On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects

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Abstract

We consider a class of dynamic advertising problems under uncertainty in the presence of carryover and distributed forgetting effects, generalizing the classical model of Nerlove and Arrow [32]. In particular, we allow the dynamics of the product goodwill to depend on its past values, as well as previous advertising levels. Building on previous work ([16]), the optimal advertising model is formulated as an infinite dimensional stochastic control problem. We obtain (partial) regularity as well as approximation results for the corresponding value function. Under specific structural assumptions we study the effects of delays on the value function and optimal strategy. In the absence of carryover effects, since the value function and the optimal advertising policy can be characterized in terms of the solution of the associated HJB equation, we obtain sharper characterizations of the optimal policy.

Keywords: stochastic control problems with delay, dynamic programming, infinite dimensional Bellman equations, optimal advertising.

1 Introduction

This paper is devoted to the study of a class of optimal control problems for linear stochastic differential equations with delay both in the state and the control term, and is a natural continuation of [16]. These problems arise in the theory of optimal advertising under uncertainty with memory structures. We approach the problem using stochastic control techniques in infinite dimensions.

In particular, in [16] we considered a controlled stochastic differential equation (SDE) with delay entering both the state and the control variable as an extension of the dynamic advertising model of Nerlove and Arrow [32]. The results of [16] are the following: we construct a controlled infinite dimensional SDE that is equivalent to the controlled SDE with delay, we prove a verification theorem, and we exhibit a simple example for which the Bellman equation associated to the control problem admits a sufficiently regular solution, hence the verification theorem can be applied. In the present manuscript, we extend [16] by developing several new sets of results. On the one hand, we provide qualitative characterization of the first- and second-order properties of the optimal value function (Section 3.1). In particular, we show that, under natural restrictions, the monotonicity of the optimal value function with respect to the initial goodwill profile still holds even in the presence of the state and control-related delay terms in the advertising dynamics (Proposition 3.6). In addition, we establish that the

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decreasing marginal influence of the attained goodwill levels on the primal profit components is retained by the optimal profit function (Proposition 3.2). As is well known, this last property is important in reducing the computational load required to solve the time- and space-discretized version of our problem by dynamic programming methods. On the other hand, in the view of intractability of the general variant of our problem, we propose approximation schemes for the optimal value function (Theorem 3.7) and for the optimal advertising policy (Propositions 3.11 and 3.12). The latter result is of particular importance since it suggests a computationally feasible approach to constructing asymptotically optimal advertising trajectories. In addition, we provide a complete characterization of the optimal advertising policy in the case when the cost function is quadratic and the reward function is linear in goodwill level (Section 3.3). For a specific instance of this case we conduct a numerical study aimed at demonstrating the importance of proper accounting of the delay effects in calculating the optimal advertising policy.

Finally, we are able to provide sharper characterization of the optimal policies in the case when the influence of advertising on the goodwill evolution is instantaneous and the delay effects are of the state-only type (Section 4). The key result in this section is Theorem 4.5 which formulates sufficient conditions ensuring that the optimal advertising policy is of the feedback type. In particular, in the case of linear cost function the optimal control takes a particularly simple "bang-bang" form (Corollary 4.10).

Optimal control problems for stochastic systems with delay in the state term admit alternative, more traditional treatments: for instance, see [10] and [25] for a more direct application of the dynamic programming principle without appealing to infinite dimensional analysis, and [23] for the linear-quadratic case. However, we would like to point out that none of the methods just mentioned apply to the control of stochastic differential equations with delay in the control term.

Analysis of advertising policies has always been occupying a front-and-center place in the marketing research. The sheer size of the advertising market (over $143 billion in the US in 2005 [31]) and the strong body of evidence of systematic over-advertising by firms across many industries (see e.g. [1], [20], [29], [30]) has caused a renewal of attention to the proper accounting for the so-called "carryover" or "distributed lag" advertising effects. The term "carryover" designates an empirically observed advertising feature under which the advertising influence on product sales or goodwill level is not immediate, but rather is spread over some period of time: according to a survey of recent empirical "carryover" research by Leone [28], delayed advertising effects can last between 6 and 9 months in different settings.

On the theoretical front, pioneering work of [34] and [32] has paved the way for the development of a number of models dealing with the optimal distribution of advertising spending over time in both monopolistic and competitive settings. A comprehensive review of the state of the advertising control literature in [11] points out that the majority of these models operate under deterministic assumptions and do not capture some of the most essential characteristics of real-world advertising phenomena. On the empirical side, one of the first and most important substreams of advertising literature was formed by the papers focused on the studies of distributed advertising lag (see e.g. [3], [5], [17]). An important early empirical result was obtained by Bass and Clark [4], who established that the initially adopted models with monotone decreasing lags (see [24]) are often inferior in their explanatory power to the models with more general lag distributions.

Despite the wide and growing empirical literature on the measurement of carryover effects, there are practically no analytical studies that incorporate distributed lag structure into the optimal advertising modeling framework in the stochastic setting. The only papers dealing with optimal dynamic advertising with distributed lags we are aware of are [5] (which provides a numerical solution to a discrete-time deterministic example), and [18], [19] (which applies a version of the maximum principle in the deterministic setting). The creation of models which incorporate the treatment of "carryover" effects in the stochastic settings have long been advocated in the advertising modeling literature (see e.g. [18], [11] and references therein).

As mentioned above, in this work we study a class of stochastic models deriving from that of Nerlove and Arrow [32], incorporating both the advertising lags as well as distributed "churn" ([33]), or "forgetting", effects. More precisely, we formulate an optimization program that seeks to maximize the goodwill level at a given time $T > 0$ net of the cumulative cost of advertising until T. This optimization problem is studied using techniques of stochastic optimal control in infinite dimensions, using the modeling approach of [16]: in particular, we specify the goodwill dynamics in terms of a controlled stochastic delay differential equation (SDDE), that can be rewritten as a stochastic differential equation (without delay) in a suitable Hilbert space. This allows us to associate to the original control problem for the SDDE an equivalent infinite dimensional control problem for the "lifted" stochastic equation.

The paper is organized as follows: in section 2 we formulate the optimal advertising problem as an optimal control problem for an SDE with delay, and we recall the equivalence result of [16]. In section 3 we prove the above mentioned results about the value function and approximate strategies in the general case, together with a detailed discussion of the effect of delays in a specific situation. Section 4 treats the case of distributed forgetting in the absence of advertising carryover.

Let us conclude this introduction fixing notation and recalling some notions that will be needed. Given a lower semicontinuous convex function $f : E \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ on a Hilbert space E with inner product $\langle \cdot, \cdot \rangle$, we denote its conjugate by $f^*(y) := \sup_{x \in E}(\langle x, y\rangle - f(x))$. Recall also that $D^- f^*(y) = \arg\max_{x\in E}(\langle x, y\rangle - f(x))$, where D^- stands for the subdifferential operator (see e.g. [2], p. 103). Throughout the paper, X will be the Hilbert space defined as

$$X = \mathbb{R} \times L^2([-r, 0], \mathbb{R}),$$

with inner product

$$\langle x, y\rangle = x_0 y_0 + \int_{-r}^{0} x_1(\xi) y_1(\xi)\, d\xi$$

and norm

$$|x| = \left(|x_0|^2 + \int_{-r}^{0} |x_1(\xi)|^2\, d\xi \right)^{1/2},$$

where $r > 0$, x_0 and $x_1(\cdot)$ denote the $\mathbb{R}$-valued and the $L^2([-r, 0], \mathbb{R})$-valued components, respectively, of the generic element x of X. Given $f : X \to \mathbb{R}$, $k \in \{0, 1\}$, we shall denote by $\partial_k f$, $D_k^- f$, respectively, the partial derivative and the subdifferential of f with respect to the k-th component. We shall use mollifiers in a standard way: for $\zeta \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$, equal to zero for $|x| > 1$ and such that $\int_{\mathbb{R}^d} \zeta(x)\, dx = 1$, we shall set $\zeta_\lambda(x) = \lambda^{-d}\zeta(\lambda^{-1}x)$ for $\lambda \neq 0$. By $a \lesssim b$ we mean that there exists a constant N such that $a \leq Nb$. If N depends on some parameter of interest p, we shall write $N(p)$ and $a \lesssim_p b$.

2 The model

We consider a monopolistic firm preparing the market introduction of a new product at some time T in the future. In defining the state descriptor for a firm to follow we use the Nerlove-Arrow framework and consider the product's "goodwill stock" $y(t)$, $0 \leq s \leq t \leq T$. The firm directly influences the rate of advertising spending $z(t)$ to induce the following trajectory for the goodwill stock:

$$\begin{cases} dy(t) = \left[a_0 y(t) + \displaystyle\int_{-r}^{0} a_1(\xi) y(t+\xi)\, d\xi + b_0 z(t) + \int_{-r}^{0} b_1(\xi) z(t+\xi)\, d\xi\right] dt \\ \qquad\qquad +\sigma\, dW_0(t), \quad s \le t \le T \\ y(s) = x_0; \quad y(s+\xi) = x_1(\xi), \ z(s+\xi) = \delta(\xi), \quad \xi \in [-r, 0], \end{cases} \tag{1}$$

where the Brownian motion W_0 is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, with $\mathbb{F}$ being the completion of the filtration generated by W_0. We assume that the advertising spending rate $z(t)$ is constrained to remain in the set $\mathcal{U}$, the space of $\mathbb{F}$-adapted processes taking values in a compact interval $U \subseteq \mathbb{R}_+$. In addition, we assume that the following conditions hold:

(i) $a_0 \le 0$;

(ii) $a_1(\cdot) \in L^2([-r, 0], \mathbb{R})$;

(iii) $b_0 \ge 0$;

(iv) $b_1(\cdot) \in L^2([-r, 0], \mathbb{R}_+)$;

(v) $x_0 \ge 0$;

(vi) $x_1(\cdot) \ge 0$, with $x_1(0) = x_0$;

(vii) $\delta(\cdot) \ge 0$.

Here a_0 and $a_1(\cdot)$ describe the process of goodwill deterioration when the advertising stops, and b_0 and $b_1(\cdot)$ provide the characterization of the effect of the current and the past advertising rates on the goodwill level. The values of x_0, $x_1(\cdot)$ and $\delta(\cdot)$ reflect the "initial" goodwill and advertising trajectories. Note that we recover the model of Nerlove and Arrow from (1) in the deterministic setting ($\sigma = 0$) in the absence of delay effects ($a_1(\cdot) = b_1(\cdot) = 0$).

Setting $X \ni x := (x_0, x_1(\cdot))$ and denoting by $y^{s,x,z}(t)$, $t \in [0, T]$, a solution of (1), we define the objective functional

$$J(s, x; z) = \mathbb{E}\left[\varphi_0(y^{s,x,z}(T)) - \int_s^T h_0(z(t))\, dt\right], \tag{2}$$

where $\varphi_0 : \mathbb{R} \to \mathbb{R}$ and $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$ are measurable utility and cost functions, respectively, satisfying the conditions

$$|\varphi_0(x)| \le K(1 + |x|)^m \qquad \forall x \in \mathbb{R}, \tag{3}$$

and

$$|h(x)| \le K \qquad \forall x \in U, \tag{4}$$

for some $K > 0$ and $m \ge 0$. In the sequel we shall often move the superscripts s, x, z to the expectation sign, with obvious meaning of the notation. Let us also define the value function V for this problem as follows:

$$V(s, x) = \sup_{z \in \mathcal{U}} J(s, x; z).$$

We shall say that $z^* \in \mathcal{U}$ is an optimal strategy if it is such that

$$V(s, x) = J(s, x; z^*).$$

The problems we will deal with are the maximization of the objective functional J over all admissible strategies $\mathcal{U}$, and the characterization of the value function V and of the optimal strategy z^*.

Throughout the paper we will always assume that the assumptions of this section hold true. In particular the constants T, m and K are fixed from now on.

2.1 An equivalent infinite dimensional Markovian representation

We shall recall a representation result (proposition 2.1 below) proved in [16], generalizing a corresponding deterministic result due to Vinter and Kwong [35].
Let us define an operator $A : D(A) \subset X \to X$ as follows:

$$\begin{aligned} A : (x_0, x_1(\xi)) \quad &\mapsto \quad \Big(a_0 x_0 + x_1(0), a_1(\xi)x_0 - \frac{dx_1(\xi)}{d\xi}\Big) \quad \text{a.e. } \xi \in [-r, 0], \\ D(A) \quad &= \quad \big\{x \in X : x_1 \in W^{1,2}([-r, 0]; \mathbb{R}),\ x_1(-r) = 0\big\}. \end{aligned}$$

Moreover, define the bounded linear control operator $B : U \to X$ as

$$B : u \mapsto \Big(b_0 u, b_1(\cdot)u\Big), \tag{5}$$

and finally the operator $G : \mathbb{R} \to X$ as $G : x_0 \mapsto (\sigma x_0, 0)$. Sometimes it will be useful to identify the operator B with the element $(b_0, b_1) \in X$.

Proposition 2.1. *Let $Y(\cdot)$ be the weak solution of the abstract evolution equation*

$$\begin{cases} dY(t) = (AY(t) + Bz(t))\, dt + G\, dW_0(t) \\ Y(s) = \bar{x} \in X, \end{cases} \tag{6}$$

with arbitrary initial datum $\bar{x} \in X$ and control $z \in \mathcal{U}$. Then, for $t \geq r$, one has, $\mathbb{P}$-a.s.,

$$Y(t) = M(Y_0(t), Y_0(t + \cdot), z(t + \cdot)),$$

where

$$\begin{aligned} M : X \times L^2([-r, 0], \mathbb{R}) \quad &\to \quad X \\ (x_0, x_1(\cdot), v(\cdot)) \quad &\mapsto \quad (x_0, m(\cdot)), \end{aligned}$$

$$m(\xi) := \int_{-r}^{\xi} a_1(\zeta)x_1(\zeta - \xi)\, d\zeta + \int_{-r}^{\xi} b_1(\zeta)v(\zeta - \xi)\, d\zeta.$$

Moreover, let $\{y(t),\ t \geq -r\}$ be a continuous solution of the stochastic delay differential equation (1), and $Y(\cdot)$ be the weak solution of the abstract evolution equation (6) with initial condition

$$\bar{x} = M(x_0, x_1, \delta(\cdot)).$$

Then, for $t \geq 0$, one has, $\mathbb{P}$-a.s.,

$$Y(t) = M(y(t), y(t + \cdot), z(t + \cdot)),$$

hence $y(t) = Y_0(t)$, $\mathbb{P}$-a.s., for all $t \geq 0$.

Using this equivalence result, we can now give a Markovian reformulation on the Hilbert space X of the problem of maximizing (2), as in [16]. In particular, denoting by $Y^{s,\bar{x},z}(\cdot)$ a mild solution of (6), (2) is equivalent to

$$J(s, x; z) = \mathbb{E}\left[\varphi(Y^{s,\bar{x},z}(T)) + \int_s^T h(z(t))\, dt\right], \tag{7}$$

with the functions $h : U \to \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ defined by

$$\begin{aligned} h(z) \quad &= \quad -h_0(z) \\ \varphi(x_0, x_1) \quad &= \quad \varphi_0(x_0). \end{aligned}$$

Hence also $V(s, x) = \sup_{z \in \mathcal{U}} J(s, x; z)$.

3 The case of delay in the state and the control term

The aims of this section are the following: to prove regularity properties of the value function, to develop an approximation scheme for the value function and the optimal strategy, and to illustrate in a numerical example the effects of the delay structures in our model. In particular, we prove that, under natural assumptions, the value function is continuous in both arguments, and monotone concave with respect to the initial goodwill profile. As already remarked, this property is essential in order to obtain computationally tractable discrete-time and discrete-state-space dynamic programming versions of our problem.

Moreover, since we cannot guarantee that the Bellman equation associated to our control problem admits a solution in general (nor do we have any information about its uniqueness and regularity), it is of primary interest to obtain approximation schemes for the optimal value function and for the optimal advertising policy. The latter result is of particular importance since it suggests a computationally feasible approach to constructing asymptotically optimal advertising trajectories.

In addition, in the last subsection we provide a complete characterization of the optimal advertising policy in the case when the cost function is quadratic and the reward function is linear in goodwill level, and for a specific instance of this case we conduct a numerical study aimed at demonstrating the importance of proper accounting of the delay effects in calculating the optimal advertising policy.

3.1 Qualitative properties of the value function

Let us first show that the value function is finite.

Proposition 3.1. *There exists a constant $N = N(T, m, K)$ such that $|V(s,x)| \le N(1+|x|)^m$ for all $s \in [0,T]$, $x \in X$.*

Proof. The estimate from below simply follows by taking a constant deterministic control. For the estimate from above we have, recalling that $h(x) \le 0$ for all $x \in U$,

$$\begin{aligned} V(s,x) &\le \sup_{z\in\mathcal{U}} \mathbb{E}^z_{s,x}\Big[\int_s^T h(z(t))\,dt + |\varphi(Y(T))|\Big] \\ &\le K \sup_{z\in\mathcal{U}} \mathbb{E}^z_{s,x}(1+|Y(T)|)^m. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathbb{E}|Y(T)|^m &\lesssim |e^{(T-s)A}x|^m + \mathbb{E}\Big|\int_0^{T-s} e^{(T-s-r)A}Bz(r)\,dr\Big|^m + \mathbb{E}\Big|\int_0^{T-s} e^{(T-s-r)A}G\,dW(r)\Big|^m \\ &\lesssim_T M_T^m|x|^m + |B|^m M_T^m \bar{z}^m + \mathbb{E}|W_A(T)|^m, \end{aligned}$$

where $\bar{z} := \max\{z : z \in U\}$, $W_A(t) := \int_0^{t-s} e^{(t-s-r)A}G\,dW(r)$, and $M_T := \sup_{t\in[0,T]}|e^{tA}|$. Recalling that GG^* is of trace class, hence

$$\mathbb{E}|W_A(T)|^2 = \mathrm{Tr}\int_0^{T-s} e^{tA}GG^*e^{tA^*}\,dt < \infty$$

(see e.g. [7], Proposition 2.2), i.e. $W_A(T)$ is a well-defined Gaussian random variable on X, we also get that $\mathbb{E}|W_A(T)|^m < \infty$. The proof is completed observing that the upper bound on $\mathbb{E}|Y(T)|^m$ is uniform over z. □

We establish now some qualitative properties of the value function that do not require studying an associated Bellman equation. The following simple result, typical of control problems with linear dynamics, asserts that the value function inherits the concavity with respect to the space variable from the reward and cost functions.

Proposition 3.2. *If φ and h are concave, then the value function $V(s,x)$ is proper concave with respect to x.*

Proof. Properness follows by the previous proposition. Moreover, let x^1, $x^2 \in X$. Then

$$\begin{aligned}
\lambda V(s,x^1) + (1-\lambda)V(s,x^2) &= \lambda \sup_{z\in\mathcal{U}} \mathbb{E}\Big[\int_s^T h(z(t))\,dt + \varphi(y^{s,x^1,z}(T))\Big] \\
&\quad + (1-\lambda)\sup_{z\in\mathcal{U}} \mathbb{E}\left[\int_s^T h(z(t))\,dt + \varphi(y^{s,x^2,z}(T))\right] \\
&= \sup_{z^1,z^2\in\mathcal{U}} \mathbb{E}\Big[\int_s^T [\lambda h(z^1(t)) + (1-\lambda)h(z^2(t))]\,dt \\
&\qquad + \lambda\varphi(y^{s,x^1,z^1}(T)) + (1-\lambda)\varphi(y^{s,x^2,z^2}(T))\Big].
\end{aligned}$$

Since $\mathcal{U}$ is a convex set and h is concave, then $z_\lambda := \lambda z^1 + (1-\lambda)z^2$ is admissible for any choice of z^1, $z^2 \in \mathcal{U}$, and one has

$$h(z_\lambda(s)) \geq \lambda h(z^1(s)) + (1-\lambda)h(z^2(s)). \tag{8}$$

Moreover, by linearity of the state equation, it is easy to prove that

$$y^{s,x_\lambda,z_\lambda}(T) = \lambda y^{s,x^1,z^1}(T) + (1-\lambda)y^{s,x^2,z^2}(T),$$

hence, by the concavity of φ,

$$\varphi(y^{s,x_\lambda,z_\lambda}(T)) \geq \lambda\varphi(y^{s,x^1,z^1}(T)) + (1-\lambda)\varphi(y^{s,x^2,z^2}(T)). \tag{9}$$

Therefore, as a consequence of (8) and (9), we obtain

$$\begin{aligned}
\lambda V(s,x^1) + (1-\lambda)V(s,x^2) &\leq \sup_{z_\lambda\in\mathcal{U}} \mathbb{E}\Big[\int_s^T h(z_\lambda(t))\,dt + \varphi(y^{s,x_\lambda,z_\lambda}(T))\Big] \\
&\leq \sup_{z\in\mathcal{U}} \mathbb{E}\Big[\int_s^T h(z(t))\,dt + \varphi(y^{s,x_\lambda,z}(T))\Big] \\
&= V(s, \lambda x^1 + (1-\lambda)x^2),
\end{aligned}$$

which proves the claim. □

As a consequence of the previous propositions we obtain the following regularity result. Of course it would be ideal to obtain a result guaranteeing that $V \in C^{0,1}([0,T]\times X)$, so that a verification theorem could be proved. Unfortunately we have not been able to obtain such result. We shall prove though that V is locally Lipschitz continuous in the X-valued variable.

Corollary 3.3. *Under the hypotheses of Proposition 3.2, the value function $V(s,x)$ is locally Lipschitz continuous with respect to $x \in X$. Moreover, the subgradient $\partial V(s,x)$ with respect to x exists for all $x \in X$ and is locally bounded.*

Proof. The first assertion comes from the fact that a concave locally bounded function is continuous in the interior of its effective domain (see e.g. Theorem 2.1.3 in [2]) and $V(s,x)$ is finite for all $x \in X$. Corollary 2.4 in [9] and the fact that $D(\phi)^\circ \subset D(\partial\phi)$ for any concave function ϕ, where A° denotes the interior of a set A, imply that $V(s,x)$ is locally Lipschitz in x, so the assertion on ∂V follows. □

Since $V(s,x) \equiv V(s,x_0,x_1)$ is a concave function of x_0 for fixed s and x_1, one can also say that V is twice differentiable almost everywhere with respect to x_0, as it follows by the Busemann-Feller theorem. A similar statement is not true regarding differentiability with respect to x_1, as the Alexandrov theorem is in general no longer true in infinite dimensions. We now prove that the value function is continuous with respect to the time variable. It is possible to prove local Lipschitz continuity of $V(s,\cdot)$ without appealing to concavity, but assuming local Lipschitz continuity of φ.

Proposition 3.4. *The value function $V(s,x)$ is continuous in s. Moreover, if $|\varphi_0(x) - \varphi_0(y)| \leq K(1+R)^m|x-y|$ for all $|x|$, $|y| \leq R$, then the function $V(s,x)$ is locally Lipschitz continuous with respect to x. Furthermore, there exists a constant $N = N(K,m)$ such that*

$$|V(s,x) - V(s,y)| \leq N(1+R)^m|x-y| \tag{10}$$

for all $|x|$, $|y| \leq R$.

Proof. Recalling that the difference of two suprema is less or equal to the supremum of the difference, we have

$$\begin{aligned} |V(s,x) - V(s,y)| &\leq \sup_{z\in\mathcal{U}} |J(s,x;z) - J(s,y;z)| \\ &\leq \sup_{z\in\mathcal{U}} \mathbb{E}|\varphi(Y^{z,s,x}) - \varphi(Y^{z,s,y})|, \end{aligned}$$

and, by Cauchy-Schwarz' inequality,

$$\begin{aligned} &\mathbb{E}|\varphi(Y^{z,s,x}(T)) - \varphi(Y^{z,s,y}(T))| \leq \\ &\qquad K\Big(\mathbb{E}(1 + |Y^{z,s,x}(T)|^m + |Y^{z,s,y}(T)|^m)^2\Big)^{1/2}\Big(\mathbb{E}|Y^{z,s,x}(T) - Y^{z,s,y}(T)|^2\Big)^{1/2}. \end{aligned}$$

Arguing as in the proof of Proposition 3.1, there exists a constant $N_1 = N_1(K,m)$ such that $\mathbb{E}|Y^{z,s,x}(T)|^m \leq N_1(1+|x|^m)$. Furthermore, a simple calculation reveals that $|Y^{z,s,x}(t) - Y^{z,s,y}(t)| = |e^{(t-s)A}(x-y)|$, hence

$$\mathbb{E}|Y^{z,s,x}(T) - Y^{z,s,y}(T)|^2 \leq M_T^2|x-y|^2,$$

and the second claim is proved. Let us now prove that $V(s,x)$ is continuous in s for a fixed x. Let $s_n \uparrow s$ be a given sequence (the case $s_n \downarrow s$ is completely similar). Bellman's principle yields

$$V(s_n,x) = \sup_{z\in\mathcal{U}} \mathbb{E}\left[\int_{s_n}^{s} h(z(r))\,dr + V(s, Y^{z,s_n,x}(s))\right], \tag{11}$$

and choosing a $1/n$-optimal strategy z_n in (11), we have

$$\begin{aligned} \limsup_{n\to\infty} V(s_n,x) - V(s,x) \quad &\leq \quad \limsup_{n\to\infty} \mathbb{E}\int_{s_n}^{s} h(z_n(r))\,dr \\ &\quad + \limsup_{n\to\infty} \mathbb{E}|V(s, Y^{z_n,s_n,x}(s)) - V(s,x)|. \end{aligned} \tag{12}$$

The first term on the right-hand side in (12) is zero because h is bounded on U. Let us show the also the second term on the right-hand side of (12) is zero: in fact we have

$$\begin{aligned} \mathbb{E}|Y^{z_n,s_n,x}(s) - x|^2 &\leq K\Big(\mathbb{E}\int_{s_n}^{s} |e^{(s-r)A}h(z_n(r))|^2\,dr + \mathbb{E}\Big|\int_0^{s-s_n} e^{(s-s_n-r)A}G\,dW(r)\Big|^2\Big) \\ &\leq K_1(s-s_n) + q_n \to 0 \end{aligned}$$

as $n \to \infty$, hecause h is bounded on U and the stochastic convolution is a Gaussian random variable with covariance operator going to 0 as $n \to \infty$. Therefore we also have $Y^{z_n,s_n,x}(s) \to x$ in probability. By (10), $V(s,x)$ is continuous in x uniformly with respect to s, hence

$$V(s, Y^{z_n,s_n,x}(s)) \to V(s,x)$$

in probability by the continuous mapping theorem. Moreover, recalling that $|V(s,x)| \le N(1+|x|^m)$ and $\mathbb{E}\sup_{t\in[0,T]} |Y(t)|^m < \infty$, we have

$$\mathbb{E}|V(s, Y^{z_n,s_n,x}(s)) - V(s,x)|^m < \infty, \tag{13}$$

hence Vitali's theorem implies

$$\mathbb{E}|V(s, Y^{z_n,s_n,x}(s)) - V(s,x)| \to 0,$$

hence $V(s_n,x) - V(s,x) \to 0$ as $n \to \infty$. Furthermore, taking any $z_0 \in U$, we have from (11),

$$V(s,x) - V(s_n,x) \le V(s,x) - (s_n - s)h(z_0) - \mathbb{E}V(s, Y^{z_0,s_n,x}),$$

which goes to zero as $n \to \infty$ by (13), and the claim is proved. □

Remark 3.5. Notice that, by the Rademacher theorem in infinite dimensions, the previous proposition implies that the value function $V(s,x)$ is differentiable in a dense subset of X. The local Lipschitz continuity of V also implies that (Clarke's) generalized gradient of $V(s,x)$ with respect to x is defined everywhere on X.

For the following proposition, which establishes a monotonicity property of the value function, we need to define the natural ordering in X: we shall write $x^1 \ge x^2$ if $x_0^1 \ge x_0^2$ and $x_1^1 \ge x_1^2$ almost everywhere. Similarly, $x^1 > x^2$ if the previous inequalities hold with the strict inequality sign.

Proposition 3.6. *If $a_1 \ge 0$ and φ_0 is increasing, then the value function $V(s,x)$ is increasing with respect to x in the sense just defined.*

Proof. The proof is completely analogous to that of proposition 4.3 below if we prove that A generates a positivity preserving semigroup. This is indeed the case: in fact, a direct calculation shows that A is the adjoint of the operator $\mathcal{A}$ defined in section 4. Since $a_1 \ge 0$, $\mathcal{A}$ generates a positivity preserving semigroup $S(t)$. It is well known that A is the generator of the adjoint semigroup $S(t)^*$. Let x, y be arbitrary positive elements of X. Then

$$0 \le \langle S(t)x, y\rangle = \langle x, S(t)^* y\rangle.$$

By the arbitrariness of x and y, $S(t)^*$ is positivity preserving. □

3.2 Approximating the value function and the optimal strategy

Let us now consider the Bellman equation on X associated to the problem of maximizing (7), which can be written as

$$\begin{cases} v_t + \dfrac{1}{2}\operatorname{Tr}(GG^* v_{xx}) + \langle Ax, v_x\rangle + H_0(v_x) = 0, \quad 0 \le t \le T \\ v(T) = \varphi, \end{cases} \tag{14}$$

where $H_0(p) = \sup_{z\in U}(\langle Bz, p\rangle + h(z))$.

The main problem with (14) is that it is not solvable with any of the techniques currently available, with the possible exception of the theory of viscosity solutions. In particular, as of now, one cannot characterize the value function as the (unique) solution, in a suitable sense, of equation (14). As a consequence, we cannot obtain an optimal strategy for the optimization problem at hand. As a (partial) remedy we develop a method to approximate the value function and to construct suboptimal feedback strategies that are asymptotically optimal, in the sense of Proposition 3.12 below. Let us also briefly recall that, if we know a priori that a smooth solution to the Bellman equation exists, then we can apply the verification theorem proved in [16], which in turns allows to obtain precise characterizations of the optimal strategy (see subsection 3.3 and section 4).

Let us begin proving some approximation results for the value function V. Let $\varepsilon \in]0,1]$ and define $G_\varepsilon : \mathbb{R}^2 \to X$ as

$$G_\varepsilon = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \varepsilon b_1 \end{bmatrix}.$$

Let W_1 be a standard real Wiener process independent of W_0, set $W = (W_0, W_1)$, and denote by $\tilde{\mathbb{F}}$ the filtration generated by W. Let $\tilde{\mathcal{U}}$ be the set of $\tilde{\mathbb{F}}$-adapted processes taking values in U.

Consider the following approximating SDE on X:

$$dY(t) = [AY(t) + Bz(t)]\,dt + G_\varepsilon\,dW(t), \quad Y(s) = \bar{x}, \tag{15}$$

where $z \in \tilde{\mathcal{U}}$ and $0 \le s \le t \le T$.

For a fixed z, let Y and Y_ε be, respectively, solutions of (6) and of (15). Moreover, let us define $\varphi_\varepsilon(x) = (\varphi_{0\varepsilon}(x), 0)$, $\varphi_{0\varepsilon} = \tilde{\varphi}_{0\varepsilon} * \zeta_\varepsilon$, $\tilde{\varphi}_{0\varepsilon}(x) = \varphi_0(x)\chi_{[-1/\varepsilon,1/\varepsilon]}(x)$, and $h_\varepsilon(x) = (-h_{0\varepsilon}(x), 0)$, $h_{0\varepsilon}(x) = h_0 * \zeta_\varepsilon(x)$. In particular $\varphi_{0\varepsilon} \in C^2_c(\mathbb{R})$, $h_{0\varepsilon} \in C^2(\mathbb{R})$. Finally, the approximate objective function and value function are defined as

$$J_{\varepsilon_1,\varepsilon_2}(s,x;z) = \mathbb{E}^z_{s,\bar{x}}\Big[\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T)) + \int_s^T h_{\varepsilon_2}(z(t))\,dt\Big], \qquad V_{\varepsilon_1,\varepsilon_2}(s,x) = \sup_{z\in\tilde{\mathcal{U}}} J_{\varepsilon_1,\varepsilon_2}(s,x;z).$$

In the following we shall set $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $\lim_{\varepsilon\to 0} := \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}$. Moreover, for $R > 0$ we set $C_R = \{x \in X : \ |x_0| \le R\}$.

Theorem 3.7. *One has $V_\varepsilon(s,x) \to V(s,x)$ as $\varepsilon \to 0$ uniformly over $s \in [0,T]$, $x \in C_R$, for all $R > 0$.*

Proof. Setting $\eta_{\varepsilon_1}(t) = Y_{\varepsilon_1}(t) - Y(t)$, one has

$$\eta_{\varepsilon_1}(t) = \varepsilon_1 \int_s^t e^{(t-r)A} B_1\,dW_1(r) =: \varepsilon_1\eta(T),$$

with $B_1 : \mathbb{R} \to X$, $B_1 : x \to (0, b_1(\cdot)x)$, and (suppressing the subscripts on the expectation sign for simplicity)

$$\begin{aligned}
|J_\varepsilon(s,x;z) - J(s,x;z)| &\le |\mathbb{E}[\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T)) - \varphi(Y(T))]| \\
&\quad + \Big|\mathbb{E}\Big[\int_s^T h_{\varepsilon_2}(z(t))\,dt - \int_s^T h(z(t))\,dt\Big]\Big| \\
&\le |\mathbb{E}[\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T)) - \varphi_{\varepsilon_2}(Y(T))]| + |\mathbb{E}[\varphi_{\varepsilon_2}(Y(T)) - \varphi(Y(T))]| \\
&\quad + \Big|\mathbb{E}\Big[\int_s^T h_{\varepsilon_2}(z(t))\,dt - \int_s^T h(z(t))\,dt\Big]\Big|
\end{aligned} \tag{16}$$

Since

$$\mathbb{E}|Y_{\varepsilon_1}(T) - Y(T)| = \varepsilon_1 \mathbb{E}\Big|\int_s^t e^{(t-r)A} B_1\,dW_1(r)\Big| \to 0,$$

then $Y_{\varepsilon_1}(T) \to Y(T)$ in probability uniformly over $x \in X$ as $\varepsilon_1 \to 0$, and by the continuous mapping theorem $\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T)) \to \varphi_{\varepsilon_2}(Y(T))$ in probability. Moreover $\varphi_{\varepsilon_2}(Y(T)) \to \varphi(Y(T))$ in probability uniformly over $x \in C_R$, for all $R > 0$, as $\varepsilon_2 \to 0$ because $\varphi_{0\varepsilon_2}(x) \to \varphi_0(x)$ dx-a.e. in $\mathbb{R}$. Let us now prove that $\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T))$ is uniformly integrable with respect to ε. First let us observe, as it is immediate to show, that there exists $\bar{K}$, independent of ε_2, such that $\varphi_{\varepsilon_2}(x) \le \bar{K}(1+|x|)^m$. Then we can write

$$\begin{aligned}
\sup_{\varepsilon \in]0,1]^2} \mathbb{E}|\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T))| &\le \sup_{\varepsilon_1 \in]0,1]} \bar{K}\mathbb{E}(1+|Y(T)+\eta_{\varepsilon_1}(T)|)^m \\
&\le K_1 + K_2 \sup_{\varepsilon_1 \in]0,1]} (\mathbb{E}|Y(T)|^m + {\varepsilon_1}^m \mathbb{E}|\eta(T)|^m) \\
&= K_1 + K_2(\mathbb{E}|Y(T)|^m + \mathbb{E}|\eta(T)|^m) < \infty, \qquad (17)
\end{aligned}$$

where we used twice the inequality $|x+y|^m \le 2^m(|x|^m+|y|^m)$ and Burkholder-Davis-Gundy's inequality. Furthermore,

$$\sup_{\varepsilon \in]0,1]^2} \mathbb{E}[|\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T))|\chi_A] \le \mathbb{E}[(K_1 + K_2(|Y(T)|^m + |\eta(T)|^m))\chi_A] \to 0 \qquad (18)$$

as $\mathbb{P}(A) \to 0$, because $K_1 + K_2(|Y(T)|^m + |\eta(T)|^m$ has finite expectation. Then (17) and (18) imply that $\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T))$ is uniformly integrable (see e.g. [22], lemma 3.10), hence

$$|\mathbb{E}\varphi_{\varepsilon_2}(Y_{\varepsilon_1}(T)) - \mathbb{E}\varphi(Y(T))| \to 0$$

as $\varepsilon \to 0$ (see e.g. [22], proposition 3.12).
Similarly, since

$$|h_{\varepsilon_2}(z(t,\omega)) - h(z(t,\omega))| \le K_1 + K_2|z(t,\omega)|^m$$

for all $t \in [0,T]$, $\omega \in \Omega$ and $\mathbb{E}\int_0^T |z(t)|^m < \infty$ (because U is compact), by the dominated convergence theorem we have

$$\mathbb{E}\int_0^T |h_{\varepsilon_2}(z(t)) - h(z(t)|\,dt \to 0$$

as $\varepsilon_2 \to 0$.

In view of (16) we have thus proved that $|J_\varepsilon(s,x;z) - J(s,x;z)| \to 0$ uniformly over $s \in [0,T]$, $x \in C_R$, for all $R > 0$ and $z \in \tilde{\mathcal{U}}$, hence also that $V_\varepsilon(s,x) \to V(s,x)$. ☐

If the cost function h_0 is continuous, one can use a different regularization, without requiring compactness of U.

Proposition 3.8. *If h is continuous, then the assertion of theorem 3.7 holds.*

Proof. Let $h_{\varepsilon,\delta}$ be the sup-inf convolution of h (in the sense of [27]), that is

$$h_{\varepsilon,\delta}(x) = \sup_{z\in X}\inf_{y \in X}\Big(\frac{|z-y|^2}{2\varepsilon} - \frac{|z-x|^2}{2\delta} + h(y)\Big), \qquad 0<\delta<\varepsilon.$$

It is known that $h_{\varepsilon,\delta}$ is differentiable with continuous derivative, that $\inf_{x\in X} h(x) \le h_{\varepsilon,\delta}(x) \le h(x)$ for all $x \in X$ and that $\lim_{\varepsilon,\delta\to 0^+} h_{\varepsilon,\delta}(x) \to h(x)$ uniformly over $x \in C_R$ (see [27]). Setting $h_{\varepsilon_2} = h_{\varepsilon_2,\varepsilon_2/2}$, proposition 3.1 and the dominated convergence theorem yield

$$\mathbb{E}\int_0^T h_{\varepsilon_2}(z(t))\,dt \xrightarrow{\varepsilon_2\downarrow 0} \mathbb{E}\int_0^T h(z(t))\,dt,$$

which implies that the third term on the right-hand side of (16) converges to 0 as $\varepsilon \to 0$. ☐

Theorem 3.7 (or its variant), together with the following result, allow one to approximate the value function $V(s,x)$ in terms of the solutions of a sequence of Bellman equations.

Proposition 3.9. *Assume that the hypotheses of theorem 3.7 are verified. Assume moreover that h_0 is strictly convex and φ_0 is concave. Then the approximate value function V_ε is the unique mild solution (in the sense of [14]) of the Bellman equation*

$$v_t + \frac{1}{2}\operatorname{Tr}(G_{\varepsilon_1} G^*_{\varepsilon_1} v_{xx}) + \langle Ax, v_x\rangle + H_{0\varepsilon_2}(v_x) = 0, \qquad v(T) = \varphi_{\varepsilon_2},$$

where $H_{0\varepsilon_2}(p) = \sup_{z\in U}(\langle Bz, p\rangle + h_{\varepsilon_2}(z))$.

Proof. Setting

$$\tilde{B}_{\varepsilon_1} : \mathbb{R} \to \mathbb{R}^2, \qquad \tilde{B}_{\varepsilon_1} = \begin{bmatrix} b_0/\sigma_0 \\ \varepsilon_1^{-1} \end{bmatrix},$$

the approximating equation (15) can be rewritten as

$$dY(t) = [AY(t) + G_{\varepsilon_1}\tilde{B}_{\varepsilon_1} z(t)]\,dt + G_{\varepsilon_1}\,dW(t), \quad Y(s) = \bar{x}. \tag{19}$$

The state equation (19), hence also (15), is covered by the FBSDE approach to semilinear PDEs in Hilbert spaces (see e.g. [13]). In order to prove the statement, we shall verify that hypothesis 7.1 in [14] holds true. In particular, G_{ε_1} is Hilbert-Schmidt because $b_1 \in L^2([-r,0],\mathbb{R}_+)$; φ_{ε_2} is Lipschitz because $\varphi_{0\varepsilon_2} \in C^2_c(\mathbb{R})$; $|\tilde{B}_{\varepsilon_1} z|_X$ is bounded for $z \in U$ because $b_1 \in L^2([-r,0],\mathbb{R})$ and U is compact; finally, since h_0 is proper and positive, it is immediate to find $\varepsilon_0 \in]0,1]$, $C \geq 0$ such that $h_{\varepsilon_2}(x) \geq -C$ and $\inf_U h_{\varepsilon_2} \leq C$, for all positive $\varepsilon_2 < \varepsilon_0$.

Since h is strictly convex, for a sufficiently small ε_2 also h_{ε_2} is strictly convex. Therefore we have

$$g_{\varepsilon_2}(p) = \arg\max_{z\in U}(\langle Bz, p\rangle + h_{\varepsilon_2}(z)) = (h'_{\varepsilon_2})^{-1}(B^* p).$$

The claim now follows from [14], theorem 7.2 provided we prove that the closed loop equation

$$dY(t) = [AY(t) + Bg(v_x(t, Y(t)))]\,dt + G_{\varepsilon_1}\,dW(t), \quad Y(s) = \bar{x}. \tag{20}$$

admits a solution. In fact this follows as in Theorem 7.2 of [12]. □

Remark 3.10. In fact the convexity of U implies that $H_{0\varepsilon_2} \in C^1(\mathbb{R})$, and hence that $V_\varepsilon \in C^{0,1}([0,T], X)$, as in corollary 4.7 below.

The above approximations do not give a way to construct approximately optimal strategies for the original problem. In fact, it is well known that the problem of constructing approximately optimal controls from the knowledge of an approximate value function is very hard, and in general unsolved. However, it is possible to construct a (suboptimal) feedback control for which we have some error control, in the sense defined below. For a map $f : [0,T] \times X \to U$ such that the equation

$$dY(t) = AY(t)\,dt + Bf(t, Y(t))\,dt + G\,dW(t), \qquad Y(s) = x,$$

admits a mild solution $Y(t)$, let us set $u_f(t) = f(t, Y(t))$ and $V^f(s,x) = J(s,x;u_f)$. Similarly we define $V^f_\varepsilon(s,x)$. Let us suppose that we can obtain a feedback law $f : [0,T] \times X \to U$, which is approximately optimal for the regularized problem, and let us write $V_\varepsilon \approx V^f_\varepsilon$ to mean that the two values differ by a small constant. Moreover, recall that $V \approx V_\varepsilon \approx V^f_\varepsilon$.

Proposition 3.11. *Let $f(t,x)$ be Lipschitz in x uniformly over t. Then $V^f_\varepsilon(s,x) \to V^f(s,x)$ as $\varepsilon \to 0$.*

Proof. Denote by Y^f and Y^f_ε, respectively, the solutions of the equations

$$\begin{aligned} dY^f(t) &= AY^f(t)\,dt + Bf(t,Y^f(t))\,dt + G\,dW(t) \\ dY^f_\varepsilon(t) &= AY^f_\varepsilon(t)\,dt + Bf(t,Y^f_\varepsilon(t))\,dt + G_\varepsilon\,dW(t), \end{aligned}$$

with $Y^f(s) = Y^f_\varepsilon(s) = x$. Let us assume, without loss of generality, $s = 0$. Let us show that $Y^f_\varepsilon(t) \to Y^f(t)$ in $L^1(\Omega,\mathbb{P})$, hence in probability, for all $t \in [0,T]$: by variation of constants we have

$$\begin{aligned} |Y^f_\varepsilon(t) - Y^f(t)| &\le \int_0^t |e^{(t-s)A}B(f(s,Y^f_\varepsilon(s)) - f(s,Y^f(s)))|\,ds \\ &\quad +\varepsilon\Big|\int_0^t e^{(t-s)A}B_1\,dW_1(s)\Big| \\ &\le \int_0^t m(s)|Y^f_\varepsilon(s) - Y^f(s)|\,ds + \varepsilon\Big|\int_0^t e^{(t-s)A}B_1\,dW_1(s)\Big|, \end{aligned}$$

where $m(s) = |e^{(t-s)A}|\,|B|\,|f|_{\mathrm{Lip}}$. Taking expectation on both sides and recalling that the stochastic convolution has finite mean, Gronwall's lemma yields

$$\mathbb{E}|Y^f_\varepsilon(t) - Y^f(t)| \le \varepsilon N e^{\int_0^T m(s)\,ds} \to 0 \tag{21}$$

as $\varepsilon \to 0$, hence $Y^f_\varepsilon(t) \to Y^f(t)$ in probability for all $t \in [0,T]$.
By the same arguments used in the proof of theorem 3.7 we obtain that

$$\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\mathbb{E}\varphi_{\varepsilon_2}(Y^f_{\varepsilon_1}(T)) = \mathbb{E}\varphi(Y^f(T)). \tag{22}$$

Similarly,

$$\begin{aligned} |h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t))) - h(f(t,Y^f(t)))| &\le |h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t))) - h_{\varepsilon_2}(f(t,Y^f(t)))| \\ &\quad +|h_{\varepsilon_2}(f(t,Y^f(t))) - h(f(t,Y^f(t)))| \end{aligned}$$

and

$$h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t))) \to h_{\varepsilon_2}(f(t,Y^f(t)))$$

in probability as $\varepsilon_1 \to 0$ for all $t \in [0,T]$, because $Y^f_{\varepsilon_1}(t) \to Y^f(t)$ in probability and $h_{\varepsilon_2}\circ f(t,\cdot)$ is continuous. Furthermore, $h_{\varepsilon_2}(f(t,Y^f(t))) \to h(f(t,Y^f(t)))$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ as $\varepsilon_2 \to 0$ because $h_{\varepsilon_2} \to h$ a.e. on $\mathbb{R}$. Since $|h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t)))| \le \sup_{x\in U} h_{\varepsilon_2}(x) < \infty$, then $h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t)))$ is uniformly integrable with respect to ε_1, hence

$$\mathbb{E}\int_0^T \Big|h_{\varepsilon_2}(f(t,Y^f_{\varepsilon_1}(t))) - h_{\varepsilon_2}(f(t,Y^f(t)))\Big|\,dt \to 0 \tag{23}$$

as $\varepsilon_1 \to 0$. Finally,

$$\mathbb{E}\int_0^T \Big|h_{\varepsilon_2}(f(t,Y^f(t)))\,dt - h(f(t,Y^f(t)))\Big|\,dt \to 0 \tag{24}$$

as $\varepsilon_2 \to 0$ by the dominated convergence theorem, taking into account that

$$\mathbb{E}\int_0^T |h_{\varepsilon_2}(f(t,Y^f(t))) - h(f(t,Y^f(t)))|\,dt < \infty,$$

because f is bounded, as follows by the compactness of U. The claim now follows by (22), (23) and (24). $\square$

The previous proposition does not obviously allow one to say that $u(t) = f(t, Y(t))$ is an approximately optimal feedback map for the original problem, as f itself in general depends on ε_1, ε_2. The next proposition gives quantitative estimates on $|V^f(s,x) - V_\varepsilon^f(s,x)|$.

Proposition 3.12. *Assume that $f(t,x)$ is Lipschitz in x uniformly over t, and that φ, h are Lipschitz continuous. Then there exist constants $N = N(|f|_{\mathrm{Lip}})$ and $\delta = \delta(\varepsilon_2)$ such that*

$$|V^f(s,x) - V_\varepsilon^f(s,x)| \leq N\varepsilon_1 + \delta(\varepsilon_2)$$

with $\lim_{\varepsilon_2 \to 0} \delta(\varepsilon_2) = 0$.

Proof. Let us write

$$\begin{aligned}
|V^f - V_\varepsilon^f| &\leq \mathbb{E}|\varphi_{\varepsilon_2}(Y_{\varepsilon_1}^f(T)) - \varphi(Y^f(T))| \\
&\quad + \mathbb{E}\int_0^T \Big| h_{\varepsilon_2}(f(t, Y^f(t)))\, dt - h(f(t, Y^f(t)))\Big|\, dt \\
&= I_1 + I_2,
\end{aligned}$$

and

$$\begin{aligned}
I_1 &\leq |\mathbb{E}[\varphi_{\varepsilon_2}(Y_{\varepsilon_1}^f(T)) - \varphi_{\varepsilon_2}(Y^f(T))]| + |\mathbb{E}[\varphi_{\varepsilon_2}(Y^f(T)) - \varphi(Y^f(T))]| \\
&= I_{11} + I_{12}.
\end{aligned}$$

Then

$$I_{11} \leq |\varphi_{\varepsilon_2}|_{\mathrm{Lip}} \mathbb{E}|Y_{\varepsilon_1}^f(T) - Y^f(T)| \leq |\varphi|_{\mathrm{Lip}}\, \varepsilon_1 e^{\int_0^T m(s)\, ds},$$

where we used the fact that mollification does not increase the Lipschitz constant. We also have

$$\begin{aligned}
I_{12} &\leq \mathbb{E}\Big[|\varphi_{0\varepsilon_2}(Y_0^f(T)) - \varphi_0(Y_0^f(T))|; |Y_0^f(T)| \leq \varepsilon_2^{-1}\Big] \\
&\quad + \mathbb{E}\Big[|\varphi_{0\varepsilon_2}(Y_0^f(T)) - \varphi_0(Y_0^f(T))|; |Y_0^f(T)| > \varepsilon_2^{-1}\Big] \\
&\leq \delta_1(\varepsilon_2) + \delta_2(\varepsilon_2),
\end{aligned}$$

where

$$\delta_1(\varepsilon_2) = \sup_{|x| \leq 1/\varepsilon_2} |\varphi_{0\varepsilon_2}(x) - \varphi_0(x)| < \infty,$$

as $\varphi_{0\varepsilon_2}$ converges to φ_0 uniformly on compact sets, and $\delta_2(\varepsilon_2)$ is defined as follows: there exist K_1, $K_2 \geq 0$ such that

$$\mathbb{E}\Big[|\varphi_{0\varepsilon_2}(Y_0^f(T)) - \varphi_0(Y_0^f(T))|; |Y_0^f(T)| > \varepsilon_2^{-1}\Big] \leq \mathbb{E}\Big[K_1 + K_2|Y_0^f(T)|^m; |Y_0^f(T)| > \varepsilon_2^{-1}\Big],$$

and

$$\begin{aligned}
Y^f(T) &\leq e^{TA}x + \int_0^T e^{(T-t)A} B f(t, Y^f(t))\, dt + \int_0^T e^{(T-t)A} G\, dW(t) \\
&\leq e^{TA}x + \int_0^T e^{(T-t)A} R\, dt + \int_0^T e^{(T-t)A} G\, dW(t) =: \mu_2 + Z_1.
\end{aligned}$$

Similarly,

$$Y^f(T) \geq e^{TA}x + \int_0^T e^{(T-t)A} r\, dt + \int_0^T e^{(T-t)A} G\, dW(t) =: \mu_1 + Z_1,$$

where $\mu_1,\ \mu_2 \in X$, $U \subseteq [r,R]$, and Z_1 is a centered X-valued Gaussian random variable. Denoting by Z the $\mathbb{R}$-valued components of Z_1, we have

$$Y_0^f(T) - Z \leq (\mu_2)_0, \qquad Y_0^f(T) - Z \geq (\mu_1)_0,$$

hence $|Y_0^f(T) - Z| \leq |(\mu_1)_0| \vee |(\mu_2)_0| =: \mu$, or equivalently $|Y_0^f(T)| \leq \mu + |Z|$. In particular, Z is a centered Gaussian random variable. Then

$$\begin{aligned}\mathbb{E}\Big[K_1 + K_2|Y_0^f(T)|^m; |Y_0^f(T)| > \varepsilon_2^{-1}\Big] &\leq K_1\mathbb{P}(|Z| + \mu > \varepsilon_2^{-1}) \\ &\quad + K_2\mathbb{E}\Big[(|Z|+\mu)^m; |Z| + \mu > \varepsilon_2^{-1}\Big] \\ &=: \delta_2(\varepsilon_2).\end{aligned}$$

Note that $\delta_2(\varepsilon_2) \to 0$ as $\varepsilon_2 \to 0$ since $\mathbb{E}(|Z|+\mu)^m < \infty$.

We have

$$\begin{aligned} I_2 &\leq \Big|h_{\varepsilon_2}(f(\cdot, Y_{\varepsilon_1})) - h_{\varepsilon_2}(f(\cdot, Y))\Big|_{L^1_T} + \Big|h_{\varepsilon_2}(f(\cdot, Y)) - h(f(\cdot, Y))\Big|_{L^1_T} \\ &= I_{21} + I_{22},\end{aligned}$$

where L^1_T stands for $L^1(\Omega \times [0,T], d\mathbb{P} \times dt)$. Recalling again that mollification does not increase the Lipschitz constant, we also have

$$I_{21} \leq |h|_{\mathrm{Lip}}|f|_{\mathrm{Lip}}|Y_{\varepsilon_1} - Y|_{L^1_T} \leq |h|_{\mathrm{Lip}}|f|_{\mathrm{Lip}}\varepsilon_1 N \int_0^T e^{\int_0^t m(s)\,ds}\,dt.$$

Finally, using again the uniform convergence on compact sets of mollified continuous functions,

$$I_{22} \leq T\delta_3(\varepsilon_2). \qquad \square$$

3.3 An example with explicit solutions

In this subsection we study in detail the optimal advertising problem with linear reward and quadratic cost. In particular, we shall assume $h(z) = -\beta z_0^2$ and $\varphi(x) = \gamma x_0$, with $\beta,\ \gamma > 0$. In [16] we proved that a solution (in integral sense) of the HJB equation (14) is of the type

$$v(t,x) = \langle w(t), x\rangle + c(t), \qquad t \in [0,T],\ x \in X,$$

where $w = (w_0, w_1) : [0,T] \to X$ and $c : [0,T] \to \mathbb{R}$ are given by

$$\begin{cases} w_0'(t) + a_0 w_0(t) + \displaystyle\int_{-r}^0 a_1(\xi) w_1(t,\xi)\,d\xi = 0, & t \in [0,T[\\ w_0(T) = \gamma, & \\ w_1(t,\xi) = w_0(t-\xi)\chi_{[0,T]}(t-\xi), & \\ c(t) = \displaystyle\int_t^T \frac{(\langle B, w(s)\rangle^+)^2}{4\beta}\,ds, & t \in [0,T]. \end{cases} \tag{25}$$

Moreover, the optimal strategy is

$$z^*(t) = \frac{\langle B, v_x(t)\rangle^+}{2\beta} = \frac{\langle B, w(t)\rangle^+}{2\beta}, \qquad t \in [0,T] \tag{26}$$

(see [16] for more details).

We extend now the analysis of this specific situation. Let us begin with a rather explicit characterization of the optimal trajectory, which could be numerically approximated simply by solving a linear ODE with delay. In particular, let $w = (w_0, w_1)$ be the solution of (25). Then, setting $z^*(t) = \frac{1}{2\beta}\langle B, w(t)\rangle^+$, the optimal trajectory is the $\mathbb{R}$-valued component Y_0 of the (mild) solution of the abstract SDE

$$dY(t) = [AY(t) + Bz^*(t)]\,dt + G\,dW(t),$$

which is given by

$$Y(t) = e^{tA}Y(0) + \int_0^t e^{(t-s)A}Bz^*(s)\,ds + \int_0^t e^{(t-s)A}G\,dW(s).$$

In particular Y is a X-valued Gaussian process with mean and covariance operator

$$\mu_t = e^{tA}Y(0) + \int_0^t e^{(t-s)A}Bz^*(s)\,ds, \quad Q_t = \int_0^t e^{(t-s)A}GG^*e^{(t-s)A^*}\,ds,$$

respectively. It follows that Y_0 is a Gaussian process itself with mean

$$\begin{aligned} \mathbb{E}Y_0(t) &= \langle \mu_t, e_1\rangle_X = \left\langle e^{tA}Y(0), e_1\right\rangle_X + \left\langle \int_0^t e^{(t-s)A}Bz^*(s)\,ds, e_1\right\rangle_X \\ &= \left\langle Y(0), e^{tA^*}e_1\right\rangle_X + \int_0^t \left\langle Bz^*(s), e^{(t-s)A^*}e_1\right\rangle_X ds, \end{aligned} \tag{27}$$

where $e_1 = (1, 0) \in X$.
Since $Y(0)$ is given as in Proposition 2.1 and $Bz^*(\cdot)$ is also easy to compute (z^* is one dimensional and B is just multiplication by a fixed vector in X), we are left with the problem of computing $e^{tA^*}e_1$. However, as one can prove by a direct calculation, the semigroup e^{tA^*} is given by

$$e^{tA^*}(x_0, x_1(\cdot)) = \Big(\phi(t), \phi(t+\xi)|_{\xi\in[-r,0]}\Big),$$

where $\phi(\cdot)$ solves the linear ODE with delay

$$\begin{cases} \dfrac{d\phi(t)}{dt} = a_0\phi(t) + \displaystyle\int_{-r}^0 a_1(\xi)\phi(t+\xi)\,d\xi, \quad 0 \le t \le T \\ \phi(0) = x_0; \quad \phi(\xi) = x_1(\xi) \;\; \forall \xi \in [-r, 0]. \end{cases} \tag{28}$$

Therefore $e^{tA^*}e_1$ is given by $(\phi(t), \phi(t+\xi)|_{\xi\in[-r,0]})$, where ϕ solves (28) with initial condition $x_0 = 1$, $x_1(\cdot) = 0$. Such ϕ can be computed numerically by discretizing (28), and then $\mathbb{E}Y_0(t)$ can be obtained by approximating the integrals in (27) with finite sums.

Analogously one can write the variance of the optimal trajectory in such a way that it can be easily approximated by numerical methods. In particular, one has

$$\begin{aligned} \operatorname{Var} Y_0(t) &= \langle Q_t e_1, e_1\rangle = \left\langle \int_0^t e^{(t-s)A}GG^*e^{(t-s)A^*}\,ds\,e_1, e_1\right\rangle \\ &= \int_0^t \left\langle e^{(t-s)A}GG^*e^{(t-s)A^*}e_1, e_1\right\rangle ds \\ &= \int_0^t \left\langle Ge^{(t-s)A^*}e_1, Ge^{(t-s)A^*}e_1\right\rangle ds \\ &= \int_0^t \left|Ge^{(t-s)A^*}e_1\right|^2 ds. \end{aligned} \tag{29}$$

Setting $\psi(s) = e^{(t-s)A^*}e_1$, which can be approximated as indicated before, one finally has

$$\operatorname{Var} Y_0(t) = \sigma^2 \int_0^t \psi(s)^2\,ds.$$

One can also perform simple comparative statics on the value function. For instance we can compute explicitly its sensitivity with respect to the (maximal) delay r:

$$\begin{aligned}
\frac{\partial V}{\partial r}(t,x;r) &= \frac{\partial}{\partial r}\langle w_1(t), x_1\rangle_{L^2([-r,0])} + \frac{\partial c}{\partial r}(t;r) \\
&= w_1(t,-r)x_1(-r) + \frac{1}{2\beta}\int_t^T \langle B, w(s)\rangle \frac{\partial}{\partial r}\left(\int_{-r}^0 b_1(\xi)w_1(s,\xi)\,d\xi\right) ds \\
&= w_1(t,-r)x_1(-r) + \frac{b_1(-r)}{2\beta}\int_t^T \langle B, w(s)\rangle w_1(s,-r)\,ds,
\end{aligned}$$

where we have used the fact that $\langle B, w(t)\rangle^+ = \langle B, w(t)\rangle$.
Note that in the above expression everything can be computed explicitly, as soon as we fix the delay kernel b_1. Let us consider, as an example, the special case of $b_1(\xi) = b_1\chi_{[-r,0]}(\xi)$, where on the right-hand side, with a slight abuse of notation, b_1 is a positive constant. One has

$$\frac{\partial V}{\partial r}(t,x;r) = w_1(t,-r)x_1(-r) + \frac{b_1}{2\beta}\int_t^T w_1(s,-r)\Big(b_0 w_0(s) + b_1\int_{-r}^0 w_1(s,\xi)\,d\xi\Big)\,ds.$$

Furthermore, if we consider the special case of delay in the control only, that is $a_1(\cdot) = 0$, we obtain, after some calculations,

$$\frac{\partial V}{\partial r}(t,x;r) = \gamma e^{a_0(T-t+r)}x_1(-r) - \frac{b_1}{4\beta a_0}\gamma^2 e^{a_0 r}\Big(b_0 - \frac{b_1}{a_0}(1-e^{a_0 r})\Big)(1-e^{2a_0(T-t)}),$$

for $t \in [r,T]$.

In the special case of $a_1(\cdot) \equiv 0$ an explicit solution of (25) is easily obtained. This solvability in closed form then "propagates" to other quantities of interest. In fact, note that (25) reduces to

$$\begin{cases} w_0'(t) + a_0 w_0(t) = 0 \\ w_0(T) = \gamma, \end{cases} \tag{30}$$

yielding

$$w_0(t) = \gamma e^{(T-t)a_0},$$

and therefore

$$w_1(t,\xi) = \gamma e^{(T-(t+\xi))a_0}\,\chi_{[0,T]}(t+\xi) \quad c(t) = \int_t^T \frac{\langle B, w(s)\rangle^2}{4\beta}\,ds.$$

That is, the last three formulae explicitly give a solution of the HJB equation (14) in our specific case.

As a consequence we can also determine the unique optimal feedback control z^* as follows:

$$z^*(t) = \frac{\langle B, v_x\rangle^+}{2\beta} = \frac{\langle B, w(t)\rangle^+}{2\beta} = \frac{\gamma e^{(T-t)a_0}}{2\beta}\left[b_0 + \int_{-r}^0 b_1(\xi)e^{-a_0\xi}\chi_{[0,T]}(t+\xi)\,d\xi\right]. \tag{31}$$

The optimal trajectory can be characterized in a completely similar way as above, with the difference that now we can explicitly write:

$$e^{tA^*}e_1 = (e^{a_0 t}, e^{a_0 t+\xi}|_{\xi\in[-r,0]}),$$

hence simplifying (27) in the present case. Even simpler is the expression for the variance of the optimal trajectory, which can be obtained by (29):

$$\mathrm{Var}\,Y_0(t) = \sigma^2\int_0^t e^{2a_0(t-s)}\,ds = \frac{\sigma^2}{2a_0}(e^{2a_0 t}-1).$$

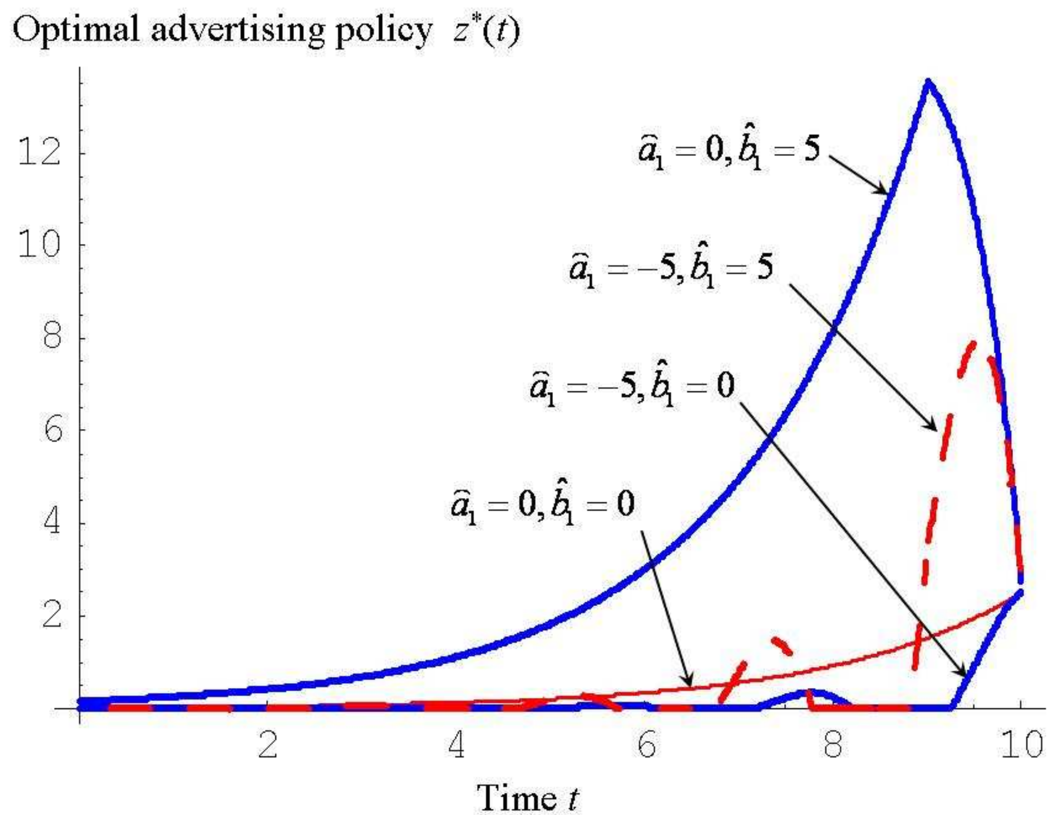

Figure 1: Optimal advertising policy in four different "churn" settings

Sharp characterizations of the optimal advertising trajectory as well as the resulting expected profit functions in the case of linear reward and quadratic cost function allow for interesting observations regarding the importance of the proper accounting for the memory effects in planning the advertising campaign. Figure 1 displays the optimal advertising spending rates $z^*(t)$, as expressed by (26), with $a_1(\xi) = \hat{a}_1 e^{-|\xi|/\delta_a}$ and $b_1(\xi) = \hat{b}_1 e^{-|\xi|/\delta_b}$ in four different settings: $a_1 = b_1 = 0$ ("no churn"), $a_1 = -5$, $b_1 = 0$ ("goodwill churn"), $a_1 = 0$, $b_1 = 5$ ("advertising churn"), $a_1 = -5$, $b_1 = 5$ ("goodwill-advertising churn"). Note that in the absence of churn, the optimal advertising trajectory, as implied by (31), is a monotone function of time with $z^*(t) = \gamma b_0/(2\beta)$. While the advertising rates are similar in all settings in the beginning of the pre-launch period as well as right before the product launch time T, the details of advertising policies differ dramatically in the middle of the pre-launch period. For example, in the presence of a strong "goodwill churn" the optimal advertising trajectory takes a characteristic "impulse" shape, while in the strong "advertising churn" setting the optimal advertising spending quickly builds up a strong goodwill level in the middle of the pre-launch region, slowing down significantly right before the product launch. When the presence of both types of "churn" is pronounced, the optimal advertising policy is represented by a set of advertising sprees with rapidly growing intensity. Figure 2 illustrates how the strong influence of memory effects on the shape of optimal advertising policies translates into performance differences between the optimal advertising policies and the policies which neglect the presence of advertising delays. In this figure we plot the relative difference

$$\frac{V(0,x) - V^0(0,x)}{V(0,x)}$$

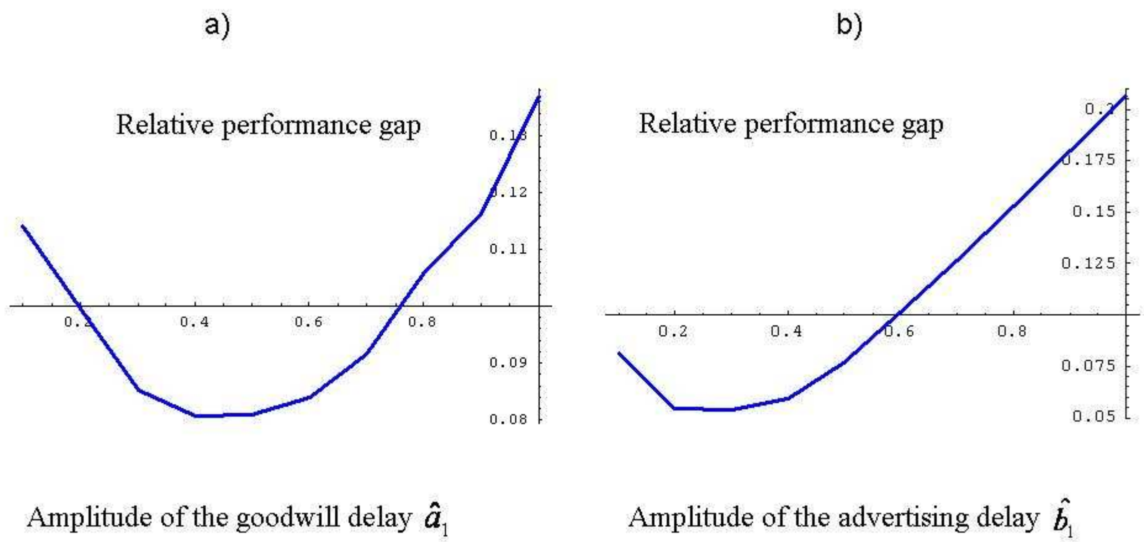

Figure 2: Relative performance gap of the "memoryless" advertising policy

between the optimal expected profit function $V(0,x)$ and the expected profit value $V^0(0,x)$ obtained by applying, for $t \in [0,T]$, the advertising policy $z^0(t) = \gamma b_0 e^{(T-t)a_0}/(2\beta)$ optimal in the absence of memory effects (i.e. in the setting $a_1 = b_1 = 0$). This relative difference is plotted as a function of the amplitude of the "goodwill churn" term a_1 (Figure 2a), and as a function of the amplitude of the "advertising churn" term b_1 (Figure 2b). The initial goodwill conditions were selected as $x_0 = 10$ and $x_1(\xi) = x_0 e^{-|\xi|}$ for $\xi \in [-r,0]$ and the advertising history $z_0(\xi)$ was set equal to 0 for $\xi \in [-r,0]$. We observe that the relative loss of efficiency associated with the use of the "memoryless" policy $z^0(t)$ can be quite significant – in the examples we use it exceeds 5% and can be as high as 20% in settings with strong "churn" effects.

4 The case of delay in the state term only

In this section we consider a model for the dynamics of goodwill with forgetting, but without lags in the effect of advertising expenditure (carryover), i.e. with $b_1(\cdot) = 0$ in (1). An analysis of this model was sketched in [16], where only an abstract existence result was given. Here we present a more refined result (see theorem 4.5 below) and obtain some qualitative properties of the value function, together with a characterization of the optimal strategies in terms of the value function in two specific cases. In particular, theorem 4.5 formulates sufficient conditions ensuring that the value function is the unique solution (in a suitable sense) of the associated Bellman equation and that the optimal advertising policy is of the feedback type. Let us recall once again that such a situation is not possible in the more general case discussed in the previous section. Moreover, in the case of linear cost function, the optimal control takes a particularly simple "bang-bang" form (Corollary 4.10).

Let us also mention that for stochastic control problem with delay terms in the state variable one can apply both the approach of Hamilton-Jacobi equations in L^2 spaces developed by Goldys and Gozzi [15], and the forward-backward SDE approach of Fuhrman and Tessitore [12]. We follow here the first approach, showing that both the value function and the optimal advertising policy can be characterized in terms of the solution of a Bellman equation in infinite dimensions.

We assume, for the sake of simplicity, that the goodwill evolves according to the following equation, where the distribution of the forgetting factor is concentrated on a point:

$$\begin{cases} dy(t) = [a_0 y(t) + a_1 y(t-r) + b_0 z(t)]\,dt + \sigma\,dW_0(t), \quad 0 \le s \le t \le T \\ y(s) = x_0; \quad y(s+\xi) = x_1(\xi), \;\; \xi \in [-r,0]. \end{cases} \tag{32}$$

The following standard infinite dimensional Markovian reformulation of this dynamics will turn out to be useful.

Let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ as

$$\mathcal{A} : (x_0, x_1(\cdot)) \mapsto (a_0 x_0 + a_1 x_1(-r), x_1'(\cdot)), \qquad D(\mathcal{A}) = \mathbb{R} \times W^{1,2}([-r,0];\mathbb{R}).$$

It is well-known (see e.g. [8]) that $\mathcal{A}$ is the generator of a strongly continuous semigroup $S(t)$ on X. More precisely, one has

$$S(t)(x_0, x_1) = (u(t), u(t+\xi)|_{\xi\in[-r,0]}),$$

where $u(\cdot)$ is the solution of the deterministic delay equation

$$\begin{cases} \dfrac{du(t)}{dt} = a_0 u(t) + a_1 u(t-r), \quad 0 \le t \le T \\ u(0) = x_0; \quad u(\xi) = x_1(\xi), \;\; \xi \in [-r,0]. \end{cases} \tag{33}$$

Furthermore, set $\bar{z} = (\sigma^{-1} b_0 z, z_1(\cdot))$, with $z_1(\cdot)$ a fictitious control taking values in $L^2([-r,0],\mathbb{R})$, and define $G : X \to X$ as

$$G : (x_0, x_1(\cdot)) \mapsto (\sigma x_0, 0).$$

Let W_1 be a cylindrical Wiener process taking values in $L^2([-r,0],\mathbb{R})$, so that $W = (W_0, W_1)$ is an X-valued cylindrical Wiener process.

Chojnowska-Michalik [6] proved the following equivalence result.

Lemma 4.1. *Let $Y = (Y_0, Y_1)$ be the unique mild solution of the following stochastic evolution equation on X:*

$$\begin{cases} dY(t) = (\mathcal{A}Y(t) + G\bar{z}(t))\,dt + G\,dW(t), \quad 0 \le s \le t \le T, \\ Y(s) = x. \end{cases} \tag{34}$$

Then $Y_0(t)$ solves the stochastic delay equation (32).

Define $h : X \to \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ as

$$\begin{aligned} h(x_0, x_1) &= -h_0(\sigma b_0^{-1} x_0) \\ \varphi(x_0, x_1) &= \varphi_0(x_0). \end{aligned}$$

Then we have, thanks to lemma 4.1,

$$J(s,x;z) = \mathbb{E}_{s,x}\left[\varphi(Y(T)) + \int_t^T h(\bar{z}(s))\,ds\right],$$

and

$$V(s,x) = \sup_{\bar z \in \mathcal{Z}} J(s,x;\bar z), \tag{35}$$

where $\mathcal{Z}$ denotes the set of all strategies $z : [0,T] \to \tilde U \times L^2([-r,0],\mathbb{R})$ adapted to the filtration generated by Y, and $\tilde U$ is the image of U under the action of the map $x \mapsto \sigma^{-1} b_0 x$.

We can now prove some qualitative properties of the value function.

Proposition 4.2. *If φ_0 is concave and h_0 is convex, then the value function $V(s,x)$ is concave with respect to x.*

Proof. Identical to the proof of proposition 3.2, thus omitted. $\square$

In the following proposition we use the ordering in X defined right before Proposition 3.6.

Proposition 4.3. *Let $a_1 \geq 0$ and φ_0 be increasing. Then the value function $V(s,x)$ is increasing with respect to x. Moreover, if φ_0 is strictly increasing, then the value function $V(s,x)$ is strictly increasing with respect to x, and $V(s,x^1) = V(s,x^2)$ if and only if $x^1 = x^2$.*

Proof. Let $x^1 \geq x^2$ in the sense just defined. One has

$$\begin{aligned} J(s,x^1;z) - J(s,x^2;z) &= \mathbb{E}\Big[\varphi(y^{s,x^1,z}(T)) - \varphi(y^{s,x^2,z}(T))\Big] \\ &= \mathbb{E}\Big[\varphi(e^{\mathcal{A}(T-s)}x^1 + \zeta) - \varphi(e^{\mathcal{A}(T-s)}x^2 + \zeta)\Big], \end{aligned}$$

where $\zeta := \int_s^T e^{\mathcal{A}(T-t)} G\bar z(t)\,dt + \int_s^T e^{\mathcal{A}(T-t)} G\,dW(t)$. The assumption $a_1 \geq 0$ together with (33) implies that the semigroup generated by $\mathcal{A}$ is positivity preserving, i.e. $x^1 \geq x^2$ implies $e^{\mathcal{A}(T-s)}x^1 \geq e^{\mathcal{A}(T-s)}x^2$. Therefore, by the monotonicity of φ_0, one also has $\varphi(e^{\mathcal{A}(T-s)}x^1 + \zeta) \geq \varphi(e^{\mathcal{A}(T-s)}x^2 + \zeta)$ a.s., hence $J(s,x^1;z) \geq J(s,x^2;z)$, and finally $V(s,x^1) = \sup_{z\in\mathcal{Z}} J(s,x^1;z) \geq \sup_{z\in\mathcal{Z}} J(s,x^2;z) = V(s,x^2)$. The other assertions follow analogously, using (33). $\square$

Remark 4.4. In the above proof the positivity preserving property of the semigroup $e^{t\mathcal{A}}$ is crucial, and the assumption $a_1 \geq 0$ is "sharp" in the following sense: if $a_1 < 0$, one can find $x > 0$ such that $e^{t\mathcal{A}}$ *inverts* the sign, i.e. $e^{t\mathcal{A}}x < 0$.

Moreover, under the assumptions of the theorem, the value function is increasing with respect to the real valued component of the initial datum. By this we mean that given $x^1 \geq x^2$ with $x_0^1 > x_0^2$ and $x_1^1 = x_1^2$ a.e., then $V(s,x^1) > V(s,x^2)$. Therefore one also has $D_0^- V \geq 0$. The subdifferential can be replaced by the derivative if we can guarantee that V is continuously differentiable with respect to x_0. Conditions for the continuous differentiability of V with respect to x are given in proposition 4.6 below.

In contrast with the general case considered in the previous section, if delay enters only the state term, then it is possible to uniquely solve the associated Bellman equation, and thus to characterize the value function and construct optimal strategies. The following result, which relies on [15], gives precise conditions for the above assertions to hold.

Theorem 4.5. *Assume that h_0 is convex, let $H_0 : \mathbb{R} \ni p \mapsto \sup_{u\in U}(pb_0u - h_0(u))$, and suppose that $a_0 < -a_1 < \sqrt{\gamma^2 + a_0^2}$, where $\gamma \coth\gamma = a_0$, $\gamma \in]0,\pi[$. Then the value function $V(s,x)$ coincides μ-a.e. with the mild solution in $L^2(X,\mu)$ (in the sense of [15]) of the equation*

$$\begin{cases} \partial_t v + \dfrac{1}{2}\sigma^2 \partial_0^2 v + (a_0 x_0 + a_1 x_1(-r))\partial_0 v + \displaystyle\int_{-r}^0 x_1'(\xi)\partial_1 v(\xi)\,d\xi + H_0(\partial_0 v) = 0 \\ v(T,x_0,x_1) = \varphi_0(x_0), \end{cases} \tag{36}$$

where μ is a measure of full support on X. Moreover, the optimal strategy admits the feedback representation

$$z^*(t) \in \sigma b_0^{-1} D_0^- H(\sigma \partial_0 v(t, Y_0^*(t), Y_1^*(t)), \tag{37}$$

with H defined in (41), provided there exists a solution $Y_0^*(t)$, $Y_1^*(t)$ of the closed-loop differential inclusion

$$\begin{cases} dY_0(t) \in \big[a_0 Y_0(t) + a_1 Y_0(t-r) + \sigma D_0^- H(\sigma \partial_0 v(t, Y_0^*(t), Y_1^*(t))\big] \, dt + \sigma \, dW_0(t) \\ dY_1(t)(\xi) = \dfrac{d}{d\xi} Y_1(t)(\xi). \end{cases} \tag{38}$$

Proof. By the usual heuristic application of the dynamic programming principle one can associate to the control problem (35) the following Hamilton-Jacobi-Bellman equation on X:

$$\begin{cases} v_t + \dfrac{1}{2} \operatorname{Tr}(GG^* v_{xx}) + \langle \mathcal{A} x, v_x \rangle + H_0(v_x) = 0, \quad 0 \le t \le T, \\ v(T, x) = \varphi(x), \end{cases} \tag{39}$$

which coincides, after some calculations, with (36). Note that the Hamiltonian H_0 can be regarded as a function of X in $\mathbb{R}$ and can be equivalently written as

$$\begin{aligned} H_0(p) = H_0(p_0) \quad &= \sup_{z \in \tilde{U} \times L^2([-r,0],\mathbb{R})} \Big(\langle Gz, p \rangle + h(z) \Big) \\ &= \sup_{z_0 \in \tilde{U}} \Big(\sigma z_0 p_0 - h_0(\sigma b_0^{-1} z_0) \Big). \end{aligned} \tag{40}$$

In order to apply the results of [15], we also need to define

$$H(q) = H(q_0) = H_0(\sigma^{-1} q_0) = \sup_{z_0 \in \tilde{U}} \Big(z_0 q_0 - h_0(\sigma b_0^{-1} z_0) \Big). \tag{41}$$

Since h_0 is bounded from below, (41) implies that H is Lipschitz continuous (in $\mathbb{R}$ and in X). Moreover, the assumption on a_0, a_1 and assumption (i) of section 2 imply that the uncontrolled version of (34), i.e.

$$dY(t) = AY(t) \, dt + G \, dW(t), \tag{42}$$

admits a unique non-degenerate invariant measure μ on X (see [8]), which is Gaussian with mean zero and covariance operator $Q_\infty = \int_0^\infty e^{sA} GG^* e^{s^* A} \, ds$. In particular, the restriction of μ on the $\mathbb{R}$-valued component of X has a density $\rho(x) = \frac{1}{\nu\sqrt{2\pi}} e^{-|x|^2/2\nu^2}$ for some $\nu > 0$. This implies that $\varphi \in L^2(X, \mu)$: in fact,

$$\int_X |\varphi(x)|^2 \, \mu(dx) = \int_{\mathbb{R}} |\varphi_0(x)|^2 \rho(x) \, dx \le K \int_{\mathbb{R}} (1 + |x|)^m e^{-|x|^2/2\nu^2} \, dx < \infty.$$

Therefore, theorems 3.7 and 5.7 of [15] yield the existence and uniqueness of a solution in $L^2(X, \mu)$ of (39), or equivalently of (36), which coincides μ-a.e. with the value function V. Finally, observing that the maximum in (41) is reached by $D_0^- H(q_0)$ (setting, if needed, $h_0(x) = +\infty$ for $x \notin U$), a slight modification of the proof of theorem 5.7 in [15] shows that the optimal strategy is given by $\bar{z}_0^*(t) \in D_0^- H(\sigma \partial_0 v(t, Y_0^*(t), Y_1^*(t))$, where $Y^* = (Y_0^*, Y_1^*)$ is a solution (if any) of the stochastic differential inclusion (38). The relation $z^*(t) = \sigma b_0^{-1} \bar{z}_0^*(t)$ thus completes the proof. □

Let us briefly comment on the previous result: the HJB equation (36) is "genuinely" infinite dimensional, i.e. it reduces to a finite dimensional one only in very special cases. For example, by the results in [26], (36) reduces to a finite dimensional PDE if and only if

$a_0 = -a_1$. However, under this assumption, we cannot guarantee the existence of a non-degenerate invariant measure for the Ornstein-Uhlenbeck semigroup associated to (42). Even more extreme would be the situation of distributed forgetting time: in this case the HJB equation is finite dimensional only if the term accounting for distributed forgetting vanishes altogether. Moreover, note that if a_1 is negative, i.e. it can be interpreted as a deterioration factor, the assumption of the theorem says that a_1 cannot be "much more negative" than a_0. On the other hand, if a_1 is positive, then the improvement effect as measured by a_1 cannot exceed the deterioration effect as measured by $|a_0|$. In essence, the condition on a_0, a_1, which is needed to ensure existence of an invariant measure for equation (42), does not impose severe restrictions on the dynamics of goodwill.

If the data of the problem are smoother, a different approach allows one to obtain regularity of the value function.

Proposition 4.6. *Assume that $\varphi_0 \in C^1(\mathbb{R})$, $|\varphi_0'(x)| \leq K(1+|x|)^m$, and $H_0 \in C^1(\mathbb{R})$. Then $V \in C^{0,1}([0,T] \times X)$.*

Proof. Follows by the regularity results for solutions of semilinear partial differential equations in Hilbert spaces obtained through the FBSDE approach. In particular, denoting by C a positive constant, boundedness of U implies that $|b_0 z| < C$, $|h_0(z)| \leq K(1+|z|)^m < C$, and finally H_0 is Lipschitz as follows by

$$\begin{aligned} |H_0(p) - H_0(q)| &= |\sup_{z \in U}(\langle Bp, z\rangle - h_0(z)) - \sup_{z \in U}(\langle Bp, z\rangle - h_0(z))| \\ &\leq \sup_{z \in U} |\langle (p-q), B^* z\rangle| \leq C|p-q|. \end{aligned}$$

Then all hypotheses of [13], theorem 4.3.1, are satisfied, which yields the claim. □

Corollary 4.7. *Let φ_0 be as in proposition 4.6 and h_0 strictly convex. Then $V \in C^{0,1}([0,T] \times X)$.*

Proof. Since convexity implies continuity in the interior of the domain, then $h_0(U)$ is bounded. Extending h_0 as $h_0(x) = +\infty$ for $x \notin U$, h_0 is clearly 1-coercive, hence H is convex and finite on the whole $\mathbb{R}$ ([21], prop. E.1.3.8). The strict convexity of h_0 implies that H_0 is continuously differentiable in the interior of its domain, i.e. on $\mathbb{R}$ ([21], thm. E.4.1.1). Then the smoothness of V follows again by [13], theorem 4.3.1. □

Remark 4.8. We should also mention that in the framework of the FBSDE approach to HJB equations ([13]), if the Hamiltonian H and the terminal condition φ satisfy some smoothness and boundedness conditions, then we do not need the assumption about the existence of an invariant measure for the uncontrolled state equation. The approach used above ([15]), while requiring the existence of the above mentioned invariant measure, allows for more singular data (for instance one could choose $\varphi_0(x) = -M$, $M \gg 0$, for $x \in \mathbb{R}_-$, and $\varphi_0(x) \geq 0$ for $x \in \mathbb{R}_+$).

In general, obtaining explicit expressions of the value function V trying to solve (36) is impossible. However, under specific assumptions on the model we can obtain stronger characterizations, at least from a qualitative point of view, of the value function and/or of the optimal strategy.

Corollary 4.9. *Assume that $h_0(x) = \beta x^2$ and $U = [0,R]$, $R < \infty$. Then the optimal strategy is given by*

$$z^*(t) = \begin{cases} 0, & D_0V^* < 0 \\ \dfrac{b_0 D_0 V^*}{2\beta}, & 0 \leq D_0V^* \leq 2b_0^{-1}\beta R \\ R, & D_0V^* > 2b_0^{-1}\beta R, \end{cases} \tag{43}$$

where $V^* := V(t, Y_0^*(t), Y_1^*(t))$.

Proof. One has

$$\begin{aligned} H_0(p) &= \sup_{0\le z_0\le \tilde R}\Big(\langle Gp, z\rangle + h(z)\Big) = \sup_{0\le z_0\le \tilde R}(\sigma p_0 z_0 - \tilde\beta z_0^2) \\ &= \frac{(\sigma p_0)^2}{4\tilde\beta} I_{\{0\le p_0\le \frac{2\tilde\beta\tilde R}{\sigma}\}} + (\sigma p_0 \tilde R - \tilde\beta \tilde R^2) I_{\{p_0 > \frac{2\tilde\beta\tilde R}{\sigma}\}} \end{aligned}$$

where $\tilde R = \sigma^{-1} b_0 R$ and $\tilde\beta := \sigma^2 b_0^{-2}\beta$. Therefore

$$H(q) = q_0^2/4\tilde\beta\, I_{\{0\le q_0\le 2\tilde\beta\tilde R\}} + (q_0\tilde R - \tilde\beta\tilde R^2)\, I_{\{q_0>2\tilde\beta\tilde R\}}$$

and

$$DH(q) = q_0/2\tilde\beta I_{\{0\le q_0\le 2\tilde\beta\tilde R\}} + \tilde R I_{\{q_0>2\tilde\beta\tilde R\}}.$$

Theorem 4.5 now yields (43). □

Note that whenever φ_0 is increasing, we get $D_0^- V^* \ge 0$, hence the optimal control is either linear in $D_0 V^*$ or constant for $D_0 V^*$ over a threshold.

Corollary 4.10. *Assume that* $h_0(x) = \beta x$. *Then the optimal strategy is of the bang-bang type and is given by*

$$z^*(t) = \begin{cases} 0, & D_0 V^* < \sigma b_0^{-2}\beta \\ \rho, & D_0 V^* = \sigma b_0^{-2}\beta \\ R, & D_0 V^* > \sigma b_0^{-2}\beta, \end{cases}$$

where ρ *is an arbitrary real number.*

Proof. Setting $\tilde R = \sigma^{-1} b_0 R$ and $\tilde\beta := \sigma^2 b_0^{-2}\beta$, one has

$$H_0(p) = \sup_{0\le z_0\le\tilde R}(\sigma p_0 z_0 - \tilde\beta z_0) = (\sigma p_0 - \tilde\beta)\tilde R\, I_{\{p_0>\tilde\beta/\sigma\}}$$

and $H(q) = (q_0 - \tilde\beta)\tilde R\, I_{\{q_0>\tilde\beta\}}$, thus

$$D^- H(q) = \mathbb{R}\, I_{\{q_0=\tilde\beta\}} + \tilde R\, I_{\{q_0>\tilde\beta\}}.$$

Theorem 4.5 yields the conclusion. □

In general, even specifying a functional form of h_0, an explicit solution of the HJB equation for arbitrary φ_0 is not available, hence the above expressions of the optimal control strategy in terms of the value function and their corresponding qualitative properties are the "best" that one can expect, at least in the cases we have considered.

5 Concluding remarks

A number of deterministic advertising models allowing for delay effects have been proposed in the literature. However, the corresponding problems in the stochastic setting have not been investigated. One of the reasons is certainly that a theory of continuous-time stochastic control with delays has only been developed recently, following two approaches. The first approach is based on the solution of an associated infinite-dimensional Hamilton-Jacobi-Bellman equation in spaces of integrable functions (see [15]). The other one relies on the analysis of an appropriate infinite-dimensional forward-backward stochastic differential equation (see [12]).

Both approaches, however, cannot be applied to problems with distributed lag in the effect of advertising.

Problems with memory effects in both the state and the control have been studied first by Vinter and Kwong [35] (in a deterministic LQ setting), and by Gozzi and Marinelli [16] (in the case of linear stochastic dynamics and general objective function). A general theory of solvability of corresponding HJB equations is currently not available, while an infinite-dimensional Markovian reformulation and a "smooth" verification theorem have been proved in [16].

In this paper we have concentrated on deriving qualitative properties of the value function (such as convexity, monotonicity with respect to initial conditions, smoothness). For specific choices of the reward and cost functions, we obtain more explicit characterizations of value function and optimal state-control pair.

While our work makes a substantial initial step in the analysis of the stochastic advertising problems with delays, more remains to be done. Potential extensions of the present work include the analysis of problems with budget constraints as well as problems of advertising through multiple media outlets with different delay characteristics.

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