

Spin-glass and related orderings in quenched random-spin systems*

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A general model in which the nearest-neighbor exchange interactions $J(\vec{x}, \vec{x}')$ are of the form $J(\vec{x}, \vec{x}') = J_1 + J_2\epsilon(\vec{x})\epsilon(\vec{x}') + J_3[\epsilon(\vec{x}) + \epsilon(\vec{x}')] + J_4\mu(\vec{x}, \vec{x}')$ is considered. Here $\epsilon(\vec{x})$ is a random site variable and $\mu(\vec{x}, \vec{x}')$ is a random bond variable. It is argued that J_4 tends to produce the Edwards-Anderson-type spin glass, whereas J_2 produces a different type of phase which is like an alloy of up and down spins. That these two phases are distinct follows from the existence of a phase boundary in the J_2 - J_4 plane when $J_1 = J_3 = 0$. A consistent, but qualitative, discussion of this model via the renormalization group is also given.

I. INTRODUCTION

The spin-glass state, which may occur in quenched random spin systems, is characterized by the existence of a frozen local moment but by the absence of long-range ferromagnetic or antiferromagnetic order. Edwards and Anderson¹ (EA) introduced by a model with a nearest-neighbor random exchange J with average value $[J]_{\text{av}}$ zero. They argued that this model has a spin-glass (SG) transition with a spin-glass order parameter $Q = [\langle \vec{s} \rangle \langle \vec{s} \rangle]_{\text{av}}$, where \vec{s} is a local m -component spin variable, $\langle \rangle$ signifies a thermal average for a given distribution of J 's and $[\]_{\text{av}}$ signifies an average over the probability distribution of J 's. This model was analyzed within mean-field theory by EA, by Sherrington and Kirkpatrick,² and others^{3,4} using the replication trick.^{1,5} [The replication trick, which we will discuss further below, results from the formal identity

$$[\ln z]_{\text{av}} = \lim_{n \rightarrow 0} (1/n) \ln [z^n]_{\text{av}}$$

as $n \rightarrow 0$.] The critical exponents, describing the divergence of the Q susceptibility, and ν describing the correlation length divergence, obtained from these calculations⁶ are $\beta=1$, $\gamma=1$, and $\nu=\frac{1}{2}$. These results can be used to predict the critical dimensionality d_c below which mean-field theory is no longer applicable. The scaling relation $2\beta + \gamma = d\nu$ is expected to hold for all $d \leq d_c$ and to hold at d_c with mean-field values for β , γ and ν . Thus, $d_c = (2\beta + \gamma)/\nu = 6$. This result leads naturally to a calculation of the critical properties of the EA model in $6 - \epsilon$ dimensions.^{6,7} Independent evidence to support this picture has been obtained recently from an analysis of the high-temperature series for the Q susceptibility near six dimensions.⁸ Not only did this analysis give $d_c = 6$, but it also gave

results for γ consistent with those obtained from the ϵ expansion.

More recently an alternative model for a "spin glass" has appeared.⁹⁻¹¹ This is a site-random model with a local variable $\epsilon(\vec{x})$ which has a value of +1 if the site \vec{x} is occupied by an A atom and -1 if it is occupied by a B atom. In the simplest model of this type the exchange $J(\vec{x}, \vec{x}')$ between spins at positions x and x' is proportional to $\epsilon(x)\epsilon(x')$. The "spin-glass" state, which we prefer to call an alloy antiferromagnetic (AAF) state, is then characterized by $[\langle \vec{s} \rangle]_{\text{av}} = 0$, and $[\langle \vec{\tau} \rangle]_{\text{av}} \equiv [\langle \epsilon \vec{s} \rangle]_{\text{av}} \neq 0$. The critical dimensionality for this model is known to be four.¹¹ The purpose of this note is to point out that this AAF model and the SG model of EA are fundamentally different, and, therefore, that results for the two models, e.g., different values for d_c , are to be expected.

II. DISCUSSION

Consider the AAF state in the site-random model. On the average all type-A spins will be aligned parallel, say, upwards and then type-B spins will be aligned downward. Thus, if an A atom at a certain site is up, we know immediately whether the spin on a site a large distance away is up or down if the type of atom on that site is specified. This can be stated analytically as follows. Let $g(0, \vec{x}) = \langle \vec{s}(0) \cdot \vec{s}(\vec{x}) \rangle_0$, where $\langle \rangle_0$ signifies the thermal average for a given configuration in the zero-temperature limit and \vec{x} labels a lattice site. In the site-random model, we know that $\lim_{x \rightarrow \infty} g(0, \vec{x})\epsilon(0)\epsilon(\vec{x}) > 0$ as $x \rightarrow \infty$, and hence that $\epsilon(x)\vec{s}(\vec{x})$ is an order parameter.

The bond model has completely different behavior. If a given spin is up, there is no analogous way of knowing whether a distant spin is up

or down. In other words, the sign of $g(\vec{0}, \vec{x})$ in the limit of large \vec{x} cannot be determined even with a knowledge of the sign of J in the vicinity of the points $\vec{0}$ and \vec{x} ; rather, the sign of $g(\vec{0}, \vec{x})$ depends in detail on the signs of J throughout the entire region between $\vec{0}$ and \vec{x} . In this respect, the behavior of $g(\vec{0}, \vec{x})$ for the bond-random problem is similar to the function measuring the probability that two sites are in the same cluster in the percolation problem (for which mean-field theory becomes invalid^{12,13} for $d < 6$).

A further difference between the two models is evident. In the site-random model, each spin $\vec{s}(\vec{x})$ aligns in its own local field $\vec{h}(\vec{x}) \sim \sum_{\vec{x}'} J(\vec{x}, \vec{x}') \vec{s}(\vec{x}')$ leading to a transition temperature, T_c , in simple cubic lattices of high dimensionality d proportional to $2dJ$. In the bond-random model, the work of EA¹ and others^{2,3} indicate that $kT_c \sim [\sum_{\vec{x}, \vec{x}'} J^2(\vec{x}, \vec{x}')]^{1/2}$ or that $kT_c \sim d^{1/2}J$. This result suggests that each spin is aligned not with the coherent field of its neighbors but rather with the fluctuation of the field of its neighbors.

Perhaps the most conclusive evidence that the SG and AAF states are distinct is obtained by considering a generalized model in which the J 's have both site-random and bond-random contributions. Consider the Hamiltonian

$$H = - \sum_{(\vec{x}, \vec{x}')} J(\vec{x}, \vec{x}') \vec{s}(\vec{x}) \cdot \vec{s}(\vec{x}'), \quad (1)$$

where the sum is over pairs of nearest neighbors, and $J(\vec{x}, \vec{x}')$ is given by

$$J(\vec{x}, \vec{x}') = J_1 + J_2 \epsilon(\vec{x}) \epsilon(\vec{x}') + J_3 [\epsilon(\vec{x}) + \epsilon(\vec{x}')] + J_4 \mu(\vec{x}, \vec{x}'), \quad (2)$$

where $\epsilon(\vec{x})$ is a random variable assuming the value +1 if \vec{x} is occupied by an A atom and -1 if \vec{x} is occupied by a B atom, and $\mu(\vec{x}, \vec{x}')$ is a random bond variable. The average of the ϵ 's and μ 's is in general nonzero and, we write

$$[\epsilon(\vec{x})]_{av} = 2p_A - 1. \quad (3)$$

Different random variables are independent. The thermodynamics of the quenched random system is obtained as usual by averaging the free energy over configuration specified by $\{\epsilon\}$ and $\{\mu\}$. Thus we write

$$\begin{aligned} Z(\{\epsilon\}, \{\mu\}) &= \text{Tr}_s e^{-\beta H}, \\ -\beta F &= [\ln Z(\{\epsilon\}, \{\mu\})]_{av}, \end{aligned} \quad (4)$$

where $\beta = (kT)^{-1}$. The model defined by Eqs. (1)–(4) reduces to the EA¹ model when $J_1 = J_2 = J_3 = 0$, and to the site-random model treated by Luttinger¹⁰ when $J_4 = 0$. In the latter case, one can make the identifications

$$\begin{aligned} J_1 &= \frac{1}{4}(J_{AA} + J_{BB} + 2J_{AB}), \\ J_2 &= \frac{1}{4}(J_{AA} + J_{BB} - 2J_{AB}), \\ J_3 &= \frac{1}{4}(J_{AA} - J_{BB}), \end{aligned} \quad (5)$$

where J_{AA} , J_{BB} , and J_{AB} are the exchanges associated, respectively, with A-A, B-B, and A-B bonds.

We wish to discuss the competition between SG and AAF ordering implied by the above model. However, since the model is so complex, we will discuss various special cases, first without and then with competition. Our eventual conclusion will be that for $J_1 = J_3 = 0$, there must be a phase boundary separating AAF and SG phases in high enough spatial dimensionality. We consider the following cases:

- (i) $J_2 = J_3 = J_4 = 0$: this corresponds to a pure system with one exchange $J = J_{AA} = J_{BB} = J_{AB}$.
- (ii) $J_1 = J_3 = J_4 = 0$: This corresponds to $J_{AA} = J_{BB} = -J_{AB}$. The transformation $\vec{\tau}(\vec{x}) = \epsilon(\vec{x})\vec{s}(\vec{x})$ produces the Hamiltonian

$$H = -J_2 \sum_{(\vec{x}, \vec{x}')} \vec{\tau}(\vec{x}) \cdot \vec{\tau}(\vec{x}'),$$

which is identical to that of case (i) with J_2 replacing J_1 and $\vec{\tau}(\vec{x})$ replacing $\vec{s}(\vec{x})$. Thus, as first observed by Mattis⁹ and further explored by Aharony and Imry,¹¹ the critical properties of $\vec{\tau}$ in this case are identical to those of \vec{s} in case (i). A striking feature of this case is that the partition function Z and the free energy F are *completely independent* of $\{\epsilon(\vec{x})\}$. On the other hand, correlation functions and susceptibilities involving \vec{s} do depend on $\{\epsilon(\vec{x})\}$. In other words, all thermodynamic properties involving $\vec{\tau}$ are completely independent of the distribution of A and B atoms. For example, such properties for a random system in which each site is occupied with an A atom with probability $p_A = \frac{1}{2}$ and a B atom with probability $p_B = \frac{1}{2}$ are identical to those for a nonrandom two-sublattice antiferromagnet. These two cases yield different behavior for correlations involving $\vec{s}(\vec{x})$, though both cases yield a cusp in the specific heat at T_c .

- (iii) $J_{AB} = 0, J_4 = 0$. In this case, there are two completely non-interacting sublattices, and care must be taken to include the possibility of percolation. The special case $J_1 = J_2, J_3 = J_4 = 0$ corresponds to $J_{AA} = J_{BB}, J_{AB} = J_4 = 0$.

(iv) $J_3 = J_4 = 0$. In this case there is competition between \vec{s} and $\vec{\tau}$ ordering. There will be \vec{s} ordering if $J_1 > J_2$ and $\vec{\tau}$ ordering if $J_1 < J_2$. The special case $J_1 = J_2$ is included in case (iii).

- (v) $J_4 = 0; J_3 \neq 0$. In this case the order parameter is a linear combination of \vec{s} and $\vec{\tau}$. The ordered state might, therefore, be called an *alloy ferromagnet*.

(vi) $J_1 = J_2 = J_3 = 0, J_4 \neq 0$. This is the EA spin-glass model.

(vii) $J_2 = J_3 = 0; J_1 \neq 0, J_4 \neq 0$. Here there is competition between ferromagnetic and SG ordering. For spatial dimensionality d large enough,^{14,15} there is a phase boundary in the (J_1, J_4) plane separating these two phases. This boundary has been treated both in mean-field theory³ and in the ϵ expansion⁷ using the replication trick.

Before considering competition between AAF and SG ordering, it is useful to identify a general symmetry property of $F(J_1, J_2, J_3, J_4)$. By setting $\tilde{\tau}(\tilde{x}) = \epsilon(\tilde{x})\tilde{s}(\tilde{x})$, one finds that the partition function with $J(\tilde{x}, \tilde{x}')$ given by Eq. (2) is identical to that for $J(\tilde{x}, \tilde{x}')$ given by

$$J(\tilde{x}, \tilde{x}') = J_2 + J_1\epsilon(\tilde{x})\epsilon(\tilde{x}') + J_3[\epsilon(\tilde{x}) + \epsilon(\tilde{x}')] + J_4\tilde{\mu}(\tilde{x}, \tilde{x}'), \quad (6)$$

where

$$\tilde{\mu}(\tilde{x}, \tilde{x}') = \epsilon(\tilde{x})\epsilon(\tilde{x}')\mu(\tilde{x}, \tilde{x}')$$

can be considered a new bond-random variable. Thus,

$$F(J_1, J_2, J_3, J_4) = F(J_2, J_1, J_3, J_4). \quad (7)$$

Now consider the phase diagram for the following:

(viii) $J_1 = J_3 = 0; J_2, J_4 \neq 0$. From Eq. (7), we know $F(0, J_2, 0, J_4) = F(J_2, 0, 0, J_4)$; but from case (vii) we know that there is a phase boundary for d large enough in the (J_2, J_4) plane separating ferromagnetic and SG phases in $F(J_2, 0, 0, J_4)$. Thus, there must be a phase boundary separating the AAF from the SG phase in $F(0, J_2, 0, J_4)$. This can be viewed as conclusive evidence that the AAF and SG phases are distinct.

(ix) $J_1, J_2, J_3, J_4 > 0$. In this case, there is competition between SG and alloy ferrimagnetic ordering.

In low enough dimension the SG and AAF states are the same. For instance, in one dimension consider the case $J_2 = J_3 = 0$ and allow only nearest-neighbor interactions. One can set $\mu(\tilde{x}, \tilde{x}') = \epsilon(\tilde{x})\epsilon(\tilde{x}')$, in which case

$$F(J_1, 0, 0, J_4) = F(J_1, J_4, 0, 0). \quad (8)$$

Thus, the random-bond model in one dimension is equivalent to the random-site model. In particular the degeneracy κ of the ground state is the same in both cases. For Ising systems, which we consider for simplicity in the following discussion, the ground state is twofold degenerate corresponding to the arbitrary alignment of a chosen spin. For d not much larger than one, κ for the bond problem probably remains two. In this case, the construction suggested by Fisher¹⁶ in analogy with the work by Binder¹⁷ is probably valid. Here one considers the bond model for

which $J_1 = J_2 = J_3 = 0$ in a particular configuration θ . If the ground state of θ is twofold degenerate, it can be described by a unique set of orientations $\epsilon_\theta(\tilde{x}) = \pm 1$. We then write

$$\mu_\theta(\tilde{x}, \tilde{x}') = \epsilon_\theta(\tilde{x})\epsilon_\theta(\tilde{x}') + 2\nu_\theta(\tilde{x}, \tilde{x}'). \quad (9)$$

To the extent that $\mu_\theta(\tilde{x}, \tilde{x}')$ can be approximated $\epsilon_\theta(\tilde{x})\epsilon_\theta(\tilde{x}')$, $\nu_\theta(\tilde{x}, \tilde{x}')$ is small. For $d=1$, we have seen that $\nu_\theta(\tilde{x}, \tilde{x}')$ can be made to vanish identically. If the effective field at a given site \tilde{x} is proportional to the number of nearest neighbors, then $\nu_\theta(\tilde{x}, \tilde{x}') = 0$. By using the transformation $\tilde{s}(x) = \epsilon_\theta(\tilde{x})\tau(\tilde{x})$, one sees that the partition function for a configuration θ is that of a ferromagnet with a concentration p_ν of antiferromagnetic bonds which occur whenever $\nu_\theta(\tilde{x}, \tilde{x}')$ is nonzero. When p_ν is sufficiently small, i.e., for d small enough, one has a weakly disordered ferromagnet. As d is raised, p_ν increases leading, we believe to the SG state. The true SG state should be characterized by a ground state with a very high degeneracy [say $\kappa \geq 0(N)$ for Ising systems where N is the number of sites]. For each ground state g in a given configuration θ , one can define a set of orientations $\epsilon_g^\theta(\tilde{x}) = \pm 1$, and "defect" bonds $\nu_g^\theta(\tilde{x}, \tilde{x}')$. For each of these ground states, the concentration of defect bonds should be large (of order $\frac{1}{2}$). We believe that there exists a d^* such that for $d < d^*$, the short-range EA model has twofold degenerate state and AAF-like behavior. However, for $d > d^*$ there is a highly degenerate ground state with true SG behavior. The value of d^* may be nonuniversal since it seems to depend on the range of interaction and on the type of lattice. For Ising systems on a square lattice, d^* appears to be two¹⁵ whereas for Heisenberg systems d^* appears to be three.¹⁴ The situation could of course be more complicated. If it is necessary for κ to diverge as $N \rightarrow \infty$ to have true SG behavior, then one could conceive of various intermediate behaviors for $2 < \kappa < 0(N)$ for Ising systems. In this case, the meaning of d^* is less clear.

III. RENORMALIZATION GROUP

In this section, we will describe how the free energy of Eqs. (1)–(4) can be cast in a Landau-Ginzburg functional form suitable for application of the ϵ expansion.^{18–20} We will then discuss difficulties that may arise in such a treatment. We will use the replication trick inasmuch as it is the only method to date which has led to non-trivial predictions for the bond problem beyond mean-field theory. We consider first the case with $J_4 = 0$, and then the general case.

To produce a Landau-Ginzburg functional formulation of a Hamiltonian such as that of Eqs. (1)–

(4), it is customary¹⁹ to replace the strong constraint that $|\tilde{s}(x)|^2$ lie on an m -dimensional unit sphere by a sloppier constraint that is manifested by the appearance in the trace operation of an integral over all values of $\tilde{s}(\tilde{x})$ with a probability weighting factor $P_0(|\tilde{s}(\tilde{x})|)$ peaked in the vicinity of $|\tilde{s}(x)|^2 = 1$. The hope is that P_0 's which are in the same universality class as the initial strong constraint can be characterized by a few parameters which will be small, at least in the vicinity of the critical dimension at which mean-field theory becomes invalid. Thus, in the usual m -component classical spin system, $P_0 \sim e^{a|\tilde{s}|^2 - b|\tilde{s}|^4}$. The initial distribution corresponds to $a, b \rightarrow \infty$ such that $a/2b = 1$. All initial values of $a > 0$, and $b > 0$, are, however, in the same universality class. This allows a calculation of critical properties in $4 - \epsilon$ dimensions with $b \sim \epsilon$. We will proceed to produce a Landau Ginzburg form for Eqs. (1) and (4) in the above spirit. We stress, however, that it will not always be possible to find a continuum distribution with a finite number of small parameters that corresponds to the initial problem defined by Eqs. (1)–(4).

As usual, we begin with the replication trick

$$\beta F = -\lim_{n \rightarrow 0} \frac{1}{n} \ln [Z^n]_{\text{av}}, \quad (10)$$

where

$$[Z^n]_{\text{av}} = \left[\int d\{\tilde{s}^\alpha\} \prod_{\alpha} P_0(|\tilde{s}^\alpha(x)|^2) \exp\left(-\beta \sum_{\alpha=1}^n H^\alpha\right) \right]_{\text{av}} \quad (11)$$

and

$$H^\alpha = - \sum_{(\tilde{x}, \tilde{x}')} J(\tilde{x}, \tilde{x}') \tilde{s}^\alpha(\tilde{x}) \cdot \tilde{s}^\alpha(\tilde{x}'),$$

where α is a replication index,

$$\int d\{\tilde{s}^\alpha\} \equiv \prod_{\tilde{x}, \alpha, i} \int d\tilde{s}_i^\alpha(\tilde{x}), \quad (12)$$

and where we have introduced the spin probability distribution $P_0(|\tilde{s}^\alpha(\tilde{x})|^2)$. When $J_4 = 0$, we wish $[Z^n]_{\text{av}}$ to be expressed in terms of the local order parameters $\tilde{s}(\tilde{x})$ and $\tilde{\tau}(\tilde{x}) = \epsilon(\tilde{x})\tilde{s}(\tilde{x})$. Accordingly, we write

$$[Z^n]_{\text{av}} = \int d\{\tilde{s}^\alpha\} d\{\tilde{\tau}^\alpha\} \prod_{\alpha} P(\tilde{s}^\alpha(\tilde{x}), \tilde{\tau}^\alpha(\tilde{x})) \times \exp\left(-\beta \sum_{\alpha=1}^n \bar{H}^\alpha\right), \quad (13)$$

where

$$\bar{H}^\alpha = - \sum_{(\tilde{x}, \tilde{x}')} \{ J_1 \tilde{s}^\alpha(\tilde{x}) \cdot \tilde{s}^\alpha(\tilde{x}') + J_2 \tilde{\tau}^\alpha(\tilde{x}) \cdot \tilde{\tau}^\alpha(\tilde{x}') + J_3 [\tilde{s}^\alpha(\tilde{x}) \cdot \tilde{\tau}^\alpha(\tilde{x}') + \tilde{\tau}^\alpha(\tilde{x}) \cdot \tilde{s}^\alpha(\tilde{x}')] \}, \quad (14)$$

when $J_4 = 0$, and

$$P(\tilde{s}^\alpha(\tilde{x}), \tilde{\tau}^\alpha(\tilde{x})) = [P(\tilde{s}^\alpha(\tilde{x}), \tilde{\tau}^\alpha(\tilde{x}), \epsilon(\tilde{x}))]_{\text{av}}, \quad (15)$$

with

$$P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon) = P_0(|\tilde{s}^\alpha|^2) \delta(\tilde{\tau}^\alpha - \epsilon \tilde{s}^\alpha). \quad (16)$$

If each site is occupied with an A atom with probability p_A and a B atom with probability $q_A = 1 - p_A$, the average of any function $A(\epsilon)$ is given by

$$[A(\epsilon)]_{\text{av}} = p_A A(+1) + q_A A(-1). \quad (17)$$

Note that since P_0 depends only on $|\tilde{s}^\alpha|^2$, it could have been replaced by $P_0(|\tilde{s}^\alpha|^2)^\lambda P_0(|\tilde{\tau}^\alpha|^2)^{1-\lambda}$ in Eq. (16). We now seek an analytic generalization of $P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon)$. To obtain the correct form, we note that

$$\delta(\tilde{\tau}^\alpha - \epsilon \tilde{s}^\alpha) \sim \lim_{v \rightarrow \infty} e^{-v|\tilde{\tau}^\alpha - \epsilon \tilde{s}^\alpha|^2}.$$

Thus, the δ function introduces $\epsilon \tilde{\tau}^\alpha \cdot \tilde{s}^\alpha$ couplings. To obtain an expression valid for $v \neq \infty$, however, we cannot merely plug in the above form of the δ function into Eq. (16). We demand that the generalized form of $P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon)$ satisfy the sum rules implied by Eq. (16):

$$\int d\tilde{\tau}^\alpha P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon) = P_0(|\tilde{s}^\alpha|^2), \quad (18a)$$

$$\int d\tilde{s}^\alpha P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon) = P_0(|\tilde{\tau}^\alpha|^2). \quad (18b)$$

These sum rules treat \tilde{s}^α and $\tilde{\tau}^\alpha$ symmetrically. We, therefore, write

$$P(\tilde{s}^\alpha, \tilde{\tau}^\alpha, \epsilon) = P_1(|s^\alpha|^2) P_1(|\tau^\alpha|^2) e^{-v\epsilon \tilde{\tau}^\alpha \cdot \tilde{s}^\alpha}. \quad (19)$$

For each value of v , there is a unique function P_1 which satisfies Eqs. (18). This function will in general be of the form $\exp(\sum a_k |\tilde{s}^\alpha|^{2k})$, with $a_k \neq 0$ for all k . Equation (19) is the simplest analytic form for P consistent with symmetry and constraints. More complicated forms (with higher powers of $\tilde{\tau}^\alpha \cdot \tilde{s}^\alpha$ appearing in the exponential for example) are of course possible. Note that the averaging process mixes different replicas as expected. After appropriate rescaling Eqs. (13) and (18) yield a Landau-Ginzburg functional form for $[Z^n]_{\text{av}}$:

$$[Z^n]_{\text{av}} = \int d\{s^\alpha\} d\{\tau^\alpha\} e^{-\beta H_{\text{eff}}}, \quad (20)$$

with

$$\beta H_{\text{eff}} = \frac{1}{2} \sum_{\alpha} \int (r_s \tilde{s}^{\alpha} \cdot \tilde{s}^{\alpha} + \nabla \tilde{s}^{\alpha} \cdot \nabla \tilde{s}^{\alpha} + r_{\tau} \tilde{\tau}^{\alpha} \cdot \tilde{\tau}^{\alpha} + \nabla \tilde{\tau}^{\alpha} \cdot \nabla \tilde{\tau}^{\alpha} + r_{s\tau} \tilde{s}^{\alpha} \cdot \tilde{\tau}^{\alpha} + K_{s\tau} \nabla \tilde{s}^{\alpha} \cdot \nabla \tilde{\tau}^{\alpha}) d^d x \\ + \int d^d x \left[u \sum_{\alpha} (|\tilde{s}^{\alpha}|^4 + |\tilde{\tau}^{\alpha}|^4) + u_1 \left(\sum_{\alpha} \tilde{\tau}^{\alpha} \cdot \tilde{s}^{\alpha} \right)^2 + \dots \right]. \quad (21)$$

Here $r_s \approx (T - T_{c,s})/T$, $r_{\tau} \approx (T - T_{c,\tau})/T$, and $K_{s\tau}, r_{s\tau} \approx J_3$, where $T_{c,s} = 2dJ$, and $T_{c,\tau} = 2dJ_2$ are the mean-field transition temperatures for s and τ ordering, respectively, when $p_A = \frac{1}{2}$.

Equations (20) and (21) cast the problem in a form convenient for renormalization group calculations using the ϵ expansion. It is important to remember that the above is a truncated Hamiltonian, and in some cases it may be necessary to keep all higher-order terms to insure satisfaction of the sum rule Eq. (18). In fact, to regain the limits $J_2 = J_3 = 0$ [case (i)], and $J_1 = J_3 = 0$ [case (ii)] discussed previously, the untruncated form of Eq. (21) must be used to insure that the sum rules produce a Hamiltonian which is n replicas of a pure Heisenberg Hamiltonian. Otherwise terms in H_{eff} like $u_1 \sum_{\alpha} (\tilde{\tau}^{\alpha} \cdot \tilde{s}^{\alpha})^2$, which couple different replicas will not vanish as they should.

Now consider some other cases. In this paragraph we will discuss the case $J_3 = J_4 = 0$. For $J_3 = 0$, one has $J_{AA} = J_{BB}$, and we set $p_A = \frac{1}{2}$, so that $r_{s\tau}$ and $K_{s\tau}$ are zero. Then Eq. (21) describes a system with competition between ordering in \tilde{s}^{α} and ordering in $\tilde{\tau}^{\alpha}$. If $J_1 > J_2$, so that $J_{AB} > 0$, then \tilde{s}^{α} will order first, and $\tilde{\tau}^{\alpha}$ can be removed to produce an effective Hamiltonian with two quartic couplings, $\sum (\tilde{s}^{\alpha} \cdot \tilde{s}^{\alpha})^2$ and $(\sum \tilde{s}^{\alpha} \cdot \tilde{s}^{\alpha})^2$, the second of which mixes replicas. This Hamiltonian is identical to that for a system with weak bond randomness.⁵ In other words, there will be a continuous transition to an ordered state with $\langle \tilde{s}^{\alpha} \rangle \neq 0$, with pure Heisenberg exponents if the pure system specific-heat exponent is negative and renormalized exponents if it is positive.^{5,21,22} If $J_2 > J_1 > 0$ ($J_{AB} < 0$), $\tilde{\tau}^{\alpha}$ will order first; \tilde{s}^{α} can be removed to produce a random bond Hamiltonian in terms of $\tilde{\tau}^{\alpha}$. In other words, if $J_2 > J_1$, there

will be a transition to a random AAF state with exponents for $\tilde{\tau}^{\alpha}$ which are the same as the exponents for \tilde{s}^{α} when $J_1 > J_2$. If $J_1 > 0$, and $J_2 = 0$, or if $J_2 > 0$, and $J_1 = 0$, the system is a pure Heisenberg one and the untruncated Hamiltonian must be employed as discussed above. One might be tempted to say that the case [case (iii)] with $J_3 = 0$, and $J_1 = J_2$, corresponds to a polycritical point with simultaneous $\tilde{\tau}^{\alpha}$ and \tilde{s}^{α} ordering with critical exponents that could be calculated using the ϵ expansion. This is not the case. The point $J_3 = 0$, $J_1 = J_2$ corresponds to a complete decoupling of the A and B atoms ($J_{AB} = 0$). To our knowledge this decoupling cannot be reproduced by any analytic form for $P(\tilde{s}^{\alpha}, \tilde{\tau}^{\alpha})$ characterized by a finite number of finite parameters. In other words, the only distribution which decouples the two atoms is the starting one with $\nu = \infty$ [Eq. (14)]. This cannot be treated using the ϵ expansion without the introduction of some other variable (such as a Potts variable¹³) describing cluster statistics.

If $J_3 \neq 0$, and/or $p_A \neq \frac{1}{2}$, $r_{s\tau}$ and $K_{s\tau}$ are nonzero. In this case the Gaussian part of the Hamiltonian is diagonalized by linear combinations of \tilde{s}^{α} and $\tilde{\tau}^{\alpha}$. Call the combination that orders first (i.e., at the higher temperature) $\tilde{\phi}^{\alpha}$. The effective Hamiltonian in terms of $\tilde{\phi}^{\alpha}$ is again the dilute random-bond Hamiltonian.⁵ Since $\tilde{\phi}^{\alpha}$ would be a ferrimagnetic order parameter in a two-sublattice system, we might call the random system an alloy ferrimagnet. It can have either pure or random exponents depending on the sign of the specific-heat exponent.

If bond randomness as well as site randomness is present, J_4 becomes nonzero and an average over $\mu(\tilde{x}, \tilde{x}')$ must be included. In this case, $\beta \bar{H}^{\alpha}$ becomes

$$\beta \bar{H}^{\alpha} = -\beta \sum_{\tilde{x}, \tilde{x}'} \{ (J_1 + J_4 [\mu]_{\text{av}}) \tilde{s}^{\alpha}(\tilde{x}) \cdot \tilde{s}^{\alpha}(\tilde{x}') + J_2 \tilde{\tau}^{\alpha}(\tilde{x}) \cdot \tilde{\tau}^{\alpha}(\tilde{x}') + J_3 [\tilde{s}^{\alpha}(\tilde{x}) \cdot \tilde{\tau}^{\alpha}(\tilde{x}') + \tilde{\tau}^{\alpha}(\tilde{x}) \cdot \tilde{s}^{\alpha}(\tilde{x}')] \} \\ - \frac{1}{2} \beta^2 J_4^2 [(\delta \mu)^2]_{\text{av}} \left(\sum_{\alpha \neq \beta} Q_{ij}^{\alpha\beta}(\tilde{x}) Q_{ij}^{\alpha\beta}(\tilde{x}') + \sum_{\alpha} [\tilde{s}^{\alpha}(\tilde{x}) \cdot \tilde{s}^{\alpha}(\tilde{x}')]^2 \right), \quad (22)$$

where $Q_{ij}^{\alpha\beta}(\tilde{x}) = s_i^{\alpha}(\tilde{x}) s_j^{\beta}(\tilde{x}')$ is the EA SG order parameter, $[(\delta \mu)^2]_{\text{av}} = [\mu^2]_{\text{av}} - [\mu]_{\text{av}}^2$, and where higher-order cumulants of μ have been omitted. Here and below we use the repeated index summation convention for the indices i, j , etc. It is evident from the above that there are three different fields that can order in principle, namely \tilde{s}^{α} , $\tilde{\tau}^{\alpha}$, and $Q_{ij}^{\alpha\beta}$. If $J_3 = 0$, the transition temperature for \tilde{s}^{α} ordering is proportional to $J_1 + J_4 [\mu]_{\text{av}}$, for $\tilde{\tau}^{\alpha}$ ordering to J_2 , and for $Q_{ij}^{\alpha\beta}$ (SG) ordering to $J_4 ([\delta \mu]_{\text{av}}^2)^{1/2}$ in high enough dimension.

$Q_{ij}^{\alpha\beta}(\tilde{x})$ can be incorporated into the Landau-Ginzburg formalism in a way that is described in detail in Ref. 7. Here, we note only that allowing J_4 to be nonzero adds additional terms to βH_{eff} in Eq. (21) of the form

$$\beta H' = \frac{1}{4} \int d^d x [r_Q Q_{ij}^{\alpha\beta}(\vec{x}) Q_{ij}^{\alpha\beta}(\vec{x}) + \nabla_i Q_{ij}^{\alpha\beta}(\vec{x}) \nabla_i Q_{ij}^{\alpha\beta}(\vec{x})] \\ + W_1 \int \text{Tr} Q^3(\vec{x}) + W_2 \int Q_{ij}^{\alpha\beta}(\vec{x}) s_i^\alpha(\vec{x}) s_j^\beta(\vec{x}) + W_3 \int Q_{ij}^{\alpha\beta}(\vec{x}) \tau_i^\alpha(\vec{x}) \tau_j^\beta(\vec{x}) + \dots \quad (23)$$

Thus the total effective Hamiltonian obtained by adding Eq. (23) to Eq. (21) is identical to that considered in Ref. 7 if $r_{s\tau}$ and $K_{s\tau} = 0$. In Ref. 7 a slightly different model in which a pure sublattice structure with random bonds was assumed so that $\vec{\tau}^\alpha$ was the staggered magnetization. The structure of the two models is identical, however. We can therefore use the results of Ref. 7 to say that when $J_3 = 0$, and $p_A = \frac{1}{2}$, the ϵ expansion predicts a SG-random ferromagnetic and an SG-random AAF transition depending on whether $J_1 > J_2$, or $J_2 > J_1$, if $[\mu]_{av} = 0$. Here random refers to the fact that the transition is in the same universality class as the weakly random m -component spin system. In the phase diagram for the model with $J_3 = 0$, and $J_4 \neq 0$, there are two multicritical points, one where the paramagnetic, random ferromagnetic and SG phases join and another where the paramagnetic, random AAF and SG phases join. These two multicritical points have the same critical exponents as those calculated in Ref. 7. If J_3 is not zero, there will be a SG-random alloy ferromagnetic transition.

The ϵ expansion in $6 - \epsilon$ dimensions is an expansion from mean-field theory and is notoriously bad at predicting things like transition temperatures. It assumes the existence of the SG state.

As such, it is not well adapted to study the suggested merging of the SG and AAF states at a lower critical dimension d^* . A more fruitful approach to study the behavior near d^* may be some modification of the Migdal-Kadanoff^{23,24} bond moving scheme.

IV. CONCLUSION

We have discussed the critical behavior of the site-random and bond-random spin glasses. In one dimension the two models are known to be equivalent, but for large dimensionality we have argued that they are different. Presumably, then, there exists a critical dimensionality d^* above which the two models are qualitatively different. It is entirely consistent with these conclusions for the critical dimension d_c above which mean-field theory is valid is to be six for the random-bond spin glass and four for the site-random spin glass. The random-bond model seems related to the percolation problem for which $d_c = 6$, which is again consistent with our conclusions.

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