ESSAYS ON MECHANISM AND INFORMATION DESIGN

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To my family and love

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ABSTRACT

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This dissertation consists of two essays that examine issues related to data - how data is generated, used and monetized. In Chapter 1, I study how intermediaries such as Amazon and Google recommend products and services to consumers for which they receive compensation from the recommended sellers. Consumers will find these recommendations useful only if they are informative about the quality of the match between the sellers' offerings and the consumer's needs. The intermediary would like the consumer to purchase the product from the recommended seller, but is constrained because consumers need not follow the recommendation. I frame the intermediary's problem as a mechanism design problem in which the mechanism designer cannot directly choose the outcome, but must encourage the consumer to choose the desired outcome. I show that in the optimal mechanism, the recommended seller has the largest non-negative virtual willingness to pay adjusted for the cost of persuasion. The optimal mechanism can be implemented via a handicap auction. I use this model to provide insights for current policy debates.

In Chapter 2, in the joint work with Mallesh Pai and Rakesh Vohra, we propose a statistical test for identifying whether a policy or an algorithm is designed by a principal with discriminatory tastes. The test can be used for identifying, for example, whether predictive policing algorithms are discriminatory against minority neighborhoods. We also argue that the marginal outcome test (Becker (1993)), the most popular test of taste-based discrimination, fails for policies. We consider a canonical setup where the principal designs a policy (algorithm) that maps signals (data) to decisions for each group, such as whether to patrol or not for each area. The principal commits to the policy, which in turn affects agents' incentives to take action, such as whether to commit a crime. In this environment, the marginal outcome test fails because the principal not only cares about the marginal benefit of catching a criminal but how patrolling changes agents' incentive to commit a crime. We propose a new statistical test that deviates from the marginal outcome test precisely as much as the incentive effect.

Contents

List of Figures

CHAPTER 1 : Optimal Recommender System Design

1.1. Introduction

As consumers increasingly use websites and digital services for shopping, online platforms play a larger role in choosing products and services. Many platforms make personalized recommendations based on past data about consumers, providing them with greater insights into which products and services best fit their needs. For example, more than 75% of Netflix selections arise from personalized recommendations derived from past viewership and stated preference. Likewise, more than 35% of Amazon's sales result from the platform's recommendations to consumers.[1]

While some platforms like Netflix focus solely on providing the best matches for users, others monetize the recommendations by collecting payments from sellers in exchange for recommending their products and services. For instance, Amazon recommends sponsored products by displaying the products at the top of search lists. Google and Facebook recommend products by displaying targeted advertisements. A unique feature of these platforms is that sellers pay for their products to be recommended, yet the platform fully designs how to recommend and the payment structure. I call this pair, of a recommendations rule and a payment rule, a *recommender system*.

In this paper, I consider a monopolistic intermediary designing a recommender system to maximize the revenue collected from sellers. There are three types of players: a representative consumer, N representative sellers and an intermediary. The consumer may choose from one of the N products or an outside option. While the consumer does not know the match values of the products, the intermediary does and monetizes this knowledge by collecting payments from sellers in exchange for recommending their products. Sellers are willing to pay for recommendations to increase their sales. The (ex-post) willingness to pay

[1] *McKinsey & Company*, "How retailers can keep up with consumers," https://www.mckinsey.com/industries/retail/our-insights/how-retailers-can-keep-up-with-consumers

is drawn from two sources: the seller's private information, such as profit margin, and the match values, which only the intermediary knows.

I frame the intermediary's problem as a revenue-maximizing mechanism design problem of allocating one unit of sales to one of multiple sellers, but with a constraint. Unlike a standard optimal auction (Myerson (1981)), the intermediary cannot directly choose an outcome of the mechanism, the sales, but must rely on the consumer to choose an outcome. The only way to influence the consumer's choice is by recommending products that are a good match so that the consumer will find it optimal to choose the recommended option. That is, the intermediary is constrained to persuade the consumer to choose the desired outcome.

The intermediary's objective of raising revenue from sellers and constraint of persuading the consumer interact in a non-trivial way. To raise revenue from sellers, the intermediary has to persuade the consumer to purchase the product from the recommended seller and the outside option if no seller is recommended. Otherwise, the consumer ignores the recommendations, and sellers would not pay for a recommendation. Persuading the consumer to take the recommended option requires recommending a product with a high match value even if its seller does not necessarily have the highest expected willingness to pay.

The presence of the consumer's outside option is important. Without it, the constraint of persuading the consumer is trivially satisfied. In the symmetric environment where products are ex-ante identical, the consumer is indifferent among all options and follows recommendations as long as they contain some information about match values. If the intermediary runs an optimal auction (with no reserves) with sellers and recommends the product of the seller with the highest virtual willingness to pay, then such recommendations are informative because sellers' virtual willingness to pay partially depends on match values. In other words, the revenue-maximizing mechanism designed ignoring the persuasion constraint trivially satisfies the constraint.

With the outside option, the constraint of persuading the consumer bites. Suppose the intermediary first runs an optimal auction with sellers and recommends the product of the seller with the highest non-negative virtual willingness to pay (Myerson (1981)). If the consumer is nearly ex-ante indifferent between the outside option and products, the consumer follows the recommendations. However, when the consumer strongly prefers his outside option over products or vice versa, the recommendations are not informative enough about match values, so the recommendations are ignored. To make recommendations informative, the intermediary adjusts the virtual willingness to pay by match values, and recommends according to the adjusted virtual willingness to pay (Theorem 1.a and Theorem 1.b). The adjusted virtual willingness to pay is larger when the product is a good match. The precise size of the adjustment is shadow price of the persuasion constraint that I call the *cost of persuasion*.

In solving the intermediary's mechanism design problem, I reformulate the problem as a Bayesian persuasion problem in which the intermediary persuades the consumer to take the recommended option by strategically releasing information about match values as well as sellers' willingness to pay. In this Bayesian persuasion problem, the intermediary has state-dependent preferences over recommendations, the state space is multidimensional and possibly infinite, and the consumer has multiple options to choose from. These features combined make the three popular approaches - concavification (Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Dworczak and Kolotilin (2019)) - hard to apply tractably. Instead, I use a guess and verify approach by focusing on a class of recommendations rules that I call *value-switching monotone*. As it tractably characterizes the binding obedience constraints and the structure of the optimal recommendations rule, I expect this approach to be useful in similar Bayesian persuasion problems.

In the second part of this paper, I use the model to examine policy questions on the regulation of platforms. The first question is whether the intermediary should be allowed to collect and use data that reflects sellers' private information, which I call *additional information*. For example, Amazon sometimes demands receipts from third-party sellers to prove their products' authenticity. The receipts may contain sensitive information such as from where and at what prices the products are purchased, which would enable Amazon to directly purchase and sell the identical products without leaving any margin to third-party sellers.[2] Google is accused of using past bidding data to estimate bids advertisers are likely to submit.[3] Regulators have initiated a series of antitrust investigations on intermediaries' use of additional information about sellers on the basis of the potential harm to consumers and sellers.[4]

I find that the intermediary's use of additional information does not necessarily harm consumers and sellers. Additional information changes the revenue gains the intermediary makes by recommending products of sellers with higher willingness to pay, and hence, the optimal recommender system (Theorem 2.a and Theorem 2.b). In particular, additional information that decreases (increases) the revenue gains benefits (harms) consumers by making the intermediary more (less) likely to recommend products based on match values (Theorem 3 and Theorem 6). The same property provides sufficient conditions under which additional information harms sellers (Theorem 4, Corollary 1, Theorem 7 and Corollary 2).

The second question is whether consumer data is protected or leaked to sellers through the recommender system.[5] I show that the intermediary cannot earn higher revenue by sharing

[2] *U.S. House Judiciary Committee's Subcommittee on Antitrust, Commercial, and Administrative Law*, "Investigation of Competition in Digital Markets," https://judiciary.house.gov/uploadedfiles/competition_in_digital_markets.pdf?utm_campaign=4493-519

[3] *The Wall Street Journal*, "Google's Secret 'Project Bernanke' Revealed in Texas Antitrust Case," https://www.wsj.com/articles/googles-secret-project-bernanke-revealed-in-texas-antitrust-case-11618097760

[4] The European Commission has launched an antitrust investigation against Amazon (https://ec.europa.eu/commission/presscorner/detail/en/ip_20_2077). Ten states led by Texas have sued Google for anti-competitive policies in online advertisement markets, including Project Bernanke. (https://www.wsj.com/articles/states-sue-google-over-digital-ad-practices-11608146817?mod=article_inline).

[5] Some intermediaries such as Facebook allow sellers to define target audience using attributes including date of birth, gender and location before they bid. Korolova (2010) demonstrates that sellers can select attributes so that they are satisfied only by a single user, effectively revealing the target consumer's demographic information that was supposed to be private. See Korolova (2010) and Venkatadri, Andreou, Liu, Mislove, Gummadi, Loiseau, and Goga (2018) for more details. This has sparked concerns about consumer data leakage through targeted advertisements, and served as one of the motivations for data protection

consumer data with sellers. In the optimal direct mechanism, the intermediary can always extract the benefit sellers would have from receiving consumer data by providing the sellers with better matches and charging more (Theorem 8). Consumer data is protected in that sellers do not learn about consumers' match values until the auction ends. However, data leakage is a feature of some indirect mechanims that implement the optimal recommender system (Theorem 9 and Theorem 10).

Lastly, I show that the welfare-maximizing mechanism increases consumer surplus but reduces the joint profit of the intermediary and sellers relative to the revenue-maximizing mechanism when the consumer's outside option is so undesirable that he always prefers products over the outside option (Theorem 11). Under the welfare maximization regime, the welfare gains by recommending products of sellers with higher willingness to pay is lower, and that by recommending better-matched products is higher, relative to the revenue gains under the revenue maximization regime. This change in gains leads the social planner to recommend products based on match values more often, increasing consumer surplus and decreasing the joint profit.

The remainder of this paper proceeds as follows. The next subsection discusses related literature. Section 1.2 provides an example to demonstrate the key properties of the revenue-maximizing recommender system. Section 1.3 describes the model. Section 1.4 characterizes the revenue-maximizing recommender system. Section 1.5 characterizes how additional information changes the optimal recommender system and the payoffs of the consumer, sellers and intermediary. Section 1.6 explores whether consumer data is protected or leaked to sellers through the recommender system. Section 1.7 discusses several extensions and relaxation of assumptions, including an alternative interpretation of the model as a search engine. Section 1.8 concludes. All proofs are collected in the Appendix.

1.1.1. Related Literature

Sales of Information

regulations such as the European Union's General Data Protection Regulation (https://gdpr.eu/).

This paper contributes to the emerging literature on the sale of information by a monopolistic information seller. Starting with Admati and Pfleiderer (1986, 1990), several papers focus on how to sell information to an information buyer who directly receives the information to make better decisions. Recent works study a monopolistic information seller selling experiments to a decision maker who has private information about the states of the world (Bergemann, Bonatti, and Smolin (2018)), statistics to a decision maker who has private information about what kinds of information it needs (Segura-Rodriguez (2021)) and consumer segments to a producer who uses it to better price-discriminate (Yang (2021)).

In contrast, in this paper, the information buyers are product sellers, and the information seller is an intermediary. Product sellers pay the intermediary in order to influence the information provided to the consumer, instead of directly receiving information. The closest to my paper is Yang (2019), which studies an intermediary who designs a recommendations rule, a transfer rule, and a pricing rule over a single product and seller. Instead, I study an intermediary who designs a recommendations rule and a transfer rule over multiple products with exogenously given prices. The consumer benefits from recommendations because the intermediary can better distinguish between ex-ante identical products. This source of consumer surplus plays a crucial role in analyzing the impact of additional information on the consumer surplus. Inderst and Ottaviani (2012), Mitchell (2021), and Aridor and Gonçalves (2021) also analyze problems of information buyers paying to influence information others receive, but the information buyers are non-strategic or do not have private information.

Regulation of Platforms

This paper is closely related to a series of papers on the use of data by platforms and their regulation. de Cornière and Taylor (2019), Hagiu, Teh, and Wright (2020) and Aridor and Gonçalves (2021) study how an intermediary uses consumer data to promote its own product when it competes with a third-party seller on prices and qualities. Madsen and Vellodi (2021) studies how the intermediary uses seller data to launch its own private-label product. Fang and Kim (2021) examines how the intermediary shares consumer data with a third-party seller when the intermediary's private-label product competes with the seller's. Hagiu and Wright (2015), Hagiu and Wright (2019) and Kang and Muir (2021) focus on how different market structures, instead of platforms' use of data, affect outcomes. While the prior literature studies how platforms and sellers interact through the downstream market competition, I focus on how platforms use data to give informative recommendations and how their regulations change the recommendations and players' welfare.

Mechanism Design

This paper combines mechanism design with Bayesian persuasion. The intermediary solves a revenue-maximizing mechanism design problem, but with a constraint that it has to persuade the consumer to take the recommended options. If the intermediary could force the consumer to take the recommended options, then the intermediary's problem reduces to a standard revenue-maximizing auction design problem (Myerson (1981)).

There are several papers that study mechanism design problems in which the mechanism designer cannot fully control the outcome. In Myerson (1982), agents choose outcomes after communicating with the mechanism designer, and it is without loss of generality to restrict the mechanism designer's attention to honest and obedient mechanisms. Myerson (1983) studies an incentive compatible communications mechanism in a Bayesian game where the outcome relies on agents' private information. Dworczak (2020) studies a problem of allocating an object to one of several agents that is followed by a black-box aftermarket that the mechanism designer cannot control. By contrast, in this paper, the intermediary (i) has private information, (ii) elicits information from one party (sellers) and recommends outcomes to the other party (the consumer), and (iii) directly interacts with the consumer who solely chooses the outcome and does not care about sellers' private information per se.

Bayesian Persuasion

This paper contributes to the Bayesian persuasion literature (Rayo and Segal (2010), Kamenica and Gentzkow (2011), Bergemann and Morris (2019)). In this literature, the persuader's preference is often given exogenously and is simplified to be independent of the states (Gentzkow and Kamenica (2016)), depend only on the posterior mean (Dworczak and Martini (2019)) or semi uppercontinuous in beliefs (Dworczak and Kolotilin (2019), Dizdar and Kováč (2020)). The state space or action space are often simplified to be finite (Kamenica and Gentzkow (2011)) or even binary (Rayo and Segal (2010), Alonso and Câmara (2016), Kolotilin (2018), Aridor and Gonçalves (2021)). Without these assumptions, the three popular tools in Bayesian persuasion are not always tractable: concavification (Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Galperti and Perego (2018), Dworczak and Kolotilin (2019), Dworczak and Martini (2019), Dizdar and Kováč (2020)). In this paper, I demonstrate that even without the above assumptions, a Bayesian persuasion problem can still be tractably analyzed by focusing on value-switching monotone recommendations rules.

Online Targeted Advertisements

Intermediaries often sell online targeted advertisements by auctioning positions of products in search results. Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) study the generalized second-price position auction and find that it has a unique perfect Bayesian equilibrium that is outcome equivalent to that under Vickery-Clark-Groves mechanism (Vickrey (1961), Clarke (1971), Groves (1973)). Positions of products in their models, however, do not convey any information about match values for the consumer. Athey and Ellison (2011) studies a position auction under which sellers that are a better match for the consumer make higher bids, so that the higher positions convey higher match values, and emphasizes the informational role of search engines. Complementary to these papers, I focus on the intermediary's role as an information provider and allow the intermediary to use a fully flexible set of mechanisms instead of the specific protocols of position auctions.

1.2. Example

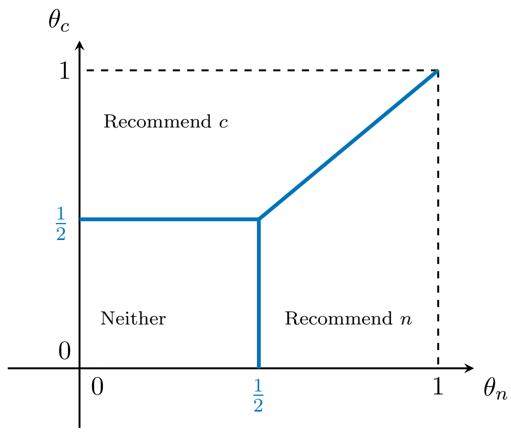
Consider a situation where a potential consumer searches *bug spray* on Amazon in a world where there are only two bug spray products on Amazon, *chemical* and *natural*. Amazon has one spot for a *sponsored product* that appears at the top of the search list.

The consumer wants to purchase a bug spray only if the product is good match for him, but without recommendations, thinks that both are unlikely to be a good match. Amazon, on the other hand, has better information about whether each product would be a good match for the consumer.[6] Formally, each product may be a good match with a probability $0 < q < \frac{1}{2}$ or a bad match with a probability $1 - q$. If the consumer chooses a product, he gets $v > 0$ if it is a good match and $-v < 0$ if a bad match. If the consumer chooses neither product, then the consumer gets 0. In the absence of additional information, the consumer does not purchase any of the products. Amazon, on the other hand, privately observes the match value $v_i \in \{v, -v\}$ for each product $i \in \{c, n\}$ where c stands for *chemical* and n for *natural*.

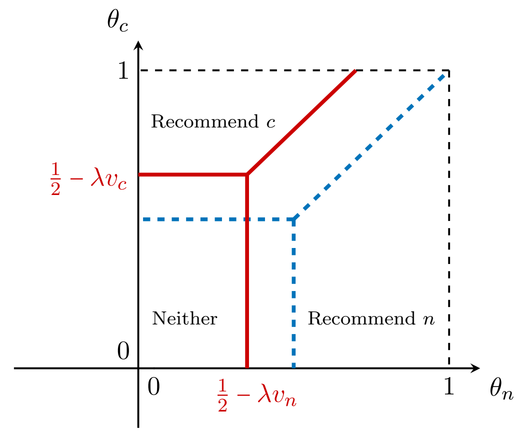
Each seller $i \in \{c, n\}$ makes a marginal profit θ_i whenever the consumer purchases seller i's product. The marginal profit θ_i is each seller's private information and is drawn from a uniform distribution over $[0, 1]$ independently of the other seller's marginal profit as well as the consumer's match values. Sellers are risk-neutral - they try to maximize their expected profits.

How should Amazon choose a sponsored product to maximize the revenue it can raise from sellers? A sensible guess is to run a second-price auction with sellers and recommend the winner's product with a reserve $\frac{1}{2}$. If sellers bid their marginal profits, the resulting recommendations rule is depicted in Figure 1a. The problem, however, is that the consumer does not purchase the recommended product because there is no information about match values in the sponsored products. For example, when the consumer sees *natural* as the

[6] For example, Amazon can infer how much the consumer will be satisfied with each product by looking up other consumers who have similar purchase histories as this particular consumer and seeing how much they are satisfied with each product.



(a) Second-price auction.



(b) Second-price auction with discounts.

Figure 1: Two auction rules for the sponsored product recommendations when $v_n > v_c$.

sponsored product, the only information that the consumer learns is that the seller of *natural* has paid more money to the intermediary. Hence, the consumer ignores the sponsored products, and sellers do not participate in the auction for sponsored products.

One way to make recommendations informative about match values is to give *discounts* to sellers based on how well their products match consumer's needs. Consider a variant of the second-price auction where the highest bidder wins the auction but is required to pay the second highest bid discounted by λv, where $\lambda > 0$. For example, if the seller *natural* bids b_n and *chemical* bids b_c with $b_n > \max\left(b_c, \frac{1}{2}\right)$, then the seller *natural* wins, but is required to pay

$$\max\left(b_c, \frac{1}{2}\right) - \lambda v_n.$$

The parameter λ governs how informative the sponsored products are about match values. The larger λ is, the greater the discount is for products that are a better match, and the more likely the sponsored product is a good match for the consumer. This encourages sellers of better products to bid higher and thus win more often. Furthermore, the discount is negative if the product is a bad match. The intermediary charges additional money when sellers of poorly matched products win the auction in order to discourage them from winning. Figure 1b depicts the resulting recommendations rule when sellers bid according to $b_i = \theta_i + \lambda v_i$ and when *natural* is a good match but *chemical* is a bad match for the consumer.

When sellers bid below the reserve, the intermediary needs to induce the consumer not to purchase any of the products. This is achieved when λ is large enough by not displaying any sponsored products. Because discounts imply that products are recommended less often when they are a worse match, when the consumer sees *no sponsored products* on his search list, the consumer understands that this is partially because the sellers did not bid high enough, but also because the products are not a good match. In the specific case in which the consumer does not buy products without recommendations, $q < \frac{1}{2}$, any $\lambda \geq 0$ successfully persuades the consumer not to purchase. When the consumer would have purchased products even without recommendations, $q \geq \frac{1}{2}$, a sufficiently high λ would persuade.

When $\lambda = 0$, displaying a sponsored product does not update the consumer's belief on match values, and the auction reduces to a standard second-price auction. When λ is positive but very small, displaying a sponsored product updates the consumer's belief about the product's match value positively, but not enough to convince him to purchase the product. There is a lowest number $\lambda^* > 0$ at which the informativeness of the sponsored product is just enough so that the consumer is indifferent between taking the sponsored product and not purchasing any of products.[7] For any $\lambda \geq \lambda^*$, the consumer purchases the sponsored product, sellers are willing to pay for sponsorship and Amazon raises positive revenue from sellers. The informativeness that maximizes Amazon's revenue is precisely λ^* that leaves the consumer indifferent between purchasing and not purchasing the sponsored product. Figure 2 depicts Amazon's revenue as a function of λ.

Note that even the lowest type seller $\theta = 0$ can win the auction if his product is a good match, and the seller receives a positive profit. Thus, Amazon can further raise its revenue

[7] As it will be shown in Section 1.4, at such λ^*, the consumer prefers the displayed sponsored product over the other product that is not sponsored.

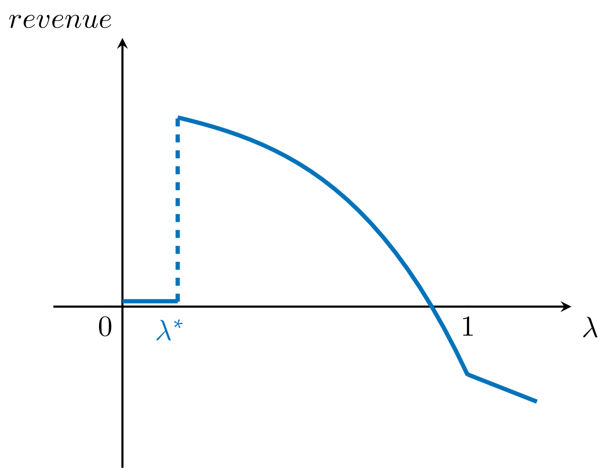


Figure 2: Amazon's revenue under the second price auction with discounts as a function of λ. When $\lambda < \lambda^*$, the consumer ignores recommendations and the sellers do not pay for recommendations. Under the optimal auction, the intermediary provides information just enough to induce the consumer to purchase recommneded products.

by collecting a *participation fee* P^* amounting to the expected profit of the lowest type $\theta = 0$ from each seller, and still induce all sellers to participate to the auction. As it will be shown in Theorem 10, the second-price auction with *discounts* $\lambda^* v$ and *participation fees* P^* as above is a revenue-maximizing mechanism in this particular setup.

1.3. Model

1.3.1. Setup

There is a consumer (he), N sellers (she) and an intermediary (it). Each seller sells one product. Each product $i \in \{1, \cdots, N\} = \mathcal{N}$ has a *match value* $v_i \in \mathbb{R}$ that is independently drawn from a common distribution F, which has a bounded support $\mathcal{V}$ with $-\infty < \inf \mathcal{V} = \underline{v} \le \bar{v} = \sup \mathcal{V} < \infty$. Only the intermediary knows the match values $\boldsymbol{v} = (v_1, \cdots, v_N) \in \boldsymbol{\mathcal{V}} = \mathcal{V}^N$ of the products; the consumer and sellers do not.

The consumer may choose from one of N products or his outside option. If the seller i's product is purchased, the consumer receives utility v_i. If the consumer does not purchase any of the products and chooses the outside option, the consumer receives utility v_0, a value commonly known to all players. The consumer's expected payoff of choosing $i \in \mathcal{N} \cup \{0\}$

with probability r_i when match values are $\boldsymbol{v}$ is

$$\sum_{i \in \mathcal{N} \cup \{0\}} v_i r_i.$$

Each seller $i \in \mathcal{N}$ has an ex-post profit

$$(\theta_i + w(v_i))r_i - t_i$$

where $\theta_i + w(v_i)$ is the seller i's (ex-post) *willingness to pay*, r_i is the probability of the consumer purchasing the product i, and t_i is a transfer that the seller i pays to the intermediary.

The willingness to pay consists of two parts. The first part θ_i is *private willingness to pay*. This is derived from the seller- or product-specific information that seller i privately knows, such as its marginal cost,[8] and is independently drawn from a common distribution G that has a support $[\underline{\theta}, \bar{\theta}] = \Theta$, where $-\infty < \underline{\theta} < \bar{\theta} < \infty$. The distribution G admits a density function g and satisfies Myerson's regularity condition, that is, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}$ is strictly increasing.

The second part $w(v_i)$ is the *value-dependent willingness to pay*, which is a part of the seller's profit that is increasing in the match value v_i. This reflects the observation that sellers prefer consumers who are a better match for their products. For example, consumers who are a better match would be more likely to repurchase the product, which increases the sellers' willingness to pay for a recommendation.[9] The value-dependent willingness to pay is a reduced-form way to capture such interactions between the consumer and sellers.

[8] For example, if each product's price p_i is public knowledge and marginal cost c_i is each seller's private knowledge, then private willingness to pay is $\theta_i = p_i - c_i$.

[9] Consumers who are a better match would be less likely to return products, which leads to higher profits and hence willingness to pay. Similarly, better matched consumers are more likely to purchase products after clicking the advertisements.

1.3.2. Recommender Systems

The intermediary knows $\boldsymbol{v}$, but does not know $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N) \in \boldsymbol{\Theta} = \Theta^N$. Before learning $\boldsymbol{v}$, the intermediary designs and commits to a recommendations rule $\boldsymbol{r}$ and a transfer $\boldsymbol{t}$. I call this pair a *recommender system*. Formally, a recommender system is

$$(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$$

such that $\sum_{i \in \mathcal{N} \cup \{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. The recommender system specifies with what probability to recommend option i, $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ and how much each seller i pays the intermediary, $t_i(\boldsymbol{v}, \boldsymbol{\theta})$, when the intermediary observes $\boldsymbol{v}$ and sellers report as $\boldsymbol{\theta}$.

Given a recommender system $(\boldsymbol{r}, \boldsymbol{t})$, when recommended with $i \in \mathcal{N} \cup \{0\}$, the consumer updates his beliefs on the expected value of each option and chooses the option with the highest expected value. The constraint for the consumer to optimally take the recommended option i over another option j is called an *obedience constraint from i to j*, which formally is written as

$$OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}). \tag{1.1}$$

where $\boldsymbol{F}(\boldsymbol{v}) = \prod_{i \in \mathcal{N}} F(v_i)$ and $\boldsymbol{G}(\boldsymbol{\theta}) = \prod_{i \in \mathcal{N}} G(\theta_i)$. Note that OB_{ij} is trivially satisfied if the intermediary does not recommend i almost surely. The recommender system $(\boldsymbol{r}, \boldsymbol{t})$ is *obedient* if all OB_{ij} are satisfied for all $i, j \in \mathcal{N} \cup \{0\}$. Since the transfer $\boldsymbol{t}$ is irrelevant for obedience, I interchangeably use the obedience of a recommender system and of the corresponding recommendations rule $\boldsymbol{r}$. For each seller $i \in \mathcal{N}$ with θ_i reporting truthfully as θ_i, her expected profit is

$$\Pi_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big((\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}) \Big) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}),$$

where $\boldsymbol{\Theta}_{-i} = \Theta^N$ and $\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i}) = \prod_{j \in \mathcal{N} \setminus \{i\}} G(\theta_j)$. The expected probability of seller i's product being recommended is

$$Q_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i}).$$

The recommender system is incentive compatible if, for all $i \in \mathcal{N}$ and $\theta_i, \theta_i' \in \Theta$,

$$IC_i : \Pi_i(\theta_i) \geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} \Big[(\theta_i + w(v_i))r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i})\Big]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}), \tag{1.2}$$

and individually rational if, for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$

$$IR_i : \Pi_i(\theta_i) \geq 0. \tag{1.3}$$

Applying the revelation principle arguments from mechanism design (Myerson (1981)) and information design (Bergemann and Morris (2019)),[10] the intermediary can restrict attention to obedient, incentive compatible and individually rational recommender systems without loss of generality. Using such recommender systems, the intermediary maximizes the expected revenue

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} t_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).$$

A defining feature of my model is that the intermediary is solving a revenue-maximizing mechanism design problem, but with a constraint. Instead of the intermediary choosing the outcome, i.e. which option to choose, the consumer chooses the best outcome for himself given the information provided by the intermediary. The intermediary designs

[10] The intermediary, in principle, may attempt to provide information in more flexible ways than recommendations. That is, the intermediary can design and commit to a pair of an information structure $(\sigma, \mathcal{S})$ where

$$\sigma : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to \Delta\mathcal{S}.$$

and a transfer $\boldsymbol{t} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to \mathbb{R}^N$, instead of a recommender system. By the revelation principle, an outcome of such an indirect mechanism can always be represented as an outcome of an obedient, incentive compatible and individually rational recommender system, so the intermediary can restrict attention to such recommender systems without loss of generality. See Bergemann and Morris (2019) for details.

recommendations to be informative enough so that the consumer chooses the outcome that intermediary wants him to choose. If the intermediary were able to choose the outcome by itself, then its problem is a standard optimal auction design problem (Myerson (1981)).

Timing of the Game

itemsep=.05mm Intermediary offers and commits to a recommender system $(\boldsymbol{r},\boldsymbol{t}):\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\to[0,1]^{N+1}\times\mathbb{R}^N$ where $\sum_{i\in\{0\}\cup\mathcal{N}} r_i(\boldsymbol{v},\boldsymbol{\theta})=1$ for all $(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}$.

iitemsep=.05mm Intermediary observes the consumer's match values $\boldsymbol{v}$. Sellers report their private information $\boldsymbol{\theta}$.

iiitemsep=.05mm Intermediary recommends an action and collects transfers according to $(\boldsymbol{r}(\boldsymbol{v},\boldsymbol{\theta}),\boldsymbol{t}(\boldsymbol{v},\boldsymbol{\theta}))$.

ivtemsep=.05mm Consumer gets a recommendation and takes an action.

1.4. Optimal Recommender System

In this section, I characterize the optimal recommender system using a class of recommendations that I call *value-switching monotone*.

1.4.1. Intermediary as a Bayesian Persuader

Notice that the transfer $\boldsymbol{t}$ is irrelevant for the consumer's obedience constraints, so that the standard characterization of incentive compatible and individually rational recommender system (Myerson (1981)) applies for any obedient recommendations rule $\boldsymbol{r}$.

Lemma 1. *An obedient $(\boldsymbol{r},\boldsymbol{t})$ recommender system is incentive compatible and individually rational if and only if, for all $i\in\mathcal{N}$ and $\theta_i\in\Theta$,*

$$Q_i(\theta_i) \text{ is increasing in } \theta_i, \tag{1.4}$$

$$\Pi_i(\theta)=\Pi_i(\underline{\theta})+\int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i, \tag{1.5}$$

$$\Pi_i(\underline{\theta})\geq 0. \tag{1.6}$$

The standard arguments of substituting the expected revenue with the virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$, dropping the incentive compatibility and individual rationality constraints, and setting the lowest type's expected profit to zero apply as well.

Lemma 2. *Suppose that a recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^N$ *maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{1.7}$$

subject to obedience constraints (1.1) *and monotonicity constraints* (1.4)*. Suppose also that*

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}) d\tilde{\theta}_i. \tag{1.8}$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ *is an optimal recommender system.*

Ignoring the monotonicity constraints, Lemma 2 recasts the intermediary's revenue maximization problem as a Bayesian persuasion problem that only uses a recommendations rule $\boldsymbol{r}$. In this Bayesian persuasion problem, the intermediary persuades the consumer to take the recommended option by strategically releasing information about $(\boldsymbol{v}, \boldsymbol{\theta})$. The problem has the following features: The intermediary's state-dependent preference over recommendations is given by its virtual willingness to pay; the consumer can choose from $N + 1$ options; the state space problem is multi-dimensional and possibly infinite.

Each feature of the problem brings a difficulty in applying those three popular approaches in Bayesian persuasion literature: concavification (Aumann and Maschler (1995), Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Galperti and Perego (2018), Dworczak and Kolotilin (2019), Dworczak and Martini (2019)).[11] Whenever departing away from the three approaches, the immediate challenge lies in identifying which of the obedience constraints bind and not

[11] Concavification has limited applicability when state space is large (Gentzkow and Kamenica (2016)). Convex function characterization necessarily assumes the sender's payoff to depend only on the expected value of the states (Gentzkow and Kamenica (2016)). Duality approach often assumes state space to be either an interval or discrete (Kolotilin (2018), Galperti and Perego (2018)).

bind at the optimal recommendations rule. With $N+1$ options, there are $\frac{N(N+1)}{2}$ obedience constraints to check, a seemingly daunting task. I overcome this challenge by applying the guess and verify method using value-switching monotone recommendations rules.

1.4.2. Value-Switching Monotone Recommendations Rule

Definition 1. A recommendations rule $\boldsymbol{r}$ is *value-switching monotone* if

1. $r_0(\boldsymbol{v},\boldsymbol{\theta})$ decreases in (v_i,θ_i) for all $i \in \mathcal{N}$.

2. $r_i(\boldsymbol{v},\boldsymbol{\theta})$ increases in (v_i,θ_i) for all $i \in \mathcal{N}$.

3. $r_i(\boldsymbol{v},\boldsymbol{\theta})$ decreases whenever v_j is switched with a larger v_i for all $i,j \in \mathcal{N}$, i.e. for all $i,j \in \mathcal{N}$, $(\boldsymbol{v}_{-ij},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}_{-ij} \times \boldsymbol{\Theta}$ and $v > v'$,

$$r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}).$$

A natural-sounding alternative to the third condition above is the standard notion of monotonicity, under which $r_i(\boldsymbol{v},\boldsymbol{\theta})$ decrease in v_j for all $j \in \mathcal{N} \setminus \{i\}$. The standard notion of monotonicity is stronger than value-switching monotonicity. It will later be shown that optimal recommendations rules are value-switching monotone, but not monotone. Value-switching monotonicity requires $r_i(\boldsymbol{v},\boldsymbol{\theta})$ to be increasing in θ_i to ensure the monotonicity constraints (1.4) satisfied, but does not require any particular behavior in respect to $\boldsymbol{\theta}_{-i}$.

The following lemma states that the intermediary can ignore the obedience constraints between products as long as the intermediary uses a value-switching monotone recommendations rule.

Lemma 3. *Any value-switching monotone recommendations rule $\boldsymbol{r}$ satisfies obedience constraints between products, i.e. OB_{ij} for all $i,j \in \mathcal{N}$.*

Lemma 3 reduces the number of possibly binding constraints to check from $\frac{N(N+1)}{2}$ to $2N$. The remaining obedience constraints are one of the two types of obedience constraints:

obedience constraints from outside option to products,

$$OB_{0i}: \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0, \tag{1.9}$$

and those from products to outside option.

$$OB_{i0}: \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) r_i(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{1.10}$$

The lemma below states that whether the remaining obedience constraints are satisfied for a given value-switching recommendations rule depends on two thresholds.

Lemma 4. *For any value-switching monotone recommendations rule $\boldsymbol{r}$, for each $i \in \mathcal{N}$, there are $-\infty \leq \underline{v}_i \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}_i \leq \infty$ such that*

1. *OB_{i0} is satisfied if and only if $v_0 \leq \bar{v}_i$,*

2. *OB_{0i} is satisfied if and only if $v_0 \geq \underline{v}_i$,*

where

$$\bar{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) & \textit{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) > 0 \\ \infty & \textit{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) = 0 \end{cases}$$

and

$$\underline{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) & \textit{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) > 0 \\ -\infty & \textit{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) = 0 \end{cases}.$$

The first part of Lemma 4 states that OB_{i0} is satisfied if and only if the outside option value is below the threshold $\bar{v}_i$. A lower outside option value provides less incentive for the consumer to take the outside option over the recommended product i, and hence, it is easier to satisfy OB_{i0}. If the intermediary recommends i with positive probability, then OB_{i0} is

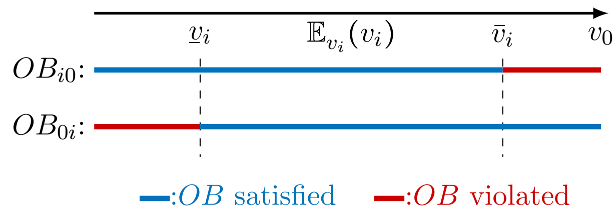


Figure 3: A graphical illustration of regions on which OB_{i0} and OB_{0i} are satisfied and violated.

equivalent to

$$v_0 \le \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i(\boldsymbol{v},\boldsymbol{\theta})) = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))},$$

which implies the threshold $\bar{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))}$. If the intermediary does not recommend i almost surely, the intermediary does not worry about keeping the consumer incentivized to purchase the recommended product i over the outside option regardless of the outside option value. In other words, OB_{i0} is trivially satisfied for all v_0, which implies the threshold $\bar{v}_i = \infty$. The second part about the other threshold $\underline{v}_i$ for OB_{0i} may be explained in a similar manner.

Lemma 4 also states that a product's ex-ante expected value $\mathbb{E}_{v_i}(v_i)$ has to be in-between the two thresholds, that is, $\underline{v}_i \le \mathbb{E}_{v_i}(v_i) \le \bar{v}_i$. To see why this has to be the case, consider $v_0 = \mathbb{E}_{v_i}(v_i)$. The consumer is ex-ante indifferent between all products and the outside option, so that he follows any recommendations as long as recommendations contain some (or no) information about match values, i.e. $Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta})) \ge 0$ and $Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta})) \le 0$. The value-switching monotonicity requires $r_i(\boldsymbol{v},\boldsymbol{\theta})$ to be increasing and $r_0(\boldsymbol{v},\boldsymbol{\theta})$ to be decreasing in v_i, which ensures that recommendations are informative about the match values. Therefore, when $v_0 = \mathbb{E}_{v_i}(v_i)$, any value-switching monotone recommendations rule satisfies all obedience constraints. All obedience constraints continue to hold as long as v_0 is close enough to $\mathbb{E}_{v_i}(v_i)$, i.e. $v_0 \in [\underline{v}_i, \bar{v}_i]$. See Figure 3 for a graphical illustration of Lemma 4.

In particular, if the intermediary runs an optimal auction with sellers ignoring the obedience constraints, the resulting recommendations rule is value-switching monotone and satisfies all obedience constraints as long as v_0 is close enough to $\mathbb{E}_{v_i}(v_i)$. Let $\boldsymbol{\rho}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ denote the resulting recommendations rule that I call by the *unconstrained optimal recommendations rule* and is given by

$$\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \mathcal{M} \text{ and } \theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{1.11}$$

where $\mathcal{M} = \{i \in \mathcal{M} \mid \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}\}$. That is, $\boldsymbol{\rho}^*$ is characterized by recommending the product of the seller with the highest non-negative virtual willingness to pay, and the outside option if all sellers' virtual willingness to pay is negative (Myerson (1981)). Note that $\boldsymbol{\rho}^*$ is symmetric,[12] so that the thresholds $\bar{v}_i$ and $\underline{v}_i$ are identical across all products $i \in \mathcal{N}$. Let $\bar{v}^*$ and $\underline{v}^*$ be the respective common thresholds.

1.4.3. Optimal Recommender System

The intermediary's problem is linear in $\boldsymbol{r}$, so that the method of Lagrangean is both necessary and sufficient for an optimal solution. The following theorem characterizes an optimal recommender system when $v_0 \in (\underline{v}, \bar{v})$.

Theorem 1.a. *Let $v_0 \in (\underline{v}, \bar{v})$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,*

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \textit{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)} + w(v_j)}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^*(\boldsymbol{v})}_{\substack{\textit{cost of} \\ \textit{persuasion}}}, 0 \right\} \\ 0 & \textit{otherwise} \end{cases} \tag{1.12}$$

[12] A recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v}, \boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$$

where $(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$ is such that $v^\iota_{\iota(i)} = v_i$ and $\theta^\iota_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v})\right\}$, *and*

$$\ell_i^*(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^*, \bar{v}^*] \\ \lambda_1^*(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^* \\ \lambda_2^*(v_0)\cdot\sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^* \end{cases} \tag{1.13}$$

where $\lambda_1^*(v_0)$ *and* $\lambda_2^*(v_0)$ *are Lagrangian multipliers for* OB_{i0} *and* OB_{0i} *that may vary depending on* v_0*, respectively. Let* $\boldsymbol{t}$ *be as in* (1.8)*. Then,* $\boldsymbol{r}^*$ *is value-switching monotone, and* $(\boldsymbol{r}^*, \boldsymbol{t})$ *is an optimal recommender system.*

The optimal recommendations rule $\boldsymbol{r}^*$ is characterized by recommending a product with the highest non-negative virtual willingness to pay adjusted for the cost of persuasion, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) - \ell_i^*(\boldsymbol{v})$, and the outside option if the adjusted virtual willingness to pay is negative for all sellers. The cost of persuasion is the shadow price of each of the binding constraints.

To gain intuition for Theorem 1.a, consider first the intermediary running an optimal auction with sellers and recommending the winner's product ignoring the obedience constraints, i.e. $\boldsymbol{\rho}^*$ that recommends only based on the virtual willingness to pay. If the consumer always follows the recommendations, $\boldsymbol{\rho}^*$ is the revenue-maximizing recommendations rule. By Lemma 4, the consumer optimally follows recommendations from $\boldsymbol{\rho}^*$ if $v_0 \in [\underline{v}^*, \bar{v}^*]$. Since none of the obedience constraints bind, the cost of persuasion ℓ_i^* is zero.

When $v_0 > \bar{v}^*$ or $v_0 < \underline{v}^*$, the unconstrained optimal recommendations rule $\boldsymbol{\rho}^*$ fails in persuading the consumer to take recommended options. To provide incentive for the consumer to take the recommended options, the intermediary needs to recommend products more often when match values are high and less often otherwise, so that the recommendations would be more informative about match values. To the extent that the intermediary cannot recommend based on virtual willingness to pay, there is a loss of revenue associated with keeping the recommendations informative. The optimal way to improve the informativness is to adjust the virtual willingness to pay with the cost of persuasion, the shadow price of the obedience constraints.

For outside option values that are always above or below the value of the products, the optimal recommender system is characterized in the following theorem.

Theorem 1.b. *1. Let $v_0 > \bar{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that*

$$r_0^*(\boldsymbol{v}, \boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. Let $\boldsymbol{t}$ be as in (1.8). Then, $\boldsymbol{r}^*$ is $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

2. Let $v_0 < \underline{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^*|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\} \\ 0 & \text{otherwise} \end{cases} \tag{1.14}$$

where $\mathcal{M}^* = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\}$ and

$$\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0 \text{ for all } i \in \mathcal{N} \text{ and } (\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \tag{1.15}$$

Let $\boldsymbol{t}$ be as in (1.8). Then, $\boldsymbol{r}^*$ is value-switching monotone, and $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

When $v_0 > \bar{v}$, the consumer always prefers the outside option over the products. For such a consumer, the only obedient recommendations rule is to always recommend the outside option. When $v_0 < \underline{v}$, the consumer always prefers products over the outside option, but does not know which product he prefers the most. The intermediary is restricted to recommend products only, but not the outside option. Consequently, the intermediary always recommends the product with the highest virtual willingness to pay, even though it may be negative.

When $v_0 = \bar{v}$, the intermediary is restricted to recommending outside option except for when $v_i = \bar{v}$ for some $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay among the products that have the valuation of $\bar{v}$. When $v_0 = \underline{v}$, the intermediary is restricted to recommending products except for when $v_i = \underline{v}$ for all $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay.

1.5. Additional Information

This section analyzes how the intermediary's use of additional information about sellers' private willingness to pay affects the consumer surplus, the intermediary's revenue and the sellers' profits. I reformulate the additional information as a change in the intermediary's preference over recommendations to provide sufficient conditions under which the additional information benefits the consumer and sellers.

1.5.1. Optimal Recommender System with Additional Information

I begin with extending the baseline model of Section 1.3 to incorporate the additional information. The intermediary observes *additional signals* $\boldsymbol{z} = (z_1, \cdots, z_N)$ about sellers' private information $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)$. Each $z_i \in \mathcal{Z} \subset \mathbb{R}$ is independently drawn from a common distribution $H(\cdot \mid \theta_i)$ conditioning on each θ_i, and is common knowledge between a seller i and the intermediary, but not known to others.[13] Let $\mathcal{H} = \{H(\cdot \mid \theta)\}_{\theta \in \Theta}$ be the *additional information*, a collection of distribution functions conditioning on each $\theta \in \Theta$. Let $\mathcal{Z}(\theta)$ be the support of $H(\cdot \mid \theta)$, and $\Theta(z)$ be the set of states at which z is generated with a positive probability. I present two examples of additional information below.

Example 1 (Perfectly revealing additional information). Additional information $\mathcal{H}$ is *per-*

[13]More generally, it may be assumed that each i observes a signal ζ_i about additional signals about others $\boldsymbol{z}_{-i}$ without affecting any of the results. The signal ζ_i may be uninformative about $\boldsymbol{z}_{-i}$ as in here, may be completely revealing or may be related with $\boldsymbol{z}_{-i}$ in any arbitrary way.

fectly revealing if $\mathcal{Z} = [\underline{\theta}, \bar{\theta}]$ and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } z \geq \theta \\ 0 & \text{if } z < \theta \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ and $\Theta(z) = \{z\}$. $\square$

Example 2 (Lower censorship additional information)**.** Additional information $\mathcal{H}$ is *lower censorship* if it reveals θ if $\theta \geq \theta^*$, but does not reveal otherwise. Formally, $\mathcal{Z} = \{z_0\} \cup [\theta^*, \bar{\theta}]$ where $z_0 < \theta^*$, and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } \theta \geq \theta^* \text{ and } z \geq \theta, \text{ or } \theta < \theta^* \text{ and } z \geq z_0 \\ 0 & \text{otherwise} \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ when $\theta \geq \theta^*$ and $\mathcal{Z}(\theta) = \{z_0\}$ when $\theta < \theta^*$, and $\Theta(z) = \{z\}$ when $z \geq \theta^*$ and $\Theta(z) = [\underline{\theta}, \theta^*]$ and $z = z_0$ $\square$

The state space is $\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. The intermediary's recommender system is

$$(\mathbf{r}, \mathbf{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$$

such that $\sum_{i \in \mathcal{N} \cup \{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. The obedience, incentive compatibility and individual rationality are defined in the standard manner. Define

$$Q_i(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i})$$

to be the probability of recommending the seller i's product when her private willingness to pay is θ_i and additional signal is z_i. Applying the standard arguments gives the following lemma.

Lemma 5. *Suppose that a recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^N$ *maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{1.16}$$

subject to obedience constraints

$$OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{1.17}$$

and monotonicity constraints, i.e. for all $i \in \mathcal{N}$, $\theta_i \in \Theta$ *and* $z_i \in \mathcal{Z}$, $Q_i(\theta_i, z_i)$ *increases in* θ_i. *Suppose also that*

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) d\tilde{\theta}_i. \tag{1.18}$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ *is an optimal recommender system.*

Similar arguments of using value-switching recommendations rules from Section 1.4 may be applied to characterize the optimal recommender system.

Theorem 2.a. *Let* $v_0 \in (\underline{v}, \bar{v})$. *Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ *be a recommendations rule such that for each* $i \in \mathcal{N}$,

$$r_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1 - G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)} + w(v_j)}_{\text{virtual willingness to pay}} - \underbrace{\ell_j^A(\boldsymbol{v})}_{\substack{\text{cost of} \\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{1.19}$$

where $\mathcal{M}^A = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j) - \ell_j^A(\boldsymbol{v})\right\}$*, and*

$$\ell_i^A(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^A, \bar{v}^A] \\ \lambda_1^A(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^A \\ \lambda_2^A(v_0)\cdot\sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^A \end{cases} \tag{1.20}$$

where $\lambda_1^A(v_0)$ *and* $\lambda_2^A(v_0)$ *are Lagrangian multipliers for* OB_{i0} *and* OB_{0i} *that may vary depending on* v_0*, respectively, and* $\bar{v}^A$ *and* $\underline{v}^A$ *are the thresholds from the unconstrained optimal recommendations rule. Let* $\boldsymbol{t}$ *be as in* (1.18)*. Then,* $\boldsymbol{r}^A$ *is value-switching monotone, and* $(\boldsymbol{r}^A, \boldsymbol{t})$ *is an optimal recommender system.*

For outside option values that are always above or below the value of the products, the optimal recommender system is characterized in the following theorem.

Theorem 2.b. *1. Let* $v_0 > \bar{v}$*. Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ *be a recommendations rule such that*

$$r_0^A(\boldsymbol{v}, \boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$*. Let* $\boldsymbol{t}$ *be as in* (1.8)*. Then,* $(\boldsymbol{r}^A, \boldsymbol{t})$ *is an optimal recommender system.*

2. Let $v_0 < \underline{v}$*. Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ *be a recommendations rule such that for each* $i \in \mathcal{N}$*,*

$$r_i^A(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j)\right\} \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

where $\mathcal{M}^A = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\right\}$ *and*

$$\ell_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0 \text{ for all } i \in \mathcal{N} \text{ and } (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \tag{1.22}$$

Let $\boldsymbol{t}$ be as in (1.18). Then, $\boldsymbol{r}^A$ is value-switching monotone and $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}^A, \boldsymbol{t})$ is an optimal recommender system.

It remains to analyze how an optimal recommendations rule with additional information $\boldsymbol{r}^A$ is different from that without additional information $\boldsymbol{r}^*$, and how does the difference impact on consumer surplus, intermediary's revenue and sellers' profits. I begin the analysis with recasting the additional information as a change in the intermediary's preference.

1.5.2. Additional Information as Change in Intermediary's Preference

A key observation is that additional information changes the intermediary's state-dependent preference over recommendations, but nothing else. The intermediary's persuasion problem without additional information, i.e. maximizing (1.7) subject to (1.1), can be reformulated as maximizing

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{1.23}$$

subject to obedience constraints (1.17). Although a recommendations rule is allowed to vary depending on additional signals $\boldsymbol{z}$, the optimal solution ignores $\boldsymbol{z}$ because the integrands of both the objective function (1.23) and the constraints (1.17) do not depend on $\boldsymbol{z}$, resulting in the same solution as maximizing (1.7) subject to (1.1).

Comparing the intermediary's persuasion problem without and with additional information, the only difference is the inverse hazard rates in the intermediary's preference. Without additional information, the intermediary's preference is given by the virtual willingness to pay,

$$\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i).$$

With additional information, the inverse hazard rate is conditioned on each additional signal,

$$\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i).$$

That is, additional information changes the intermediary's preference through inverse hazard rates, but nothing else.

The following definition is useful in capturing the change in the intermediary's preference caused by additional information.

Definition 2. Let $\mathcal{H}$ be additional information. A *θ-revenue difference* for $\theta > \theta'$ without additional information is

$$\Delta(\theta, \theta') = \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right)$$

A θ-revenue difference with additional signals $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$ is

$$\Delta^{z,z'}(\theta, \theta') = \left(\theta - \frac{1 - G(\theta | z)}{g(\theta | z)}\right) - \left(\theta' - \frac{1 - G(\theta' | z')}{g(\theta' | z')}\right).$$

The θ-revenue difference without additional information measures an increase in virtual willingness to pay by recommending a product with higher θ over that with lower θ' holding others fixed. In other words, this measures how much the revenue increases as θ increases. By Myerson's regularity,

$$\Delta(\theta, \theta') > 0.$$

The θ-revenue difference with additional information measures the same except that the additional signals z and z', each corresponding to θ and θ', may be different from each other.

Example 3. To understand why θ-revenue difference is useful, consider an environment with 2 products $\{i, j\}$ where $\mathcal{V} = [\underline{v}, \bar{v}]$ and $\Theta = [\underline{\theta}, \bar{\theta}]$. Suppose that $w(v)$ strictly increases in v. Each product i is characterized by a pair (v, θ). The area inside the dashed square in Figure 4 is the space of all possible pairs for the product i. Let (v_j, θ_j) be the pair for the product j. An iso-revenue curve at (v_j, θ_j) is a set of points (v_i, θ_i) that gives the same revenue, the virtual willingness to pay, as (v_j, θ_j), and is drawn as a blue curve in Figure 4. Since the virtual willingness to pay increases in v and θ, the increasing direction

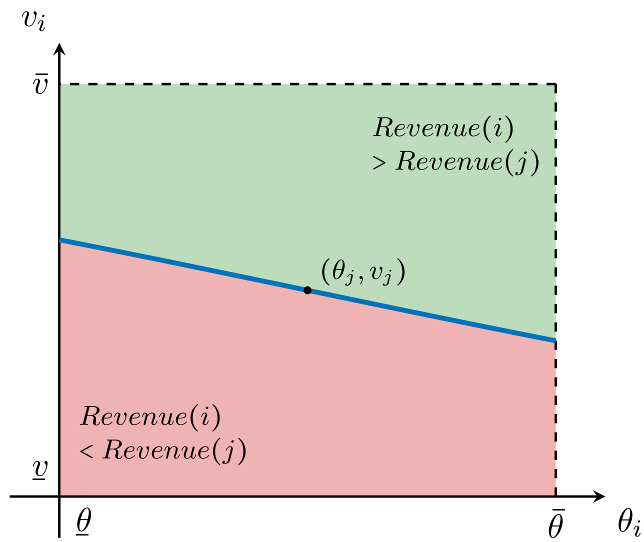


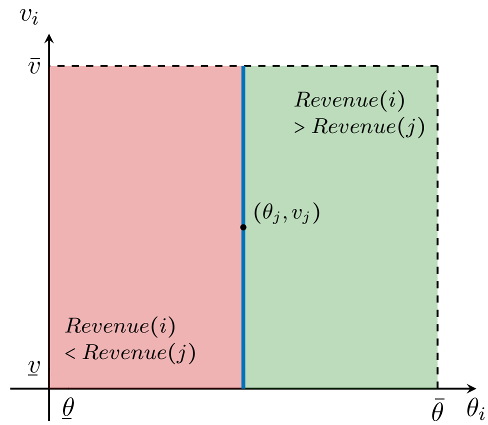
Figure 4: Iso-revenue curve.

of the iso-revenue curve is northeast. If the product i's pair (v_i, θ_i) is above the indifference curve, then the intermediary gets more revenue by recommending i over j; if below, then otherwise.

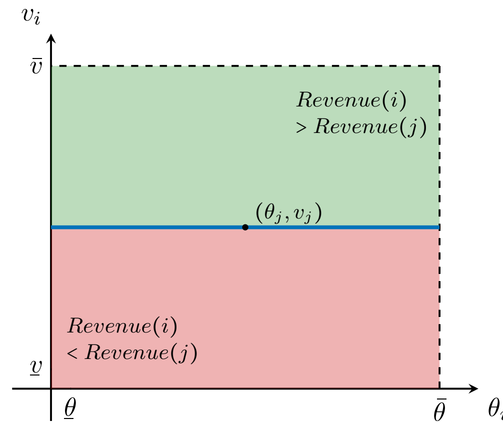
Assume, for simplicity, that the intermediary always recommends products based on the revenue and the consumer always follows the recommendations.

The higher the slope of the iso-revenue curve is, the more likely to recommend a product with higher θ, the lower consumer surplus is. One extreme case is in Figure 5a where the slope is so high that the iso-revenue is a vertical line. Under this iso-revenue curve, the intermediary recommends whichever product has the highest θ. If the consumer follows the recommendation, the consumer payoff is low because the recommendations do not reflect match values at all. Another extreme case is in Figure 5b where the slope is so low that the iso-revenue is a horizontal line. The intermediary recommends whichever has the highest v. If the consumer follows the recommendation, the consumer payoff is high because the recommendations are made only based match values.

Whether additional information benefits the consumer depends on whether additional in-



(a) Vertical Iso-revenue Curve



(b) Horizontal Iso-revenue Curve

Figure 5: Extreme cases.

formation decreases the slope of the iso-revenue curve. It can be shown that the additional information decreases the slope for every point $(\boldsymbol{v}, \boldsymbol{\theta})$ if it decreases θ-revenue difference, i.e. for all $\theta, \theta' \in \Theta$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') \geq \Delta(\theta, \theta'),$$

and hence, benefits the consumer. The opposite holds as well: Additional information increases the slope if it increases θ-revenue difference, and hence, harms the consumer. $\square$

The above example illustrates the main intuitions behind how additional information changes the intermediary's preference and the consumer surplus through θ-revenue difference. However, there are two caveats. First, the graphical analysis only applies to how recommendations change between products, not between a product and the outside option. Second, the consumer in this example is assumed to always follow the recommendations. These caveats motivate a class of additional information and environment under which the intuition well-applies.

1.5.3. Consumer Surplus under Small Inverse Hazard Rates Environment

Definition 3. Additional information $\mathcal{H}$ is *well-behaving* if it

1. satisfies *generalized Myerson's regularity* if for all $\theta > \theta'$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') > 0.$$

2. *increases (decreases) θ-revenue differences* if for all $\theta > \theta'$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') \geq (\leq) \Delta(\theta, \theta').$$

The first condition requires the virtual willingness to pay to be strictly increasing in θ no matter the additional signals. The second condition requires θ-revenue to be uniformly increasing or decreasing for all pairs of θ and z. Together, it increases or decreases the downward sloping iso-revenue curve as in Example 3.

Example 1, cont. Let $\mathcal{H}$ be a perfectly revealing additional information. The perfectly revealing additional information is well-behaving, and decreases (increases) θ-revenue difference if and only if $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.

□

Example 2, cont. Let $\mathcal{H}$ be a lower censorship additional information with θ^*. Let G be a distribution that has a decreasing inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ on $[\underline{\theta}, \bar{\theta}]$ and has a density function such that for some neighborhood $B(\underline{\theta})$ of $\underline{\theta}$, $\inf_{\theta \in B(\underline{\theta})} g(\theta) > 0$ and $\sup_{\theta \in B(\underline{\theta})} g^{'}(\theta) < \infty$. This nests a rich class of distributions including uniform distribution, linear virtual valuation distribution, (truncated) normal distribution, (truncated) exponential distribution and unimodal distribution with appropriate restrictions.

For sufficiently small θ^*, the lower censorship additional information is well-behaving, and always decreases θ-revenue difference. □

Definition 4. A triple $(G, \mathcal{H}, w)$ is said to have *small inverse hazard rates* if for all $i \in \mathcal{N}$

$$\inf_{v_i \in \mathcal{V}, \theta_i \in \Theta} \theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) > 0$$

and

$$\inf_{v_i \in \mathcal{V}, \theta_i \in \Theta, z_i \in \mathcal{Z}} \theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) > 0.$$

A small inverse hazard rates environment is likely to arise when sellers' marginal profit through recommender systems are high relative to their costs. For example, when online targeted advertisements often have better returns than other media (Hu, Shin, and Tang (2016)) or generate more revenue per ad and higher conversion rates than non-targeted ads (Howard (2010)), the environment is likely to have small inverse hazard rates.

In a small inverse hazard rates environment, the intermediary always prefer recommending products over the outside option. That is, the intermediary does not recommend the outside option unless doing so is necessary for the persuasion. Recommending the outside option is required only when the outside option value is very high, under which OB_{i0} binds and the consumer surplus is zero with and without additional information. When the outside option value is lower, the intermediary always recommends products over the outside option, and the graphical analysis from Example 3 applies.

The consumer surplus under a recommendations rule $\boldsymbol{r}$ at v_0 is

$$CS(v_0; \boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left[\sum_{i \in \{0\} \cup \mathcal{N}} (v_i - u^*) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$

where $u^* = \max(v_0, \mathbb{E}_{v_i}(v_i))$ is the consumer's optimal payoff without the recommendations.

Theorem 3. *Consider a small inverse hazard rates environment. Let $\mathcal{H}$ be any well-behaving additional information. Additional information increases (decreases) consumer surplus for all v_0 if it decreases (increase) θ-revenue difference.*

Example 1, cont. By Theorem 3, the perfectly revealing additional information increases (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.

$\square$

Recall that many 'natural' distributions (uniform, normal, exponential, log-concave, etc.) have decreasing $\frac{1-G(\theta)}{g(\theta)}$. Therefore, for natural distributions, the perfectly revealing additional information *increases* the consumer surplus for all consumers. This is a surprising result, as one of the grounds for restricting platforms from collecting seller data is the potential for consumer harm.[14] Instead, restricting the intermediary from collecting the most precise seller data harms all consumers by adding an information friction between the intermediary and sellers.

Example 2, cont. By Theorem 3, lower censorship additional information always increases consumer surplus for all v_0. $\square$

1.5.4. Sellers' Profits in Small Inverse Hazard Rates Environment

Additional information about sellers does not necessarily harm sellers' profits. Additional information reduces information rents, which in turn reduces sellers' profits conditioning on recommending products. However, the reduced information rent also allows the intermediary to recommend products when information rents restrained it from doing so, increasing the chance of recommending products, and hence, sellers' profits.

Example 4. Consider an environment where there is only one seller whose private willingness to pay θ is drawn from a uniform distribution over $[0,1]$. Match value v is drawn from a uniform distribution over $\{\underline{v}, \bar{v}\}$. The consumer's outside option value is $v_0 = \frac{1}{2}(\underline{v} + \bar{v})$, so that the consumer follows the intermediary's recommendations as long is it is value-switching monotone. There is no value-dependent willingness to pay $w(v_i) = 0$, so that the

[14] *European Commission*, "Antitrust: Commission sends Statement of Objections to Amazon for the use of non-public independent seller data and opens second investigation into its e-commerce business practices," https://ec.europa.eu/commission/presscorner/detail/en/ip_20_2077

seller's virtual willingness to pay is

$$\theta - \frac{1 - G(\theta)}{g(\theta)} + w(v) = 2\theta - 1.$$

Without additional information, the intermediary recommends the product if and only if $\theta \le \frac{1}{2}$ and the consumer follows the recommendations.

Consider *partitional* additional information that informs whether θ is above or below $\frac{1}{2}$, i.e. $\mathcal{Z} = \{z^L, z^H\} \subset \mathbb{R}^1$ with $z^L < z^H$ such that

$$H(z_i \mid \theta_i) = \begin{cases} 1 & \theta < \frac{1}{2} \text{ and } z \ge z^L \text{ and } \theta \ge \frac{1}{2} \text{ and } z \ge z^H \\ 0 & \text{otherwise} \end{cases}.$$

Conditioning on $z = z^H$, the seller's virtual willingness to pay $2\theta_i - 1$ is the same as before, so the intermediary recommends in the same manner and the seller gets the same profit. Conditioning on $z = z^L$, the intermediary learns that the seller has $\theta \le \frac{1}{2}$, which reduces the inverse hazard rates to $\frac{1}{2} - \theta$ and increases the virtual willingness to pay to

$$\theta - \left(\frac{1}{2} - \theta\right) = 2\theta - \frac{1}{2}.$$

With the increased virtual willingness to pay, the intermediary recommends the product for θ that it used to recommend the outside option, $\frac{1}{4} \le \theta \le \frac{1}{2}$, increasing the seller's profit. Since every type of seller earns the same or more profit than before, the additional information increases the seller's ex-ante profit.[15] $\square$

The θ-revenue difference continues to play an important role determining whether additional information harms sellers. A seller i's ex-ante expected profit without additional information is

$$\Pi_i^* = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i)}{g(\theta_i)} r_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{1.24}$$

[15] Note that the additional information is Pareto-improving in this example.

and with additional information is

$$\Pi_i^A = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1-G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} r_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{1.25}$$

Notice that two objects change from (1.24) to (1.25): The recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$ and inverse hazard rates from $\frac{1-G(\theta_i)}{g(\theta_i)}$ to $\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}$. It is helpful to separate the change in total profit by each of the changes. To this end, define a fictitious expected profit function obtained by fixing the inverse hazard rates at $\frac{1-G(\theta_i)}{g(\theta_i)}$ but changing the recommendations rule changes from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$

$$\Pi_i^F = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{1.26}$$

The total change in the seller's profit $\Pi_i^A - \Pi_i^*$ can be decomposed into two terms,

$$\underbrace{\Pi_i^A - \Pi_i^*}_{\text{total change}} = \underbrace{(\Pi_i^A - \Pi_i^F)}_{\text{inverse hazard rates effect}} + \underbrace{(\Pi_i^F - \Pi_i^*)}_{\substack{\text{recommendations}\\ \text{rule effect}}}$$

where *recommendations rule effect* $\Pi_i^F - \Pi_i^*$ captures the change in profit caused by a change in recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$, and *inverse hazard rates effect* $\Pi_i^A - \Pi_i^F$ captures the change in profit caused by a change in inverse hazard rates with and without additional information. I say the recommendations rule effect increases (decreases) all sellers' profits if $\Pi_i^F - \Pi_i^* \geq (\leq)\ 0$. The following theorem characterizes how each effect changes the profit.

Theorem 4. *Consider a small inverse hazard rates environment. Let $\mathcal{H}$ be any well-behaving additional information and $v_0 \leq \mathbb{E}_{v_i}(v_i)$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*

   (a) *Additional information increases (decreases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ increases (decreases) in θ_i.*

(b) *Additional information decreases (increases) θ-revenue difference and inverse hazard rates* $\frac{1-G(\theta)}{g(\theta_i)}$ *decreases (increases) in* θ_i.

2. *Inverse hazard rates effect decreases all sellers' profits if for all* $z_i \in \mathcal{Z}$ *and* $\theta_i \in \Theta(z_i)$,

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \leq \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

A similar intuition from Example 3 applies. An increased θ-revenue difference increases the slope of the iso-revenue curve, so that the intermediary is more likely to recommend products with higher private willingness to pay θ instead of those with higher match values v. This change increases sellers' profits if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is increasing in θ_i, but decreases if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is decreasing in θ_i. Consequently, one sufficient condition for additional information to increase sellers' profits is for it to increase θ-revenue difference and $\frac{1-G(\theta_i)}{g(\theta_i)}$ to be increasing in θ_i. By a similar argument, if additional information decreases θ-revenue difference, then it increases sellers' profits if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is decreasing in θ_i.

Let us say additional information *reduces inverse hazard rates* if for all $z_i \in \mathcal{Z}$ and $\theta_i \in \Theta(z_i)$,

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \leq \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

This is a sufficient condition for sellers' profits to be decreased through the inverse hazard rates effect. Both perfectly revealing and lower censorship additional information reduces inverse hazard rates, but not all well-behaving additional information does so.

Sufficient conditions under which additional information decreases sellers' profits are provided below.

Corollary 1. *Consider a small inverse hazard rates environment. Let* $\mathcal{H}$ *be any well-behaving additional information. Let* $v_0 \leq \mathbb{E}_{v_i}(v_i)$. *Additional information decreases sellers' profits if it reduces inverse hazard rates and one of the following conditions is satisfied:*

1. *Additional information increases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*

2. *Additional information decreases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

Example 1, cont. Perfectly revealing additional information always decreases sellers' profits to 0. $\square$

Example 2, cont. By Corollary 1, lower censorship additional information decreases sellers' profits if $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. $\square$

1.5.5. General Environment

This section examines the impact of additional information without small inverse hazard rates assumption.

Additional information always increases the intermediary's revenue, because the intermediary can always choose to ignore additional information.

Theorem 5. *Let $\mathcal{H}$ be any additional information and $v_0 \in \mathcal{R}^1$. Additional information always increases the intermediary's revenue.*

For consumers with outside option values lower than $\underline{v}$, the intermediary is restricted to recommend products, so the same analysis from the small inverse hazard rates environment applies for the consumer surplus. For consumers with outside options values higher than $\bar{v}$, the intermediary is restricted to recommend the outside option, so additional information is irrelevant.

Theorem 6. *Let $\mathcal{H}$ be any well-behaving additional information.*

1. *Let $v_0 < \underline{v}$. Additional information increases (decreases) consumer surplus for all v_0 if it decreases (increase) θ-revenue difference.*

2. *Let $v_0 > \bar{v}$. Additional information does not change the consumer surplus.*

Example 1, cont. By Theorem 3, perfectly revealing additional information *increases* (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ *decreases* (increases) in θ. $\square$

Example 2, cont. Since lower censorship additional information always decreases θ-revenue difference, by Theorem 6, the additional information increases consumer surplus for $v_0 \le \underline{v}$, but does not change consumer surplus for $v_0 \ge \bar{v}$. □

For consumers with outside option values lower than $\underline{v}$, the intermediary is restricted to recommend products, so the same analysis from the small inverse hazard rates environment applies for the sellers' profits.

Theorem 7. *Let $\mathcal{H}$ be any well-behaving additional information and $v_0 < \underline{v}$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*

   (a) *Additional information increases (decreases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ increases (decreases) in θ_i.*

   (b) *Additional information decreases (increases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ decreases (increases) in θ_i.*

2. *Inverse hazard rates effect decreases all sellers' profits if for all $z_i \in \mathcal{Z}$ and $\theta_i \in \Theta(z_i)$,*

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \le \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

Corollary 2. *Let $\mathcal{H}$ be any well-behaving additional information. Let $v_0 < \underline{v}$. Additional information decreases sellers' profits if it reduces inverse hazard rates and one of the following conditions is satisfied:*

1. *Additional information increases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*

2. *Additional information decreases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

1.6. Consumer Data Protection

This section explores whether consumer data is protected or leaked to sellers through the recommender system. I find that the intermediary does not earn a higher revenue by sharing the consumer data with sellers under the optimal direct mechanism. However, there are indirect mechanisms that implement the optimal recommender system and leak consumer data to sellers.

For concreteness, I consider the environment from Section 1.3 with $v_0 \in (\underline{v}, \bar{v})$, but all results extend to any other environments from this paper.

1.6.1. Sharing Consumer Data

The analysis so far has assumed that that the intermediary cannot directly communicate any information about the consumer's match values to sellers. Consumer data is protected in that sellers do not learn about the consumer's match value $\boldsymbol{v}$ until the game ends. The intermediary could potentially earn a higher revenue by sharing some consumer data with sellers.

A data sharing policy is a pair $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ where $\boldsymbol{\mathcal{Y}} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$ and $\boldsymbol{Y} : \boldsymbol{\mathcal{V}} \to \Delta \boldsymbol{\mathcal{Y}}$ that privately sends $y_i \in \mathcal{Y}_i$ to each seller i before reporting θ_i. The distribution $\boldsymbol{Y}$ can potentially be asymmetric across sellers. The intermediary's problem is to choose a pair of a data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ and a recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Y}} \to [0,1]^{N+1} \times \mathbb{R}^N$.

Fix a data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$. For each seller i with (θ_i, y_i), her expected profit is

$$\Pi_i^Y(\theta_i, y_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Y}}_{-i}} \Big((\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \Big) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{\mathcal{Y}}_{-i}(d\boldsymbol{y}_{-i} \mid y_i) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}),$$

where $\boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) = \frac{\boldsymbol{Y}(d\boldsymbol{y} | \boldsymbol{v}) \boldsymbol{F}(d\boldsymbol{v})}{\int_{\boldsymbol{\mathcal{V}}} \boldsymbol{Y}(d\boldsymbol{y} | \boldsymbol{v}) \boldsymbol{F}(d\boldsymbol{v})}$, and the probability of getting recommended is

$$Q_i^Y(\theta_i, y_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Y}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{\mathcal{Y}}_{-i}(d\boldsymbol{y}_{-i} \mid y_i) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}).$$

Data sharing signals $\boldsymbol{y}$ do not affect on sellers' incentive to report $\boldsymbol{\theta}$ truthfully. For each given $\boldsymbol{y}$, the incentive compatibility and individual rationality may be characterized in the standard way.

Lemma 6. *Let $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ be a data sharing policy. An obedient $(\boldsymbol{r}, \boldsymbol{t})$ recommender system is incentive compatible and individually rational if and only if for all $i \in \mathcal{N}$, $\theta_i \in \Theta$ and $y_i \in \mathcal{Y}_i$,*

$$
\begin{aligned}
&Q_i^Y(\theta_i, y_i) \text{ is increasing in } \theta_i, \\
&\Pi_i^Y(\theta_i, y_i) = \Pi_i^Y(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^Y(\tilde{\theta}_i, y_i) d\tilde{\theta}_i, \\
&\Pi_i^Y(\underline{\theta}, y_i) \geq 0.
\end{aligned}
$$

Applying the standard arguments, for a given data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$, the intermediary's problem is to maximize

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Y}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{Y}(d\boldsymbol{y}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{1.27}$$

subject to obedience constraints, for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Y}}} (v_i - v_j) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{Y}(d\boldsymbol{y}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{1.28}$$

Note that the integrands of both the objective function (1.27) and the constraints (1.28) do not depend on $\boldsymbol{y}$, so that the optimal recommendations rule ignores $\boldsymbol{y}$. Therefore, the optimal recommender system remains the same regardless of the data sharing policy.

Even in the absence of data sharing, the intermediary already extracts all of sellers' potential benefit from having a better estimate about value-dependent willingness to pay $w(v_i)$ by recommending better based on $\boldsymbol{v}$ but charging more accordingly. Another potential incentive for strategic data sharing is to affect sellers' incentive to report their private information. This channel is muted by the additive separability between θ_i and $w(v_i)$, and would have been important if sellers' profits were not additively separable, $\theta_i w(v_i)$, for example. How the non-separability would affect on the optimal data sharing policy is yet an open question.

Let $(\boldsymbol{r}^*, \boldsymbol{t}^*)$ be the optimal recommender system without data sharing. The intermediary does not share consumer data with sellers if $\boldsymbol{\mathcal{Y}} = \varnothing$. When data is not shared, the consumer data is protected.

Theorem 8. *When data sharing is allowed, the intermediary's optimal recommender system is the same regardless of the data sharing policy* $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$*. In particular, the recommender system* $(\boldsymbol{r}^*, \boldsymbol{t}^*)$ *without data sharing is optimal.*

1.6.2. Implementation

Under a mild condition, a variant of handicap auction (Eső and Szentes (2007)) implements the optimal recommender system. In a special environment with linear private virtual willingness to pay, a second-price auction with discounts and participation fees implement the optimal recommender system. This includes the example in Section 1.2 as a special case. For concreteness, I consider the environment from Section 1.3 with $v_0 \in (\underline{v}, \bar{v})$. Both versions of the implementation extend to any other environments that I consider in this paper.

Handicap Auction

Let $v_0 \in (\underline{v}, \bar{v})$.[16] The handicap auction consists of two rounds. In the first round, each seller i with θ_i chooses a price premium $p_i \in \mathbb{R}^1$ at a fee $C_i(p_i)$ from the menu of price premia $(p, C_i(p))_{p \in \mathbb{R}^1}$ proposed by the intermediary. The price premium chosen by each seller is known only to the seller and the intermediary, but not to other sellers and the consumer. After the first round and before the second, the intermediary discloses $\boldsymbol{v}$ to sellers and announces the cost of persuasion $\ell_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ for each $i \in \mathcal{N}$. In the second round, a second-price auction with zero reservation price, price premia and costs of persuasion follows. The seller with the highest bid wins the auction, but is required to pay the second highest pay plus the price premium and the cost of persuasion. That is, if others bid $(b_j)_{j \in \mathcal{N} \setminus \{i\}}$, the

[16] When $v_0 > \bar{v}$, the optimal recommender system from Theorem 1.b is implemented by a handicap auction with positive infinite reservation prices and $\ell_i^*(\boldsymbol{v}) = 0$ for all i and $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$. When $v_0 < \underline{v}$, the optimal recommender system from the same theorem is implemented by a handicap auction with negative infinite reservation prices and $\ell_i^*(\boldsymbol{v}) = 0$ for all i and $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$ as in (1.15).

price premium is p_i and the cost of persuasion is $\ell_i(\boldsymbol{v})$ for each i, then if the seller i bids $b_i > \max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0)$, then she wins the auction and is required to pay

$$\max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0) + p_i + \ell_i(\boldsymbol{v}).$$

If she bids $b_i < \max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0)$, then she loses and pays nothing.

Arguments below closely follow Eső and Szentes (2007). For the completeness, I present the full arguments here. I begin with characterizing an equilibrium in the second round auction: it is weakly dominant strategy for each seller i to bid his willingness to pay minus the price premium.

Lemma 7. *Suppose each seller i is informed with $\boldsymbol{v}$ and is charged with a price premium p_i. In the second round of the handicap auction, it is a weakly dominant strategy for seller i to bid $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$.*

From here on, sellers are assumed to play according to their weakly dominant strategy in the second round. The handicap auction is represented by a triple of of functions $p_i : \Theta \to \mathbb{R}^1$, $c_i : \Theta \to \mathbb{R}^1$ and $\ell_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ for each $i \in \mathcal{N}$ where $p_i(\theta_i)$ is the price premium that θ_i chooses at the fee of $c_i(\theta_i) = C_i(p_i(\theta_i))$.

Given a handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$, by Lemma 7, if the seller θ_i reports non-truthfully as $\hat{\theta}_i$ in the first round, the seller bids $\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v})$ in the second round. Denote each seller's equilibrium bid assuming that she bids truthfully in each stage by

$$b_i^*(\boldsymbol{v}, \theta_i) = \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}).$$

Let $b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) = \max_{j\in\mathcal{N}\setminus\{i\}}(b_j^*(\boldsymbol{v}, \theta_j), 0)$ be the equilibrium highest bid and reservation price excluding i's bid in the second round at each given state $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

The handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$ is incentive compatible if every seller i optimally reports its true type in the first round. Seller θ_i's expected profit after reporting $\hat{\theta}_i$ assuming others report truthfully is

$$\pi_i^H(\theta_i,\hat{\theta}_i) = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\Big[(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))$$
$$\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\Big] - c(\hat{\theta}_i),$$

and the seller wins the auction with probability

$$Q_i^H(\theta_i,\hat{\theta}_i) = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\left[\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right].$$

The incentive compatibility constraint of handicap auctions is characterized in the following lemma.

Lemma 8. *A handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$ is incentive compatible if and only if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$,*

$$\pi_i^H(\theta_i,\theta_i) = \pi_i^H(\underline{\theta},\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \tag{1.29}$$

and for all $\theta_i', \theta_i'' \in \Theta$ such that $\theta_i' < \theta_i < \theta_i''$,

$$Q_i^H(\theta_i,\theta_i') \leq Q_i^H(\theta_i,\theta_i) \leq Q_i^H(\theta_i,\theta_i''). \tag{1.30}$$

Inequality (1.30) states that if each seller reports his type to be higher (lower) in the first round, then he is more (less) likely to win the auction in the second round. Since misreporting in the first round only changes the price premium, (1.30) is satisfied if the price premium $p_i(\boldsymbol{v},\theta_i)$ decreases in θ_i.

Theorem 9. *Let $v_0 \in (\underline{v},\bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ decreases in θ_i. The intermediary can implement the optimal recommendations rule (1.12) and attain the same revenue via a handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$ where*

$$p_i^*(\boldsymbol{v},\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}, \tag{1.31}$$

$$
\begin{aligned}
c_i^*(\theta_i) =& \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\left[(\theta_i + w(v_i) - p_i^*(\theta_i) - \ell_i^*(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - p_i^*(\theta_i) - \ell_i^*(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] \\
&- \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i, \qquad (1.32)
\end{aligned}
$$

and the cost of persuasion $(\ell_i^*)_{i\in\mathcal{N}}$ *is as in* (1.13).

Note that the premium $p_i^*(\boldsymbol{v}, \theta_i)$ decreases in θ_i, so that (1.30) is satisfied. The first term in the fee schedule c_i^* is the seller i's expected profit from the second round. The second term is the information rent. By paying the fee, the seller's expected profit conditioning on θ_i is exactly the information rent, so that (1.29) is satisfied with $\pi_i^H(\underline{\theta}, \underline{\theta}) = 0$. By Lemma 8, the handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$ is incentive compatible. The handicap auction is individually rational and attains attains the same revenue as the optimal recommender system because $\pi_i^H(\underline{\theta}, \underline{\theta}) = 0$ and the second round auction implements the optimal recommendations rule (1.12).

Second-Price Auction with Discounts and Participation Fees

Consider an environment where the virtual private willingness to pay is linear

$$\alpha\theta_i - \beta \text{ for some } \alpha > 1, \beta > 0$$

with support $\left[\underline{\theta}, \frac{\beta}{\alpha-1}\right]$ where $0 \leq \underline{\theta} < \frac{\beta}{\alpha-1}$, i.e. $w(v) = 0$. The class of distributions with linear virtual private willingness to pay includes includes uniform, exponential distribution, Pareto distribution and log-logistic distribution.

A second-price auction with discounts and participation fees is represented by a pair of a discount function $d_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ and a participation fee $P_i \in \mathbb{R}^1$ for each $i \in \mathcal{N}$. Each seller first decides whether to participate by paying the fee P_i. Once having participated, each seller is informed of the discount $d_i(\boldsymbol{v})$ as well as $\boldsymbol{v}$. The seller with the highest bid wins, but is required to pay the second-highest bid minus a discount. With appropriately chosen discounts, participation fees and reserve prices, the auction implements the optimal recommender system.

Theorem 10. *Let $v_0 \in (\underline{v}, \bar{v})$. Suppose that $\theta_i - \frac{1-G(\theta_i)}{g_i(\theta_i)} = \alpha\theta_i - \beta$ with $\alpha, \beta > 0$ and $w(v_i) = 0$ for all $v_i \in \mathcal{V}$. The intermediary can implement the optimal recommendations rule (1.12) and attain the same revenue via a second price auction with discounts and participation fees $(d_i^*, P_i^*)_{i\in\mathcal{N}}$ where*

$$d_i^*(\boldsymbol{v}) = -\frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) - \frac{\alpha-1}{\alpha}w(v_i),$$

and

$$P_i^* = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\Bigg[\Bigg(\underline{\theta} + \frac{1}{\alpha}w(v_i) - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) - \max_{j\in\mathcal{N}\setminus\{i\}}\Big(\theta_j + \frac{1}{\alpha}w(v_j) - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha}\Big)\Bigg) \cdot \mathbf{1}_{\{\underline{\theta}+\frac{1}{\alpha}w(v_i)-\frac{1}{\alpha}\ell_i^*(\boldsymbol{v})>\max_{j\in\mathcal{N}\setminus\{i\}}(\theta_j+\frac{1}{\alpha}w(v_j)-\frac{1}{\alpha}\ell_j^*(\boldsymbol{v}),\frac{\beta}{\alpha})\}}\Bigg]$$

with reservation price $\frac{\beta}{\alpha}$.

Conditioning on participation, each seller's weakly dominant strategy is to bid

$$\theta_i + w(v_i) + d_i^*(\boldsymbol{v}) = \theta_i + \frac{1}{\alpha}w(v_i) - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}).$$

If all sellers of every type participate, this auction implements the optimal recommendations rule (1.12).

It remains to make sure that the lowest type's expected profit is 0. Note that the lowest private willingness to pay type $\underline{\theta}$ may have a positive probability of winning the auction when he sells a product that is high match. When he wins the auction, he always gets a non-negative profit. The participation fee P_i^* is exactly the expected profit of the lowest type without the fee, making the expected profit of $\underline{\theta}$ to be is 0. Consequently, all sellers of all types participate, and the intermediary attains the same revenue as under the optimal recommender system.

1.6.3. Discussion on Consumer Data Protection

Under the direct mechanism, consumer data is protected in that sellers do not learn about the consumer's match values $\boldsymbol{v}$ until the game ends. Furthermore, the intermediary does not earn higher revenue by sharing information about match values with sellers. However, both the handicap auction and the second-price auction with discounts and participation fees involve disclosing $\boldsymbol{v}$ to sellers. The consumer data is leaked in that sellers learn about $\boldsymbol{v}$ before the game ends.

1.7. Discussion and Additional Results

1.7.1. Constrained Welfare Maximization

Define α-welfare, a weighted sum of the consumer welfare and the joint profit of the intermediary and sellers,

$$\alpha \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) + (1-\alpha)\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} (\theta_i + w(v_i)) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}),$$

where $\alpha \in (0,1)$. A recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$ is a *constrained α-welfare maximizing* recommender system if it maximizes the α-welfare subject to obedience constraints (1.1), incentive compatibility (1.2) and individual rationality (1.3). In this section, I characterize α-welfare maximizing recommender system and its implication on the consumer surplus. Throughout this section, I assume that $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ and $v_0 < \underline{v}$.

Note that transfer $\boldsymbol{t}$ is irrelevant for the constrained α-welfare maximization problem as long as it makes a given recommendations rule incentive compatible and individually rational. In particular, in characterizing the constrained α-welfare maximizing recommendations rule and α-welfare, it is without loss of generality to assume $\boldsymbol{t}$ to be (1.8) and drop incentive compatibility and individual rationality constraints. Subtracting a constant αv_0 from the α-welfare gives the following lemma.

Lemma 9. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i + w(v_i) + \frac{\alpha}{1-\alpha}(v_i - v_0)\right) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \tag{1.33}$$

subject to obedience constraints (1.1) *and monotonicity constraints* (1.4)*. Suppose also that*

$$t_i(\boldsymbol{v},\boldsymbol{\theta}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i})d\tilde{\theta}_i. \tag{1.34}$$

Then, $(\boldsymbol{r},\boldsymbol{t})$ *is a constrained* α*-welfare recommender system.*

Comparing the α-welfare maximization problem in Lemma 9 to the intermediary's revenue maximization problem in Lemma 5, the only difference is the integrand in each objective function is changed from

$$\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) \tag{1.35}$$

to

$$\theta_i + w(v_i) + \frac{\alpha}{1-\alpha}(v_i - v_0). \tag{1.36}$$

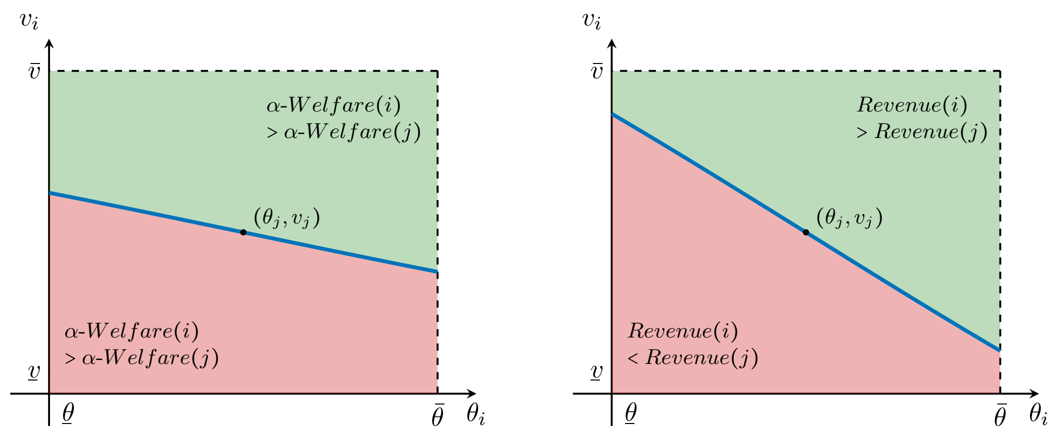
The identical graphical analysis from Example 3 applies. Unlike in Section 1.5, however, the transition from (1.35) to (1.36) changes not only the term related to θ_i, but also the term related to v_i as well. The difference terms are defined for both θ as well as v.

Definition 5. A *θ-revenue difference* for $\theta > \theta'$ is

$$\Delta(\theta,\theta') = \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right)$$

and *v-revenue difference* for $v > v'$ is

$$D(v,v') = w(v) - w(v')$$



(a) Iso-welfare Curve

(b) Iso-revenue Curve

Figure 6: Iso-welfare and iso-revenue curves.

Similarly, a θ*-welfare difference* for $\theta > \theta'$ is

$$\Delta^{\alpha}(\theta, \theta') = \theta - \theta'$$

and v*-welfare difference* for $v > v'$ is

$$D^{\alpha}(v, v') = \left(w(v) + \frac{\alpha}{1-\alpha} v \right) - \left(w(v') + \frac{\alpha}{1-\alpha} v' \right).$$

Note that $\Delta^{\alpha}(\theta, \theta') \leq \Delta(\theta, \theta')$ for all $\theta > \theta'$ and $D^{\alpha}(v, v') \geq D(v, v')$ for all $v > v'$. Relative to the revenue-maximization regime, the additional gain from recommending a product with a higher θ decreases and that with a higher v increases under the α-maximization regime. In other words, the iso-welfare curve has a lower slope than the iso-revenue curve as in Figure 6. Consequently, α-welfare-maximizing recommendations rule recommends products with higher match values more often, which increases the consumer surplus and decreases the joint profit.

Theorem 11. *Suppose that* $\frac{1-G(\theta_i)}{g(\theta_i)}$ *decreases in* θ_i *and* $v_0 < \underline{v}$*. Relative to the revenue maximizing recommender system* $(\boldsymbol{r}^*, \boldsymbol{t}^*)$*, under the* α*-welfare maximizing recommender system* $(\boldsymbol{r}^{\alpha}, \boldsymbol{t}^{\alpha})$*,*

1. *Consumer surplus is higher.*

2. *Joint profit of the intermediary and sellers is lower.*

1.7.2. Relaxing v_0 as Common Knowledge

The value of the outside option v_0 has been assumed to be a constant that is commonly known to all players. One way to relax this assumption is to assume that v_0 is common knowledge between the intermediary and consumer, but not to the sellers. Sellers instead believe that v_0 is drawn from a distribution F_0. All results continue to hold identically under this assumption. Below I explore two different ways to relax the symmetric knowledge of v_0 between the intermediary and the consumer: One under which v_0 is the intermediary's private information; the other under which v_0 is the consumer's private information.

Value of Outside Option as Private Knowledge of Intermediary

Suppose that only the intermediary observes v_0 that is drawn from a distribution F_0. For simplicity,[17] assume that $F_0(v_0) = F(v_0 - \mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is a mean shifter of F_0. If $\mu_0 = 0$, then the distribution for the outside option F_0 is identical to those of other products F; if $\mu_0 > 0$, then F_0 first-order stochastically dominates F; if $\mu_0 < 0$, then F_0 is first-order stochastically dominated by F. All of the results in Section 1.4 and Section 1.5 continues to hold after replacing v_0 with μ_0.

Value of Outside Option as Private Knowledge of Consumer

Suppose that only the consumer observes v_0 that is drawn from a distribution F_0 that has a full support on the real line. Suppose that the intermediary offers the same recommender system to the consumer of all types, and the consumer of each type decides whether to follow the recommendation. This is equivalent to the intermediary designing the set $\mathcal{V}_0^{NR} \subset \mathbb{R}^1$ of the consumer types who will obey recommendations on top of designing a recommender system itself. For any given recommender system, a consumer with high v_0 disobeys when recommended with a product; a consumer with low v_0 disobeys when recommended with the outside option. Consequently, the intermediary's problem reduces to designing a rec-

[17]All results here can be generalized to any family distributions $F_0(v_0;\mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is an index such that $F_0(v_0;\mu_0)$ first-order stochastically dominates $F_0(v_0;\mu_0')$ whenever $\mu_0 > \mu_0'$.

ommender system and picking up two thresholds $\underline{v}^{NR} \leq \bar{v}^{NR}$ such that the consumer obeys if and only if $v_0 \in [\underline{v}^{NR}, \bar{v}^{NR}]$.

The intermediary faces another layer of trade-offs, setting target population, on top of the trade-off between raising revenue and keeping the consumer incentivized to obey for each consumer $v_0 \in \mathcal{V}_0^{NR}$. Which population to target depends on whether the intermediary chooses to recommend the outside option with positive probability or not.

If the intermediary does not recommend the outside option almost surely, then any consumer with bad enough outside option is obedient to the recommender system. In particular, an optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is characterized by $\bar{v}^{NR} \geq \mathbb{E}_{v_i}(v_i)$ such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = (-\infty, \bar{v}^{NR}]$, and

$$r_i^{NR}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{|\mathcal{M}^{NR}|} \text{ if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)} + w_j \right\} \tag{1.37}$$

where $\mathcal{M}^{NR} = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j \right\}$. Note that there is no cost of persuasion because none of the obedience constraints bind. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i^{NR}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{w}) d\tilde{\theta}_i. \tag{1.38}$$

If the intermediary recommends the outside option with positive probability, then the optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is characterized by $\underline{v} \leq \underline{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v}, \boldsymbol{\theta}) = 1) \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v}, \boldsymbol{\theta}) = 1) \leq \bar{v}^{NR} \leq \bar{v}$such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = [\underline{v}^{NR}, \bar{v}^{NR}]$. Furthermore, OB_{0i} binds at $\underline{v}^{NR}$ and OB_{i0}

binds at $\bar{v}^{NR}$, so that

$$r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^{NR}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^{NR}(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{1.39}$$

where $\mathcal{M}^{NR} = \arg\max_{j\in\mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^{NR}(\boldsymbol{v}) \right\}$, and

$$\ell_i^{NR}(\boldsymbol{v}) = \lambda_1(\bar{v}^{NR})(\bar{v}^{NR} - v_i) - \lambda_2(\underline{v}^{NR}) \sum_{k\in\mathcal{N}} (\bar{v}^{NR} - v_k)$$

where $\lambda_1^{NR}(v_0)$ and $\lambda_2^{NR}(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^{NR}$ and $\bar{v}^{NR}$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given as in (1.38).

Theorem 12. *Suppose that the consumer privately observes v_0 that is drawn from a distribution F_0. An optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ takes one of the following two structures:*

1. *The intermediary always recommends one of products. An optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is as in (1.37) and (1.38). The consumer is obedient if and only if $v_0 \leq \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

2. *The intermediary sometimes recommends the outside option. An optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is as in (1.39) and (1.38). The consumer is obedient if and only if $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \leq v_0 \leq \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

1.7.3. Relaxing Intermediary's Private Knowledge of $w(v)$

The value-dependent willingness to pay $w(v)$ has been assumed to be a deterministic function of v_i. Combined with the assumption that only the intermediary knows $\boldsymbol{v}$, this entails that the intermediary privately knows sellers' value-dependent willingness to pay $w(v)$ that sellers themselves do not know.

This assumption may be relaxed in two different ways. The first is to assume that each seller learns the value of $w(v_i)$ even though he does not know v_i. With the strict monotonicity of $w(v_i)$, learning the value of $w(v_i)$ is equivalent to learning v_i and hence having v_i as common knowledge between the intermediary and seller i. This does not change the optimal recommendations rule: the intermediary discloses v_i to and extract the entire value-dependent willingness to pay $w(v_i)$ from each seller i in the baseline model under which the seller does not know v_i. Therefore, the optimal recommendations rule letting seller to learn $w(v_i)$ does not change the optimal recommender system.

Another way to relax the assumption is to assume that each seller privately observes a value-dependent willingness to pay in the following way similar to Eső and Szentes (2007): Suppose that the intermediary can disclose[18] v_i to a seller i. Upon disclosure, the seller i privately observes w_i independently drawn from a common distribution $W(. \mid v_i)$ conditioning on v_i, where $W(. \mid v)$ first-order dominates $W(. \mid v')$ whenever $v > v'$. Without disclosure, the seller does not learn any information about v_i and w_i.

Following Eső and Szentes (2007), it can be shown that the intermediary completely discloses its private information $\boldsymbol{v}$ under an optimal recommender system and obtains the same expected revenue as if the intermediary could observe $\boldsymbol{w}$ using the modified handicap auction obtained by replacing $w(v_i)$ with w_i from Theorem 9. A sketch of the proof is presented here.

Let $v_0 \in (\underline{v}, \bar{v})$ and G be such that $\frac{1-G(\theta)}{g(\theta)}$ decreases.. Suppose that the intermediary has disclosed v_i to each seller i. A recommender system is now extended to $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}} \to [0,1]^{N+1} \times \mathbb{R}^N$ with $\sum_{i \in \mathcal{N}\{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}}$ where $\mathcal{W}$ is a

[18] An alternate setup under which the intermediary can provide any arbitrary information to sellers, instead of being restricted to disclosing or not, leads to the same conclusion. The revenue from (1.41) and (1.42) still is an upper bound of the intermediary's revenue which can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running the modified handicap auction from Theorem 9

support of $W(.\mid.)$ and $\boldsymbol{\mathcal{W}} = \mathcal{W}^N$. An optimal recommendations rule is given by

$$r_i^W(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^W|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^W(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{1.40}$$

where $\mathcal{M}^W = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^W(\boldsymbol{v})\right\}$, and

$$\ell_i^W(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^W, \bar{v}^W] \\ \lambda_1^W(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^W \\ \lambda_2^W(v_0)\cdot\sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^W \end{cases} \tag{1.41}$$

where $\lambda_1^W(v_0)$ and $\lambda_2^W(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^W$ and $\bar{v}^W$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{w})d\tilde{\theta}_i \tag{1.42}$$

for each i.

The revenue from (1.41) and (1.42) clearly is an upper bound of the intermediary's revenue. This revenue can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running a modified handicap auction as the following.

Theorem 13. *Let $v_0 \in (\underline{v},\bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ weakly decreases in θ_i. The intermediary can implement the optimal recommendations rule with the same revenue via a handicap auction $(c,p,\ell_i)_{i\in\mathcal{N}}$ where*

$$p(\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}, \tag{1.43}$$

$c(\theta_i)$ is defined by

$$\begin{aligned} c(\theta_i) =& E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - p(\theta_i) - \ell_i(\boldsymbol{v}) - b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - p(\theta_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] \\ &- \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i},\boldsymbol{v}}\left[\mathbf{1}_{\{\tilde{\theta} + w(v_i) - p(\tilde{\theta}) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] d\tilde{\theta} \end{aligned} \tag{1.44}$$

and the cost of persuasion $(\ell_i^W)_{i \in \mathcal{N}}$ from (1.41).

1.8. Conclusion

In this paper, I study a monopolistic intermediary designing a recommender system. I frame the intermediary's problem as a revenue-maximizing mechanism design problem of allocating one unit of sales to one of multiple sellers, but with a constraint of having to rely on the consumer to choose the outcome. I reformulate the intermediary's revenue-maximizing mechanism design problem as a Bayesian persuasion problem. Using value-switching monotone recommendations rules, I find that the intermediary recommends the product of the seller with the highest virtual willingness to pay adjusted by the cost of persuasion.

I use this model to explore policy-relevant questions. First, I characterize the types of seller data that benefit or harm consumers and sellers. Second, I find that the optimal direct mechanism protects consumer privacy, but consumer data is leaked to sellers under other implementations. Lastly, I show that the welfare-maximizing recommender system increases consumer surplus, but reduces the joint profit of the intermediary and sellers.

There are several directions for future works. To start, endogenizing prices leads to a number of economic and technical questions. Should the prices be set by the intermediary or by sellers, and at what timing? How does the recommender system affects price competition among sellers? How should the pricing and recommendations rule jointly be determined? Furthermore, allowing consumers to have ex-ante asymmetric, privately known preferences over products would also be an interesting direction. In this extension, the consumer has multi-dimensional private information about his preference. The key challenge lies in tractably characterizing the recommendations rules that make consumers report their type truthfully. Finally, considering competition among intermediaries is another important direction for both theory and practice. Although analyzing competition among mechanism designers and persuaders is generally difficult, characterizing the optimal recommender system in this environment would provide valuable insights into the strategic interactions of intermediaries with growing capabilities and influence.

CHAPTER 2 : Outcome Test for Policies

This chapter is co-authored with Mallesh Pai and Rakesh Vohra.

2.1. Introduction

There is much interest in evaluating the "fairness" of various socioeconomic institutions, e.g. criminal justice, access to employment/credit/education etc. In practice, this often boils down to focusing on a specific binary decision,[19] and comparing if this differs across various demographics e.g. black vs white defendants, male vs female job applicants. Within the economics literature, and more generally, the "gold standard" is the *marginal outcome test*, originally due to Becker (1957). A failure of this test is interpreted as evidence of discrimination by the decision maker (see e.g. Hull (2021), Bohren, Haggag, Imas, and Pope (2019)). It has been applied to a wide variety of settings.[20]

In this paper, we revisit the question: when is the marginal outcome test valid? We identify a natural class of settings of interest where a "fair" principal would choose a rule that fails the marginal outcome test, and identify the correct test for such settings. Specifically, these are settings where the principal is choosing a *policy*, and agents are responding *strategically* to the chosen policy. Settings that satisfy the desiderata we describe are easily motivated in practice. The idea that agents' relevant choices may be strategic and may respond to policy choices of the decision maker is of course standard in economics, and has long been considered in related settings (e.g. the design of affirmative action policy, see e.g. Coate and Loury (1993), Foster and Vohra (1992) or Fryer Jr and Loury (2013)) but largely absent from the literature on evaluating fairness. Settings where the adjudicator is making a policy choice also abound. For example, in the case of traffic stops it may amount to guidance issued by the leadership directing troopers on whom to stop. Similarly, as decision-making

[19] For example a judge choosing whether to acquit/ convict a defendant, a bank choosing whether or not to extend a loan to a loan applicant, an employer deciding whether or not to employ a job candidate.

[20] Some notable examples include: in the context of lending (Ferguson and Peters, 1995), judicial decision making (Arnold, Dobbie, and Yang (2018), Alesina and La Ferrara (2014)), traffic stop/ search decisions (Knowles, Persico, and Todd, 2001; Anwar and Fang, 2006; Antonovics and Knight, 2009) etc.

gets increasingly automated by the use of computers/ machine learning/ AI, it may be the choice of algorithm by the institution (e.g. the use of automated rules to determine who gets issued a loan in a banking setting, or the use of resume scanning software by an employer to determine which applicants get called back for an interview).

To fix ideas, let us first outline the model that the marginal outcome test implicitly assumes, focusing on the example of checking for racial bias in traffic stops by state troopers for contraband. A set of motorists each has a payoff-relevant attribute that is not directly observed by the decision maker (whether or not they are carrying contraband). The decision maker observes information about the motorist (including their race) and makes a binary decision on whether to interdict. Once the decision is made, this attribute is observed (i.e. upon conducting a traffic stop, the trooper learns whether the motorist was carrying contraband). The null hypothesis of no discrimination is that conditioned on being *marginal*, i.e. conditioned on the information seen by the decision maker being such that they are indifferent, the distribution of outcomes should be similar across races—after all, ceteris paribus, a decision maker should be indifferent at roughly the same rate of successful interdiction. Differences are either the result of a preference by the decision maker to pull over e.g. black drivers at a higher rate ("taste-based discrimination") or of an incorrect statistical model that causes the decision maker to over-estimate the risk of (marginal) black drivers ("incorrect statistical discrimination"). The underlying economic logic of the test is clear and uncontroversial, and therefore seemingly universally applicable.[21]

Formally, we show that marginal outcome tests may fail when the outcome of the agent is not exogenously determined, but instead depends on a strategic choice made by the agent (e.g. in our running example, the agents choose whether or not to carry contraband). In particular, suppose the decision maker chooses and commits to a decision policy *a priori*, and the agent understands this policy at the time of their own choice. In the language of

[21] Operationally, one still needs to (correctly) identify the marginal agent which can be difficult in practice. The marginal outcomes test may also fail in richer models, see e.g. Canay, Mogstad, and Mountjoy (2020) which we discuss below.

Game Theory, the decision maker is a Stackelberg leader, or, equivalently, in the language of mechanism design, the decision maker has commitment. The agent's choice is thus based on a cost-benefit calculation given decision maker's policy (e.g. both the benefits of carrying contraband, and the associated risk of being apprehended). Therefore, the decision maker announces a policy that optimizes an objective function, taking into account that agents will respond to the underlying policy.

Our main positive result shows how to test for discrimination in such settings. At a high level, the intuition for this test can be described thusly: Under mild assumptions (Assumption 2), we show that the principal's optimal policy remains a group specific threshold on the signal. We show that agents of the two groups who generate a signal exactly equal to this threshold (i.e., the analog of the marginal agent at the standard marginal outcomes test) nevertheless will have different distributions of outcomes. This is precisely because the choice of threshold by the principal also affects agents' incentives. Since the optimal policy of the principal accounts also for how it affects the choices of the agents, we can derive a novel test statistic that a "fair" principal would equate across the groups.

The remainder of the paper is organized as follows—Section 2.2 outlines the general model, identifying in Section 2.2.1 some examples of special interest. Section 2.3 presents our results and discusses some of the key assumptions. Section 2.4 concludes with a discussion of the related literature.

2.2. Model

There is a set of agents. For each, the principal must take a binary decision $\in = \{0, 1\}$. This decision is the object of study—it could be, for example, traffic stops of motorists, loan approval/denial decisions, or job interview callback/rejection decisions etc.

Each agent belongs to a group $g \in$. A group corresponds to an observable characteristic of the agent, for instance race or gender, with respect to which we wish to evaluate the fairness of the principal's decision. We will concern ourselves with two groups, i.e. $= \{1, 2\}$, the extension to more than two groups is obvious.

Unobserved by the principal is a binary action choice by the agent $\in = \{0, 1\}$. This action affects both the principal and the agent. In the traffic stop example, a is the choice of the agent on whether or not to carry contraband. In the case of employment, a might represent the choice of an agent to invest in human capital.

Prior to making their decision the principal observes the group identity of the agent. The principal also observes other information about the agent. Instead of directly modeling the information observed by the principal, we summarize this as a signal $\in \subseteq \mathfrak{R}$ which is informative of the agent's action. The distribution of the signal depends only on the agent's chosen action and possibly their group. In particular, the signal is distributed according to CDF F^{g} (with pdf f^{g}) for an agent of group g who has taken action $\in$.[22] We assume that the signals are informative in the same direction across groups, formally:

Assumption 1. *We assume that the distributions* $\{f^{g}\}_{\in}$ *satisfy the Monotone Likelihood Ratio Property (MLRP), i.e.* $\frac{f_g^1()}{f_g^0()}$ *is non-decreasing in for all groups.*[23]

The principal has a utility function $u : \times \to \mathfrak{R}$. By assumption therefore, the principal's decision and the agent's action are payoff relevant to the principal. Other observables, such as the agent's group identity and the signal they generate are payoff irrelevant by assumption (though of course they are informationally relevant in choosing an appropriate decision).

The choice for the principal is a *decision rule*, i.e., what decision $\in$ they make as a function of what they observe $(,) \in \times$. We denote the decision rule by $\beta_g :\to$, i.e. $\beta_g()$ denotes the decision on an agent of group g for whom signal was observed.

As we presaged above, the action choice of the agent is endogenous, and depends on the decision rule chosen by the principal. To be precise, agents also have preferences over action and decision, $v : \times \times \Theta \to \mathfrak{R}$, where Θ are payoff relevant types. The distribution of types Θ

[22] Note that we implicitly assume that the distribution of signals admits a density. Distributions with atoms etc. can be accommodated at some notational cost.

[23] If the distribution of signals is the same across groups, then this assumption is vacuous—it can be achieved by e.g. renaming signals appropriately.

in group g is given by μ_g. An agent of group g with type θ, facing the principal's decision rule β_g, chooses the action that maximizes their expected utility, i.e.

$$a_g^*(\theta, \beta_g) = \arg\max_{\in} \int v(\beta_g(s), a, \theta) f_g^a(s) \mathrm{d}s.$$

The principal's problem then is to solve for each group:

$$\max_{\beta_g : \to} \int_\Theta \left(\int_S u(\beta_g(s), a^*(\theta, \beta_g))\, f_g^{a^*(\theta,\beta_g)}(s) \mathrm{d}s \right) \mathrm{d}\mu_g(\theta)\ . \tag{OPT-g}$$

It will be useful, at this stage, to be clear about timing and observability. First, a principal, announces and commits to $\beta_g :\to$ for each group $g \in$. Then, each agent of group g privately observes their type θ (drawn according to distribution μ_g). The agent then chooses a utility maximizing action $a_g^*(\theta, \beta_g)$. Finally, for each agent, the principal observes the agent's group identity g and signal s (which depends on their chosen action), and takes the corresponding action, $\beta_g(s)$.

It will be useful to put some mild restrictions on the preferences of the principal and the agent to add structure to the model.

Assumption 2. *We make the following assumptions on the preferences of the principal and agent:*

1. *Agent prefers decision* 1*: For any agent of any group g, type θ and action a,* $v(1, a, \theta) \geq v(0, a, \theta)$*.*

2. *Principal prefers action* 1*: Ceteris paribus, the principal would prefer that agents take action* 1*, i.e., for any decision d,* $u(d, 1) \geq u(d, 0)$*.*

3. *Principal prefers to match action and decision:* $u(1, 1) \geq u(0, 1)$ *and* $u(0, 0) \geq u(1, 0)$*.*

These assumptions are weak and capture the applications of interest: part (1) simply says that, ceteris paribus, decision 1 is the desirable decision from the agent's perspective (e.g.

getting a loan, getting admitted to school, getting a job, not being pulled over in a traffic stop, etc). Similarly, part (2) says that from the principal's perspective, inducing action 1 by the agents is desirable (e.g. investing in human capital, not carrying contraband etc.). Finally, Part (3) says that the principal would like to match action and decision as much as possible. For example, in the traffic stop application, if action 1 is the agent's choice to not carry contraband (and 0 denotes the choice to carry contraband), the assumption simply says that the for an agent carrying contraband, the principal would prefer to interdict, while for an agent not carrying contraband, the principal would prefer not to interdict.

As we detail in examples below, this still allows flexibility. For instance this model accommodates the principal preferring that the agent take a particular action (e.g. the design of education policy to maximize human capital investment by groups as in Fryer Jr and Loury (2013)).

2.2.1. Examples

Before proceeding to our results, we list some concrete examples of our model.

Example 5 (Fixed Actions)**.** Our model subsumes the special case where agent actions are non-strategic, or equivalently for the purposes of the principal, the agent takes the action before the principal chooses their decision rule.

This can be achieved by giving agents a dominant action as a function of their type (i.e. their preferences over actions are independent of how the principal decides among agents). Formally, suppose $\Theta = \mathfrak{R}$, with,

$$v(d, 1, \theta) = \theta, \quad \text{and} \quad v(d, 0, \theta) = -\theta.$$

□

Example 6 (Strategic Agents)**.** Of course, more pertinent for our model is the case where agent's actions are strategically chosen to maximize the agents' expected utility given the principal's decision rule. A specific example of this is where $\Theta \subseteq \mathbb{R}_+$, and a given $\theta = (\theta_1, \theta_2)$

consists of two elements, where θ_1 represents the strength of the agent's preference to have decision $d = 1$ taken (e.g. additional value of getting a job), and θ_2 her net disutility of taking action $a = 1$ (e.g. disutility of investing in human capital). An agent of type θ has preferences given by:

$$v(d, a, \theta) = \chi_{\{d=1\}}\theta_1 - \chi_{\{a=1\}}\theta_2.$$

□

Example 7 (Consequentialist preferences)**.** A special case that is relevant for some applications is where the principal only has preferences over the agent's action when they take decision $d = 1$. By a (slight) abuse of terminology, we call these consequentialist preferences: for example an employer only cares about the agent's choice of human capital investment if they choose to employ them ($d = 1$) but are otherwise indifferent. Formally, consequentialist preferences are preferences of the form $u(0,0) = u(0,1)(= 0)$, while $u(1,0) \neq u(1,1)$. □

Example 8 (Paternalistic Preferences)**.** Another natural special case to consider is one where the principal purely cares about the action taken by the agent—the decision is purely instrumental to incentivize the agent to take the desired action. For example, continuing with the employment/ human-capital application, these preferences might reflect those of a benevolent social planner wishing to maximize the fraction of agents who choose to invest in human capital. Formally, paternalistic preferences are of the form $u(d,1) = 1$, $u(d,0) = 0$. □

2.3. Results and Discussion

To begin our analysis, note that the assumption on preferences (Assumption 2) combined with the assumption that the distribution of signals satisfies MLRP (Assumption 1) simplifies the principal's problem into a single threshold.

Lemma 10. *Under Assumptions 1 and 2, for each group g, the solution β_g to (OPT-g) simply specifies a threshold s_g^* such that $\beta_g(s) = 1 \iff s \geq s_g^*$.*

Proof. To see this, fix a group g and a decision rule of the principal $\beta_g(\cdot)$. Observe that any decision rule induces an effective probability p_g^a that an agent who takes action a receives decision 1, and correspondingly probability $1 - p_g^a$ of receiving decision 0. Note further by observation that agent's incentives are determined purely by p_g^1, p_g^0— any two decision rules that induce the same p_g^1, p_g^0 induce the same actions by the agent.

Next, note that by Assumption 1, for any feasible probabilities (p_g^1, p_g^0) that can be delivered by some decision rule, there exists a threshold rule which induces probabilities $(p_g^{1\prime}, p_g^{0\prime})$ such that $p_g^{1\prime} \geq p_g^1$ and $p_g^{0\prime} \leq p_g^0$. By Assumption 2 part (1), weakly more types of the agent take action 1 under this threshold rule than the original decision rule. By Assumption 2 part (2) and (3), this threshold rule can only be better in terms of the principal's objective (OPT-g) than the original. □

Since the type θ of the agent is payoff irrelevant to the principal, as a function of the principal's threshold s_g^*, we can summarize the distribution of the agent's actions by a single number $\pi_g(s_g) \in [0,1]$. Here, $\pi_g(s_g^*)$ is the fraction of agents in group g that take action 1 when the principal uses a decision rule with threshold s_g^*. For the rest of this paper, we will assume that $\pi_g(\cdot)$ is a differentiable function.

In light of these simplifications, we can write the principal's problem as,

$$\begin{aligned}\max_{s_g^*} \quad & u(1,1)(1 - F_g^1(s_g^*))\pi_g(s_g^*) + u(1,0)(1 - F_g^0(s_g^*))(1 - \pi_g(s_g^*)) \\ & + u(0,1)F_g^1(s_g^*)\pi_g(s_g^*) + u(0,0)F_g^0(s_g^*)(1 - \pi_g(s_g^*))\end{aligned} \tag{Simple-Opt-g}$$

This gives us the following (well-known) result, which justifies the validity of marginal outcome tests in settings where the actions of agents are fixed/ exogenously given (e.g. Example 5):

Suppose agent's actions are fixed, i.e. $\pi_g(s_g^*) = \pi_g$ constant. Then, taking first-order conditions, the optimal threshold for the principal must satisfy

$$\begin{aligned}
0 &= -u(1,1)f_g^1(s_g^*)\pi_g - u(1,0)f_g^0(s_g^*)(1-\pi_g) \\
&\quad + u(0,1)f_g^1(s_g^*)\pi_g + u(0,0)f_g^0(s_g^*)(1-\pi_g). \\
\implies \frac{f_g^1(s_g^*)\pi_g}{f_g^0(s_g^*)(1-\pi_g)} &= \frac{u(0,0)-u(1,0)}{u(1,1)-u(0,1)}.
\end{aligned}$$

The latter equation is the foundation of the marginal outcome test—after all the left hand side is the ratio of agents revealed to be taking action 1 to action 0 among marginal agents; i.e. those that generate signal s_g^* where the principal is different between either decision. The first order condition asserts that this quantity must be equal across groups, since the right hand side is a quantity that is independent of group identity.

However, in the general setting, the principal must also account for how their choice of threshold s_g^* affects an agent's behavior. The optimal thresholds for the principal's problem (Simple-Opt-g) will not equate marginal outcomes in general. Formally,

Theorem 14 (Failure of the Marginal Outcome Test). *Let $\{s_g^*\}_{g\in}$ be the solution to the principal's problem* (Simple-Opt-g)*. If $\pi_g'(s_g^*) \neq 0$, then for any other group g', we have:*

$$\frac{f_g^1(s_g^*)\pi_g}{f_g^0(s_g^*)(1-\pi_g)} \neq \frac{f_{g'}^1(s_{g'}^*)\pi_{g'}}{f_{g'}^0(s_{g'}^*)(1-\pi_{g'})}.$$

In words, our theorem says that the standard statistic that is compared for marginal outcome tests may be different for different groups when agents' choices are endogenous and the principal's test is designed taking into account agents' responses. This despite the maintained assumption (by fiat) that the principal's preferences are independent of group identity.

So, in terms of positive results, what can we say about testing for discrimination in such a setting? As a first result, note that (Simple-Opt-g) already gives us a straightforward necessary first-order condition.

Theorem 15. *Under the maintained assumptions, the solution to the principal's problem* (Simple-Opt-g) *for each group g must be a threshold s_g^* such that*

$$\begin{aligned}0 = (u(0,1) - u(1,1))\left(f_g^1(s_g^*)\pi_g(s_g^*) + F_g^1(s_g^*)\pi_g'(s_g^*)\right)\\ + (u(0,0) - u(1,0))\left(f_g^0(s_g^*)(1 - \pi_g(s_g^*)) - F_g^0(s_g^*)\pi_g'(s_g^*)\right)\\ + (u(1,1) - u(1,0))\,\pi_g'(s_g^*).\end{aligned} \tag{FOC}$$

Proof of Theorems 14, 15. Theorem 15 follows from the assumption that $\pi_g(\cdot)$ is a differentiable function so that (FOC) is a necessary condition of optimality for (Simple-Opt-g). Theorem 14 follows by observation of (FOC) □

Note that we can rewrite (FOC) as:

$$\begin{aligned}0 = (u(0,1) - u(1,1))\frac{\mathrm{d}F_g^1(s_g^*)\pi_g(s_g^*)}{\mathrm{d}s_g^*} + (u(0,0) - u(1,0))\frac{\mathrm{d}F_g^0(s_g^*)(1 - \pi_g(s_g^*))}{\mathrm{d}s_g^*}\\ + (u(1,1) - u(1,0))\,\pi_g'(s_g^*).\end{aligned} \tag{FOC2}$$

Observe that this already provides a testable restriction if the econometrician knows the stated utility function of the principal. Testing this across groups therefore requires the econometrician to estimate, for each group g, quantities $\frac{\mathrm{d}F_g^1(s_g^*)\pi_g(s_g^*)}{\mathrm{d}s_g^*}$, $\frac{\mathrm{d}F_g^0(s_g^*)(1-\pi_g(s_g^*))}{\mathrm{d}s_g^*}$ and $\pi_g'(s_g^*)$. We discuss the possibility of such estimation in what follows. However, before this, we derive some implications for special cases.

Corollary 3. *Suppose the principal has consequentialist preferences of the form described in Example 7, i.e. $u(0,\cdot) = 0$. Then, for a principal applying the optimal policy, the optimal threshold s_g^* for any group solves*

$$u(1,1)\frac{\mathrm{d}(1 - F_g^1(s_g^*))\pi_g(s_g^*)}{\mathrm{d}s_g^*} - u(1,0)\left(\pi_g'(s_g^*) + \frac{\mathrm{d}F_g^0(s_g^*)(1 - \pi_g(s_g^*))}{\mathrm{d}s_g^*}\right) = 0 \tag{2.1}$$

i.e., under the maintained assumption about the nature of the principal's preferences, the *ratio of*

$$\frac{\mathrm{d}(1-F_g^1(s_g^*))\pi_g(s_g^*)}{\mathrm{d}s_g^*} \quad \textit{and} \quad \pi_g'(s_g^*) + \frac{\mathrm{d}F_g^0(s_g^*)(1-\pi_g(s_g^*))}{\mathrm{d}s_g^*}$$

is equal across groups.

Corollary 4. *Suppose the principal has paternalistic preferences of the form described in Example 8, i.e $u(\cdot,0)=0$. For a principal applying the optimal policy, the optimal threshold s_g^* for any group solves*

$$0=(u(0,1)-u(1,1))\frac{\mathrm{d}F_g^1(s_g^*)\pi_g(s_g^*)}{\mathrm{d}s_g^*}+u(1,1)\pi_g'(s_g^*). \tag{2.2}$$

i.e., under the maintained assumption of paternalistic preferences, the ratio of

$$\frac{\mathrm{d}F_g^1(s_g^*)\pi_g(s_g^*)}{\mathrm{d}s_g^*} \ \textit{and} \ \pi_g'(s_g^*)$$

is equal across groups.

2.3.1. Discussion

Estimation

Our corollaries provide analogs of the marginal outcomes test under the assumption of strategic agents and a "mechanism designer" principal. The possibility to execute such a test depends on the ability to estimate the relevant quantities, i.e. $\pi_g'(s_g^*)$, $\frac{\mathrm{d}F_g^1(s_g^*)\pi_g(s_g^*)}{\mathrm{d}s_g^*}$, and $\frac{\mathrm{d}F_g^0(s_g^*)(1-\pi_g(s_g^*))}{\mathrm{d}s_g^*}$. It is worth discussing what these quantities correspond to in terms of the underlying model.

Let us start with the first: $\pi_g'(s_g^*)$. This is the derivative of the fraction of group g agents taking the action 1 with respect to the principal's threshold for that group s_g^*. As we will see this is the novel term that would need to be estimated (relative to a traditional marginal outcomes test). Estimating this would either require further modeling of the agents' incentives (i.e. a structural model of their choices), or, e.g., identifying variation (e.g. different principals who use slightly different thresholds) that can be exploited. Of course, any such estimation would be nontrivial, so we do not speculate further here.

Given an estimate of $\pi'_g(s^*_g)$, the second term $\frac{\mathrm{d}F^1_g(s^*_g)\pi_g(s^*_g)}{\mathrm{d}s^*_g}$ is easier to estimate. Note that by an application of the product rule, it can be written as

$$f^1_g(s^*_g)\pi_g(s^*_g) + F^1_g(s^*_g)\pi'_g(s^*_g).$$

Here, the first term, $f^1_g(s^*_g)\pi_g(s^*_g)$ corresponds to the fraction of the agents at the principal's threshold (s^*_g) who have taken action 1—this is exactly the numerator of the standard marginal outcomes test (recall Observation 2.3) and can be estimated similarly. The second term is the product of $F^1_g(s^*_g)$ (the false negative rate implied by the principal's threshold) and the previously estimated $\pi'_g(s^*_g)$. Analogously, the third term $\frac{\mathrm{d}F^0_g(s^*_g)(1-\pi_g(s^*_g))}{\mathrm{d}s^*_g}$, by an appeal to the product rule, can be written as

$$f^0_g(s^*_g)(1 - \pi_g(s^*_g)) - F^0_g(s^*_g)\pi'_g(s^*_g).$$

Here again, the first term is the denominator of the standard marginal outcomes test, while the second is the product of the previously estimated $\pi'_g(s^*_g)$ and 1− the false positive rate implied by the principal's threshold.

In short, therefore, relative to the standard marginal outcome test, three new quantities need to be estimated for each group: the previously discussed $\pi'_g(s^*_g)$; and the False Positive and False Negative rates for the group . Note that the first of these has no analog in the setting considered by the standard marginal outcome test. The latter two, i.e. the False Positive/ False Negative Rate of the principal's decision rule are well understood quantities economically: interestingly however these are precisely the quantities that the marginal outcomes test eschewed. The reason for their inclusion is simple: when agents' actions are endogenous, their incentives depend on the entire distribution of the principal's decisions, not just the principal's decisions at the margin.

The Role of Commitment

It may be useful at this stage to clarify the role of the two modeling assumptions we made, i.e. (1) commitment to the classification rule by the principal and (2) a relevant action being taken by strategic agents *after learning the principal's decision rule.*

First, as discussed, (2) is critical—if agents' actions are exogenously fixed then the marginal outcome test is valid independent of (1) (recall Observation 2.3).

As we show in what follows, (1) is also critical, i.e. in the absence of commitment, again, the marginal outcome test is valid. To see this, observe that in the absence of commitment, the principal can only take the sequentially rational decision at the time of deciding (i.e. the action that maximizes their expected utility conditional on the observed signal). Formally, a principal who sees signal s, in the absence of commitment takes $d = 1$ over $d = 0$ if:

$$u(1,1)\pi_g f_g^c(s) + u(1,0)(1-\pi_g) f_g^o(s) > u(0,1)\pi_g f_g^c(s) + u(0,0)(1-\pi_g) f_g^o(s).$$

Here π_g is the fraction of agents in group g who take action 1, since this is determined at the time the principal takes their action. The principal is indifferent if

$$u(1,1)\pi_g(s_g) f_g^c(s) + u(1,0)(1-\pi_g(s_g)) f_g^o(s) = u(0,1)\pi_g(s_g) f_g^c(s) + u(0,0)(1-\pi_g(s_g)) f_g^o(s)$$

which is equivalent to

$$\frac{f_g^1(s_g^*)\pi_g}{f_g^0(s_g^*)(1-\pi_g)} = \frac{u(0,0)-u(1,0)}{u(1,1)-u(0,1)}.$$

Under the MLRP assumption (assumption 1), the principal follows a threshold rule of decision $d = 1$ for $s > s_g^*$ and $d = 0$ otherwise. By observation, this is the same as the case of exogenously fixed actions, and the marginal outcome test remains valid. As we describe below, this case of endogenous actions but without commitment was considered in Knowles et al. (2001) and Anwar and Fang (2006).

2.4. Related Literature

The original marginal outcome test is generally attributed to Becker (1957). More recently, Hull (2021) and Bohren et al. (2019) revisit the marginal outcomes test and provide formal models in which the test is valid. On the flip side, Canay et al. (2020) point out that there are natural models in which the marginal outcome test fails in both directions, i.e. differences in marginal outcome are possible despite a principal who by assumption has no discriminatory preferences; and vice versa. Critically, they allow an agent's observable characteristics to directly enter the principal's preferences. This may be reasonable in some settings, nevertheless we follow the majority of the literature in assuming that other observables are informative for the principal but do not directly affect their preferences. Our "negative result" (i.e., Theorem 14) therefore is for conceptually different reasons.

As we pointed out earlier, a major difficulty operationalizing the marginal outcome test is correctly identifying the marginal agent so as to do the appropriate comparison. Various approaches have been taken to get around this. Closest in spirit to our paper is the paper of Knowles et al. (2001) on detecting racial bias in traffic stops (see also the extensions in the appendix of Anwar and Fang (2006))—they construct an equilibrium model in which both agents and police officers are strategic. In the taxonomy of our model, these papers consider a setting with out commitment to a policy, i.e., one where the police officers take a sequentially rational action given the information they observe rather than committing *a priori* to a policy. Operationally, in the equilibrium of their model, the marginal and average outcomes for agents are the same (since agents are observationally homogeneous to police officers beyond their race). This allows them to construct a test based on the (easy to observe) average outcomes. A majority of the papers however take a non-structural approach. In particular they use quasi-experimental approaches to identity the marginal agent, for example the random assignment of judges to cases— see e.g. Arnold et al. (2018), Feigenberg and Miller (2020), Grau, Vergara, et al. (2020).

More recently, there has been progress towards more robust tests: see e.g. Marx (2018) or Martin and Marx (2021). These papers construct tests based on necessary implications of unbiased decision making— i.e. passing the test does not necessarily imply unbiased decision, but failing the test is (strong) evidence of prejudice.

The idea of commitment to a policy, though not stated as such, also arises when thinking of the design of e.g., a machine learning algorithm to automatically classify agents. Computer scientists have grown increasingly concerned about whether and how even seemingly neutral algorithms can treat different demographic groups differently. This has resulted in literatures studying the incompatibility of various formal notions of fairness (see Chouldechova (2017), Kleinberg, Mullainathan, and Raghavan (2016)). A subsequent literature has proposed (or criticized) notions of fairness based on ethical/ normative grounds and discussed the possibility of algorithms that are fair with respect to such notions (see e.g. Dwork, Hardt, Pitassi, Reingold, and Zemel (2012); Hardt, Price, and Srebro (2016); Corbett-Davies and Goel (2018); Corbett-Davies, Pierson, Feller, Goel, and Huq (2017); Feller, Pierson, Corbett-Davies, and Goel (2016); Friedler, Scheidegger, and Venkatasubramanian (2016); Kearns, Neel, Roth, and Wu (2018); Hébert-Johnson, Kim, Reingold, and Rothblum (2018); Liu, Simchowitz, and Hardt (2019)). Perhaps the closest to the present paper is the paper of Jung, Kannan, Lee, Pai, Roth, and Vohra (2020) who study the design of optimal policy with respect to a specific objective function (in our terminology, a principal with paternalistic preferences, Example 8), and derive the optimal classification rule.

Finally, as we pointed out earlier, there has also been a literature in economic theory trying to understand the design of (e.g. affirmative action) policy taking into account differing incentives in differing groups to take a relevant action (e.g., invest in human capital)— see e.g. Loury et al. (1977), Coate and Loury (1993), Foster and Vohra (1992) or Fryer Jr and Loury (2013). The broader literature is surveyed in Fang and Moro (2011). Even outside the context of fairness/ discrimination, several papers study the provision of incentives in hiring/ admission settings. For an example of the former see Hatfield, Kojima, and Kominers (2014) or Hatfield, Kojima, and Kominers (2018) who point out that in employment matching settings, workers need to get the ex-post marginal product of their labor to align their incentives to undertake the ex-ante efficient investment in human capital. In the latter setting, Frankel and Kartik (2019) consider a setting where applicants have both a underlying ability and an ability to "game" the signal observed by the decision maker. They show that a decision maker wishing to match on underlying ability may wish to commit to a policy that conditions less strongly on the observed signal so as to disincentivize gaming. Finally, the work of Frankel (2021) studies a setting where a principal must delegate to an agent of unknown bias, and has limited control over the agent. For example, relevant to the present context, this could be a city hiring traffic police officers of unknown bias. The paper shows that the principal hires and delegates using a rule such that *marginal* police officer hired conducts traffic stops which would satisfy the marginal outcomes test.

APPENDIX

A.1. Proofs for Section 1.3 and Section 1.4

A.1.1. Proof for Lemma 2

When seller $i \in \mathcal{N}$ with θ_i reports truthfully as θ_i, her expected profit is

$$\Pi(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \left[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i}),$$

and the expected probability of her product i being recommended is

$$Q_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i}).$$

The following lemma characterize incentive compatible and individually rational recommender systems.

Lemma 11. *A recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ is incentive compatible, individually rational and obedient if and only if for each $i \in \mathcal{N}$, for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$,*

$$Q_i(\theta_i) \text{ is increasing in } \theta_i, \tag{A.1}$$

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i, \tag{A.2}$$

$$\Pi_i(\underline{\theta}) \geq 0 \tag{A.3}$$

and for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(dv) \boldsymbol{G}(d\boldsymbol{\theta}) \\ &\geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i r_j(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(dv) \boldsymbol{G}(d\boldsymbol{\theta}). \end{aligned} \tag{A.4}$$

Proof. **Necessity:** Let $\theta_i > \hat{\theta}_i$. Let

$$\pi_i(\hat{\theta}_i;\theta_i) = \int_{\mathcal{V}\times\boldsymbol{\Theta}_{-i}} \left[(\theta_i + w(v_i))r_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i})\right]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit of seller i with θ_i when she reports as $\hat{\theta}_i$. Note that

$$\pi_i(\hat{\theta}_i;\theta_i) = \Pi_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i)$$

Similarly,

$$\pi_i(\theta_i;\hat{\theta}_i) = \Pi_i(\theta_i) + (\hat{\theta}_i - \theta_i)Q_i(\theta_i).$$

Incentive compatibility implies $\Pi_i(\theta_i) \geq \pi_i(\hat{\theta}_i;\theta_i)$ and $\Pi_i(\hat{\theta}_i) \geq \pi_i(\theta_i;\hat{\theta}_i)$, which in turn implies

$$(\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \leq \Pi_i(\theta_i) - \Pi_i(\hat{\theta}_i) \leq (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (A.2).

Individual rationality is equivalent to $\Pi_i(\theta_i) \geq 0$ for all $i \neq 0$ and $\theta_i \in [\underline{\theta},\bar{\theta}]$, from which $\Pi_i(\underline{\theta}) \geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (A.2) and the monotonicity of Q_i,

$$\begin{aligned}\Pi_i(\theta_i) &= \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i \\ &\geq \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i)d\tilde{\theta}_i \\ &= \Pi_i(\hat{\theta}) + (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \\ &= \pi_i(\hat{\theta}_i;\theta_i) \qquad \text{(A.5)}\end{aligned}$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (A.5) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (A.2) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. $\square$

By Lemma 14, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$,

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i.$$

The expected transfer of the seller i with θ_i is

$$T_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i - \Pi_i(\underline{\theta}). \tag{A.6}$$

By the usual argument of the change of variables,

$$\begin{aligned}
&\int_{\Theta} T_i(\theta_i) G(d\theta_i) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)} \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) - \Pi_i(\underline{\theta}),
\end{aligned}$$

so that the intermediary's expected revenue is

$$\begin{aligned}
&\sum_{i \in \mathcal{N}} \int_{\Theta} T_i(\theta_i) G(d\theta_i) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)} \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) - \sum_{i \in \mathcal{N}} \Pi_i(\underline{\theta}).
\end{aligned} \tag{A.7}$$

The intermediary's problem is to maximize (A.7) using a recommender system $(\boldsymbol{r}, \boldsymbol{t})$ subject to monotonicity constraints (A.1), payoff equivalence constraints (A.2), non-negativity constraints (A.3) and obedience constraints (A.4). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (A.1) and (A.4), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta}) = 0$ for all $i \in \mathcal{N}$ and whose interim transfer satisfies (A.6) maximizes (1.7) while satisfying (A.2) and (A.3). Transfer (1.8) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta}) = 0$ for all $i \in \mathcal{N}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes

(1.7) subject to (A.1) and (A.4), together with the corresponding transfer (1.8), maximizes the intermediary's expected revenue subject to (A.1), (A.2), (A.3) and (A.4).

A.1.2. Proof for Lemma 3

The obedience constraint from a product $i \in \mathcal{N}$ to another product $j \in \mathcal{N}$ is

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_j) r_i(v_i, v_j, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0$$

which can be written as

$$\int_{v>v'} \int_{\boldsymbol{\mathcal{V}}_{-ij}\boldsymbol{\Theta}} \Big[(v - v') r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \boldsymbol{F}_{ij}(d(v, v')) \\ + (v' - v) r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \boldsymbol{F}_{ij}(d(v', v)) \Big] \boldsymbol{F}_{-ij}(d\boldsymbol{v}_{-ij}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0 \quad \text{(A.8)}$$

By symmetry, $\boldsymbol{F}_{ij}(d(v, v')) = \boldsymbol{F}_{ij}(d(v', v))$, so that (A.9) can be written as

$$\int_{v>v'} \int_{\boldsymbol{\mathcal{V}}_{-ij}\boldsymbol{\Theta}} (v - v') (r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) - r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta})) \\ \boldsymbol{F}_{ij}(d(v, v')) \boldsymbol{F}_{-ij}(d\boldsymbol{v}_{-ij}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0 \quad \text{(A.9)}$$

where the inequality follows from $v > v'$ and the value-switching monotonicity of $\boldsymbol{r}$, that is, $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) - r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq 0$.

A.1.3. Proof for Lemma 4

1. Let $i \in \mathcal{N}$. OB_{i0} is

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \quad \text{(A.10)}$$

   Define $R_i(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_i(v_i, \boldsymbol{v}_{-i}, \boldsymbol{\theta}) \boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i}) \boldsymbol{G}(d\boldsymbol{\theta})$ which is increasing in v_i, where $\mathbb{E}_{v_i}(R_i(v_i)) = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v}, \boldsymbol{\theta}))$. Then, (A.10) is equivalent to

$$Cov_{v_i}(v_i - v_0, R_i(v_i)) + (\mathbb{E}_{v_i}(v_i) - v_0) \mathbb{E}_{v_i}(R_i(v_i)) \geq 0. \quad \text{(A.11)}$$

The first term $Cov_{v_i}(v_i - v_0, R_i(v_i)) = Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))$ is non-negative since both $v_i - v_0$ and $R_i(v_i)$ are increasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) = 0$, then (A.11) always holds, so that $\bar{v}_i = \infty$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) > 0$, then (A.11) holds if and only if $v_0 \leq \bar{v}_i$, where

$$\bar{v}_i = \mathbb{E}_{v_i}(v_i) + Cov_{v_i}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) \geq \mathbb{E}_{v_i}(v_i).$$

In either way, there is $\bar{v}_i \geq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \leq \bar{v}_i$.

2. Let $i \in \mathcal{N}$. OB_{0i} is

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{A.12}$$

Define $R_0(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_0(v_i, \boldsymbol{v}_{-i}, \boldsymbol{\theta}) \boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i}) \boldsymbol{G}(d\boldsymbol{\theta})$ which is decreasing in v_i, where $\mathbb{E}_{v_i}(R_0(v_i)) = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}))$. Then, (A.12) is equivalent to

$$-Cov_{v_i}(v_i - v_0, R_0(v_i)) - (\mathbb{E}_{v_i}(v_i) - v_0)\mathbb{E}_{v_i}(R_0(v_i)) \geq 0. \tag{A.13}$$

The first term $Cov_{v_i}(v_i - v_0, R_0(v_i)) = Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta}))$ is non-positive since $v_i - v_0$ is increasing in v_i but $R_0(v_i)$ is decreasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) = 0$, then (A.13) always holds, so that $\underline{v}_i = 0$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) > 0$, then (A.13) holds if and only if $v_0 \geq \underline{v}_i$ where

$$\underline{v}_i = \mathbb{E}_{v_i}(v_i) + Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) \leq \mathbb{E}_{v_i}(v_i).$$

In either way, there is $\underline{v}_i \leq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \geq \bar{v}_i$.

A.1.4. Proof for Theorem 1.a

I first show that a symmetric recommender system attains the optimal revenue. Recall that recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^{\iota},\boldsymbol{\theta}^{\iota})$$

where $(\boldsymbol{v}^{\iota}, \boldsymbol{\theta}^{\iota})$ is such that $v^{\iota}_{\iota(i)} = v_i$ and $\theta^{\iota}_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

Lemma 12. *For each obedient recommendations rule $\boldsymbol{r}$, there is a symmetric recommendations rule $\boldsymbol{r}^0$ that is obedient and attains the same revenue as $\boldsymbol{r}$.*

Proof. Let us first construct a symmetric recommendations rule $\boldsymbol{r}^0$ from any given obedient recommendations rule $\boldsymbol{r}$. Let $\boldsymbol{r}$ be a recommendations rule. Let

$$\mathcal{I}^{\mathcal{N}} = \{\iota^{\dagger} \mid \iota : \mathcal{N} \to \mathcal{N} \text{ is a bijective function}\}$$

be a set of all permutation functions on $\mathcal{N}$. For each $\iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}$, let $\boldsymbol{r}^{\iota^{\dagger}}$ be a recommendations rule obtained by permutating $\boldsymbol{r}$ according to $\iota^{\dagger}$, i.e. for each $i \in \mathcal{N}$,

$$r_i^{\iota^{\dagger}}(\boldsymbol{v}, \boldsymbol{\theta}) = r_{\iota^{\dagger}(i)}(\boldsymbol{v}^{\iota^{\dagger}}, \boldsymbol{\theta}^{\iota^{\dagger}}).$$

Then, $\boldsymbol{r}^{\dagger}$ satisfies obedience constraints as well. Define another recommendatiosn rule $\boldsymbol{r}^0$ such that for each $i \in \mathcal{N}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{N!} \sum_{\iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}} r^{\iota^{\dagger}}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{N!} \sum_{\iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}} r_{\iota^{\dagger}(i)}(\boldsymbol{v}^{\iota^{\dagger}}, \boldsymbol{\theta}^{\iota^{\dagger}}). \tag{A.14}$$

To prove that $\boldsymbol{r}^0$ is symmetric, it is sufficient to show that for any bijection $\iota \in \mathcal{I}^{\mathcal{N}}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v}, \boldsymbol{\theta}) = r^0_{\iota(i)}(\boldsymbol{v}^{\iota}, \boldsymbol{\theta}^{\iota}). \tag{A.15}$$

To show this, note that

$$r^0_{\iota(i)}(\boldsymbol{v}^{\iota}, \boldsymbol{\theta}^{\iota}) = \frac{1}{N!} \sum_{\iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}} r_{\iota^{\dagger} \circ \iota(i)}(\boldsymbol{v}^{\iota^{\dagger} \circ \iota}, \boldsymbol{\theta}^{\iota^{\dagger} \circ \iota}) \tag{A.16}$$

where $\iota^{\dagger} \circ \iota$ is a composition of two permuntation functions. Note that there is a bijective mapping between $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^{\dagger} \circ \iota \mid \iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}\}$. To show this, since both $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^{\dagger} \circ \iota \mid \iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}\}$ are finite sets, it is sufficient to show that $\mathcal{I}^{\mathcal{N}} = \{\iota^{\dagger} \circ \iota \mid \iota^{\dagger} \in \mathcal{I}^{\mathcal{N}}\}$. To show the equality, first note that $\iota^\dagger \circ \iota$ is a composition of two bijective mappings and hence is a bijection, i.e. $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \subset \mathcal{I}^{\mathcal{N}}$. To show the other inclusion, let $\tilde{\iota} \in \mathcal{I}^{\mathcal{N}}$. Since both $\tilde{\iota}$ and ι are bijection over the same finite space, I can define $\iota^\dagger = \tilde{\iota} \cdot \iota^{-1}$ which is a composition of two bijections and hence a well-defined bijection over $\mathcal{N}$. By construction, for each $j \in \mathcal{N}$, $\iota^\dagger \circ \iota(j) = \iota^\dagger(\iota(j)) = \tilde{\iota}(j)$, and hence $\tilde{\iota} = \iota^\dagger \circ \iota$ for some $\iota^\dagger \in \mathcal{I}^{\mathcal{N}}$. In other words, $\tilde{\iota} \in \{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and hence $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \supset \mathcal{I}^{\mathcal{N}}$, which gives the desired equality. That there is a bijective mapping between $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and $\mathcal{I}^{\mathcal{N}}$ implies that the right-hand side of (A.16) is

$$\frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger \circ \iota(i)}(\boldsymbol{v}^{\iota^\dagger \circ \iota}, \boldsymbol{\theta}^{\iota^\dagger \circ \iota}) = \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}) = r_i^0(\boldsymbol{v}, \boldsymbol{\theta}),$$

and therefore, (A.15) holds.

It remains to show that $\boldsymbol{r}^0$ is obedient and attains the equal revenue. These results follow from the linearity of the revenue and obedience constraints. For each $i, j \in \mathcal{N}$, the obedience constraint from i to j for $\boldsymbol{r}^0$ is

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} (v_i - v_j) r_i^0(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} (v_i - v_j) \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} (v^{\iota^\dagger}_{\iota^\dagger(i)} - v^{\iota^\dagger}_{\iota^\dagger(j)}) r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}) \boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger}) \boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger}) \\
&\geq 0
\end{aligned} \tag{A.17}$$

where the third equality follows from the definition that $v^{\iota^\dagger}_{\iota^\dagger(i)} = v_i$ and $\theta^{\iota^\dagger}_{\iota^\dagger(i)} = \theta_i$, and the last inequality follows from the fact that $\boldsymbol{r}^{\iota^\dagger}$ is obtained by permutating $\boldsymbol{r}$ which is obedient, and hence, so is $\boldsymbol{r}^{\iota^\dagger}$.

For $i \in \mathcal{N}$, the obedience constraint from i to 0 is

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) r_i^0(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) && \text{(A.18)}\\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&= \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v^{\iota^\dagger}_{\iota^\dagger(i)} - v_0) r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}) \boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\
&\geq 0 && \text{(A.19)}
\end{aligned}$$

where the last inequality follows from OB_{i0} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{i0} is satisfied for $\boldsymbol{r}^0$. The obedience constraint from 0 to i is

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0^0(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) && \text{(A.20)}\\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) \left(1 - \sum_{j\in\mathcal{N}} r_j^0(\boldsymbol{v},\boldsymbol{\theta})\right) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) \left(1 - \sum_{j\in\mathcal{N}} \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&= \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v^{\iota^\dagger}_{\iota^\dagger(i)}) \left(1 - \sum_{j\in\mathcal{N}} r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right) \boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&= \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v^{\iota^\dagger}_{\iota^\dagger(i)}) r_0(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}) \boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\
&\geq 0 && \text{(A.21)}
\end{aligned}$$

where the last inequality follows from OB_{0i} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{0i} is satisfied for $\boldsymbol{r}^0$. By (A.17), (A.19) and (A.21), $\boldsymbol{r}^0$ is obedient.

It remains to verify that $\boldsymbol{r}$ and $\boldsymbol{r}^0$ attain the same revenue. Note that every $\boldsymbol{r}^{\iota^\dagger}$ has the same revenue as $\boldsymbol{r}$ because $\boldsymbol{r}^{\iota}$ is obtained by permutating $\boldsymbol{r}$ according to $\iota^\dagger$. Consequently, their average must be the same as the revenue obtained by $\boldsymbol{r}$ as shown below.

$$
\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta^{\iota^\dagger}_{\iota^\dagger(i)} - \frac{1-G(\theta^{\iota^\dagger}_{\iota^\dagger(i)})}{g(\theta^{\iota^\dagger}_{\iota^\dagger(i)})} + w(v^{\iota^\dagger}_{\iota^\dagger(i)})\right) r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger}) \\
&= \frac{1}{N!} \sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).
\end{aligned}
$$

□

By Lemma 12, there always exists a symmetric recommendations rule that maximizes the revenue. From here on, I focus on symmetric recommendations rules. The following lemma states that the following symmetric recommendations rule is value-switching monotone and recommends one of the options with certainty almost surely.

Lemma 13. *For each $i\in\mathcal{N}$, let $\xi_i:\mathcal{V}\times\Theta\to\mathbb{R}^1$ be any function that is strictly increasing in (v_i,θ_i), and $\xi_0:\mathcal{V}\times\Theta\to\mathbb{R}$ be any function that is symmetric and increases in $\boldsymbol{v}$, which could possibly be 0. Define*

$$\psi_i(\boldsymbol{v},\boldsymbol{\theta}) = \xi_i(v_i,\theta_i) + \xi_0(\boldsymbol{v}).$$

Ignoring ties, let

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} 1 & \text{if } i = \arg\max_{j\in\mathcal{N}}\left(\psi_j(\boldsymbol{v},\boldsymbol{\theta}),0\right) \\ 0 & \text{otherwise} \end{cases},$$

for $i\in\mathcal{N}$, and $r_0(\boldsymbol{v},\boldsymbol{\theta}) = 1-\sum_{i\in\mathcal{N}} r_i(\boldsymbol{v},\boldsymbol{\theta})$. Then, $\boldsymbol{r}$ is value-switching monotone almost surely and $r_j(\boldsymbol{v},\boldsymbol{\theta})=1$ for some $j\in\mathcal{N}\cup\{0\}$ almost surely.

Proof. Since ξ_i strictly increases and ξ_0 increases in (v_i, θ_i), almost surely, for each $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$, there is $i \in \mathcal{N}$ $\psi_i(\boldsymbol{v}, \boldsymbol{\theta}) > \max_{j \in \mathcal{N} \setminus \{i\}}(\psi_j(\boldsymbol{v}, \boldsymbol{\theta}), 0)$ or $0 > \max_{j \in \mathcal{N}} \psi_j(\boldsymbol{v}, \boldsymbol{\theta})$, so that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \{0\} \cup \mathcal{N}$ almost surely.

Let $\boldsymbol{\mathcal{W}} = \{(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \mid r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1 \text{ for some } i \in \mathcal{N} \cup \{0\}\}$. Since $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ almost surely, to show almost sure value-switching monotonicity, it is sufficient to show establish that for any $i \in \mathcal{N}$, $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i})$, $(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ such that $(v_i, \theta_i) \geq (v_i', \theta_i')$,

1. $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$ implies $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$,

2. $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 1$ implies $r_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 1$,

and for any $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$,

3. $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ implies $r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ for any $v > v'$.

To show the first item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$. By definition, $\max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$. Increasing from (v_i', θ_i') to (v_i, θ_i) increases $\psi_j(\boldsymbol{v}, \boldsymbol{\theta})$ for all $j \in \mathcal{N}$, so that $\max_{j \in \mathcal{N}} \psi_j(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \geq \max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$, and hence, $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$.

To show the second item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}) = 1$.

To show the last item, let $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$ and $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$. □

By Lemma 12, there is a symmetric optimal recommender system. The rest of the proof focuses on constructing a symmetric optimal recommender system.

Let $v_0 \in [\underline{v}^*, \bar{v}^*]$. The unconstrained optimal recommendations rule obtained ignoring the obedience constraints $\boldsymbol{\rho}^*$ as in (1.11) is obedient and hence optimal. In other words, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0$ for all $i \in \mathcal{N}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

Let $v_0 > \bar{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{i0}. Also, since $v_0 > \bar{v}^* \geq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{i0} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{i0}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda(v_0 - v_j), 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{A.22}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v})\right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

Let $v_0 < \underline{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{0i}. Also, since $v_0 < \underline{v}^* \leq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{0i} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{0i}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda \sum_{k\in\mathcal{N}}(v_0 - v_k), 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{A.23}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v})\right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

A.2. Proofs for Section 1.5

A.2.1. Preliminary Works to Section 1.5

Obedient, Incentive Compatible and Individual Rational Recommender System

This section characterizes incentive compatible, individually rational and obedient recommender system, and recast the intermediary's problem to a Bayesian persuasion problem.

For a seller $i \in \mathcal{N}$ with (θ_i, z_i) reporting truthfully as θ_i, let

$$\Pi(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} \left[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z})\right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit, and

$$Q(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected probability of recommending i's product.

Lemma 14. *A recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ is incentive compatible, individually rational and obedient if and only if for each $i \in \mathcal{N}$, for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,*

$$Q_i(\theta_i, z_i) \text{ is increasing in } \theta_i, \tag{A.24}$$

$$\Pi_i(\theta_i, z_i) = \Pi_i(\underline{\theta}, z_i) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i, \tag{A.25}$$

$$\Pi_i(\underline{\theta}) \geq 0 \tag{A.26}$$

and for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}} r_i(\boldsymbol{v}, \theta_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \\ \geq &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}} r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \end{aligned} \tag{A.27}$$

Proof. **Necessity:** Let $\theta_i > \hat{\theta}_i$ and $z_i \in \mathcal{Z}$. Let

$$
\begin{aligned}
&\pi_i(\hat{\theta}_i; \theta_i, z_i) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} \left[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \hat{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \hat{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})
\end{aligned}
$$

be the expected profit of the seller with (θ_i, z_i) when he reports as $\hat{\theta}_i$. Note that

$$
\pi_i(\hat{\theta}_i; \theta_i, z_i) = \Pi_i(\hat{\theta}_i, z_i) + (\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i, z_i)
$$

Similarly,

$$
\pi_i(\theta_i; \hat{\theta}_i, z_i) = \Pi_i(\theta_i, z_i) + (\hat{\theta}_i - \theta_i) Q_i(\theta_i, z_i).
$$

Incentive compatibility $\Pi_i(\theta_i, z_i) \geq \pi_i(\hat{\theta}_i; \theta_i, z_i)$ and $\Pi_i(\hat{\theta}_i, z_i) \geq \pi_i(\theta_i; \hat{\theta}_i, z_i)$, which in turn implies

$$
(\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i) \leq \Pi_i(\theta_i) - \Pi_i(\hat{\theta}_i) \leq (\theta_i - \hat{\theta}_i) Q_i(\theta_i, z_i).
$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (A.25).

Individual rationality is equivalent to $\Pi_i(\theta_i) \geq 0$ for all $i \neq 0$ and $\theta_i \in [\underline{\theta}, \bar{\theta}]$, from which $\Pi_i(\underline{\theta}) \geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (A.25) and the monotonicity of Q_i,

$$
\begin{aligned}
\Pi_i(\theta_i, z_i) &= \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i \\
&\geq \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i, z_i) d\tilde{\theta}_i \\
&= \Pi_i(\hat{\theta}, z_i) + (\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i, z_i) \\
&= \pi_i(\hat{\theta}_i; \theta_i, z_i) \qquad \text{(A.28)}
\end{aligned}
$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (A.28) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (A.25) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. □

Lemma 15. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \tag{A.29}$$

subject to OB and monotonicity. Suppose also that

$$t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})d\tilde{\theta}_i. \tag{A.30}$$

Then, $(\boldsymbol{r},\boldsymbol{t})$ is an optimal recommender system.

Proof. By Lemma 14, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,

$$\Pi_i(\theta_i, z_i) = \Pi_i(\underline{\theta}, z_i) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i)d\tilde{\theta}_i.$$

The expected transfer of the seller i with (θ_i, z_i) is

$$\begin{aligned} T_i(\theta_i, z_i) = &\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}_{-i}} (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i})\boldsymbol{H}(d\boldsymbol{z}_{-i}) \\ &- \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i)d\tilde{\theta}_i - \Pi_i(\underline{\theta}, z_i). \end{aligned} \tag{A.31}$$

By the usual argument of the change of variables, for each $z_i \in \mathcal{Z}$, we have

$$\begin{aligned} &\int_{\Theta\times\mathcal{Z}} T_i(\theta_i, z_i)G(d\theta_i \mid z_i)H(dz_i) \\ &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \left(\theta_i + w(v_i) - \frac{1-G(\theta_i)}{g(\theta_i)}\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - \int_{\mathcal{Z}} \Pi_i(\underline{\theta}, z_i)H(dz_i), \end{aligned}$$

so that the intermediary's expected revenue is

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}} \int_{\Theta\times\mathcal{Z}} T_i(\theta_i, z_i) G(d\theta_i \mid z_i) H(dz_i) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)}\right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \\
&- \int_{\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} \Pi_i(\underline{\theta}, z_i) \boldsymbol{H}(d\boldsymbol{z}).
\end{aligned} \tag{A.32}
$$

The intermediary's problem is to maximize (A.32) using a recommender system $(\boldsymbol{r}, \boldsymbol{t})$ subject to monotonicity constraints (A.24), payoff equivalence constraints (A.25), non-negativity constraints (A.26) and obedience constraints (A.27). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (A.24) and (A.27), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta}, z_i) = 0$ for all $i \in \mathcal{N}$ and $z_i \in \mathcal{Z}$ and whose interim transfer satisfies (A.31) maximizes (A.32) while satisfying (A.25) and (A.26). Transfer (A.30) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta}, z_i) = 0$ for all $i \in \mathcal{N}$ and $z_i \in \mathcal{Z}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes (A.29) subject to (A.24) and (A.27), together with the corresponding transfer (A.30), maximizes the intermediary's expected revenue subject to (A.24), (A.25), (A.26) and (A.27). □

Four Equivalent Representations of Intermediary's Problems without Additional Information

This section presents four equivalent ways to express the intermediary's problem without additional information, each interpreted as: 1. the intermediary's problem without additional information; 2. the intermediary's problem with additional information but with invariance constraints; 3. additional information as a change in preference; 4. additional information as relaxation of invariance constraints.

Lemma 16. *The followings are solution equivalent (after adjusting for invariance constraints related notations):*

1. *A recommendations rule without additional infromation* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ *that maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{A.33}$$

subject to monotonicity constraints without additional information, for all $i \in \mathcal{N}$ *and* $\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \theta_i, \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \\ \geq &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \end{aligned} \tag{A.34}$$

and obedience constraints without additional information, for all $i, j \in \mathcal{N} \cup \{0\}$*,*

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \\ \geq &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}). \end{aligned} \tag{A.35}$$

2. *A recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ *that maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{A.36}$$

subject to monotonicity constraints (A.24)*, obedience constraints* (A.27) *and invariance constraints*

$$r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}') \text{ for all } \boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}. \tag{A.37}$$

3. *A recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ *that maximizes* (A.36) *subject to monotonicity constraints* (A.24) *and obedience constraints* (A.27)*.*

4. *A recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ *that maximizes* (A.29) *subject to monotonicity constraint* (A.24)*, obedience constraints* (A.27) *and invariance constraints* (A.37)*.*

The first is the intermediary's problem without additional information after substituting the expected transfer with virtual willingness to pay using the standard arguments. The second is a reformulation of the first under the setup with additional information. The third is stating that invariance constraints (A.37) are redundant in the second, because the integrands both in the objective function and the constraints do not depend on $\boldsymbol{z}$.

Note that the set of constraints are identical under the third and the intermediary's problem with additional information. The only difference between the two problems is the objective functions. In other words, additional information changes the intermediary's objective function from (A.36) to (A.29) subject to the *same* constraints, i.e. 'additional information as a change in the intermediary's preference,' the idea used for the consumer surplus analysis.

This means that the baseline problem without additional information can be understood as maximizing the same objective function but with added invariance constraints in relative to . That is, additional information is a *deletion* of invariance constraints withe the *same* objective function, i.e. 'additional information as a deletion of invariance constraints,' the idea used for the intermediary's revenue analysis.

Proof. **1** $\iff$ **2**: Once restricting attention to the recommendations rule satisfying the invariance constraints, the first problem and the second problem are identical, and hence, their solutions must be solution-equivalent.

2 $\iff$ **3**: The solution to the third problem is

$$r_i^{P3}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } i = \arg\max_{j\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) + \ell_i(\boldsymbol{v}), 0\right)$$

where $\ell_i(\boldsymbol{v})$ is a cost of persuasion. Note that $\boldsymbol{r}^{P3}$ does not vary depending on $\boldsymbol{z}$, and hence, satisfies the invariance constraints. This is because neither the objective function (A.36) nor the obedience constraints (A.27) have integrands that depend on $\boldsymbol{z}$ whereas the monotonicity constraints (A.24) are automatically satisfied. Therefore, the solution to the third problem $\boldsymbol{r}^{P3}$ solves the second problem $\boldsymbol{r}^{P2}$.

2 $\iff$ **4**: Note that

$$\int_{\mathcal{Z}} \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} G(d\theta_i \mid z_i) H(dz_i) = 1 - \int_{\mathcal{Z}} G(\theta_i \mid z_i) H(dz_i) = 1 - G(\theta_i) = \frac{1 - G(\theta_i)}{g(\theta_i)} G(d\theta_i). \tag{A.38}$$

By (A.38), restricting attention to the recommendations rule satisfying the invariance constraints (A.37), the objective function in the fourth problem (A.29) becomes

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta})
\end{aligned}$$

so that the fourth problem $P4$ becomes the same as the second problem $P2$. □

A.2.2. Proof for Example 1

Let $\mathcal{H}$ be perfectly revealing additional information. For all $z \in \mathcal{Z}$, $\Theta = \{z\}$, so that $\frac{1-G(\theta|z)}{g(\theta|z)} = 0$ for all $\theta \in \Theta(z)$. Consequently, for any $\theta > \theta'$,

$$\Delta^{z,z'}(\theta, \theta') = \theta - \theta' > 0,$$

and therefore, satisfies the generalized Myerson's regularity. Note for any $\theta > \theta'$,

$$\Delta(\theta, \theta') = \left(\theta - \frac{1 - G(\theta)}{g(\theta)} \right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')} \right) \geq (\leq) \Delta^{z,z'}(\theta, \theta') = \theta - \theta'$$

if and only if

$$\frac{1 - G(\theta')}{g(\theta')} \geq (\leq) \frac{1 - G(\theta)}{g(\theta)},$$

that is, $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ. Therefore, the additional information increases θ-revenue difference if $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.

A.2.3. Proof for Example 2

Let $\mathcal{H}$ be lower censorship additional information. Let G be a twice continuously differentiable distribution that has a decreasing $\frac{1-G(\theta)}{g(\theta)}$ on Θ, and $g(\theta) > 0$ and $0 \leq g^{'}(\theta) < \infty$ on a neighborhood of $\underline{\theta}$. Note that these imply that $|g^{'}(\theta)| < M$ and $g(\theta) > m$ for some $0 < m < M < \infty$ on a neighborhood of $\underline{\theta}$.

For $z \in [\theta^*, \bar{\theta}]$, the signal fully reveals the state, $\Theta(z) = \{\theta\}$, and hence, $\frac{1-G(\theta|z)}{g(\theta|z)} = 0$. For $z = z_0$, the signal informs that $\theta \in \Theta(z) = [\underline{\theta}, \theta^*)$, and the inverse hazard rate is

$$\frac{1 - G(\theta \mid z_0)}{g(\theta \mid z_0)} = \frac{\int_\theta^{\bar{\theta}} g(\tilde{\theta}) 1_{\tilde{\theta} \in [\underline{\theta}, \theta^*)} d\tilde{\theta}}{g(\theta) 1_{\theta \in [\underline{\theta}, \theta^*)}} = \frac{G(\theta^*) - G(\theta)}{g(\theta)}.$$

Let us first show the lower censorship additional information satisfies the generalized Myerson's regularity. Let $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$ such that $\theta > \theta'$. If $\theta > \theta' \geq \theta^*$, then $\frac{1-G(\theta|z)}{g(\theta|z)} = 0 = \frac{1-G(\theta'|z')}{g(\theta'|z')}$, so that

$$\Delta^{z,z'}(\theta, \theta') = \theta - \theta' > 0. \tag{A.39}$$

If $\theta^* \geq \theta' > \theta$, then their additional signal is z_0, so that

$$\begin{aligned} \Delta^{z,z'}(\theta, \theta') &= \left(\theta - \frac{G(\theta^*) - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{G(\theta^*) - G(\theta')}{g(\theta')}\right) \\ &= (\theta - \theta')\left(1 - \frac{-g^2(\theta'') - (G(\theta^*) - G(\theta''))g^{'}(\theta'')}{g^2(\theta'')}\right) \\ &= (\theta - \theta')\left(2 + (G(\theta^*) - G(\theta''))\frac{g^{'}(\theta'')}{g^2(\theta'')}\right) \end{aligned} \tag{A.40}$$

where the second equality follows from the Mean Value Theorem and θ'' is some value between θ and θ'. Since $|g^{'}(\theta)| < M$ and $g(\theta) > m$ for some $0 < m < M < \infty$ on a neighborhood of $\underline{\theta}$ by assumption, the ratio $|\frac{g^{'}(\theta)}{g^2(\theta)}| < M'$ on a neighborhood of $\underline{\theta}$ for some $M' < \infty$. Since $\theta'' \in (\theta', \theta)[\underline{\theta}, \theta^*]$, as $\theta^* \to \underline{\theta}$, $G(\theta^*) - G(\theta'') \to 0$. Therefore, as $\theta^* \to \underline{\theta}$, $(G(\theta^*) - G(\theta''))\frac{g^{'}(\theta'')}{g^2(\theta'')} \to 0$, and hence, (A.40) is positive for any $\theta^* \geq \theta > \theta'$ for sufficiently small θ^*.

Lastly, if $\theta \geq \theta^* \geq \theta'$, then

$$\Delta^{z,z'}(\theta,\theta') = \theta - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) = (\theta - \theta') + \frac{1-G(\theta')}{g(\theta')} > 0. \tag{A.41}$$

By (A.42), (A.40) and (A.41), the lower censorship additional information satisfies the generalized Myerson's regularity.

Let us now prove that the lower censorship additional information decreases θ-revenue difference. If $\theta > \theta' \geq \theta^*$, then

$$\begin{aligned}
\Delta^{z,z'}(\theta,\theta') &= \theta - \theta' \\
&= \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) + \left(\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}\right) && \text{(A.42)}\\
&\leq \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) \\
&= \Delta(\theta,\theta') && \text{(A.43)}
\end{aligned}$$

where the last inequality follows from the assumption that $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.

If $\theta^* \geq \theta > \theta'$, then

$$\begin{aligned}
\Delta^{z,z'}(\theta,\theta') &= \left(\theta - \frac{G(\theta^*)-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{G(\theta^*)-G(\theta')}{g(\theta')}\right) \\
&= \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) + (1-G(\theta^*))\left(\frac{1}{g(\theta)} - \frac{1}{g(\theta')}\right) \\
&\leq \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) \\
&\leq \left(\theta - \frac{1-G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1-G(\theta')}{g(\theta')}\right) && \text{(A.44)}
\end{aligned}$$

where the last inequality follows from the assumption that $g(\theta)$ increases in θ on a neighborhood of $\underline{\theta}$, so that $\frac{1}{g(\theta)} - \frac{1}{g(\theta')} \leq 0$ if θ^* is sufficiently small.

If $\theta \geq \theta^* \geq \theta'$, then

$$
\begin{aligned}
\Delta^{z,z'}(\theta,\theta') &= \theta - \left(\theta' - \frac{G(\theta^*) - G(\theta')}{g(\theta')}\right) \\
&= \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right) + \left(\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta^*)}{g(\theta')}\right) \\
&\leq \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right) + \left(\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta^*)}{g(\theta^*)}\right) \\
&\leq \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right) \\
&\leq \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right) \qquad \text{(A.45)}
\end{aligned}
$$

where the first inequality follows from the assumption that $g(\theta)$ is increasing on a neighborhood of $\underline{\theta}$ so that $-\frac{1-G(\theta)}{g(\theta')} \leq -\frac{1-G(\theta^*)}{g(\theta^*)}$ if θ^* is small enough, and the second inequality follows from the assumption that $\frac{1-G(\theta)}{g(\theta)}$ is decreasing in θ so that $\frac{1-G(\theta)}{g(\theta)} \leq \frac{1-G(\theta^*)}{g(\theta^*)}$. Therefore, by (A.43), (A.44) and (A.45), the lower censorship additional information decreases θ-revenue difference.

A.2.4. Proof for Theorem 3

The following lemma provides a sufficient condition under which the consumer surplus under one recommendations rule $\tilde{\boldsymbol{r}}$ is higher or lower than that under the other $\boldsymbol{r}^\dagger$. The lemma states that if $\boldsymbol{r}^\dagger$ almost surely recommends an option that is at least (at most) as good as options recommended by $\tilde{\boldsymbol{r}}$, then the consumer surplus under $\boldsymbol{r}^\dagger$ is higher (lower) than that under $\tilde{\boldsymbol{r}}$.

Define the consumer surplus under $\boldsymbol{r}$ at v_0 as

$$
CS(v_0;\boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \left[v_0 r_0(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) + \sum_{i\neq 0} v_i r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)
$$

where

$$
u^*(v_0) = \max(v_0, \mathbb{E}(v_i))
$$

is the consumer's optimal payoff without recommendations.

Lemma 17. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } v_j \geq (\leq)\ v_i \text{ for } i,j \in \mathcal{N}\cup\{0\},$$

almost surely, then $CS(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ CS(v_0;\boldsymbol{r}^{\dagger})$*.*

Proof. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } v_j \geq\ v_i \text{ for } i,j \in \mathcal{N}\cup\{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}$. Then,

$$\sum_{i\in\mathcal{N}\cup\{0\}} v_i\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_i \leq \min_{j:r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_j \leq \sum_{j\in\mathcal{N}\cup\{0\}} v_j r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})$$

so that

$$\begin{aligned}
CS(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&= CS(v_0;\boldsymbol{r}^{\dagger}).
\end{aligned}$$

The other inequality may be shown similarly. □

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (A.36) subject to monotonicity constraints (A.24) and $\boldsymbol{\rho}^A$ maximizes (A.29) subject to monotonicity constraints (A.24).

Lemma 18. *Let* v_0 *at which both* $\boldsymbol{\rho}^*$ *and* $\boldsymbol{\rho}^A$ *are obedient. If additional information is* *well-behaving and increases (decreases) θ-revenue difference, then*

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \\
&\geq (\leq) \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).
\end{aligned} \tag{A.46}$$

Proof. Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the environment has small inverse hazard rates, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. To use Lemma 17, it is sufficient to show:

> if additional information increases (decreases) θ-revenue difference, $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $v_j \leq (\geq)\ v_i$ almost surely.

Notice that both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ always recommend products over the outside option only based on virtual willingness to pays. Since ties in the virtual willingness to pays happen with probability zero, we may restrict our attention to $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})$ such that no virtual willingness to pays tie.

Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which no virtual willingness to pays tie, and $i \in \mathcal{N}$ and $j \in \mathcal{N}$ be chosen with positive probability under $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$, respectively.

Suppose additional information increases θ-revenue difference. Suppose $v_j > v_i$. For i to be chosen under $\boldsymbol{\rho}^*$ with positive probability, i must yield a higher virtual willingness to pay than j, that is,

$$\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right) > w(v_j) - w(v_i),$$

which also implies that $\theta_i > \theta_j$ by Myerson's regularity. Since additional information increases θ-revenue difference,

$$\begin{aligned}
\left(\theta_i - \frac{1-G(\theta_i\mid z_i)}{g(\theta_i\mid z_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j\mid z_j)}{g(\theta_j\mid z_j)}\right) &\geq \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right) \\
&> w(v_j) - w(v_i),
\end{aligned}$$

that is, j yields a strictly lower virtual willingness to pay than i with additional information, implying that j cannot be chosen with positive probability under $\boldsymbol{\rho}^A$, a contradiction. Therefore, if additional information increases θ-revenue difference, then $v_j \leq v_i$ almost surely. By Lemma 17, $CS(v_0; \boldsymbol{\rho}^*) \geq CS(v_0; \boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations $u^*(v_0)$ is identical under both problems without and with additional information, this is equivalent to

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$
$$\geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}),$$

which proves the first part of Lemma 18.

To prove the second part of Lemma 18, suppose additional information decreases θ-revenue difference. Suppose $v_j < v_i$. For j to be chosen under $\boldsymbol{\rho}^A$ with positive probability, j must yield a higher virtual willingness to pay than i, that is,

$$\left(\theta_j - \frac{1 - G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)}\right) - \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)}\right) > w(v_i) - w(v_j),$$

which also implies that $\theta_j > \theta_i$ by generalized Myerson's regularity. Since additional information decreases θ-revenue difference,

$$\left(\theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)}\right) - \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)}\right) \geq \left(\theta_j - \frac{1 - G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)}\right) - \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)}\right)$$
$$> w(v_i) - w(v_j),$$

that is, i yields a strictly lower virtual willingness to pay than j with additional information, implying that j cannot be chosen with positive probability under $\boldsymbol{\rho}^*$, a contradiction. Therefore, if additional information increases θ-revenue difference, then $v_j \leq v_i$ almost surely. By Lemma 17, $CS(v_0; \boldsymbol{\rho}^*) \leq CS(v_0; \boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations $u^*(v_0)$ is identical under both problems without and with additional information, this is equivalent to

$$\int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$
$$\leq \int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}),$$

which proves the second part of Lemma 18. □

Let $\bar{v}^* = \mathbb{E}_{v_i}(v_i) + Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, \rho_i^*(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}))$. For $v_0 \leq \bar{v}^*$, $\boldsymbol{\rho}^*$ satisfies obedience constraints and hence $\boldsymbol{\rho}^*$ is optimal, that is, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $CS(v_0; \boldsymbol{\rho}^*) > 0$. For $v_0 > \bar{v}^*$, $\boldsymbol{\rho}^*$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^*$ so that $CS(v_0; \boldsymbol{r}^*) = 0$. In particular, at $v_0 = \bar{v}^*$, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $CS(v_0; \boldsymbol{r}^*) = 0$, implying that

$$\int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) = \bar{v}^*.$$

The consumer surplus without additional information is

$$CS(v_0; \boldsymbol{r}^*) = \begin{cases} \int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - \mathbb{E}_{v_i}(v_i) & \text{if } v_0 < \mathbb{E}_{v_i}(v_i) \\ \int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - v_0 & \text{if } v_0 \in [\mathbb{E}_{v_i}(v_i), \bar{v}^*] \\ 0 & \text{if } v_0 > \bar{v}^* \end{cases}$$

A similar analysis is applied for $\boldsymbol{r}^A$. Let $\bar{v}^A = \mathbb{E}_{v_i}(v_i) + Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, \rho_i^A(\boldsymbol{v},\boldsymbol{\theta})) \big/ \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_i^A(\boldsymbol{v},\boldsymbol{\theta}))$. Note that $\bar{v}^A \geq \mathbb{E}_{v_i}(v_i)$. For $v_0 \leq \bar{v}^A$, $\boldsymbol{\rho}^A$ satisfies obedience constraints and hence $\boldsymbol{\rho}^A$ is optimal, that is, $\boldsymbol{r}^A = \boldsymbol{\rho}^A$ and $CS(v_0; \boldsymbol{r}^A) > 0$. For $v_0 > \bar{v}^A$, $\boldsymbol{\rho}^A$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^A$ so that $CS(v_0; \boldsymbol{r}^A) = 0$. In particular, at $v_0 = \bar{v}^A$, $\boldsymbol{r}^A = \boldsymbol{\rho}^A$ and $CS(v_0; \boldsymbol{r}^A) = 0$, implying that

$$\int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) = \bar{v}^A.$$

The consumer surplus under the problem with additional information problem is

$$CS(v_0;\boldsymbol{r}^A)=\begin{cases}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})-\mathbb{E}_{v_i}(v_i) & \text{if } v_0<\mathbb{E}_{v_i}(v_i)\\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})-v_0 & \text{if } v_0\in[\mathbb{E}_{v_i}(v_i),\bar{v}^A]\\ 0 & \text{if } v_0>\bar{v}^A\end{cases}$$

For $v_0>\max(\bar{v}^*,\bar{v}^A)$, $CS(v_0;\boldsymbol{r}^*)=CS(v_0;\boldsymbol{r}^A)=0$. For $v_0\le\min(\bar{v}^*,\bar{v}^A)$, since the best value without recommendations $u^*(v_0)$ is identical without and with additional information,

$$CS(v_0;\boldsymbol{r}^*)\ge(\le)\ CS(v_0;\boldsymbol{r}^A)$$

if and only if

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\\ &\ge(\le)\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).\end{aligned}\tag{A.47}$$

For $v_0\in(\min(\bar{v}^*,\bar{v}^A),\max(\bar{v}^*,\bar{v}^A)]$, if $\bar{v}^*\le\bar{v}^A$, then $CS(v_0;\boldsymbol{r}^*)=0\le CS(v_0;\boldsymbol{r}^A)$; if $\bar{v}^*\ge\bar{v}^A$, then $CS(v_0;\boldsymbol{r}^*)\ge 0=CS(v_0;\boldsymbol{r}^A)$. Therefore, $CS(v_0;\boldsymbol{r}^*)\ge(\le)\ CS(v_0;\boldsymbol{r}^A)$ if and only if $\bar{v}^*\ge(\le)\ \bar{v}^A$ which is equivalent to (A.47). Therefore, for any $v_0\in\mathbb{R}$, $CS(v_0;\boldsymbol{r}^*)\ge(\le)\ CS(v_0;\boldsymbol{r}^A)$ if and only if (A.47). By Lemma 18, if additional information increases (decreases) rate of substitution, then (A.47) holds, and hence, $CS(v_0;\boldsymbol{r}^*)\ge(\le)\ CS(v_0;\boldsymbol{r}^A)$.

A.2.5. Proof for Theorem 4

Define a seller i's profit under recommendations rule $\boldsymbol{r}$ at v_0 by

$$\Pi_i(v_0;\boldsymbol{r})=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\frac{1-G(\theta_i)}{g(\theta_i)}r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).$$

and sum of all sellers' expected profits by

$$\Pi(v_0;\boldsymbol{r})=\sum_{i\in\mathcal{N}}\Pi_i(v_0;\boldsymbol{r}).$$

Lemma 19. *Let $\tilde{\boldsymbol{r}}$ and $\boldsymbol{r}^{\dagger}$ be recommendations rule that never recommends the outside option almost surely.*

1. *Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } \theta_j \geq (\leq)\ \theta_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

   almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

2. *Suppose $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ. If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } \theta_j \leq (\geq)\ \theta_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

   almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \geq (\leq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

Proof. Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } \theta_j \geq\ \theta_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. Then,

$$\begin{aligned}
\sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)}\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) &\leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_i)}{g(\theta_i)} \\
&\leq \min_{j:r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_j)}{g(\theta_j)} \\
&\leq \sum_{j\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_j)}{g(\theta_j)} r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})
\end{aligned}$$

so that

$$\begin{aligned}
\Pi(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)} \tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)\\
&\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)\\
&= \Pi(v_0;\boldsymbol{r}^{\dagger}).
\end{aligned}$$

The other inequalities may be shown similarly. □

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (A.36) subject to monotonicity constraints (A.24) and $\boldsymbol{\rho}^A$ maximizes (A.29) subject to monotonicity constraints (A.24). Note that under small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ does not recommend the outside option almost surely.

Lemma 20. *Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Suppose the additional information is regular. Then,*

$$\Pi(v_0;\boldsymbol{\rho}^*) \leq (\geq)\ \Pi(v_0;\boldsymbol{\rho}^A). \tag{A.48}$$

if one of the following conditions is satisfied:

1. *Additional information increases (decreases) θ-revenue difference and inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ increases (decreases) in θ.*
2. *Additional information decreases (increases) θ-revenue difference and inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.*

Proof. I prove that if additional information increases θ-revenue difference and inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ increases in θ, then $\Pi(v_0;\boldsymbol{\rho}^*) \geq \Pi(v_0;\boldsymbol{\rho}^A)$. Others may be shown in a similar way.

Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the environment has small inverse hazard rates, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. To use Lemma 19, it is sufficient to show:

> if additional information increases θ-revenue difference and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ, $\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $\theta_j \geq \theta_i$ almost surely.

Notice that both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ always recommend products over the outside option only based on virtual willingness to pays. Since ties in the virtual willingness to pays happen with probability zero, we may restrict our attention to $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z})$ such that no virtual willingness to pays tie.

Let $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which no virtual willingness to pays tie, and $i \in \mathcal{N}$ and $j \in \mathcal{N}$ be chosen with positive probability under $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$, respectively.

Suppose additional information increases θ-revenue difference. Suppose $\theta_j < \theta_i$. For i to be chosen under $\boldsymbol{\rho}^*$ with positive probability, i must yield a higher virtual willingness to pay than j, that is,

$$\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right) > w(v_j) - w(v_i).$$

Since additional information increases θ-revenue difference,

$$\begin{aligned}\left(\theta_i - \frac{1-G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)}\right) &\geq \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right) \\ &> w(v_j) - w(v_i),\end{aligned}$$

that is, j yields a strictly lower virtual willingness to pay than i with additional information, implying that j cannot be chosen with positive probability under $\boldsymbol{\rho}^A$, a contradiction. Therefore, if additional information increases θ-revenue difference, then $v_j \leq v_i$ almost surely. By Lemma 19, $\Pi(v_0; \boldsymbol{\rho}^*) \geq \Pi(v_0; \boldsymbol{\rho}^A)$. □

Let $v_0 \le \mathbb{E}_{v_i}(v_i)$. Under a small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient, so that Lemma 20 applies straightforwardly to induce the desired results.

A.2.6. Proof for Theorem 5

Note that for each $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$,

$$\begin{aligned}
\int_{\mathcal{Z}} \left(\theta - \frac{1 - G(\theta \mid z)}{g(\theta \mid z)}\right) G(d\theta \mid z) H(dz) &= \theta g(\theta) - \int_{\mathcal{Z}} (1 - G(\theta \mid z)) H(dz) \\
&= \theta g(\theta) - \left(1 - \int_{\mathcal{Z}} \Pr(\tilde{\theta} \le \theta, z) dz\right) \\
&= \theta g(\theta) - (1 - G(\theta)) \\
&= \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) G(d\theta)
\end{aligned}$$

and

$$\int_{\mathcal{Z}} v G(d\theta \mid z) H(dz) = v G(d\theta).$$

This means that the intermediary's problem without additional information, which is to maximize

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta)} + w(v_i)\right) r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta})$$

subject to obedience constraints

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_i r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_j r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \text{ for all } i, j \in \mathcal{N} \cup \{0\}$$

is identical to maximizing the intermediary's problem with additional information

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i)\right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$

subject to obedience constraints

$$\int_{\mathcal{V}\times\Theta\times\mathcal{Z}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})$$
$$\geq \int_{\mathcal{V}\times\Theta\times\mathcal{Z}} v_j r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \text{ for all } i,j\in\mathcal{N}\cup\{0\}$$

and invariance constraints

$$\boldsymbol{r}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = \boldsymbol{r}^{\dagger}(\boldsymbol{v},\boldsymbol{\theta}) \text{ for some } \boldsymbol{r}^{\dagger} \text{ for all } \boldsymbol{z}\in\boldsymbol{\mathcal{Z}}.$$

The intermediary's problem with additional information is the same but without the invariance constraints. Since both have the same objective function but there is another set of constraints in the problem without the additional information, by revealed preference, the intermediary's revenue is higher.

A.2.7. Proof for Theorem 6

Suppose $v_0 < \underline{v}$. Since the consumer always prefers products over the outside option always, any obedient recommendations rule must always recommend one of the products. An optimal recommendations rule with this constraint but without additional information is given by for each $i\in\mathcal{N}$

$$\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = \frac{1}{|\mathcal{M}^*|} \text{ if } i\in\mathcal{M}^* \tag{A.49}$$

where $\mathcal{M}^* = \arg\max_{j\in\mathcal{N}}\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}$, and that with additional information is given by

$$\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = \frac{1}{|\mathcal{M}^A|} \text{ if } i\in\mathcal{M}^A \tag{A.50}$$

where $\mathcal{M}^A = \arg\max_{j\in\mathcal{N}}\{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j)\}$. Both recommendations rules are completely determined by θ-revenue difference, so that whether additional information increases the consumer surplus or not is completely determined by whether additional information decreases or increases the θ-revenue difference. Applying similar arguments as in Theorem 3 gives the desired result.

Suppose $v_0 > \bar{v}$. Since the consumer always prefers the outside option over products, any obedient recommendations rule must always recommend the outside option, without and with additional information, under which the consumer surplus is always 0. Therefore, the additional information does not change the consumer surplus.

A.3. Proofs for Section 1.6

A.3.1. Proof for Lemma 7

Let $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. Suppose each seller $j \in \mathcal{N} \setminus \{i\}$ bids b_j and the seller i is charged with a price premium p_i and a cost of persuastion $\ell_i(\boldsymbol{v})$. If the seller i bids b_i and wins, then her ex-post payoff is

$$\theta_i + w(v_i) - \Big(\max_{j \in \mathcal{N} \setminus \{i\}} (b_j, 0) + p_i + \ell_i(\boldsymbol{v}) \Big).$$

Now I show that $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ is a weakly dominant strategy.

If $\theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v}) > \max_{j \in \mathcal{N} \setminus \{i\}} (b_j, 0)$, then the seller i's ex-post payoff after winning is positive, so that she prefers winning over losing and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ results in winning.

If $\theta_i + w(v_i) - p_i(\theta_i, \boldsymbol{v}) - \ell_i(\boldsymbol{v}) < \max_{j \in \mathcal{N} \setminus \{i\}} (b_j, 0)$, then the seller i's ex-post payoff after winning is negative, so that she prefers losing over winning and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ results in losing.

If $\theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v}) = \max_{j \in \mathcal{N} \setminus \{i\}} (b_j, 0)$, then the seller i's ex-post payoff after winning is zero, so that she is indifferent among winning, losing and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i$ results in any of those.

Therefore, it is weakly dominant to bid $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$.

A.3.2. Proof for Lemma 8

For reference, let

$$\Pi_i^H(\theta_i) = \Pi_i^H(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \text{ for all } i \neq 0 \text{ and } \theta_i \in \Theta \tag{A.51}$$

and

$$Q_i(\theta_i,\theta_i') \leq Q_i(\theta_i,\theta_i) \leq Q_i(\theta_i,\theta_i'') \text{ for all } \theta_i',\theta_i'' \in [\underline{\theta},\bar{\theta}] \text{ such that } \theta_i' < \theta_i < \theta_i''. \tag{A.52}$$

Necessity (→):

Note that an incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{\theta}_i < \theta_i,\ \pi_i^H(\theta_i,\theta_i) \geq \pi_i^H(\theta_i,\hat{\theta}_i) \text{ and } \pi_i^H(\hat{\theta}_i,\hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i,\theta_i).$$

Without loss of generality, we assume $\hat{\theta}_i < \theta_i$. Define for $x, y \in [\underline{\theta},\bar{\theta}]$,

$$\begin{aligned}\Delta(x,y) = E_{\theta_{-i},\boldsymbol{v}}[&(x + w(v_i) - p_i(y) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\\ &\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) - w(v_i) - p_i(y) + \ell_i(\boldsymbol{v}) \in (\min(x,y),\max(x,y))\}}]\Big]\end{aligned}$$

We can rewrite $\pi_i^H(\theta_i,\hat{\theta}_i)$ as

$$\begin{aligned}\pi_i^H(\theta_i,\hat{\theta}_i) = E_{\theta_{-i},\boldsymbol{v}}\Big[&(\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\\ &\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\Big] - c(\hat{\theta}_i)\\ &+ E_{\theta_{-i},\boldsymbol{v}}\Big[(\theta_i - \hat{\theta}_i)\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}\}}\Big]\\ &+ E_{\theta_{-i},\boldsymbol{v}}\Big[(\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\\ &\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) \geq \hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v})\}}\Big]\\ = \Pi_i^H(\hat{\theta}_i) + Q_i^H(\hat{\theta}_i,\hat{\theta}_i)(\theta_i - \hat{\theta}_i) + \Delta_i(\theta_i,\hat{\theta}_i).&\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{\theta}_i, \theta_i) = \Pi_i^H(\hat{\theta}_i) - Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(\theta_i - \hat{\theta}_i) - \Delta_i(\theta_i, \hat{\theta}_i).$$

Incentive compatibility is equivalent to, for all i and $\hat{\theta}_i < \theta_i$,

$$Q_i^H(\hat{\theta}_i, \hat{\theta}_i) + \frac{\Delta_i(\theta_i, \hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i) + \frac{\Delta_i(\hat{\theta}_i, \theta_i)}{\theta_i - \hat{\theta}_i}$$

Note that $\Delta_i(x, y) \geq 0$ if and only if $x \geq y$, so that $\Delta_i(\hat{\theta}_i, \theta_i) \leq 0 \leq \Delta_i(\theta_i, \hat{\theta}_i)$. Therefore, incentive compatibility implies $\mathrm{Q}_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i)$. Since $Q_i^H(\theta_i, \theta_i)$ increases in θ_i, it is integrable, which implies (A.51).

It remains to prove that (A.52) holds. Suppose $\hat{\theta}_i < \theta_i$. If $p(\hat{\theta}_i) \geq p(\theta_i)$, then $Q_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \theta_i)$. Suppose $p(\hat{\theta}_i) < p(\theta_i)$. Let

$$\begin{aligned}\epsilon_i(x, y) = \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}\Big[&\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) \in (\min(x - p_i(x) - \ell_i(\boldsymbol{v}), p(y) + \ell_i(\boldsymbol{v})), \max(x - p(x) - \ell_i(\boldsymbol{v}), p(y) + \ell_i(\boldsymbol{v})))\}} \\ &\cdot (x + w(v_i) - p_i(y) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\Big].\end{aligned}$$

Rewrite

$$\begin{aligned}\pi_i^H(\theta_i, \hat{\theta}_i) =& \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})} \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}) \\ &(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))] - c(\theta_i) \\ &- \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})} \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\})(p(\theta_i) - p(\hat{\theta}_i))] + c(\theta_i) - c(\hat{\theta}_i) \\ &+ \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}\Big[\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) > \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})\}} \\ &\cdot (\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\Big] \\ =& \Pi_i^H(\theta_i) + Q_i^H(\theta_i, \theta_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + c(\theta_i) - c(\hat{\theta}_i) + \epsilon_i(\theta_i, \hat{\theta}_i).\end{aligned}$$

Similarly,

$$\pi_i^H(\theta_i, \hat{\theta}_i) = \Pi_i^H(\hat{\theta}_i) + Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + c(\hat{\theta}_i) - c(\theta_i) - \epsilon_i(\hat{\theta}_i, \theta_i).$$

By incentive compatibility, $\pi_i^H(\theta_i, \hat{\theta}_i) - \pi_i^H(\theta_i, \theta_i) \le 0 \le \pi_i^H(\hat{\theta}_i, \hat{\theta}_i) - \pi_i^H(\hat{\theta}_i, \theta_i)$, which is equivalent to

$$\pi_i^H(\theta_i, \theta_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + \epsilon_i(\theta_i, \hat{\theta}_i) \le c(\hat{\theta}_i) - c(\theta_i) \le \pi_i^H(\hat{\theta}_i, \hat{\theta}_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + \epsilon_i(\hat{\theta}_i, \theta_i). \tag{A.53}$$

Since $\hat{\theta}_i < \theta_i$ and $p(\hat{\theta}_i) < p(\theta_i)$, it follows that $\epsilon_i(\hat{\theta}_i, \theta_i) \le 0 \le \epsilon_i(\theta_i, \hat{\theta}_i)$. Then, $\epsilon_i(\theta_i, \hat{\theta}_i) = \epsilon_i(\hat{\theta}_i, \theta_i) = 0$; otherwise, (A.53) implies $Q_i^H(\theta_i, \theta_i) < Q_i^H(\hat{\theta}_i, \hat{\theta}_i)$, contradicting (A.3.2). Since $\epsilon_i(\theta_i, \hat{\theta}_i) = 0$,

$$\mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \ge b^{**}_{-i\,-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) > \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})\}}) = 0,$$

which is equivalent to $\Pi_i^H(\theta_i, \hat{\theta}_i) = \Pi_i^H(\theta_i, \theta_i)$. Therefore, $\hat{\theta}_i < \theta_i$ implies $\Pi_i^H(\theta_i, \hat{\theta}_i) \le \Pi_i^H(\theta_i, \theta_i)$ whether $p(\theta_i) \le p(\hat{\theta}_i)$ or not, so that the first inequality of (A.52) holds. The other inequality may be shown similarly, which establishes (A.52).

Sufficiency ($\leftarrow$): Suppose that the seller i with θ_i has reported itself as $\hat{\theta}_i$ when purchasing the premium. In the second stage, after learning $\boldsymbol{v}$, the payoff of reporting as θ_i' is

$$\begin{aligned} &U_i(\theta_i, \theta_i'; \hat{\theta}_i, \boldsymbol{v}) \\ &= E_{\theta_{-i}}\left[(\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i' + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \ge b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right]. \end{aligned}$$

Since the second price auction in the second stage is incentive compatible, for any $\theta_i \in (\underline{\theta}, \bar{\theta})$

$$\frac{\partial U_i^H}{\partial \theta_i}(\theta_i; \hat{\theta}_i, \boldsymbol{v}) = Q_i^H(\theta_i \mid \hat{\theta}_i, \boldsymbol{v})$$

where $U_i^H(\theta_i;\hat{\theta}_i,\boldsymbol{v}) = U_i(\theta_i,\theta_i;\hat{\theta}_i,\boldsymbol{v})$ and $Q_i^H(\theta_i;\hat{\theta}_i,\boldsymbol{v}) = E_{\theta_{-i}}\left[\mathbf{1}_{\{\theta_i'+w(v_i)-p_i(\hat{\theta}_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]$.
For any $\theta_i,\theta_i'\in(\underline{\theta},\bar{\theta})$,

$$U_i^H(\theta_i;\hat{\theta}_i,\boldsymbol{v}) = U_i^H(\theta_i';\hat{\theta}_i,\boldsymbol{v}) + \int_{\theta_i'}^{\theta_i} Q_i^H(\tilde{\theta};\hat{\theta}_i,\boldsymbol{v})d\tilde{\theta}.$$

The seller's interim payoff function in the first stage may be expressed as

$$\begin{aligned}
\pi_i^H(\theta_i,\hat{\theta}_i) &= E_{\boldsymbol{v}}\left[U_i^H(\theta_i;\hat{\theta}_i,\boldsymbol{v})\right] - c(\hat{\theta}_i)\\
&= E_{\boldsymbol{v}}\left[U_i^H(\hat{\theta}_i;\hat{\theta}_i,\boldsymbol{v})\right] + E_{\boldsymbol{v}}\left[\int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i;\hat{\theta}_i,\boldsymbol{v})d\tilde{\theta}_i\right] - c(\hat{\theta}_i)\\
&= \Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta},\hat{\theta}_i)d\tilde{\theta}.
\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{\theta}_i,\theta_i) = \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta},\theta_i)d\tilde{\theta}.$$

Note that the incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{\theta}_i<\theta_i,\ \pi_i^H(\theta_i,\theta_i)\geq\pi_i^H(\theta_i,\hat{\theta}_i) \text{ and } \pi_i^H(\hat{\theta}_i,\hat{\theta}_i)\geq\pi_i^H(\hat{\theta}_i,\theta_i).$$

Note that

$$\begin{aligned}
&\pi_i^H(\theta_i,\theta_i)\geq\pi_i^H(\theta_i,\hat{\theta}_i)\\
&\text{iff } \Pi_i^H(\theta_i)\geq\Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)d\tilde{\theta}_i\\
&\text{iff } \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \geq \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)d\tilde{\theta}_i \qquad \text{(By (A.51))}
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)\geq Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)$ for all $\tilde{\theta}_i\geq\hat{\theta}_i$ by (A.52).

Similarly,

$$\begin{aligned}
&\pi_i^H(\hat\theta_i,\hat\theta_i) \geq \pi_i^H(\hat\theta_i,\theta_i)\\
&\text{iff } \Pi_i^H(\hat\theta_i) \geq \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat\theta_i} Q_i^H(\tilde\theta_i,\theta_i)d\tilde\theta_i\\
&\text{iff } \int_{\theta_i}^{\hat\theta_i} Q_i^H(\tilde\theta_i,\tilde\theta_i)d\tilde\theta_i \geq \int_{\theta_i}^{\hat\theta_i} Q_i^H(\tilde\theta_i,\theta_i)d\tilde\theta_i \qquad\qquad \text{(By (A.51))}\\
&\text{iff } \int_{\hat\theta_i}^{\theta_i} Q_i^H(\tilde\theta_i,\tilde\theta_i)d\tilde\theta_i \leq \int_{\hat\theta_i}^{\theta_i} Q_i^H(\tilde\theta_i,\theta_i)d\tilde\theta_i
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde\theta_i,\tilde\theta_i) \leq Q_i^H(\tilde\theta_i,\theta_i)$ for all $\tilde\theta_i \leq \theta_i$ by (A.52).

A.3.3. Proof for Theorem 9

Given the handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$, if every seller of every type reports its willingness to pay truthfully under the handicap auction, then the intermediary recommends product i if and only if

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1 \text{ iff } \theta_i + w(v_i) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_i^*(\boldsymbol{v}) > \max_{j\neq i,0}\left(\theta_j + w(v_j) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_j^*(\boldsymbol{v}), 0\right)$$

which is the same recommendations rule as in the revenue maximizing recommender system.

It remains to verify that the handicap auction (1.31) and (1.32) is incentive compatible and individually rational. Since p_i weakly decreases in θ_i, the monotonicity condition (A.52) holds. The interim payoff of the seller i with θ_i receives is

$$\begin{aligned}
\Pi_i^H(\theta_i) =& E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]\\
&- c(\theta_i)
\end{aligned}$$

which implies that $\Pi_i^H(\underline{\theta}) = 0$. The handicap auction gives the same revenue as the revenue maximizing recommender system because the recommendation rules are identical and the lowest type's payoff is the same by $\Pi_i^*(\underline{\theta}) = 0 = \Pi_i^H(\underline{\theta})$, and therefore, $\Pi_i^*(\theta_i) = \Pi_i^H(\theta_i)$, which in turn implies that the expected payment from each recommender system must be identical between the two. The individual rationality trivially follows from the observation that the lowest payoff each seller gets $\Pi_i(\underline{\theta})$ is 0.

A.3.4. Proof for Theorem 10

In a virtual private willingness to pay environment,

$$\alpha\theta_i - \beta \text{ for some } \alpha > 1, \beta > 0,$$

the optimal recommendations rule is given by for each $i \in \mathcal{N}$,

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j + \frac{1}{\alpha}w(v_j) - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha}\right\} \\ 0 & \text{otherwise} \end{cases} \tag{A.54}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j + \frac{1}{\alpha}w(v_j) - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v})\right\}$, and

$$\ell_i^*(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^*, \bar{v}^*] \\ \lambda_1^*(v_0) \cdot (v_0 - v_i) & \text{if } v_0 > \bar{v}^* \\ \lambda_2^*(v_0) \cdot \sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^* \end{cases} \tag{A.55}$$

Consider the second-price auction with discounts

$$d_i^*(\boldsymbol{v}) = -\frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) - \frac{\alpha - 1}{\alpha}w(v_i),$$

and

$$P_i^* = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\Bigg[\left(\underline{\theta} + \frac{1}{\alpha}w(v_i) - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) - \max_{j\in\mathcal{N}\setminus\{i\}}\left(\theta_j + \frac{1}{\alpha}w(v_j) - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha}\right)\right)$$
$$\cdot \mathbf{1}_{\{\underline{\theta} + \frac{1}{\alpha}w(v_i) - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) > \max_{j\in\mathcal{N}\setminus\{i\}}(\theta_j + \frac{1}{\alpha}w(v_j) - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha})\}}\Bigg]$$

with reservation price $\frac{\beta}{\alpha}$.

After paying the participation fee P_i^*, if seller i bids b_i while others bid b_{-i}, her payoff is

$$\theta_i + w(v_i) - \left(\max_{j \in \mathcal{N} \setminus \{i\}} \left(b_j, \frac{\beta}{\alpha} \right) - d_i^* \right) \tag{A.56}$$

if $b_i > \max_{j \in \mathcal{N} \setminus \{i\}} \left(b_j, \frac{\beta}{\alpha} \right)$, and 0 if $b_i < \max_{j \in \mathcal{N} \setminus \{i\}} \left(b_j, \frac{\beta}{\alpha} \right)$. She prefers winning (losing) over losing (winning) and being drawn if and only if (A.56) is positive (negative), which is equivalent to

$$\theta_i + w(v_i) + d_i^* > (<) \max_{j \in \mathcal{N} \setminus \{i\}} \left(b_j, \frac{\beta}{\alpha} \right).$$

Therefore, it is a weakly dominant strategy to bid

$$b_i = \theta_i + w(v_i) + d_i^* = \theta_i + \frac{1}{\alpha} w(v_i) - \frac{1}{\alpha} \ell_i^*(\boldsymbol{v}). \tag{A.57}$$

Combined with the reservation price $\frac{\alpha}{\beta}$, the recommendations rule under the weakly dominant strategy equilibrium under which all sellers bid according to (A.57) is (A.54). Given the weakly dominant strategy equilibrium, seller i's expected payoff is

$$\mathbb{E}_{\boldsymbol{v}, \theta_{-i}} \Bigg[\left(\theta_i + \frac{1}{\alpha} w(v_i) - \frac{1}{\alpha} \ell_i^*(\boldsymbol{v}) - \max_{j \in \mathcal{N} \setminus \{i\}} \left(\theta_j + \frac{1}{\alpha} w(v_j) - \frac{1}{\alpha} \ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha} \right) \right)$$
$$\cdot \mathbf{1}_{\{\theta_i + \frac{1}{\alpha} w(v_i) - \frac{1}{\alpha} \ell_i^*(\boldsymbol{v}) > \max_{j \in \mathcal{N} \setminus \{i\}} (\theta_j + \frac{1}{\alpha} w(v_j) - \frac{1}{\alpha} \ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha})\}} \Bigg] - P_i^*,$$

which becomes 0 if $\theta_i = \underline{\theta}$. Therefore, the second-price auction with discounts and participation fees implement the optimal recommendations rule and attains the same revenue.

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