension c such that $d < c$ and G and H are of the same size. Let ℓ be the length of the dimensions of G, and a be the greatest common divisor of c and d. Assume that c is not divisible by d. Then G can be embedded in H with a dilation cost of $2\ell^{(d-a)/c}$ if G is a torus of odd size and H is a mesh, and with a dilation cost of $\ell^{(d-a)/c}$ otherwise.* □

Proof. We construct an embedding of G in H through an intermediate graph G' for which the shape of G' is an expansion of the shape of G and the shape of H is a general reduction of the shape of G'. We first consider the case in which G and H are meshes. Let m be the length of the dimensions of H. Let $u = d/a$, and $v = c/a$. Since u and v are relatively prime, and $\ell^{u/v}$ is an integer, by Lemma 50, $\ell^{1/v}$ is also an integer. Let G' be a mesh of dimension vd and with the length of the dimensions equal to $\ell^{1/v}$. The mesh G' has the same size as G, and the shape of G' is an expansion of the shape of G with an expansion factor of

$$\underbrace{(\mathcal{R}, \ldots, \mathcal{R})}_{d} \quad \text{where} \quad \mathcal{R} = \underbrace{(\ell^{1/v}, \ldots, \ell^{1/v})}_{v}.$$

By Theorem 32, the mesh G can be embedded in G' with unit dilation cost.

Next we construct an embedding of G' in H. The dimension of G', which is vd, can be written as $(c/a)d = cu$. By definitions of u and v, we have $d = au$ and $c = av$. Since a is the greatest common divisor of d and c, and since by the assumption of the theorem, c is not divisible by d, we have $u > 1$. The dimension of G' is thus greater than the dimension of H. Since G' and H are square and of the same size, by Theorem 51, G' can be embedded in H with a dilation cost of $(\ell^{1/v})^{(vd-c)/c} = \ell^{(d-a)/c}$. Therefore, the embedding sequence $G \to G' \to H$ gives an embedding of G in H with a dilation cost of $\ell^{(d-a)/c}$.

We next consider the case in which G is a torus and H is a mesh. We define a mesh G' the same way as in the preceding case. If the size of G is odd, then by Theorem 32, the torus G can be embedded in the mesh G' with a dilation cost of 2. If the size of G is even, then $\ell^{1/v}$ must also be even. Furthermore, since $d < c$ and a is the greatest common divisor of d and c, it follows that v, which is c/a, must be greater than 1. Thus again by Theorem 32, G can be embedded in G' with unit dilation cost. Therefore, the embedding sequence $G \to G' \to H$ gives an embedding of G in H with a dilation cost of $2\ell^{(d-a)/c}$ if the size of G is odd, and a dilation cost of $\ell^{(d-a)/c}$ otherwise.

The proofs of the other cases of G and H are similar and thus omitted. □

In summary, our embeddings for square toruses and square meshes are all defined using the generalized embeddings defined in Section 4. For lowering dimension cases, if the dimension of G is divisible by the dimension of H, then the shape of H is a simple reduction of the shape of G. Otherwise, G can be embedded in H through a sequence of intermediate graphs in which every pair of successive graphs have shapes satisfying the condition of general reduction. In either case, our embeddings have dilation costs optimal to within a constant for fixed values of d and c. For increasing dimension cases, if the dimension of H is divisible by the dimension of G, then H is always an expansion of G, and an embedding of G in H can be immediately constructed by applying the results from Section 4. Furthermore, this embedding is always optimal. If the dimension of H is not divisible by the dimension of G, then an embedding of G in H is constructed through an intermediate graph G' such that the shape of G' is an expansion of the shape of G and the shape of H is a general reduction of the shape of G'. This embedding, however, may not be optimal in general.

6 Conclusion

This paper studies the embeddings among toruses and meshes of the same size. All of the results are based on several basic embeddings from a line or a ring in a torus or a mesh. The results for basic embeddings are all optimal. For generalized embeddings for which at least one of the two graphs is not square, our results are restricted only to those cases in which the shapes of the two graphs satisfy the condition of expansion for increasing dimension cases and the condition of reduction for lowering dimension cases. The results for lowering dimension cases are not optimal in general. On the other hand, the results for increasing dimension cases are all optimal except when G is a torus of even size and H is a mesh. For this case, we provide an embedding with a dilation cost of 2, and under certain condition, an embedding with optimal unit dilation cost.

For increasing dimension cases, if the graph H is a hypercube, the condition of expansion can always be satisfied; similarly, for lowering dimension cases, if the graph G is a hypercube, the condition of simple reduction can always be satisfied. Consequently, our results for generalized embeddings can always be applied if one of the two graphs is a hypercube.

Furthermore, our results can always be applied if both graphs are square. For increasing dimension cases, these embeddings are optimal when the dimension of H is divisible by that of G. For lowering dimension cases, the embeddings are all optimal to within a constant for fixed values of d and c; by comparing with the several known optimal results in the literature, we have further shown that some of these embeddings are truly optimal.

Given any argument in the corresponding domains of our embedding functions, the numbers of operations needed to evaluate the functions are all proportional to the dimension of H.

7 Appendix

In this appendix, we prove that for all positive integers d, $\sum_{k=0}^{d-1}\binom{k}{\lfloor k/2\rfloor}$ can be written as $\varepsilon_{d-1}2^{d-1}$, where $\varepsilon_{d-1} > 0$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$, and for all $d \geq 3$, $\varepsilon_{d-1} > \varepsilon_d$.

For all positive integer k, let

$$C_{k-1} = \begin{cases} \prod_{j=1}^{(k-1)/2}(1 - 1/(2j+2)), & \text{for } k-1 \text{ even and } k-1 \geq 0; \\ \prod_{j=2}^{k/2}(1 - 1/(2j)), & \text{for } k-1 \text{ odd and } k-1 \geq 1. \end{cases}$$

Proposition 1. For all positive integers k, $\binom{k}{\lfloor k/2 \rfloor} = 2^{k-1}C_{k-1}$.

Proof. We use induction on odd k's and even k's.

Case 1. k is even.

Basis. $k = 2$.

We have $\binom{2}{1} = 2 = 2C_1$.

Induction hypothesis. Assume that the proposition is true for all positive, even integers $k \leq a$, where a is an even number.

Induction step. Prove for $k = a + 2$.

$$\begin{aligned} \binom{a+2}{\lfloor (a+2)/2 \rfloor} &= \binom{a+2}{(a+2)/2} \\ &= \frac{(a+2)!}{((a+2)/2)!((a+2)/2)!} \\ &= 2^2(1 - \frac{1}{a+2})\binom{a}{a/2} \\ &= 2^2(1 - \frac{1}{a+2})2^{(a-1)}C_{a-1} \\ &= 2^{(a+2)-1}(1 - \frac{1}{a+2})\prod_{j=2}^{a/2}(1 - \frac{1}{2j}) \\ &= 2^{(a+2)-1}C_{(a+2)-1}. \end{aligned}$$

Case 2. k is odd.

The proof is similar to the proof for Case 1 and is omitted.

□

Proposition 2. For all positive integers k, $C_k \leq C_{k-1}$.

Proof. We consider two cases:

Case 1. k is odd.

Since $C_{k-1} = \prod_{j=1}^{(k-1)/2}(1 - 1/(2j+2)) = \prod_{j'=2}^{(k+1)/2}(1 - 1/(2j'))$, we have $C_{k-1} = C_k$.

Case 2. k is even.

Since $C_k = \prod_{j=1}^{k/2}(1 - 1/(2j+2)) = \prod_{j'=2}^{k/2+1}(1 - 1/(2j')) = (1 - 1/(k+2))C_{k-1}$, we have $C_k < C_{k-1}$. □

Proposition 3. Let m be a positive integer, and $t_m = \sum_{k=0}^{m}\binom{k}{\lfloor k/2 \rfloor}$. Then $t_m = \varepsilon_m 2^m$, where $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$, and for all $m \geq 3$, $\varepsilon_m = (\varepsilon_{m-1} + C_{m-1})/2$ and $C_{m-1} < \varepsilon_m < \varepsilon_{m-1}$.

Proof. We only prove the case with $m \geq 3$; the proof for the case with $m < 3$ is obvious. We use induction on m.

Basis. $m = 3$.

Since $t_3 = 7$, $\varepsilon_2 = 1$, and $C_2 = 3/4$, we have $t_3 = \varepsilon_3 2^3$, where $\varepsilon_3 = 7/8 = (\varepsilon_2 + C_2)/2$, and $C_2 < \varepsilon_3 < \varepsilon_2$.

Induction hypothesis. Assume that the proposition is true for all positive integers $m \leq a$.

Induction step. Prove for $m = a + 1$.

Since $t_{a+1} = \sum_{k=0}^{a+1}\binom{k}{\lfloor k/2 \rfloor}$, by proposition 1, we have $t_{a+1} = t_a + 2^a C_a = 2^{a+1}(\varepsilon_a + C_a)/2$. Thus, $t_{a+1} = \varepsilon_{a+1}2^{a+1}$, where $\varepsilon_{a+1} = (\varepsilon_a + C_a)/2$.

Since $C_{a-1} < \varepsilon_a < \varepsilon_{a-1}$ by induction hypothesis, and $C_a \leq C_{a-1}$ by proposition 2, we have $C_a < \varepsilon_a$. Hence, $\varepsilon_{a+1} < \varepsilon_a$ and $C_a < \varepsilon_{a+1}$. Therefore, $C_a < \varepsilon_{a+1} < \varepsilon_a$. □

From proposition 3, we thus have $\sum_{k=0}^{d-1}\binom{k}{\lfloor k/2 \rfloor} = \varepsilon_{d-1}2^{d-1}$, where $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$, and for all $d \geq 3$, $\varepsilon_{d-1} > \varepsilon_d$. From the recurrence relation $\varepsilon_m = (\varepsilon_{m-1} + C_{m-1})/2$ for all $m \geq 3$, we also have $\varepsilon_m = (1/2)^{m-2} + \sum_{k=2}^{m-1}(1/2)^{m-k}C_k$, for all $m \geq 3$.

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