

# Robot Navigation Functions on Manifolds with Boundary

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This paper concerns the construction of a class of scalar valued analytic maps on analytic manifolds with boundary. These maps, which we term *navigation functions*, are constructed on an arbitrary *sphere world*—a compact connected subset of Euclidean  $n$ -space whose boundary is formed from the disjoint union of a finite number of  $(n - 1)$ -spheres. We show that this class is invariant under composition with analytic diffeomorphisms: our sphere world construction immediately generates a navigation function on all manifolds into which a sphere world is deformable. On the other hand, certain well known results of S. Smale guarantee the existence of smooth navigation functions on any smooth manifold. This suggests that analytic navigation functions exist, as well, on more general analytic manifolds than the deformed sphere worlds we presently consider. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

Consider the following problem in robotics. A kinematic chain—a sequence of mutually constrained actuated rigid bodies—is allowed to move in a cluttered workspace. Contained within the *joint space*—an analytic manifold which forms the configuration space of the kinematic chain—is the *free space*,  $\mathcal{F}$ —the set of all configurations which do not involve intersection with any of the “obstacles” cluttering the workspace. Given any destination point in the interior of  $\mathcal{F}$  to which it is desired to move the robot, find a curve in  $\mathcal{F}$  from an arbitrary initial point to the desired destination.

The negative gradient vector field of a scalar valued function which is transverse (exterior directed) on the boundary of the free space, and which has a single minimum at the destination point gives rise to a flow which moves almost all initial conditions toward that desired point. Thus, a

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suitably chosen scalar valued "cost" function solves the geometric problem of finding paths to the destination in free space. Moreover, interpreting the cost function as an artificial potential energy, it can be shown that a gradient vector field on  $\mathcal{F}$  "lifts naturally" to a Lagrangian vector field on  $T\mathcal{F}$  describing the robot's Newtonian dynamics when subjected to a suitable feedback compensating control law [11]. Under certain additional regularity conditions, the Lagrangian system "inherits" the limit properties of the gradient system, and an explicitly specified open neighborhood about  $F \times 0$  in  $T\mathcal{F}$  is positive invariant with respect to the lifted flow [12]. Thus, a further constrained cost function solves the robot navigation and the attendant control problems simultaneously.

The geometric problem of constructing a path between two points in a space obstructed by sets with arbitrary polynomial boundary (given perfect information) has already been completely solved [21]. Moreover, a near optimally efficient solution has recently been offered for this class of problems as well [4]. The motivation for the present direction of inquiry (beyond its apparent academic interest) is the desire to incorporate explicitly aspects of the control problem—the construction of feedback compensators for a well characterized class of dynamical systems in the presence of well characterized constraints—in the planning phase of robot navigation problems. That is, the geometrical "find path" problem is generalized to the search for a family of paths in  $\mathcal{F}$  (the one-parameter group of the gradient flow), which provides a feedback control law for the physical robot as well. The idea of using "potential functions" for the specification of robot tasks with a view of the control problems in mind was pioneered by Khatib [8] in the context of obstacle avoidance. Fundamental work of Hogan [7] in the context of force control further advanced the interest in this approach. The methodology has been developed independently by Arimoto in Japan [2], and by Soviet investigators as well [20].

This paper concerns the construction of analytic "navigation functions" on "sphere worlds." In the next section, we motivate and define these notions. In the final section, we present an explicit one-parameter family of functions defined on an arbitrary sphere world and prove that its members are navigation functions for all parameter values greater than an integer determined by the boundary locations.

## 2. NAVIGATION FUNCTIONS ON SPHERE WORLDS

We seek an analytic real valued map whose gradient vector field, if integrated, produces curves to the destination point (from any starting point) that never leave the free space. In Section 2.1 we make our notion

of “obstacle” precise by defining sphere worlds and their deformation classes. In Section 2.2, we observe that smooth (hence, analytic) vector fields are too tightly constrained to permit strict navigation on any homeomorph of a sphere world. This leads to the slightly relaxed definition of a “navigation function” in Section 2.3, where we also discuss our preference for analytic as opposed to merely smooth functions. In Section 2.4 we use results of S. Smale to show that smooth navigation functions exist on every smooth manifold. Finally, in Section 2.5, we show that the navigation properties are invariant under diffeomorphism, so that a construction on a “model space” immediately generates a navigation function on any manifold into which it is deformable.

### 2.1. Sphere Worlds and Their Deformations

A *sphere world* is a compact connected subset of  $E^n$  whose boundary is formed from the disjoint union of a finite number, say  $M + 1$ , of  $(n - 1)$ -spheres. It follows that there is one large sphere which bounds the *workspace*,

$$\mathcal{W} \triangleq \{q \in E^n: \|q\|^2 \leq \rho_0^2\},$$

and  $M$  smaller spheres which bound the *obstacles*,

$$\mathcal{O}_j \triangleq \{q \in E^n: \|q - q_j\|^2 < \rho_j^2\}, \quad j = 1 \dots M.$$

Note that the spheres are represented by listing their  $M + 1$  positive radii,  $\{\rho_j\}_{j=0}^M$ , and  $M$  center points,  $\{q_i\}_{i=1}^M$ . For ease of exposition we refer to  $E^n - \mathcal{W}$  as the zero<sup>th</sup> obstacle, and center the workspace at the origin of our coordinate system.

The *free space* remains after removing all the obstacles from the workspace,

$$\mathcal{F} \triangleq \mathcal{W} - \bigcup_{j=1}^M \mathcal{O}_j.$$

For  $\mathcal{F}$  to be a valid sphere world we must impose the additional constraint that all obstacle closures are contained in the interior of the workspace,

$$\rho_0 > 0 \quad \text{and} \quad \|q_i\| + \rho_i < \rho_0; \quad 1 \leq i \leq M,$$

and that none of them intersect,

$$\|q_i - q_j\| > \rho_i + \rho_j; \quad 1 \leq i, j \leq M.$$

Clearly, sphere worlds constitute a trivial task domain: there are more intuitive navigation schemes for such models whose proof of correctness

would proceed more simply than ours. However, we will demonstrate that the properties enjoyed by the navigation functions on sphere worlds remain invariant under diffeomorphism. Thus, our slightly more complicated construction on a “model space” automatically induces a correct solution for much more interesting robot navigation problems. The actual construction of analytic diffeomorphisms from this simple model to “real worlds” is the topic of another paper [1].

## 2.2. Strict Global Navigation Is Not Possible

For reasons that will be made clear in the next section, we restrict our attention to non-degenerate vector fields which are transverse on the boundary of  $\mathcal{F}$ . Given those constraints, we now show that a smooth vector field on any sphere world which has a unique attractor, must have at least as many saddles as there are obstacles. Thus a globally attracting equilibrium state is topologically impossible.

It is readily apparent that continuity arguments constrain the possible gradient vector fields: any continuous function,  $\varphi$ , on a compact set,  $\mathcal{F}$ , must attain its maximum and minimum on that set. If there is a maximum in the interior of  $\mathcal{F}$  then the conditions for strict global navigation have already been violated: the minimum will no longer be globally attracting, since gradient motion which starts exactly at the maximum will stay there forever. It might be imagined, however, that a sufficiently clever construction will not only be transverse on the boundary, but will also attain its maximum on the boundary (the obstacles), leaving the destination point to be the only singularity in  $\mathcal{F}$ . This we now show to be impossible. The various technical terms mentioned in the sequel are defined in Appendix A.

**PROPOSITION 2.1.** *The Euler characteristic of any sphere world homeomorphic with  $M$  obstacles is*

$$\chi(\mathcal{F}) = 1 - (-1)^n M.$$

*Proof.* First note that the closed disk,  $\overline{\mathcal{D}^n}$ , has an Euler characteristic of unity. To see this, distinguish a point on its boundary,  $\partial\overline{\mathcal{D}^n}$ , so that the punctured  $(n-1)$ -sphere resulting from its removal is homeomorphic to  $\mathcal{D}^{(n-1)}$ . This constitutes a finite cellular decomposition [15] of  $\mathcal{F}$ , containing one  $n$ -cell, one  $(n-1)$ -cell, and one 0-cell, from which it follows, according to the definition of Euler characteristic<sup>1</sup>

$$\chi(\overline{\mathcal{D}^n}) = 1 + (-1)^{n-1} + (-1)^n = 1.$$

<sup>1</sup>Let  $\mathcal{X}$  be a finite cellular decomposition of a compact manifold  $\mathcal{X}$ . For any integer  $q \geq 0$ , let  $\alpha_q$  denote the number of  $q$ -cells of  $\mathcal{X}$ . The Euler characteristic of  $\mathcal{X}$ , denoted by  $\chi(\mathcal{X})$ , is the integer  $\sum_{q \geq 0} (-1)^q \alpha_q$  [15].

Now observe that the closed disk is the union of the free space,  $\mathcal{F}$ , with all obstacles re-introduced,

$$\mathcal{D}^n = \mathcal{F} \cup \bigcup_{i=1}^M \mathcal{O}_i,$$

and that each obstacle is an open  $n$ -disk—an  $n$ -cell. Thus, a finite cellular decomposition of  $\mathcal{F}$  in conjunction with the set of obstacles constitutes a finite cellular decomposition of  $\mathcal{D}^n$ . It follows that

$$\chi(\mathcal{D}^n) = \chi(\mathcal{F}) + (-1)^n M. \quad \square$$

The immediate implication of these facts is an unequivocal refutation of the possibility of strict global navigation using nondegenerate smooth (much less, analytic) vector fields.

**COROLLARY 2.2.** *There is no smooth nondegenerate vector field,  $f$ , on the free space,  $\mathcal{F}$ , with  $M > 0$  obstacles, which is transverse on  $\partial\mathcal{F}$ , such that the flow induced by*

$$\dot{x} = -f,$$

*admits a globally asymptotically stable equilibrium state.*

*Proof.* According to the Poincaré–Hopf theorem the sum over the indices of the equilibrium states of a vector field which points outward on the boundaries of  $\mathcal{F}$ , must equal the Euler characteristic  $\chi(\mathcal{F})$ . If  $(-f)$  has a single globally attracting equilibrium state,  $x^*$ , in the interior of  $\mathcal{F}$ , and  $f$  is transverse on  $\partial\mathcal{F}$ , then  $f$  points outward on that set—otherwise  $x^*$  could not be the positive limit set of the boundary under the flow of  $(-f)$ . If  $f$  is nondegenerate, then its jacobian at the equilibrium state,  $Df(x^*)$ , has all positive eigenvalues (otherwise  $x^*$  is not an attractor under the flow of  $(-f)$ ), from which it follows that the index of  $f$  at  $x^*$  is  $+1$ .

Pursuing the assumption that  $f$  has no other equilibrium state, it follows that the sum of its indices is unity as well. But the Euler characteristic,  $\chi(\mathcal{F}) = 1 - (-1)^n M$ , as computed in Proposition 2.1, cannot be unity unless  $M = 0$ —a contradiction.  $\square$

**COROLLARY 2.3.** *Let  $f$  be a smooth nondegenerate vector field on the free space,  $\mathcal{F}$ , with  $M > 0$  obstacles, which is transverse on  $\partial\mathcal{F}$ . Suppose that  $(-f)$  has a unique attracting equilibrium point. Then each obstacle introduces at least one saddle point of  $f$ .*

*Remark.* This result provides a lower bound— $M$ , on the number of saddles necessitated by  $M$  obstacles. Our construction demonstrates that this bound may actually be attained; that is, a function with exactly  $M$  saddles of index  $n - 1$  exists on any  $n$ -dimensional sphere world.

*Proof.* Letting  $i(x)$  denote the vector field index of  $(+f)$  at  $x$ , if  $x^*$  is an attractor of  $(-f)$  then  $i(x^*) = 1$ . If  $x^*$  is the only attractor, according to the Poincaré–Hopf theorem [18],

$$1 + \sum_{x \in (f^{-1}(0) - x^*)} i(x) = \chi(\mathcal{F}),$$

and, from Proposition 2.1, this implies

$$\sum_{x \in (f^{-1}(0) - x^*)} i(x) = -(-1)^n M.$$

Since there must be additional zeros of  $f$ , assume that  $p$  of them are local attractors of  $(+f)$ . The index of each is  $(-1)^n$  [18, Lemma 6.4]. We now have

$$\sum_{x \in (\text{saddles of } f)} i(x) = -(-1)^n (M + p),$$

so there must be at least  $(M + p)$  saddles, and the result follows.  $\square$

### 2.3. Navigation Functions

Having defined the class of sphere worlds, and seen that they “defeat” the strict navigation capabilities of smooth vector fields, we must now relax the criterion of navigation. At the same time, we wish to add constraints reflecting the ultimate use in a control algorithm that respects the Lagrangian dynamics of actuated kinematic chains. Before discussing in detail the resulting relaxed but “dynamically sound” class of cost functions, we define this class using technical terms that will be discussed in the text below.

**DEFINITION 1.** Let  $\mathcal{F} \subset E^n$  be a compact connected analytic manifold with boundary. A map  $\varphi: \mathcal{F} \rightarrow [0, 1]$ , is a *navigation function* if it is:

1. Analytic on  $\mathcal{F}$ ;
2. Polar on  $\mathcal{F}$ , with minimum at  $q_d \in \mathcal{F}$ ;
3. Morse on  $\mathcal{F}$ ;
4. Admissible on  $\mathcal{F}$ .

The intuitive motivation for this definition is most simply provided by reference to the following fact which obtains from elementary properties of gradient vector fields, for example, as discussed in [6].

**PROPOSITION 2.4** [12]. *Let  $\varphi$  be a smooth Morse function on the compact Riemannian manifold,  $\mathcal{F}$ . Suppose that  $\nabla\varphi$  is transverse and directed*

away from the interior of  $\mathcal{J}$  on the boundary of that set. Then the negative gradient flow has the following properties:

- (i)  $\mathcal{J}$  is a positive invariant set;
- (ii) the positive limit set of  $\mathcal{J}$  consists of the critical points of  $\varphi$ ;
- (iii) there is a dense open set,  $\tilde{\mathcal{J}} \subset \mathcal{J}$ , whose limit set consists of the local minima of  $\varphi$ .

*Proof.* Since the vector field is directed toward the interior of  $\mathcal{J}$  on its boundary by hypothesis, it follows that this set is positive invariant. The limit set for any trajectory of a gradient system on a compact manifold is an equilibrium point [6], hence, in this case, a minimum, maximum, or saddle of  $\varphi$  in the interior of  $\mathcal{J}$ . Clearly, a maximum may constitute the positive limit set of no initial condition in  $\mathcal{J}$  other than itself. Now suppose that there is some open set of initial conditions in  $\mathcal{J}$  whose positive limit set is a saddle point. This would imply that the saddle has a local stable manifold of dimension equal that of  $\mathcal{J}$ —a contradiction, since the Hessian is non-degenerate by assumption.  $\square$

Using the terminology of M. Morse, we say that  $\varphi$  is *polar* if it has a unique minimum on  $\mathcal{F}$  [19]. If  $\mathcal{F}$  is disconnected it is clearly impossible to construct a continuous function which is polar. Supposing, however, that the free space is connected, that  $\varphi$  has a unique minimum at  $q_d$ , and that the other hypotheses of Proposition 2.4 hold, then all initial conditions away from a set of measure zero are successfully brought to  $q_d$  without running into the free space boundary ("hitting any obstacle"). It has been shown in the previous section that one cannot do better than this with smooth vector fields: topological obstructions prohibit the existence of vector fields which take *every* point in  $\mathcal{F}$  to  $q_d$ .

Property 3 in Definition 1 is added to match the hypothesis of the Proposition. In consequence, it is impossible for any submanifold of codimension 1 not attracted to  $q_d$  to disconnect  $\mathcal{F}$  and "block" the flow toward  $q_d$ . For, this would imply that some maximum or saddle has a attracting domain which includes an open set—contradicting the fact that a non-degenerate unstable equilibrium state has a stable manifold of dimension less than  $n$ . The condition permits, as well, a straightforward proof that the desirable limiting behavior of the gradient flow is "inherited" by the ultimate closed loop mechanical system formed by using  $\nabla\varphi$  directly as a feedback control law for the robot's actuators [12].

Using the terminology of M. Hirsch [5], we say that a scalar valued function is *admissible* if all boundary components have the same maximal height—that is,  $\partial\mathcal{F} = \varphi^{-1}(1)$ . This requirement, Property 4 in Definition 2, while sufficient to guarantee that  $\nabla\varphi$  is transverse to the boundary of  $\mathcal{F}$  (as additionally required by Proposition 2.4), is a much stronger condition

imposed to ensure that the transients of the resulting closed loop mechanical system "inherit" the desirable properties of the gradient flow which prevent collisions with the boundary. Obviously, a careful discussion of the control theoretic aspects of this work is beyond the scope of the present paper, and the reader is referred to [9–11] for details.

Finally, it might be said that Property 1 in Definition 1 reflects the authors' "ideological" perspective that closed form mathematical expressions are a preferable encoding of actuator commands to algorithms which include logical decisions. Mathematically, we require merely  $C^2$  functions, but even smooth ( $C^\infty$ ) functions may still be defined by "patching together" different closed form expressions on different portions of the space leading to the kind of branching and looping in the ultimate control algorithm that we would like to avoid as much as possible. Analytic navigation functions will be harder to construct, but once defined, yield a provably correct control algorithm directly by "parsing" the symbolic expression into its gradient.<sup>2</sup> Unquestionably, real world scenes will often not admit even a smooth, much less an analytic representation, and it may well turn out (the theoretical recourse to ever more accurate analytic approximations notwithstanding) that any serious attempt to extend this work beyond the class of ball obstacles requires a relaxation of Property 1. Until such a time, we prefer to remain within the category of analytic maps on analytic manifolds.

#### 2.4. "Almost" Global Navigation Is Possible

When does a compact manifold with boundary admit a navigation function? In the initial approach to this problem, the first author was led to apply certain elementary tests from Morse theory which could reveal obstructions to the desired goal, but not provide a definitive judgement otherwise [10]. In the course of further reading [3] and subsequent conversations with M. Hirsch, we have become aware of a body of relatively recent mathematical results which has much more direct bearing upon this question. Smale proved the generalized "Poincaré's conjecture" in higher dimensions roughly three decades ago. In so doing, he was led to develop a number of results concerning gradient systems of which the most important to us is the following (in this section "index" of a critical point denotes Morse index—see Appendix A).

**THEOREM 1** (Smale, 1961 [23, Theorem C]). *Let  $\mathcal{M}$  be a compact  $n$ -dimensional  $C^\infty$  manifold with  $\partial\mathcal{M}$  equal to the disjoint union of  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , each  $\mathcal{V}_i$  closed in  $\partial\mathcal{M}$ . Then there exists a  $C^\infty$  function  $\phi$  on  $\mathcal{M}$  with*

<sup>2</sup>We presume, as well, that such considerations will play an important role with respect to verifiability of implementations in complicated environments.

non-degenerate critical points, regular on  $\partial\mathcal{M}$ ,  $\varphi(\mathcal{V}_1) = -\frac{1}{2}$ ,  $\varphi(\mathcal{V}_2) = n + \frac{1}{2}$  and at a critical point  $p$  of  $\varphi$ ,  $\varphi(p) = \text{index } p$ .

Smale calls such a function,  $\varphi$ , a *nice* function for  $\mathcal{M}$  [22]. He obtains a number of important results with this construction, including a generalization of the somewhat earlier result of Morse which demonstrates that every smooth manifold with no boundary admits a smooth polar non-degenerate function [19]. For our purposes, this result is important if it can be extended to the general case with boundary.

The desired extension obtains by applying the notion of "cancellation" of adjacent (in index) critical points that Morse and Smale developed in the course of their independent investigations. A (reasonably) self-contained exegesis upon these techniques is provided by Milnor [16], whose version may be rendered as follows. Suppose that  $\varphi$  is a smooth Morse function on  $\mathcal{M}$  with two distinct interior critical points,  $p_1$  and  $p_2$ , with indices  $\lambda_1, \lambda_2$ , respectively, possessing the properties  $\lambda_1 \neq \lambda_2$  and  $\varphi(p_1) \neq \varphi(p_2)$ . These two points may be *cancelled* if there exists another smooth Morse function,  $\varphi'$ , on  $\mathcal{M}$ , which agrees with  $\varphi$  everywhere away from a neighborhood of  $\varphi^{-1}[\varphi(p_1)]$  and  $\varphi^{-1}[\varphi(p_2)]$  in  $\mathcal{M}$ , yet which has two fewer critical points—one less critical point of index  $\lambda_1$ ; one less critical point of index  $\lambda_2$ . It turns out that pairs of index 0 and index 1 critical points may be cancelled if the "lower boundary" has the right homology type.

**THEOREM 2** (Index 0 cancellation theorem [16, Theorem 8.1]). *Let  $\mathcal{M}$  be a smooth compact manifold, with boundary formed from the disjoint union of  $\mathcal{V}_1, \mathcal{V}_2$ , two smooth manifolds with no boundary. Suppose that  $\varphi$  is a "nice" function for  $\mathcal{M}$  with  $\mathcal{V}_1 = \varphi^{-1}(-\frac{1}{2})$ . If  $H_0(\mathcal{M}, \mathcal{V}_1) = 0$ , then the critical points of index 0 can be cancelled against an equal number of critical points of index 1.*

Moreover, note that there are "enough" index 1 critical points to cancel all the minima if the manifold is connected: the proof was suggested by W. Massey.

**PROPOSITION 2.5** [14]. *Let  $\varphi$  be a "nice" function for the manifold,  $\mathcal{M}$ , of the previous theorem. If the manifold is connected then there are at least as many index 1 critical points as index 0 critical points of  $\varphi$  in  $\mathcal{M}$ .*

*Proof.* Supposing the contrary, apply Theorem 2 to obtain nice function,  $\varphi'$ , which has at least one minimum and no index 1 critical points. For all sufficiently small  $\varepsilon$ , the set  $\varphi'^{-1}[-\frac{1}{2}, \varepsilon]$  has the homotopy type of  $\varphi'^{-1}[-\frac{1}{2}, -\varepsilon]$  with a 0-cell attached [17, Theorem I.3.2]. This implies that  $\varphi'^{-1}[-\frac{1}{2}, \varepsilon]$  is a disconnected set. Since there are no index 1 critical points, the rest of the manifold,  $\mathcal{M} = \varphi'^{-1}[-\frac{1}{2}, n + \frac{1}{2}]$ , obtains from

attaching cells of dimension two or greater to  $\varphi'^{-1}[-\frac{1}{2}, \varepsilon]$  [17, Theorem I.3.5]. Now “attaching” a cell is defined to be the continuous identification of its boundary with a subset of the target set [17, p. 3], and the boundary of a  $k$ -cell is a  $(k - 1)$ -sphere—a connected set for all  $k \geq 2$ . Thus, the former may be attached to only one of the disconnected components, and the manifold cannot be connected, in contradiction to the hypothesis.  $\square$

We are now in a position to apply these results in the present setting: the following argument was suggested by M. Hirsch.

**THEOREM 3.** *For every smooth compact connected manifold with boundary,  $\mathcal{M}$ , and any point,  $x_0 \in \mathring{\mathcal{M}}$ , there exists a  $C^\infty$  navigation function.*

*Proof.* Let  $\mathcal{N}_0$  be an open disc about  $x_0$  in the interior of  $\mathcal{M}$ . Thus,  $\mathcal{V}_0 \triangleq \partial \mathcal{N}_0$  is a boundary of  $\mathcal{M}' \triangleq \mathcal{M} - \mathcal{N}_0$ . Moreover, defining  $\mathcal{V}_1 \triangleq \partial \mathcal{M}$ , the boundary of  $\mathcal{M}'$  is exactly the disjoint union of  $\mathcal{V}_0$  and  $\mathcal{V}_1$ . Now let  $\varphi$  be a “nice” function on  $\mathcal{M}'$  guaranteed to exist by Smale’s theorem.

Note that both  $\mathcal{M}'$  and  $\mathcal{V}_0$  are connected, thus,  $H_0(\mathcal{M}', \mathcal{V}_0) = 0$  [15]. It now follows from Theorem 2 that  $\varphi$  may be replaced with a new function,  $\varphi'$ , which agrees with  $\varphi$  on  $\mathcal{V}_0$  yet which has no critical points of index 0.

Finally, extend  $\varphi'$  to  $\mathcal{M}$ , by defining a cost function,  $\pi$ , on some open neighborhood of  $\overline{\mathcal{N}_0}$  that agrees with  $\varphi'$  at the boundary,  $\mathcal{V}_0$ , and has a unique critical point, a minimum, at  $x_0$ . This may be done since  $\mathcal{N}_0$  is diffeomorphic to  $\mathcal{D}^n$ .  $\square$

## 2.5. Navigation Properties Are Invariant Under Deformation

While the results of Section 2.2 apply to any homeomorph of the sphere worlds, in this section we restrict our attention to analytic diffeomorphs of a sphere world. We now show that the navigation properties are invariant under diffeomorphism of both the range and the domain spaces. In the latter context, we regard the particular free space of Definition 1 as a simplified “model,”  $\mathcal{M}$ , of a family of spaces which are “deformable” into it. The following statement, suggested by M. Hirsch, constitutes a formal guarantee of the existence of analytic navigation functions over every space in the analytic diffeomorphism class of a given model.

**PROPOSITION 2.6.** *Let  $\varphi: \mathcal{M} \rightarrow [0, 1]$  be a navigation function on  $\mathcal{M}$ , and  $h: \mathcal{F} \rightarrow \mathcal{M}$  be an analytic diffeomorphism. Then*

$$\tilde{\varphi} \triangleq \varphi \circ h,$$

*is a navigation function on  $\mathcal{F}$ .*

*Proof.* Applying the chain rule yields

$$\nabla \tilde{\varphi} = Dh^T((\nabla \varphi) \circ h).$$

According to the hypothesis,  $Dh^T$  is never singular, hence

$$\mathcal{C}_{\tilde{\varphi}} = h(\mathcal{C}_{\varphi}).$$

Moreover,<sup>3</sup>

$$\begin{aligned} \nabla^2 \tilde{\varphi}|_{\mathcal{C}_{\tilde{\varphi}}} &= \left( Dh^T[(\nabla^2 \varphi) \circ h] Dh + [((\nabla \varphi) \circ h)^T \otimes I] D(Dh)^s \right) \Big|_{\mathcal{C}_{\tilde{\varphi}}} \\ &= Dh^T(\nabla^2 \varphi|_{\mathcal{C}_{\varphi}}) Dh. \end{aligned}$$

Since  $Dh$  is non-singular, it follows that  $\nabla^2 \tilde{\varphi}|_{\mathcal{C}_{\tilde{\varphi}}}$  and  $\nabla^2 \varphi|_{\mathcal{C}_{\varphi}}$  have the same rank, hence, that  $\tilde{\varphi}$  is a Morse function.

It follows as well that for any  $v \in E^n$  there exists a  $u \in E^n$ ,  $u = [Dh]v$ , with the property that

$$v^T(\nabla^2 \tilde{\varphi}|_{\mathcal{C}_{\tilde{\varphi}}})v = u^T(\nabla^2 \varphi|_{\mathcal{C}_{\varphi}})u,$$

hence,

$$\text{index}(\tilde{\varphi})|_{\mathcal{C}_{\tilde{\varphi}}} = \text{index}(\varphi)|_{\mathcal{C}_{\varphi}}.$$

The induced cost function has exactly one minimum, at  $p_d \triangleq h^{-1}(q_d)$ , since  $h$  is injective. The last two statements show that  $\hat{\varphi}$  inherits the polar property from  $\varphi$ .

Admissibility of  $\tilde{\varphi}$  follows from the admissibility of  $\varphi$ , since it can be shown that  $h(\partial \mathcal{F}) = \partial \mathcal{H}$  [15].  $\square$

Deformation of the range space will be used explicitly in this paper. It will serve to deform a given cost function on  $\mathcal{F}$ , to a navigation function. Specifically, it will be used in Section 3 to make a cost function  $\hat{\varphi}$  admissible on  $\mathcal{F}$ , and to change  $q_d$  to a non-degenerate critical point.

**PROPOSITION 2.7.** *Let  $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathbb{R}$  be intervals,  $\hat{\varphi}: \mathcal{F} \rightarrow \mathcal{I}_1$  and  $\sigma: \mathcal{I}_1 \rightarrow \mathcal{I}_2$  be analytic. Define the composition  $\varphi: \mathcal{F} \rightarrow \mathcal{I}_2$ , to be*

$$\varphi \triangleq \sigma \circ \hat{\varphi}.$$

*If  $\sigma$  is monotonically increasing on  $\mathcal{I}_1$ , then the set of critical points of  $\hat{\varphi}$  and  $\varphi$  coincide,*

$$\mathcal{C}_{\varphi} = \mathcal{C}_{\hat{\varphi}},$$

<sup>3</sup>Using the identity  $Ax = (x^T \otimes I)A^s$ , where  $A, I \in \mathbb{R}^{n \times n}$ , and  $\otimes, (\cdot)^s$  denote the Kronecker product and the stack operation, respectively.

and the index of each point is identical,

$$\text{index}(\varphi)|_{\mathcal{C}_\varphi} = \text{index}(\hat{\varphi})|_{\mathcal{C}_{\hat{\varphi}}}.$$

*Proof.*  $\nabla\varphi = \nabla(\sigma \circ \hat{\varphi}) = (((d/dx)\sigma) \circ \hat{\varphi})\nabla\hat{\varphi}$ , by the chain rule. But  $\sigma$  is monotonically increasing, hence  $(d/dx)\sigma > 0$ , which implies that

$$\mathcal{C}_\varphi = \mathcal{C}_{\hat{\varphi}}.$$

Also,

$$\nabla^2\varphi|_{\mathcal{C}_\varphi} = \left( \left( \frac{d}{dx}\sigma \right) \circ \hat{\varphi} \right) \nabla^2\varphi|_{\mathcal{C}_{\hat{\varphi}}} + \nabla\hat{\varphi}|_{\mathcal{C}_{\hat{\varphi}}} \left( \nabla \frac{d}{dx}\sigma \right)^\top = \left( \left( \frac{d}{dx}\sigma \right) \circ \hat{\varphi} \right) \nabla^2\hat{\varphi}|_{\mathcal{C}_{\hat{\varphi}}},$$

and for the same reason,

$$\text{index}(\varphi)|_{\mathcal{C}_\varphi} = \text{index}(\hat{\varphi})|_{\mathcal{C}_{\hat{\varphi}}}. \quad \square$$

In other words, the composition with  $\sigma$  neither changes the set of critical points, nor their type (minimum, maximum, or a saddle) or degeneracy.

### 3. THE CONSTRUCTION

The proposed navigation function,  $\varphi: \mathcal{F} \rightarrow [0, 1]$ , is a composition of three functions:

$$\varphi \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}.$$

The function  $\hat{\varphi}$  is polar, almost everywhere Morse, and analytic; it attains a uniform height on  $\partial\mathcal{F}$  by blowing up there. Its image is “squashed” by the diffeomorphism,  $\sigma$ , of  $[0, \infty)$  into  $[0, 1]$ , where

$$\sigma(x) \triangleq \frac{x}{1+x},$$

resulting in a polar, admissible, and analytic function which is non-degenerate on  $\mathcal{F}$  except at one point—the destination. This last flaw is repaired by  $\sigma_d$ .

We distinguish between “good” and “bad” subsets of  $\mathcal{F}$ . When a point belongs to the “good” set, we expect the negative gradient lines to lead to it (here it is just the destination  $\{q_d\}$ ). The “bad” subset includes all the boundary points of the free space, and we expect the cost at such a point

to be high. Let  $\gamma$  and  $\beta$  denote analytic real valued maps whose zero-levels, i.e.,  $\gamma^{-1}(0)$ ,  $\beta^{-1}(0)$ , are respectively, the “good” and “bad” sets. We define  $\hat{\varphi}$  to be

$$\hat{\varphi} \triangleq \frac{\gamma}{\beta},$$

where  $\gamma: \mathcal{F} \rightarrow [0, \infty)$  is

$$\gamma \triangleq \gamma_d^k, \quad k \in \mathbb{N}; \quad \gamma_d \triangleq \|q - q_d\|^2,$$

and  $\beta: \mathcal{F} \rightarrow [0, \infty)$  is

$$\beta \triangleq \prod_{j=0}^M \beta_j,$$

where

$$\beta_0 \triangleq \rho_0^2 - \|q\|^2; \quad \beta_j \triangleq \|q - q_j\|^2 - \rho_j^2, \quad j = 1 \dots M.$$

In the sequel we will denote the “omitted product” by the symbol

$$\bar{\beta}_i \triangleq \prod_{j=0, j \neq i}^M \beta_j.$$

Due to the parameter  $k$  in  $\hat{\varphi}$ , the destination point is a *degenerate* critical point. To counteract this effect, the “distortion”  $\sigma_d: [0, 1] \rightarrow [0, 1]$ ,

$$\sigma_d(x) \triangleq (x)^{1/k}, \quad k \in \mathbb{N},$$

is introduced, to change  $q_d$  to a non-degenerate critical point.

The following theorem is the main contribution of this paper.

**THEOREM 4.** *If the free space,  $\mathcal{F}$ , is a valid sphere world (as defined in Section 2.1), then there exists a positive integer  $N$  such that for every  $k \geq N$ , for any finite number of obstacles, and for any destination point in the interior of  $\mathcal{F}$ ,*

$$\varphi = \sigma_d \circ \sigma \circ \hat{\varphi} = \left( \frac{\gamma_d^k}{\gamma_d^k + \beta} \right)^{1/k}, \quad (1)$$

is a navigation function on  $\mathcal{F}$ .

*Remark.* While  $\hat{\varphi}$  is analytic only on the interior of  $\mathcal{F}$ ,  $\sigma$  is analytic on  $[0, \infty)$ , and  $\sigma_d$  is analytic only on  $(0, \infty)$ , their composition,  $\varphi$ , is analytic on the entirety of some open neighborhood containing  $\mathcal{F}$ .

*Remark.* In the proof that follows, a constructive formula for  $N$  is given; it has the “schematic” form

$$N = \max_{\{\mathcal{O}_i\}_{i=0}^M} N_i \left( q_d, \begin{bmatrix} q_0 \\ \rho_0 \end{bmatrix}, \dots, \begin{bmatrix} q_M \\ \rho_M \end{bmatrix} \right),$$

where  $\mathcal{O}_i$  is the  $i^{\text{th}}$  obstacle. The functions  $N_i$  are given explicitly in Appendix B.

### 3.1. Proof of Correctness

Let  $\varepsilon > 0$ , define  $\mathcal{B}_i(\varepsilon) \triangleq \{q \in E^n: 0 < \beta_i < \varepsilon\}$  (i.e., an  $n$ -ball “without a core”). In the proof that follows, the free space is partitioned into five subsets:

1. the destination point,

$$\{q_d\};$$

2. the free space boundary,

$$\partial \mathcal{F} = \beta^{-1}(0);$$

3. the set “near the obstacles,”

$$\mathcal{F}_0(\varepsilon) \triangleq \bigcup_{i=1}^M \mathcal{B}_i(\varepsilon) - \{q_d\};$$

4. the set “near the workspace boundary,”

$$\mathcal{F}_1(\varepsilon) \triangleq \mathcal{B}_0(\varepsilon) - (\{q_d\} \cup \mathcal{F}_0(\varepsilon));$$

5. the set “away from the obstacles,”

$$\mathcal{F}_2(\varepsilon) \triangleq \mathcal{F} - (\{q_d\} \cup \partial \mathcal{F} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon)).$$

We assume, to begin with, that  $\varepsilon$  is sufficiently small to guarantee

$$\mathcal{F}_0(\varepsilon) \subset \mathcal{F}.$$

This assumption is interpreted algebraically as

$$\varepsilon < (\|q_i - q_j\| - \rho_j)^2 - \rho_i^2, \quad i, j \in \{1, \dots, M\}, i \neq j \quad (2)$$

and

$$\varepsilon < (\rho_0 - \|q_i\|)^2 - \rho_i^2, \quad i \in \{1, \dots, M\}. \quad (3)$$

Note that in practicality  $\varepsilon$  is expected to be small enough so that the exclusion of  $\{q_d\}$  from  $\mathcal{F}_0(\varepsilon)$  and  $\mathcal{F}_1(\varepsilon)$  is redundant.

We will begin by showing that  $q_d$  is a non-degenerate local minimum and that  $\varphi$  has no critical points on  $\partial\mathcal{F}$ , using the navigation function itself. Then, since Proposition 2.7 applies to  $\mathcal{F} - \partial\mathcal{F} - \{q_d\}$ , it will suffice to assert the theorem in consideration of  $\mathcal{F}_0(\varepsilon)$ ,  $\mathcal{F}_1(\varepsilon)$ , and  $\mathcal{F}_2(\varepsilon)$ , using  $\hat{\varphi}$ , which is simpler to deal with.

The following technical lemma gives formulas for the gradient and Hessian of a rational function at a critical point, to which we will continually refer in the sequel.

LEMMA 3.1. *Let  $\nu, \delta \in C^{(2)}[E^n, \mathbb{R}]$ , and define*

$$\rho \triangleq \nu/\delta.$$

Then

$$\nabla^2 \rho|_{\mathcal{C}_\rho} = \frac{1}{\delta^2} [\delta \nabla^2 \nu - \nu \nabla^2 \delta]. \quad (4)$$

*Proof.* Since

$$\nabla \rho = \frac{1}{\delta^2} (\delta \nabla \nu - \nu \nabla \delta), \quad (5)$$

we have

$$\nabla^2 \rho = \frac{1}{\delta^2} [\delta \nabla^2 \nu + \nabla \nu \nabla \delta^T - \nabla \delta \nabla \nu^T - \nu \nabla^2 \delta] + \delta^2 \nabla \rho \left( \nabla \frac{1}{\delta^2} \right)^T.$$

But at a critical point  $\nabla \rho = 0$  and  $\nabla \nu = \rho \nabla \delta$ ; hence

$$\nabla^2 \rho|_{\mathcal{C}_\rho} = \frac{1}{\delta^2} [\delta \nabla^2 \nu - \nu \nabla^2 \delta]. \quad \square$$

### 3.2. The Destination and the Boundary of $\mathcal{F}$

PROPOSITION 3.2. *If the workspace is valid, the destination point,  $q_d$ , is a non-degenerate local minimum of  $\varphi$ .*

*Proof.* Applying Eq. (5) to the definition of  $\varphi$ , given in Eq. (1),

$$\nabla \varphi(q_d) = \frac{1}{(\gamma_d^k + \beta)^{2/k}} \left( (\gamma_d^k + \beta)^{1/k} \nabla \gamma_d - \gamma_d \nabla (\gamma_d^k + \beta)^{1/k} \right) \Big|_{q_d} = 0,$$

since both  $\gamma_d$  and  $\nabla \gamma_d$  vanish at  $q_d$ . Using Eq. (4) and the fact that

$$\nabla^2 \gamma_d = 2I,$$

$$\begin{aligned} (\nabla^2 \varphi)(q_d) &= \frac{1}{(\gamma_d^k + \beta)^{2/k}} \left[ (\gamma_d^k + \beta)^{1/k} 2I - \gamma_d \nabla^2 (\gamma_d^k + \beta)^{1/k} \right] \Big|_{q_d} \\ &= 2\beta^{-1/k} I, \end{aligned}$$

which implies that  $q_d$  is a non-degenerate local minimum of  $\varphi$ .  $\square$

**PROPOSITION 3.3.** *If the workspace is valid, all the critical points of  $\varphi$  are in the interior of the free space.*

*Proof.* Let  $q_0$  be a point in  $\partial \mathcal{F}$ . By construction  $\beta_i(q_0) = 0$  for some  $i \in \{0, \dots, M\}$ . If the workspace is valid, it follows that  $\beta_j > 0$  for all  $j \in \{0, \dots, M\}$ ,  $j \neq i$ . Applying again Eq. 5 to the definition of  $\varphi$ ,

$$\begin{aligned} \nabla \varphi(q_0) &= \frac{1}{(\gamma_d^k + \beta)^{2/k}} \left( (\gamma_d^k + \beta)^{1/k} \nabla \gamma_d - \gamma_d \nabla (\gamma_d^k + \beta)^{1/k} \right) \Big|_{q_0} \\ &= \frac{1}{\gamma_d} \left( \nabla \gamma_d - \frac{1}{k} \gamma_d^{1-k} (k \gamma_d^{k-1} \nabla \gamma_d + \nabla \beta) \right) \Big|_{q_0} \\ &= -\frac{1}{k} \gamma_d^{-k} \left( \prod_{j=0, j \neq i}^M \beta_j \right) \nabla \beta_i \neq 0. \quad \square \end{aligned}$$

### 3.3. The Absence of Minima in the Interior of $\mathcal{F}$

From now on, we will assert the theorem using  $\hat{\varphi}$ . The trick is to use  $k$  in  $\hat{\varphi}$  as a tuning parameter. Intuitively,  $\nabla \hat{\varphi}$  (see Eq. 5) consists of the terms  $\nabla \gamma$  and  $\nabla \beta$ . By increasing  $k$ , the first term dominates, forcing  $-\nabla \hat{\varphi}$  to be directed toward  $q_d$  and have a larger magnitude. The overall effect will be to shift the critical points of  $\hat{\varphi}$  toward the obstacle boundaries. But we may as well expect that when  $k$  is high enough, each critical point is *not* a local minimum, since the overall behavior of  $\hat{\varphi}$  tends to that of  $\gamma$ . In such a case any test direction which is parallel to the “nearest” obstacle boundary should prove that this critical point is not a local minimum.

The proof that follows has two steps: first we show that all the critical points can be shifted arbitrarily close to the boundary of the free space. Then we find a test direction along which  $D^2 \hat{\varphi}$  has a negative eigenvalue at any critical point. As a result,  $q_d$  is the unique minimum of  $\hat{\varphi}$ . The following proposition shows that  $\mathcal{F}_2(\varepsilon)$ , the set “away from the obstacles,” can be “cleaned” of critical points.

**PROPOSITION 3.4.** *For every  $\varepsilon > 0$  there exists a positive integer  $N(\varepsilon)$  such that if  $k \geq N(\varepsilon)$  then there are no critical points of  $\hat{\varphi}$  in  $\mathcal{F}_2(\varepsilon)$ .*

*Proof.* At a critical point,  $q \in \mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_2(\varepsilon)$ , according to Eq. (5) we have

$$k\beta \nabla \gamma_d = \gamma_d \nabla \beta.$$

Taking the magnitude of both sides yields

$$2k\beta = \sqrt{\gamma_d} \|\nabla \beta\|,$$

since  $\|\nabla \gamma_d\| = 2\sqrt{\gamma_d}$ . A sufficient condition for the above equality not to hold is given by

$$\frac{1}{2} \frac{\sqrt{\gamma_d} \|\nabla \beta\|}{\beta} < k \quad \text{for all } q \in \mathcal{F}_2(\varepsilon).$$

An upper bound on the left side<sup>4</sup> is given by

$$\begin{aligned} \frac{1}{2} \frac{\sqrt{\gamma_d} \|\nabla \beta\|}{\beta} &\leq \frac{1}{2} \sqrt{\gamma_d} \sum_{i=0}^M \frac{\bar{\beta}_i}{\beta} \|\nabla \beta_i\| \\ &< \frac{1}{2} \frac{1}{\varepsilon} \max_{\mathcal{F}} \{\sqrt{\gamma_d}\} \sum_{i=0}^M \max_{\mathcal{F}} \{\|\nabla \beta_i\|\} \triangleq N(\varepsilon), \end{aligned} \quad (6)$$

since  $\beta_j \geq \varepsilon$ ,  $j \in \{0, \dots, M\}$ .  $\square$

In the proof of Proposition 3.6, it will prove important to have an upper bound for

$$\nu_i \triangleq \frac{1}{4} \nabla \beta_i \cdot \nabla \gamma_d - \gamma_d$$

over the closure of  $\mathcal{B}_i(\varepsilon)$ , the set  $\overline{\mathcal{B}_i(\varepsilon)} \triangleq \{q: 0 \leq \beta_i(q) \leq \varepsilon\}$ . This is readily obtained using Lagrange multipliers.

LEMMA 3.5.  $\max_{\overline{\mathcal{B}_i(\varepsilon)}} \{\nu_i\} = (\sqrt{\varepsilon + \rho_i^2} - \|q_d - q_i\|) \|q_d - q_i\|.$

*Proof.* Expanding  $\nu_i$  yields

$$\begin{aligned} \frac{1}{4} \nabla \beta_i \cdot \nabla \gamma_d - \gamma_d &= (q - q_i) \cdot (q - q_d) - (q - q_d) \cdot (q - q_d) \\ &= (q_d - q_i) \cdot (q - q_d). \end{aligned}$$

Since  $\nu_i$  is affine, it follows that its maximum over any compact set is attained on the boundary, in this case  $\partial \overline{\mathcal{B}_i(\varepsilon)} = \beta_i^{-1}(\varepsilon) \cup \beta_i^{-1}(0)$ . Further-

<sup>4</sup>See Appendix for  $\max_{\mathcal{F}}(\cdot)$ .

more, imagine the “filled” set

$$\text{fill}(\overline{\mathcal{B}_i(\varepsilon)}) \triangleq \{q: -\rho_i^2 \leq \beta_i \leq \varepsilon\} = \{q: 0 \leq \|q - q_i\|^2 \leq \rho_i^2 + \varepsilon\};$$

it is a compact set,  $\text{fill}(\overline{\mathcal{B}_i(\varepsilon)}) \supseteq \overline{\mathcal{B}_i(\varepsilon)}$ , and  $\nu_i$  attains its maximum on  $\partial \text{fill}(\overline{\mathcal{B}_i(\varepsilon)}) = \beta_i^{-1}(\varepsilon)$ . It follows then, that  $\nu_i$  attains its maximum on  $\beta_i^{-1}(\varepsilon)$ , the “outer boundary.” At that maximum,  $q^*$ , we have

$$\lambda \nabla \nu_i = \nabla \beta_i$$

or

$$\lambda(q_d - q_i) = (q^* - q_i)$$

for some  $\lambda \in \mathbb{R}$ . Which implies that

$$q^* = (1 - \lambda)q_i + \lambda q_d \quad \text{and} \quad \beta_i = \lambda^2 \|q_d - q_i\|^2 - \rho_i^2; \quad (7)$$

hence,

$$\nu_i(q^*) = (\lambda - 1)\|q_d - q_i\|^2.$$

Solving for  $\lambda$  in Eq. (7) yields

$$\lambda = \pm \frac{\sqrt{\beta_i + \rho_i^2}}{\|q_d - q_i\|}$$

or

$$\nu_i(q^*) = \left( \pm \frac{\sqrt{\beta_i + \rho_i^2}}{\|q_d - q_i\|} - 1 \right) \|q_d - q_i\|^2.$$

Choosing for the maximum the “+” option and substituting  $\varepsilon$  for  $\beta_i(q^*)$ , it follows that

$$\max_{\overline{\mathcal{B}_i(\varepsilon)}} \{\nu_i\} = \left( \sqrt{\varepsilon + \rho_i^2} - \|q_d - q_i\| \right) \|q_d - q_i\|. \quad \square$$

Note that  $\max_{\overline{\mathcal{B}_i(\varepsilon)}} \{\nu_i\}$  is negative for  $\varepsilon$  small enough, in consequence of the assumption that  $q_d$  is not inside the obstacle  $\mathcal{O}_i$ .

The following proposition shows that for  $\varepsilon$  small enough, the set “near the obstacles,”  $\mathcal{F}_0(\varepsilon)$ , is free of local minima.

**PROPOSITION 3.6.** *For any valid workspace, there exists an  $\varepsilon_0 > 0$  such that  $\hat{\phi}$  has no local minimum in  $\mathcal{F}_0(\varepsilon)$ , as long as  $\varepsilon < \varepsilon_0$ .*

*Proof.* If  $q \in \mathcal{F}_0(\varepsilon) \cap \mathcal{C}_{\hat{\varphi}}$ , then  $q \in \mathcal{B}_i(\varepsilon)$  for at least one  $i \in \{1, \dots, M\}$ —i.e.,  $q$  is very close to some obstacle boundary. We will use a unit vector orthogonal to  $\nabla \beta_i$  at  $q$  as a test direction to demonstrate that  $(\nabla^2 \hat{\varphi})(q)$  has at least one negative eigenvalue. Using Eq. (4),

$$\begin{aligned} (\nabla^2 \hat{\varphi})(q) &= \frac{1}{\beta^2} (\beta \nabla^2 \gamma_d^k - \gamma_d^k \nabla^2 \beta) \\ &= \frac{\gamma_d^{k-2}}{\beta^2} (k\beta [\gamma_d \nabla^2 \gamma_d + (k-1) \nabla \gamma_d \nabla \gamma_d^T] - \gamma_d^2 \nabla^2 \beta). \end{aligned} \quad (8)$$

At a critical point,  $k\beta \nabla \gamma_d = \gamma_d \nabla \beta$ , according to Eq. (5). Hence, taking the outer-product of both sides,

$$(k\beta)^2 \nabla \gamma_d \nabla \gamma_d^T = \gamma_d^2 \nabla \beta \nabla \beta^T.$$

Substituting for  $k(k-1)\beta \nabla \gamma_d \nabla \gamma_d^T$  ( $q \neq q_d$ ) in Eq. (8) yields

$$(\nabla^2 \hat{\varphi})(q) = \frac{\gamma_d^{k-1}}{\beta^2} \left( k\beta \nabla^2 \gamma_d + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \nabla \beta \nabla \beta^T - \gamma_d \nabla^2 \beta \right). \quad (9)$$

If  $A$  is a matrix, let  $(A)_s$  denote its symmetric part— $\frac{1}{2}(A + A^T)$ . Recalling that  $\bar{\beta}_i = \Pi_{j=0, j \neq i}^M \beta_j$ , note that

$$\begin{aligned} (\nabla^2 \hat{\varphi})(q) &= \frac{\gamma_d^{k-1}}{\beta^2} \left( k\beta \nabla^2 \gamma_d + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \right. \\ &\quad \cdot [\beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T + 2\beta_i \bar{\beta}_i (\nabla \bar{\beta}_i \nabla \beta_i^T)_s + \bar{\beta}_i^2 \nabla \beta_i \nabla \beta_i^T] \\ &\quad \left. - \gamma_d [\beta_i \nabla^2 \bar{\beta}_i + 2(\nabla \bar{\beta}_i^T \nabla \beta_i)_s + \bar{\beta}_i \nabla^2 \beta_i] \right). \end{aligned}$$

Evaluating the quadratic form associated with  $(\nabla^2 \hat{\varphi})(q)$  at  $\hat{v} \triangleq \nabla(\beta_i(q_c)/\|\nabla \beta_i(q_c)\|)^{\perp}$  yields

$$\begin{aligned} \frac{\beta^2}{\gamma_d^{k-1}} \hat{v}^T (\nabla^2 \hat{\varphi})(q) \hat{v} &= 2k\beta - 2\gamma_d \bar{\beta}_i \\ &\quad + \hat{v}^T \left[ \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \beta_i^2 \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T - \gamma_d \beta_i \nabla^2 \bar{\beta}_i \right] \hat{v}, \end{aligned} \quad (10)$$

since  $\nabla^2 \gamma_d = \nabla^2 \beta_i = 2I$ . Now take the inner-product of both sides of the equation  $k\beta \nabla \gamma_d = \gamma_d \nabla \beta$  with  $\nabla \gamma_d$  to obtain

$$\begin{aligned} 4k\beta &= \nabla \beta \cdot \nabla \gamma_d \\ &= \bar{\beta}_i \nabla \beta_i \cdot \nabla \gamma_d + \beta_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d. \end{aligned}$$

Substituting this for  $2k\beta$  in Eq. (10) and grouping the terms which are proportional to  $\beta_i$ , we have

$$\begin{aligned} & \frac{\beta^2}{\gamma_d^{k-1}} \hat{v}^T (\nabla^2 \hat{\phi})(q) \hat{v} \\ &= 2\bar{\beta}_i \overbrace{\left( \frac{1}{4} \nabla \beta_i \cdot \nabla \gamma_d - \gamma_d \right)}^{\nu_i(q)} \\ &+ \beta_i \left( \frac{1}{2} \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{v}^T \left[ \left( 1 - \frac{1}{k} \right) \frac{1}{\bar{\beta}_i} \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T - \nabla^2 \bar{\beta}_i \right] \hat{v} \right). \quad (11) \end{aligned}$$

The second term is proportional to  $\beta_i$ , and can be made arbitrarily small by a choice of  $\varepsilon$ , but it can still be positive, so the first term should be strictly negative. According to Lemma 3.5, this is guaranteed by the condition

$$\varepsilon < \|q_d - q_i\|^2 - \rho_i^2 \triangleq \varepsilon'_{0i}, \quad i \in \{1, \dots, M\}. \quad (12)$$

In order to assure the inequality  $\hat{v}^T (\nabla^2 \hat{\phi})(q) \hat{v} < 0$ , it now follows from Eq. 11 that  $\varepsilon$  must be further constrained to satisfy

$$\varepsilon < \frac{2(-\nu_i) \bar{\beta}_i^2}{(1/2) \bar{\beta}_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{v}^T [(1 - 1/k) \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T - \bar{\beta}_i \nabla^2 \bar{\beta}_i] \hat{v}},$$

for which it will suffice that

$$\varepsilon < \frac{\min_{\mathcal{B}_i(\varepsilon)} \{2|\nu(q)| \bar{\beta}_i^2\}}{\max_{\mathcal{B}_i(\varepsilon)} \{(1/2) \bar{\beta}_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{v}^T [(1 - 1/k) \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T - \bar{\beta}_i \nabla^2 \bar{\beta}_i] \hat{v}\}}.$$

Consider the right-hand side of the above inequality to be a scalar-valued function  $\zeta(\varepsilon)$ . If  $\varepsilon < \varepsilon'$  then  $\mathcal{B}_i(\varepsilon) \subseteq \mathcal{B}_i(\varepsilon')$ , and it follows that  $\zeta(\varepsilon) \geq \zeta(\varepsilon')$ . Hence it will also suffice that

$$\begin{aligned} \varepsilon &< \frac{\min_{\mathcal{B}_i(\varepsilon_{0i})} \{2|\nu(q)| \bar{\beta}_i^2\}}{\max_{\mathcal{B}_i(\varepsilon_{0i})} \{(1/2) \bar{\beta}_i \nabla \bar{\beta}_i \cdot \nabla \gamma_d + \gamma_d \hat{v}^T [(1 - 1/k) \nabla \bar{\beta}_i \nabla \bar{\beta}_i^T - \bar{\beta}_i \nabla^2 \bar{\beta}_i] \hat{v}\}} \\ &\triangleq \varepsilon''_{0i}. \end{aligned}$$

By making  $\varepsilon_0 = \min\{\varepsilon'_{0i}, \varepsilon''_{0i}\}$ ,  $i \in \{1, \dots, M\}$ , the proof is completed.  $\square$

We now consider the set  $\mathcal{F}_1(\varepsilon)$ . By adjusting  $\varepsilon$ , a point in this set can be made so close to the workspace boundary that  $\nabla\beta_0$  dominates any obstacle gradient. We will show that such a point cannot be a critical point of  $\hat{\phi}$ , provided that it is far enough from any obstacle.

**PROPOSITION 3.7.** *If  $k \geq N(\varepsilon)$ , then there exists an  $\varepsilon_1 > 0$  such that  $\hat{\phi}$  has no critical points on  $\mathcal{F}_1(\varepsilon)$ , as long as  $\varepsilon < \varepsilon_1$ .*

*Proof.* It is first convenient to bound  $\mathcal{B}_0(\varepsilon)$  away from the ball of radius given by the destination point  $q_d$ , as follows. If

$$\varepsilon < (\rho_0)^2 - \|q_d\|^2,$$

and  $\beta_0 < \varepsilon$ , then

$$\beta_0 = (\rho_0)^2 - \|q\|^2 < \varepsilon;$$

hence

$$\|q\| > \|q_d\|, \quad \text{for all } q \in \mathcal{F}_1(\varepsilon).$$

This is a sufficient condition for  $\nabla\beta_0$  to point away from the destination—i.e.,  $\nabla\gamma_d \cdot \nabla\beta_0 < 0$  on  $\mathcal{B}_0(\varepsilon)$ —because

$$\frac{1}{4} \nabla\gamma_d \cdot \nabla\beta_0 = -(q - q_d) \cdot q = q \cdot q_d - \|q\|^2 \leq \|q\|(\|q_d\| - \|q\|) < 0.$$

Now,  $\nabla\hat{\phi}$  is non-vanishing on  $\mathcal{F}_1(\varepsilon)$ , since its inner-product with  $\nabla\gamma_d$ , according to Eq. (5), is given by

$$\begin{aligned} \nabla\hat{\phi} \cdot \nabla\gamma_d &= \frac{\gamma_d^k}{\beta^2} (4k\beta - \nabla\beta \cdot \nabla\gamma_d) \\ &= \frac{\gamma_d^k}{\beta^2} (4k\beta - (\beta_0 \nabla\bar{\beta}_0 \cdot \nabla\gamma_d + \bar{\beta}_0 \nabla\beta_0 \cdot \nabla\gamma_d)) \\ &> \beta_0 \frac{\gamma_d^k}{\beta^2} (4k\bar{\beta}_0 - \nabla\bar{\beta}_0 \cdot \nabla\gamma_d). \end{aligned}$$

If  $k$  is large enough,

$$k > \frac{1}{4} \frac{\nabla\bar{\beta}_0 \cdot \nabla\gamma_d}{\bar{\beta}_0}, \quad \text{for all } q \in \mathcal{F}_1(\varepsilon),$$

the term  $\nabla\hat{\phi} \cdot \nabla\gamma_d$  will be positive. But  $k \geq N(\varepsilon)$  is sufficient for this to be

true, since

$$\begin{aligned}
 \frac{1}{4} \frac{\nabla \bar{\beta}_0 \cdot \nabla \gamma_d}{\bar{\beta}_0} &\leq \frac{1}{2} \frac{\|\nabla \bar{\beta}_0\| \sqrt{\gamma_d}}{\bar{\beta}_0} \\
 &\leq \frac{1}{2} \sqrt{\gamma_d} \sum_{i=1}^M \frac{\bar{\beta}_i}{\beta} \|\nabla \beta_i\| \\
 &\leq \frac{1}{2} \frac{1}{\varepsilon} \max_{\mathcal{N}} \{\sqrt{\gamma_d}\} \sum_{i=1}^M \max_{\mathcal{N}} \{\|\nabla \beta_j\|\} < N(\varepsilon).
 \end{aligned}$$

Since by definition of  $\mathcal{F}_1(\varepsilon)$ ,  $\beta_i \geq \varepsilon$  for  $i \in \{1, \dots, M\}$ . The proof is completed by choosing

$$\varepsilon_1 \triangleq (\rho_0)^2 - \|q_d\|^2. \quad \square$$

### 3.4. Non-degeneracy of Critical Points in the Interior of $\mathcal{F}$

The proof that  $\hat{\varphi}$  is polar was completed in the previous section. We now show that it is also Morse. The following lemma, which will be used in Proposition 3.9, asserts that the non-singularity of a linear operator follows from the fact that its associated quadratic form is sign definite on complementary subspaces of  $E^n$ .

LEMMA 3.8. *Let  $E^n = \mathcal{P} \oplus \mathcal{N}$ , and let the symmetric matrix  $Q \in \mathbb{R}^{n \times n}$  define a quadratic form on  $E^n$*

$$\xi(v) \triangleq v^T Q v.$$

*If  $\xi|_{\mathcal{P}}$  is positive definite and  $\xi|_{\mathcal{N}}$  is negative definite, then  $Q$  is non-singular and*

$$\text{index}(Q) = \dim(\mathcal{N}).$$

A proof can be found in [13].

Let  $\xi_q(v)$  denote the quadratic form associated with the Hessian of  $\hat{\varphi}$ ,  $(\nabla^2 \hat{\varphi})(q)$ , on the tangent space to the set “near the obstacles” at  $q \in \mathcal{F}_0(\varepsilon)$ , denoted as  $T_q \mathcal{F}_0(\varepsilon)$ .

PROPOSITION 3.9. *There exists an  $\varepsilon_2 > 0$  such that for every  $\varepsilon < \varepsilon_2$  at each critical point of  $\hat{\varphi}$  in  $\mathcal{F}_0(\varepsilon)$ ,  $q \in \mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_0(\varepsilon)$ , there is a direct sum decomposition  $T_q \mathcal{F}_0(\varepsilon) = \mathcal{P}_q \oplus \mathcal{N}_q$ , where  $\dim(\mathcal{P}_q) = 1$ , for which  $\xi_q|_{\mathcal{P}_q}$  is positive definite and  $\xi_q|_{\mathcal{N}_q}$  is negative definite.*

According to Lemma 3.8, this implies that all the critical points of  $\hat{\varphi}$  are non-degenerate.

*Proof.* Without loss of generality, assume that  $q \in \mathcal{B}_i(\varepsilon)$ , where  $\mathcal{B}_i(\varepsilon) = \{q: 0 < \beta_i < \varepsilon\}$ . Define  $\mathcal{P}_q \triangleq \text{span}\{\nabla\beta_i(q)\}$ , and let  $\mathcal{N}_q$  be the orthogonal complement of  $\mathcal{P}_q$  in  $T_q\mathcal{F}_0(\varepsilon)$ . In the proof of Proposition 3.6, it was shown that  $\xi_q|_{\mathcal{N}_q}$  is negative definite, as long as  $\varepsilon < \varepsilon_0$ . It remains to show that  $\xi_q|_{\mathcal{P}_q} > 0$ . Taking the squared norm of both sides of Eq. 5 yields

$$(k\beta)^2 \|\nabla\gamma_d\|^2 = \gamma_d^2 \|\nabla\beta\|^2,$$

and this implies

$$2k\beta = \frac{\gamma_d}{2k\beta} \|\nabla\beta\|^2.$$

Substituting for  $2k\beta$  in Eq. (9),

$$\begin{aligned} \frac{\beta^2}{\gamma_d^{k-1}} \widehat{\nabla\beta_i}^T (\nabla^2 \hat{\phi})(q) \widehat{\nabla\beta_i} &= \frac{\gamma_d}{2k\beta} \|\nabla\beta\|^2 + \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} (\nabla\beta \cdot \widehat{\nabla\beta_i})^2 \\ &- \gamma_d \widehat{\nabla\beta_i}^T \nabla^2 \beta \widehat{\nabla\beta_i}; \quad \text{where } \widehat{\nabla\beta_i} = \nabla\beta_i / \|\nabla\beta_i\|. \end{aligned}$$

Expanding  $\|\nabla\beta\|^2 = \|\nabla(\bar{\beta}_i\beta_i)\|^2$  and  $(\nabla\beta \cdot \widehat{\nabla\beta_i})^2 = (\nabla(\bar{\beta}_i\beta_i) \cdot \widehat{\nabla\beta_i})^2$  yields

$$\begin{aligned} \frac{\gamma_d}{2k\beta} &\left( \underbrace{\beta_i^2 \|\nabla\bar{\beta}_i\|^2 + 2\beta \nabla\beta_i \cdot \nabla\bar{\beta}_i + \bar{\beta}_i^2 \|\nabla\beta_i\|^2}_{*} \right) \\ &+ \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left( \underbrace{\beta_i^2 (\widehat{\nabla\beta_i} \cdot \nabla\bar{\beta}_i)^2}_{*} + \underbrace{2\beta \|\nabla\beta_i\| \widehat{\nabla\beta_i} \cdot \nabla\bar{\beta}_i}_{**} + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 \right) \\ &- \gamma_d \widehat{\nabla\beta_i}^T \nabla^2 \beta \widehat{\nabla\beta_i}. \end{aligned}$$

Noting that  $\|\nabla\beta_i\| \widehat{\nabla\beta_i} \cdot \nabla\bar{\beta}_i = \nabla\beta_i \cdot \nabla\bar{\beta}_i$ , enter the term (\*\*), which multiplies  $-(1/k)(\gamma_d/\beta)$  inside the term (\*), inside the term  $\widehat{\nabla\beta_i}^T \nabla^2 \beta \widehat{\nabla\beta_i} = \widehat{\nabla\beta_i}^T \nabla^2(\bar{\beta}_i\beta_i) \widehat{\nabla\beta_i}$ :

$$\begin{aligned} \frac{\gamma_d}{2k\beta} &(\beta_i^2 \|\nabla\bar{\beta}_i\|^2 - 2\beta \nabla\beta_i \cdot \nabla\bar{\beta}_i + \bar{\beta}_i^2 \|\nabla\beta_i\|^2) \\ &+ \left(1 - \frac{1}{k}\right) \frac{\gamma_d}{\beta} \left( \beta_i^2 (\widehat{\nabla\beta_i} \cdot \nabla\bar{\beta}_i)^2 + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 \right), \\ &+ \underbrace{2\gamma_d \|\nabla\beta_i\| \widehat{\nabla\beta_i} \cdot \nabla\bar{\beta}_i}_{\dagger} - \gamma_d \widehat{\nabla\beta_i}^T \left[ \beta_i \nabla^2 \bar{\beta}_i + \underbrace{2(\nabla\bar{\beta}_i \nabla\beta_i^T)_s}_{\ddagger} + 2\bar{\beta}_i I \right] \widehat{\nabla\beta_i}. \end{aligned}$$

The term ( $\dagger$ ) is the leftover from (\*\*), and it is canceled by the term ( $\ddagger$ ). Since both  $\beta_i^2 \|\nabla\bar{\beta}_i\|^2 - 2\beta \nabla\beta_i \cdot \nabla\bar{\beta}_i + \bar{\beta}_i^2 \|\nabla\beta_i\|^2 = (\beta_i \|\nabla\bar{\beta}_i\| - \bar{\beta}_i \|\nabla\beta_i\|)^2$

and  $\beta_i^2(\widehat{\nabla\beta_i} \cdot \nabla\bar{\beta_i})^2$  are non-negative,

$$\begin{aligned} & \frac{\beta^2}{\gamma_d^{k-1}} \widehat{\nabla\beta_i}^T (\nabla^2\hat{\varphi})(q) \widehat{\nabla\beta_i} \\ & \geq \frac{\gamma_d}{\beta_i} \left( \left(1 - \frac{1}{k}\right) \bar{\beta}_i \|\nabla\beta_i\|^2 - \beta_i^2 \widehat{\nabla\beta_i}^T \nabla^2\bar{\beta}_i \widehat{\nabla\beta_i} - 2\beta_i \bar{\beta}_i \right). \end{aligned}$$

Recalling the hypothesis that  $q \in \mathcal{B}_i(\varepsilon) \cap \mathcal{C}_{\hat{\varphi}}$ ,

$$\begin{aligned} & \frac{\beta^2}{\gamma_d^{k-1}} \widehat{\nabla\beta_i}^T (\nabla^2\hat{\varphi})(q) \widehat{\nabla\beta_i} \\ & \geq \frac{\gamma_d}{\beta_i} \left( \left(1 - \frac{1}{k}\right) \bar{\beta}_i \|\nabla\beta_i\|^2 - \varepsilon^2 |\widehat{\nabla\beta_i}^T \nabla^2\bar{\beta}_i \widehat{\nabla\beta_i}| - 2\varepsilon \bar{\beta}_i \right), \end{aligned}$$

which can be conveniently rearranged as

$$\begin{aligned} & \frac{\gamma_d}{\beta_i} \left( \underbrace{\left\{ \frac{1}{2} \left(1 - \frac{1}{k}\right) \bar{\beta}_i \|\nabla\beta_i\|^2 - 2\varepsilon \bar{\beta}_i \right\}}_{*} \right. \\ & \quad \left. + \underbrace{\left\{ \frac{1}{2} \left(1 - \frac{1}{k}\right) \bar{\beta}_i \|\nabla\beta_i\|^2 - \varepsilon^2 |\widehat{\nabla\beta_i}^T \nabla^2\bar{\beta}_i \widehat{\nabla\beta_i}| \right\}}_{**} \right). \end{aligned}$$

Assuming that  $k \geq 2$ , a sufficient condition for the term (\*) to be positive,<sup>5</sup>

$$\varepsilon < \frac{1}{8} \min_{\mathcal{B}_i(\varepsilon)} \{\|\nabla\beta_i\|^2\},$$

or, substituting for  $\min_{\mathcal{B}_i(\varepsilon)} \{\|\nabla\beta_i\|^2\}$ , using Eq. (16) in the Appendix,

$$\varepsilon < \frac{1}{2} \rho_i^2 \triangleq \varepsilon'_{2i}.$$

A sufficient condition for the (\*\*) term to be positive is

$$\varepsilon < \frac{1}{4} \frac{\min_{\mathcal{B}_i(\varepsilon)} \{\sqrt{\bar{\beta}_i} \|\nabla\beta_i\|\}}{\max_{\mathcal{B}_i(\varepsilon)} \{\sqrt{|\hat{\varphi}^T \nabla^2\bar{\beta}_i \hat{\varphi}|}\}},$$

which, by the same reasoning detailed in the proof of Proposition 3.6, is

<sup>5</sup>See Appendix for  $\min_{\mathcal{B}_i(\varepsilon)} \{\cdot\}$ ,  $\max_{\mathcal{B}_i(\varepsilon)} \{\cdot\}$ .

satisfied if

$$\varepsilon < \frac{1}{4} \frac{\min_{\mathcal{B}_i(\varepsilon_{2i})} \{\sqrt{\beta_i} \|\nabla \beta_i\|\}}{\max_{\mathcal{B}_i(\varepsilon_{2i})} \{\sqrt{|\hat{v}^T \nabla^2 \beta_i \hat{v}|}\}} \triangleq \varepsilon_{2i}''. \quad (13)$$

To complete the proof, choose

$$\varepsilon_2 \triangleq \min\{\varepsilon_{2i}', \varepsilon_{2i}''\}, \quad i \in \{1, \dots, M\}. \quad \square$$

Finally, if we will choose  $N(\varepsilon) = N(\varepsilon_{\min})$  in Eq. (6), where

$$\varepsilon_{\min} \triangleq \frac{1}{2} \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\},$$

the proof of Theorem 4 is completed.

#### 4. CONCLUSION

Presented with the geometrical model described, the task of navigating a point robot toward an arbitrary destination while avoiding the obstacles is captured in a cost function. This representation is correct in the sense that if one computes the parameter  $k$  according to the formula given in Eq. (6), the resulting gradient vector field has a unique attractor at the destination and is directed away from the interior of the free space on its boundary. The cost function immediately gives rise to a correct feedback control law for a torque actuated mechanical system as well. The issue of numerical stability will be discussed in a future paper, in which we will present a numerical procedure to implement this algorithm.

#### APPENDIX A: NOTATION AND TERMINOLOGY

Given a topological space,  $\mathcal{X}$ , we will denote the *closure* of a set,  $\mathcal{S} \subseteq \mathcal{X}$  by  $\bar{\mathcal{S}}$ , and its interior by  $\mathring{\mathcal{S}}$ . Let  $\mathcal{D}_\rho^n(x)$  denote the open Euclidean  $n$ -disk of radius  $\rho$  about the point  $x$ ,

$$\mathcal{D}_\rho^n(x) \triangleq \{y \in E^n: \|y - x\| < \rho\}.$$

In the special case that  $\rho = 1$ ,  $x = 0$ , this will be written  $\mathcal{D}^n$ .

If  $h: E^n \rightarrow E^m$  then  $Dh$  denotes the *Jacobian*—that is, the matrix of partial derivatives of  $h$ . If  $[Dh](x)$  is not surjective then  $x \in E^n$  is a *critical point* of  $h$ ; otherwise it is a *regular point*. If  $h^{-1}(y)$  contains a critical point, then  $y \in E^m$  is a *critical value*.

If  $m = 1$  then the row matrix  $Dh$  is the “differential one-form” induced by the scalar valued map,  $h$ . Since we will always use the Euclidean metric in  $E^n$ , the *gradient vector field* induced by  $h$  is simply the column matrix,

$$\nabla h \triangleq (Dh)^T.$$

The *Hessian* is the symmetric square matrix,

$$D^2h \triangleq D \nabla h.$$

Let  $\varphi \in C^2[E^n, E]$ . The set of critical points of  $\varphi$  will be denoted by  $\mathcal{C}_\varphi$ . A critical point of  $\varphi$  (a zero of the gradient vector field) is *non-degenerate* if the Hessian,  $D^2\varphi$ , has full rank at that point. The scalar valued function,  $\varphi$ , is called a *Morse function* if all its critical points are non-degenerate. The *Morse index* of  $\varphi$  at a critical point,  $x$ , is the dimension of the subspace of  $E^n$  spanned by eigenvectors of the Hessian with negative eigenvalues:

$$\lambda_\varphi(x) \triangleq \dim\{y \in E^n: y^T[D^2\varphi](x)y < 0\}.$$

Each critical point of  $\varphi$  is a strict local minimum or maximum (defined in the standard way—index 0 or  $n$ , respectively), or a saddle—any non-degenerate critical point which is neither a minimum nor a maximum.  $\varphi$  is said to be *polar on*  $\mathcal{X}$  *at*  $q_d$  if it has exactly one minimum, at  $q_d$  [17]. Finally, it is *admissible* on  $\mathcal{X}$  if  $\varphi(\partial\mathcal{X}) = 1$ , and at any other point in the interior of  $\mathcal{X}$   $0 \leq \varphi < 1$  [5].

## APPENDIX B: COMPUTATION OF THE NAVIGATION FUNCTION PARAMETER

### B.1. Bounding Each Obstacle Cost

To obtain a practical lower bound for  $\varepsilon_{\min}$ , and consequently an upper bound for  $N(\varepsilon_{\min})$ , the following “tokens” have to be found,

$$\gamma_{d_{\min}}^i, \beta_{j_{\min}}^i, \gamma_{d_{\max}}, \beta_{i_{\max}}; \quad i, j \in \{0, \dots, M\}, i \neq j, \quad (14)$$

and their normed gradients,

$$\|\nabla \gamma_{d_{\min}}^i\|, \|\nabla \beta_{j_{\min}}^i\|, \|\nabla \gamma_{d_{\max}}\|, \|\nabla \beta_{i_{\max}}\|; \quad i, j \in \{0, \dots, M\}, \quad (15)$$

where

$$(\cdot)_{\min}^i \triangleq \min_{\mathcal{B}(\varepsilon)} \{(\cdot)\} \quad \text{and} \quad (\cdot)_{\max} \triangleq \max_{\mathcal{X}} \{(\cdot)\}.$$

Note that

$$\gamma_d = \beta_i|_{q_i=q_d, \rho_i=0};$$

it follows then that  $\{\gamma_{d_{\min}}^i, \gamma_{d_{\max}}, \|\nabla \gamma_{d_{\max}}\|\}$  need not be considered as a special case.

LEMMA B.1.  $\beta_{i_{\max}} = (\rho_0 + \|q_i\|)^2 - \rho_i^2 \quad i \in \{1, \dots, M\}$ .

*Proof.* At any point  $q \in \mathcal{W}$ ,

$$\|q\|^2 \leq (\rho_0)^2,$$

adding to both sides the term  $\|q_i\|^2 - 2q \cdot q_i - \rho_i^2$ ,

$$\begin{aligned} \|q - q_i\|^2 - \rho_i^2 &\leq (\rho_0)^2 - 2q \cdot q_i + \|q_i\|^2 - \rho_i^2 \\ &\leq (\rho_0)^2 + 2\rho_0 \cdot \|q_i\| + \|q_i\|^2 - \rho_i^2; \end{aligned}$$

hence,

$$\beta_{i_{\max}} = (\rho_0 + \|q_i\|)^2 - \rho_i^2. \quad \square$$

And of course,

$$\beta_{0_{\max}} = (\rho_0)^2.$$

Turning our attention to  $\beta_{j_{\min}}^i \quad i \in \{1, \dots, M\}$ , we will find the minimum of  $\beta_j$  over  $\overline{\mathcal{B}_i(\varepsilon)}$ , using Lagrange multipliers.

LEMMA B.2.  $\beta_{j_{\min}}^i = \min_{\overline{\mathcal{B}_i(\varepsilon)}} \{\beta_j\} = (\sqrt{\varepsilon + \rho_i^2} - \|q_i - q_j\|)^2 - \rho_j^2$ .

*Proof.* By the assumption of a valid sphere world the obstacles do not intersect, which implies that  $q_j$  cannot be inside the obstacle  $\mathcal{O}_i$ . Each  $\beta_j$  is a quadratic function, which has exactly one critical point, at  $q = q_j$ . The case of  $q_j \in \overline{\mathcal{B}_i(\varepsilon)}$ , which is interpreted algebraically as

$$\beta_i(q_j) \leq \varepsilon,$$

can be excluded if

$$\varepsilon < \|q_i - q_j\|^2 - \rho_i^2, \quad i, j \in \{1, \dots, M\}, i \neq j,$$

which is the case by the assumptions detailed in Eq. (2) and Eq. (3). It follows that the argument given in the proof of Lemma 3.5 applies also here, namely, that  $\beta_j$  attains its minimum in  $\beta_i^{-1}(\varepsilon)$ .

At the minimum,  $q^*$ , we have

$$\lambda \nabla \beta_i = \nabla \beta_j$$

or

$$\lambda(q^* - q_i) = q^* - q_j,$$

for some  $\lambda \in \mathbb{R}$ . Which implies that

$$q^* = \frac{\lambda}{\lambda - 1} q_i - \frac{1}{\lambda - 1} q_j.$$

Hence,

$$\beta_j(q^*) = \left( \frac{\lambda}{\lambda - 1} \right)^2 \|q_i - q_j\|^2 - \rho_j^2$$

and

$$\beta_i(q^*) = \frac{1}{(\lambda - 1)^2} \|q_i - q_j\|^2 - \rho_i^2.$$

Solving for  $\lambda$  in the above equation yields

$$\lambda = 1 \pm \frac{\|q_i - q_j\|}{\sqrt{\beta_i(q^*) + \rho_i^2}}$$

or

$$\beta_j(q^*) = \left( \sqrt{\beta_i(q^*) + \rho_i^2} \pm \|q_i - q_j\| \right)^2 - \rho_j^2,$$

choosing the “−” option for the minimum and substituting  $\varepsilon$  for  $\beta_i(q^*)$ ,

$$\beta_{j_{\min}}^i = \min_{\mathcal{D}_i(\varepsilon)} \{\beta_j\} = \left( \sqrt{\varepsilon + \rho_i^2} - \|q_i - q_j\| \right)^2 - \rho_j^2. \quad \square$$

Following the above proof almost identically, it can be readily found that

$$\beta_{0_{\min}}^i = \min_{\mathcal{D}_i(\varepsilon)} \{\beta_0\} = \rho_0^2 - \left( \sqrt{\varepsilon + \rho_i^2} + \|q_i\| \right)^2.$$

*Remark.* The assumption that the “extended obstacles” do not intersect, and in the interior of  $\mathcal{W}$ , detailed algebraically in Eqs. (2) and (3),

guarantees that

$$\beta_{j_{\min}}^i > 0, \quad i \in \{1, \dots, M\}, j \in \{0, \dots, M\}, i \neq j.$$

Finally, the minimum of the normed gradients is readily obtained from the above results, since

$$\|\nabla \beta_j\| = 2\sqrt{\beta_j + \rho_j^2}, \quad j \in \{1, \dots, M\},$$

which implies that

$$\|\nabla \beta_{j_{\min}}^i\| = 2\sqrt{\beta_{j_{\min}}^i + \rho_j^2} = \begin{cases} 2(\sqrt{\varepsilon + \rho_i^2} - \|q_i - q_j\|) & \text{if } i \neq j \\ 2\rho_i & \text{if } i = j, \end{cases}$$

$i, j \in \{1, \dots, M\}. \quad (16)$

## B.2. Bounding the “ $\varepsilon$ Limiters”— $\varepsilon_0, \varepsilon_2$

In general, if  $\varphi_1, \varphi_2$  are non-negative scalar valued functions on a compact set  $\mathcal{X}$ , then

$$\min_{\mathcal{X}} \{\varphi_1\} \min_{\mathcal{X}} \{\varphi_2\} \leq \min_{\mathcal{X}} \{\varphi_1 \varphi_2\},$$

and, of course, the same applies to the  $\max\{\cdot\}$ .

Using this fact, with the Schwartz and triangular inequalities, we can trivially obtain a bound on each “ $\varepsilon$  limiter” in terms of the “tokens” detailed in Eqs. (14) and (15). The only term which deserves attention is  $|\hat{v}^T \nabla^2 \bar{\beta}_i \hat{v}|$ , which appears in  $\varepsilon_{0i}''$  and  $\varepsilon_{2i}''$ .

$$\nabla \bar{\beta}_i = \sum_{j=0, j \neq i}^M \left( \prod_{l=0, l \neq i, j}^M \beta_l \right) \nabla \beta_j,$$

which implies that

$$\begin{aligned} \nabla^2 \bar{\beta}_i &= \sum_{j=0, j \neq i}^M \left( \prod_{l=0, l \neq i, j}^M \beta_l \right) \nabla^2 \beta_j \\ &+ \sum_{j=0, j \neq i}^M \sum_{l=0, l \neq i, j}^M \left( \prod_{m=0, m \neq i, j, l}^M \beta_m \right) \nabla \beta_j^T \nabla \beta_l, \end{aligned}$$

since  $|\nabla^2 \beta_j| = 2I$ . It follows then, that an upper bound on  $|\hat{\sigma}^T \nabla^2 \bar{\beta}_i \hat{\sigma}|$  is

$$\begin{aligned} \max_{\hat{\sigma}_i(\epsilon)} \{|\hat{\sigma}^T \nabla^2 \bar{\beta}_i \hat{\sigma}|\} &\leq 2 \sum_{j=0, j \neq i}^M \left( \left( \prod_{l=0, l \neq i, j}^M \max_{\hat{\sigma}_l(\epsilon)} \{\beta_l\} \right) \right. \\ &\quad \left. + \sum_{l=0, l \neq i, j}^M \left( \prod_{m=0, m \neq i, j, l}^M \max_{\hat{\sigma}_m(\epsilon)} \{\beta_m\} \right) \right) \\ &\quad \cdot \max_{\hat{\sigma}_j(\epsilon)} \{\|\nabla \beta_j\|\} \max_{\hat{\sigma}_i(\epsilon)} \{\|\nabla \beta_i\|\}. \end{aligned} \quad (17)$$

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