Convergence Analysis of Blind Equalization Algorithms Using Constellation-Matching

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***Abstract*—Two modified blind equalization algorithms are analyzed for performance. These algorithms add a constellation-matched error term to the cost functions of the generalized Sato and multimodulus algorithms. The dynamic convergence behavior and steady-state performance of these algorithms, and of a related version of the constant modulus algorithm, are characterized. The analysis establishes the improved performance of the proposed algorithms.**

***Index Terms*—Adaptive equalizer, blind equalization algorithms, convergence analysis.**

I. Introduction

THE best known algorithms for blind equalization include the generalized Sato algorithm (GSA), the constant modulus algorithm (CMA), and the multimodulus algorithm (MMA) [1]–[3]. Various extensions have been suggested to improve equalizer performance. The modified CMA (MCMA) [4] adds a constellation-matched error (CME) function to the CMA cost function. This letter analyses similar modifications of the GSA and MMA. Our analysis leads to new results for these CME-enhanced blind equalization algorithms for square QAM signalling, characterizing their superior transient as well as steady-state performance. Although the modified algorithms and analysis are for square QAM signals in this letter, the approach is applicable more generally.

II. Modified GSA and Modified MMA

Consider an i.i.d. data sequence $\{s_k\}$ transmitted through an FIR channel with impulse response $[h_{-L} \quad h_{-L+1} \quad \cdots \quad h_L]$. At the receiver an FIR equalizer produces output

$$z_k = \sum_{l=-K}^{K} w_l\left(k\right) x_{k-l} = \mathbf{w}_k^T \mathbf{x}_k. \tag{1}$$

Here $\mathbf{w}_k = [w_{-K}(k) \quad w_{-K+1}(k) \quad \cdots \quad w_K(k)]^T$ is the equalizer weight vector, and $\mathbf{x}_k = [x_{k+K}(k) \quad x_{k+K-1}(k) \quad \cdots \quad x_{k-K}(k)]^T$ is the equalizer input vector, which can be expressed as (2) (see next page). In (2) $\mathbf{H}$ is the $N \times (M+N-1)$ channel matrix, $\mathbf{s}_k$ is the $(M+N-1)$-element transmitted symbol vector where $M = 2L+1$ and $N = 2K+1$, and $\boldsymbol{\nu}_k$ is the N-element channel additive white Gaussian noise vector. A general form of equalizer weight update can be expressed as:

$$\mathbf{w}_{k+1} = \mathbf{w}_k - \mu \widetilde{e}_k \mathbf{x}_k^* \tag{3}$$

where $\widetilde{e}_k$ is an error function arising from the instantaneous gradient of a particular cost function, and μ is the adaptation step size.

For the square QAM constellation and a cosine-square CME function, the modified CMA (MCMA) cost function becomes (4) (see next page), where $2d$ is the minimum distance between the constellation symbols, β is a weighting factor governing the relative importance of the CMA and CME errors, and subscripts r and i denote real and imaginary components, respectively. As in the case of the MCMA, we can add a CME term to the cost functions of the GSA and MMA for QAM signaling [5]–[7]. This yields the modified GSA (MGSA) and modified MMA (MMMA) with cost functions (5) and (6) (see next page), where $csgn\left(z_k\right) = \text{sgn}\left(z_{kr}\right) + \text{jsgn}\left(z_{ki}\right)$. The equalizer weights are then updated according to (3) with respective error functions

$$\begin{aligned}\widetilde{e}_{k,MGSA} \quad &= \quad z_k - R_{GSA} csgn\left(z_k\right) \\ &\quad -\beta\frac{\pi}{2d}\left[\sin\left(\frac{z_{kr}}{d}\pi\right) + j\sin\left(\frac{z_{ki}}{d}\pi\right)\right],\end{aligned} \tag{7}$$

$$\begin{aligned}\widetilde{e}_{k,MMMA} \quad &= \quad z_{kr}\left(z_{kr}^2 - R_{MMA}\right) + jz_{ki}\left(z_{ki}^2 - R_{MMA}\right) \\ &\quad -\beta\frac{\pi}{2d}\left[\sin\left(\frac{z_{kr}}{d}\pi\right) + j\sin\left(\frac{z_{ki}}{d}\pi\right)\right].\end{aligned} \tag{8}$$

The GSA/MMA term provides initial convergence, and the CME term (last term on right-hand-side of (7) and (8)) improves the subsequent local convergence performance. Note that for the MGSA and MMMA, the CME terms work without explicit phase compensation.

III. MSE Convergence Analysis for MGSA, MCMA and MMMA

For small β in the CME term, the initial convergence of the MGSA/MCMA/MMMA is determined primarily by the original GSA/CMA/MMA term. When the equalizer error has become small enough, the CME term begins to contribute towards further convergence improvement and lowering of steady-state mean-square error (MSE). Based on this understanding, we separate our dynamic MSE *analysis* into two regions: global (initial) convergence region and local (final) convergence region.

Let $\mathbf{R} = E\left\{\mathbf{x}_k\mathbf{x}_k^H\right\}$ be the observation covariance matrix, which is eigendecomposed as $\mathbf{R} = \mathbf{U}^H\boldsymbol{\rho}\mathbf{U}$, where $\boldsymbol{\rho} = \text{diag}\left[\lambda_1, \cdots, \lambda_n\right]$ with $\mathbf{U}$ an orthonormal matrix. To simplify the analysis, input vector $\mathbf{x}_k$ is transformed to $\mathbf{y}_k = \mathbf{U}\mathbf{x}_k$ [8].

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$$\mathbf{x}_k = \begin{bmatrix} h_{-L} & h_{-L+1} & \cdots & h_L & \cdots & 0 \\ 0 & h_{-L} & \cdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & h_{L-1} & h_L \end{bmatrix} \begin{bmatrix} s_{k+K+L} \\ s_{k+K+L-1} \\ \vdots \\ s_{k-K-L} \end{bmatrix} + \begin{bmatrix} \nu_{k+K} \\ \nu_{k+K-1} \\ \vdots \\ \nu_{k-K} \end{bmatrix} = \mathbf{H}\mathbf{s}_k + \boldsymbol{\nu}_k \tag{2}$$

$$J_{MCMA} = E\left\{ \frac{1}{4}\left(|z_k|^2 - R_{CMA} \right)^2 + \beta \left(\cos^2\left(\frac{z_{kr}}{2d}\pi \right) + \cos^2\left(\frac{z_{ki}}{2d}\pi \right) \right) \right\}, R_{CMA} = \frac{E\left\{ |s_k|^4 \right\}}{E\left\{ |s_k|^2 \right\}} \tag{4}$$

$$J_{MGSA} = E\left\{ \frac{1}{2}\left| z_k - R_{GSA} csgn\left(z_k \right) \right|^2 + \beta \left(\cos^2\left(\frac{z_{kr}}{2d}\pi \right) + \cos^2\left(\frac{z_{ki}}{2d}\pi \right) \right) \right\}, R_{GSA} = \frac{E\left\{ s_{kr}^2 \right\}}{E\left\{ |s_{kr}| \right\}} = \frac{E\left\{ s_{ki}^2 \right\}}{E\left\{ |s_{ki}| \right\}} \tag{5}$$

$$\begin{aligned} J_{MMMA} &= E\left\{ \frac{1}{4}\left(z_{kr}^2 - R_{MMA} \right)^2 + \frac{1}{4}\left(z_{ki}^2 - R_{MMA} \right)^2 + \beta \left(\cos^2\left(\frac{z_{kr}}{2d}\pi \right) + \cos^2\left(\frac{z_{ki}}{2d}\pi \right) \right) \right\}, \\ R_{MMA} &= \frac{E\left\{ s_{kr}^4 \right\}}{E\left\{ s_k^2 \right\}} = \frac{E\left\{ s_{ki}^4 \right\}}{E\left\{ s_{ki}^2 \right\}} \end{aligned} \tag{6}$$

Let $\mathbf{c}_k \triangleq \mathbf{U}^* \mathbf{w}_k = \left[c_{-K}(k) \quad c_{-K+1}(k) \quad \cdots \quad c_K(k) \right]^T$, then (3) can be expressed as:

$$\mathbf{c}_{k+1} = \mathbf{c}_k - \mu \widetilde{e}_k \mathbf{y}_k^*. \tag{9}$$

A list of the primary assumptions and approximations for a convergence analysis are given in [8]–[10]. One important assumption is that the tap weight vector $\mathbf{w}_k$ is independent of the equalizer input vector $\mathbf{x}_k$. Using this assumption and (9), the MSE can be expressed as

$$\begin{aligned} \sigma_e^2(k) &= E\left\{ |e_k|^2 \right\} = E\left\{ |z_k - s_k|^2 \right\} \\ &= \boldsymbol{\rho}_1^T \boldsymbol{\Gamma}_k + P_s - 2P_s \mathrm{Re}\left[\mathbf{M}_k^T \boldsymbol{\eta} \right] \end{aligned} \tag{10}$$

where $\boldsymbol{\rho}_1 = [\lambda_1, \lambda_2, \cdots, \lambda_N]^T$, $P_s = E\left\{ |s_k|^2 \right\}$, $\mathbf{M}_k \triangleq \left[E\{c_{-K}(k)\} \quad E\{c_{-K+1}(k)\} \quad \cdots \quad E\{c_K(k)\} \right]^T$, $\boldsymbol{\Gamma}_k \triangleq \left[E\left\{ |c_{-K}(k)|^2 \right\} \ E\left\{ |c_{-K+1}(k)|^2 \right\} \ \cdots \ E\left\{ |c_{-K}(k)|^2 \right\} \right]^T$, $\boldsymbol{\eta} = \mathbf{U}\mathbf{H}\mathbf{e}_{K+L+1}$ and $\mathbf{e}_j$ is the zero vector except for a single 1 in the jth component. Finding recursive relations for $\mathbf{M}_k$ and $\boldsymbol{\Gamma}_k$ is the main task in obtaining the MSE trajectories expressed by (10). Two analysis methods have been used to derive the MSE trajectories. In the *conditional Gaussian* MSE analysis, the property that conditioned on s_k and $\mathbf{c}_k$, the quantities z_k and $\mathbf{y}_k$ are jointly Gaussian is used extensively. The conditional Gaussian MSE analysis for the GSA, CMA and MMA is given in [7]–[10]. The *Taylor series-based approximation* was used by Garth [10] to derive the MSE of the GSA, CMA and MMA; it gives simpler but less accurate MSE expressions.

Let $\mathbf{w}_{opt}$ be the optimum (in the MSE sense) equalizer coefficient vector, given by $\mathbf{w}_{opt} = P_s \left(\mathbf{R}^{-1}\mathbf{H}\mathbf{e}_{K+L+1} \right)^*$ [10]. Define the orthogonally transformed weight error vector $\boldsymbol{\varepsilon}_k = \mathbf{U}^* \left(\mathbf{w}_k - \mathbf{w}_{opt} \right) = \mathbf{c}_k - \mathbf{c}_{opt}$. Let $\mathbf{M}_\varepsilon(k) = E\{\boldsymbol{\varepsilon}_k\}$, $\Gamma_{\varepsilon_l}(k) = E\left\{ |\varepsilon_l(k)|^2 \right\}$, $\Gamma_{c_{opt_l}} = |c_{opt_l}|^2$. The MSE expression in (10) becomes (11) (see next page). In the Taylor series-based MSE analysis, a first order Taylor expansion is used to approximate the error function $\widetilde{e}_k$ in (9) about the optimal equalizer output $\widetilde{z}_k = \mathbf{w}_{opt}^T \mathbf{x}_k$:

$$\widetilde{e}_k \approx \widetilde{e}\left(\widetilde{z}_k \right) + \widetilde{e}'\left(\widetilde{z}_k \right) \mathbf{y}_k^T \boldsymbol{\varepsilon}_k. \tag{12}$$

Based on (12), recursive expressions for $\mathbf{M}_\varepsilon(k)$ and $\boldsymbol{\Gamma}_\varepsilon(k)$ are given in [10].

The conditional Gaussian analysis is complicated due to the sinusoidal CME function, while a first order linear approximation of the nonlinear error function, given by (12), is more applicable when the equalizer output is closer to the optimal equalizer output $\widetilde{z}_k$. We therefore use the accurate conditional Gaussian MSE analysis for initial convergence analysis, and the simple Taylor series-based MSE analysis for the local convergence region. The transition point between these two convergence analysis regions may be estimated as the point at which σ_e^2 given by (10) first satisfies $\sigma_e^2 \leq d^2$. At around this point, the CME term begins to provide useful feedback for the convergence process, and from this point on the Taylor series-based analysis is applicable. The parameter β has to be set not too large so that the initial convergence process will essentially depend on the unmodified GSA/CMA/MMA error term. A reasonable β value will make the CME error contribution no larger than that of the unmodified error term. For example, for a 64-QAM constellation, we should have $\beta < 300d^4/\pi$ for the MMMA [5]. As long as such a condition holds, the MSE will tend to decrease during initial convergence, and when it becomes low enough (e.g. $\sigma_e^2 \leq d^2$), the Taylor series-based analysis becomes valid.

The CME error function of the MGSA, MCMA and MMA is a sinusoidal function. To simplify the analysis, we use the following approximation (13) (see next page), where $\sigma_e(k)$ is the MSE at time k. Here $\sigma_e(k)$ is an approximation for $z_{kr} - s_{kr}$ and $z_{ki} - s_{ki}$. We then obtain the derivatives of the error functions $\widetilde{e}(z_k)$ for the modified algorithms as (14) (see next page). The second term on the right side of (14) is the weighted approximate derivative from (13). This simplification

$$\sigma_e^2(k) = \boldsymbol{\rho}_1^T \left(\boldsymbol{\Gamma}_\varepsilon(k) + \boldsymbol{\Gamma}_{c_{opt}} \right) + P_s + 2\mathrm{Re}\left[\boldsymbol{\rho}_1^T \mathbf{M}_\varepsilon(k)\,\mathbf{c}_{opt}^* - P_s \left(\mathbf{M}_\varepsilon(k) + \mathbf{c}_{opt} \right)^T \boldsymbol{\eta} \right] \tag{11}$$

$$\begin{aligned}
\widetilde{e}_{CME}(z_k) &= -\frac{\pi}{2d}\left[\sin\left(\frac{z_{kr}-s_{kr}+s_{kr}}{d}\pi\right) + j\sin\left(\frac{z_{ki}-s_{ki}+s_{ki}}{d}\pi\right)\right] \\
&= \frac{\pi}{2d}\left[\frac{\sin\left(\frac{z_{kr}-s_{kr}}{d}\pi\right)}{(z_{kr}-s_{kr})}(z_{kr}-s_{kr}) + j\frac{\sin\left(\frac{z_{ki}-s_{ki}}{d}\pi\right)}{(z_{ki}-s_{ki})}(z_{ki}-s_{ki})\right] \\
&\approx \frac{\pi}{2d}\left[\frac{\sin\left(\frac{\sigma_e(k)}{d}\pi\right)}{\sigma_e(k)}(z_{kr}-s_{kr}) + j\frac{\sin\left(\frac{\sigma_e(k)}{d}\pi\right)}{\sigma_e(k)}(z_{ki}-s_{ki})\right] \\
&= \frac{\pi}{2d}\frac{\sin\left(\frac{\sigma_e(k)}{d}\pi\right)}{\sigma_e(k)}(z_k - s_k)
\end{aligned} \tag{13}$$

$$\widetilde{e}'_{MGSA/MCMA/MMMA}(z_k) \approx \widetilde{e}'_{GSA/CMA/MMA}(z_k) + \beta\frac{\pi}{2d}\frac{\sin\left(\frac{\sigma_e(k)}{d}\pi\right)}{\sigma_e(k)} \tag{14}$$

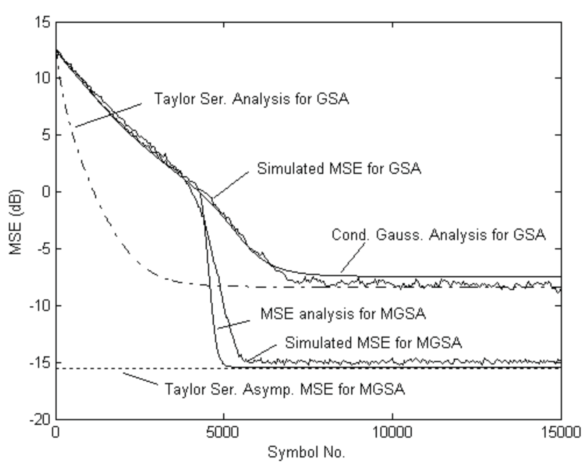

Fig. 1. MSE trajectories from simulation and analysis for GSA and MGSA in the voice-band communication channel.

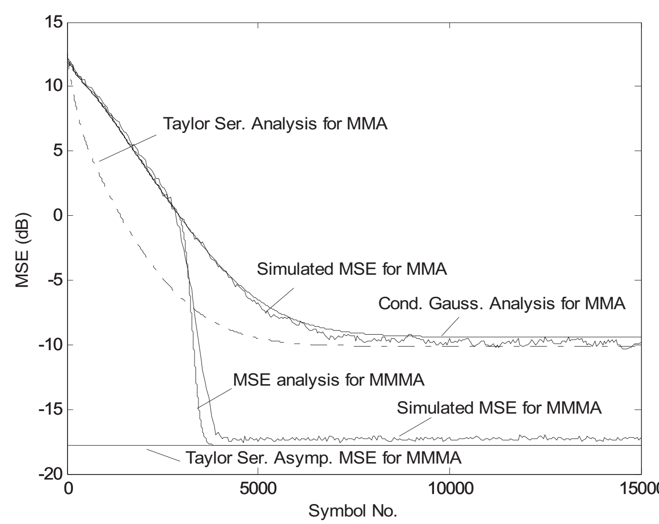

Fig. 2. MSE trajectories from simulation and analysis for MMA and MMMA in the voice-band communication channel.

follows because the sinc function in the result of (13) has derivative which is small near the origin and the difference $z_k - s_k$ is also small in the local convergence region. From (14) we can obtain the required coefficients values for the MGSA/MCMA/MMMA in the local convergence region from the coefficients for the GSA/CMA/MMA, given in [10], as [6]:

$$\begin{aligned}
\widetilde{F}_{M,ml}(k) &= \widetilde{F}_{ml} + \beta\frac{\pi}{2d}\frac{\sin\left(\frac{\sigma_e(k)}{d}\pi\right)}{\sigma_e(k)} \\
f_M(0) &= f(0) + \beta\frac{\pi^2}{2d^2}, \quad f_M(1) = f(1) + \beta\frac{\pi^2}{2d^2} \\
g_M(0) &= g(0), \quad g_M(1) = g(1)
\end{aligned} \tag{15}$$

Here the subscript M represents the coefficients for the modified algorithms, and $\widetilde{F}_{ml}$, $f(0)$, $f(1)$, $g(0)$ and $g(1)$ are the coefficients given in [10] for the Taylor series-based MSE analysis for the original GSA/CMA/MMA. Using these coefficients, we can obtain the MSE trajectories of the modified algorithms in the local convergence region.

IV. Simulations

We consider two channels: a typical voice-band communication channel [7], and a three-component real multipath channel with coefficients [0.27 1 0.27], which is a poor contender for the conditional Gaussian approximation. The transmitted signal is from a 64-QAM constellation and the minimum distance between symbols is 2. The equalizer is a 9-tap FIR filter, initialized with $\mathbf{w}_0$ all zeros except for a 1 in the center tap. The start of the local convergence process is characterized by the first occurrence of MSE $\sigma_e^2 \leq 1$. For the initial convergence process the conditional Gaussian MSE analysis is used. When σ_e^2 first becomes less than or equal to 1, the Taylor series-based analysis for the modified algorithms is applied. In all simulations, the MSE trajectories were averaged over 150 trials.

Figs. 1 and 2 show the simulated and calculated MSE of the GSA/MGSA and MMA/MMMA in the voice-band

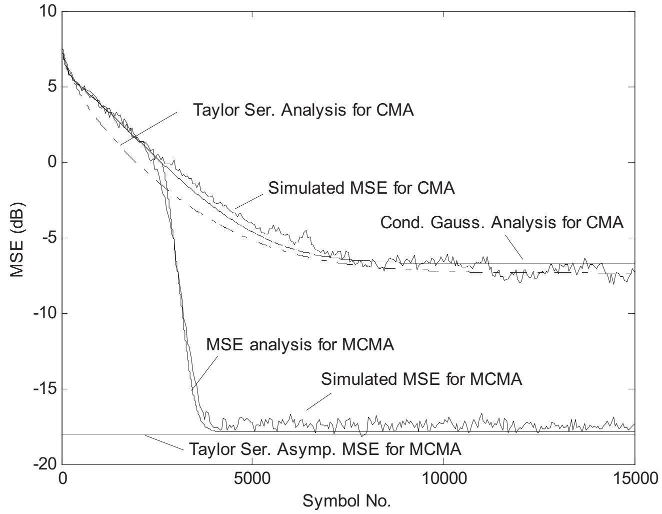

Fig. 3. MSE trajectories from simulation and analysis for CMA and MCMA; three-component channel.

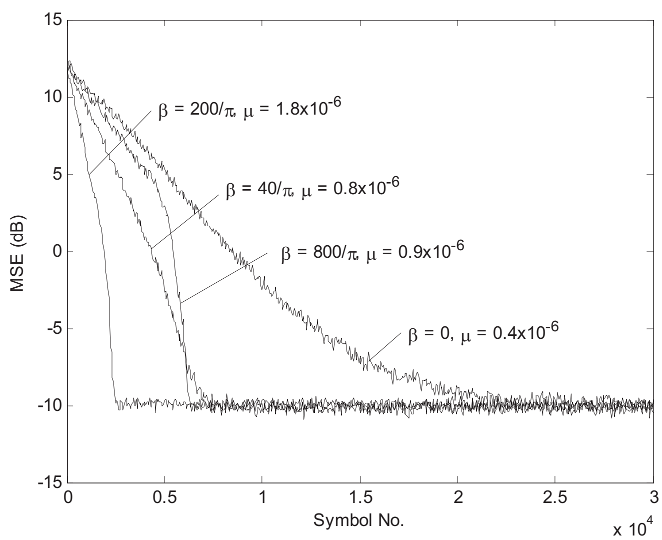

Fig. 4. Simulated MSE performance for different values of weighting factor and step sizes (SNR=30 dB).

communication channel. We set $\mu = 5 \times 10^{-5}$ for the GSA/MGSA, $\mu = 1.2 \times 10^{-6}$ for the MMA/MMMA, $\beta = 4/\pi$ for the MGSA, and $\beta = 200/\pi$ for the MMMA. For the three-component channel, we got similar results. For the MCMA, we simulated it in the three-component real channel to avoid the effect of phase recovery in the MSE analysis for the CMA. The results are given in Fig. 3 with $\mu = 10^{-6}$ and $\beta = 200/\pi$. It is obvious that the modified versions improve performance with faster convergence and lower residual errors. These representative simulation results confirm the significance and accuracy of our analytical approximations. Fig. 4 gives the simulated MSE trajectories of the MMMA for the voice channel with SNR = 30 dB. It shows the effects of the weighting factor β on equalizer performance for the modified algorithms. Other simulations ([4]–[6]) with different channels and noise have also shown the improved performance of the MGSA/MCMA/MMMA over the GSA/CMA/MMA.

V. Conclusion

In this letter we modified the GSA and MMA by adding constellation information in their cost functions to improve equalizer performance. We analyzed the dynamic convergence process of the MGSA, MCMA and MMMA, and derived MSE expressions by using a combined conditional Gaussian and Taylor series-based approximation. Computer simulation results established the significance and accuracy of the analytical approximations.

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