Notes On Hierarchies And Inductive Inference

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The following notes rework a discussion due to Kevin Kelly on the application of topological notions in the context of learning (see Kelly (1990)). All the results except for (2), (4) and (9) are due to Kelly, but are proved differently.

1 Preliminaries

1.1 Hierarchies

Fix a countable set $\mathbf{O}$. We suppose $\mathbf{O}$ is equipped with adequate structure to allow the definition of recursive relations on the product spaces $\mathbf{O}^{<\omega} \times \omega^n$ for $n \in \omega$. We assume familiarity with the definitions and basic properties of the relativized arithmetical hierarchy and the (finite levels of) the Borel hierarchy (see Hinman (1978), Chapter 3, Section 1). We use the following notation: $\Sigma_n[\beta]$ is the n-th level (on the Σ side) of the arithmetical hierarchy relativized to $\beta \in \omega^\omega$; similarly, $\mathbf{\Sigma}_n = \bigcup_{\beta \in \omega^\omega} \Sigma_n[\beta]$ is the n-th level (on the Σ side) of the Borel hierarchy. $\Pi_n[\beta]$ and $\mathbf{\Pi}_n$ are defined similarly. We make use of the following fact which is a consequence of the normal form theorem for relativized arithmetical relations.

(1) FACT: For every $X \subseteq \mathbf{O}^\omega$, $X \in \Sigma_2[\beta]$ if and only if there is a recursive in β relation $R \subseteq \mathbf{O}^{<\omega} \times \omega^2$ such that $\forall t \in \mathbf{O}^\omega (t \in X \leftrightarrow \exists m \forall n R(\bar{t}(n), n, m))$.

Let $\mathcal{K} \subseteq \mathbf{O}^\omega$ be given. Define $\Sigma_n[\beta](\mathcal{K}) = \{X \cap \mathcal{K} \mid X \subseteq \mathbf{O}^\omega \wedge X \in \Sigma_n[\beta]\}$. $\mathbf{\Sigma}_n(\mathcal{K})$, $\Pi_n[\beta](\mathcal{K})$, and $\mathbf{\Pi}_n(\mathcal{K})$ are defined similarly. Define $\Sigma_n(\mathcal{K}) = \Sigma_n[\beta](\mathcal{K})$ for β recursive. $\Pi_n(\mathcal{K})$ is defined similarly.

1.2 Detection

Let $\mathcal{K} \subseteq \mathbf{O}^\omega$, $P \subseteq \mathcal{K}$, and $\Psi : \mathbf{O}^{<\omega} \to \{0,1\}$ be given. For $t \in \mathbf{O}^\omega$ we write $\Psi(t) = 1$ if $\Psi(\bar{t}(n)) = 1$ almost everywhere, and similarly for $\Psi(t) = 0$. Ψ *semi-detects* P *in* $\mathcal{K}$ just in

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case for all $t \in \mathcal{K}$, $\Psi(t) = 1$ iff $t \in P$. In this case P is *semi-detectable in* $\mathcal{K}$. If there is such a Ψ which is recursive [in β], P is [β-]*effectively semi-detectable in* $\mathcal{K}$.

2 Characterization results

The following proposition characterizes relativized effective semi-detectablity in terms of the relativized arithmetical hierarchy; it is a relativized version of Theorem 5 of Gold (1965), p. 38.

(2) PROPOSITION: Let $\mathcal{K} \subseteq \mathbf{O}^\omega$ be given. $P \subseteq \mathcal{K}$ is β-effectively semi-detectable in $\mathcal{K}$ iff $P \in \Sigma_2[\beta](\mathcal{K})$.

Proof: only if. Let Ψ be recursive in β and semi-detect P in $\mathcal{K}$. Define $R \subseteq \mathbf{O}^{<\omega} \times \omega^2$ by $R(\sigma, n, m) \leftrightarrow ((\mathit{length}(\sigma) = n \wedge n > m) \rightarrow \Psi(\sigma) = 1)$. Then, R is recursive in β and $P = \{t \in \mathbf{O}^\omega \mid \exists m \forall n R(\bar{t}(n), n, m))\} \cap \mathcal{K}$. Hence, by (1), $P \in \Sigma_2[\beta](\mathcal{K})$.

if. Suppose that $P \in \Sigma_2[\beta](\mathcal{K})$. Then, by (1), there is an $R \subseteq \mathbf{O}^{<\omega} \times \omega^2$ such that R is recursive in β and $P = \{t \in \mathbf{O}^\omega \mid \exists m \forall n R(\bar{t}(n), n, m))\} \cap \mathcal{K}$. Define $f : \mathbf{O}^{<\omega} \longrightarrow \omega$ by $f(\sigma) = \mu m < \mathit{length}(\sigma)(\forall \tau \subseteq \sigma R(\tau, \mathit{length}(\tau), m))$, if such an m exists; $= \mathit{length}(\sigma)$, otherwise. Define $\Psi : \mathbf{O}^{<\omega} \longrightarrow \{0, 1\}$ as follows.

$$\Psi(\sigma) = \begin{cases} 1 & \text{if } f(\sigma) = f(\sigma-), \text{ where } \sigma- \text{ is } \sigma \text{ without its last member.} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that Ψ is recursive in β and semi-detects P in $\mathcal{K}$. ■

The following corollary to Proposition (2) combines the results of Kelly (1990), Theorem 2.2, pp. 6-8, and Theorem 3, pp. 15-16.

(3) COROLLARY: Let $\mathcal{K} \subseteq \mathbf{O}^\omega$ be given.

(a) $P \subseteq \mathcal{K}$ is semi-detectable in $\mathcal{K}$ iff $P \in \mathbf{\Sigma}_2(\mathcal{K})$.

(b) $P \subseteq \mathcal{K}$ is effectively semi-detectable in $\mathcal{K}$ iff $P \in \Sigma_2(\mathcal{K})$.

3 An application of the characterization results to first-order logic

Fix a countable, first-order language $\mathcal{L}$ with formulas $\mathcal{L}_{form}$ and sentences $\mathcal{L}_{sen}$. We will suppose a formulation of first-order languages in which the set of variables which are allowed to occur free in formulas is disjoint from the set of variables which are allowed to occur bound in formulas. We reserve the symbols $v_0, v_1, \ldots$ for variables of the first kind.

For purposes of this section, let $\mathbf{O}$ be the (countable) set of basic formulas of $\mathcal{L}$, that is, formulas which are either atomic formulas or negations of atomic formulas of $\mathcal{L}$; if $e \in \mathbf{O}^\omega$, we

say that e is an *environment.* Let $\mathcal{S}$ be a structure for $\mathcal{L}$. Any mapping $g : \{v_0, v_1, \ldots\} \longrightarrow |\mathcal{S}|$ is an *assignment to* $\mathcal{S}$. We say g is a *complete assignment to* $\mathcal{S}$ just in case $range(g) = |\mathcal{S}|$. We say $e \in \mathbf{O}^\omega$ is an *environment for* $\mathcal{S}$ *via* g just in case for every basic $\beta \in \mathcal{L}_{form}$, $\beta \in range(e)$ if and only if $\mathcal{S} \models \beta[g]$.

Fix theory $T \subseteq \mathcal{L}_{sen}$. Given $\theta \in \mathcal{L}_{form}$ and environment e, we say that e *sustains* θ just in case there is a structure $\mathcal{S}$ and complete assignment g to $\mathcal{S}$ such that:

(a) e is for $\mathcal{S}$ via g

(b) $\mathcal{S} \models \theta[g]$

We let $E(\theta) = \{e \mid e \text{ sustains } \theta\}$. We let $E(T) = \{e \mid e \text{ sustains every } \theta \in T\} = \{e \mid e \text{ is for a structure that satisfies } T\}$.

Let Σ_n^T, Π_n^T be the quantifier-alternation hierarchy for $\mathcal{L}_{form}$ over the theory T, that is, $\theta \in \Sigma_n^T$ iff θ is equivalent over T to a prenex normal form formula whose quantifier prefix begins with an existential quantifier and contains $n-1$ alternations of blocks of quantifiers of like kind and similarly for Π_n^T with "universal" in place of "existential." The following proposition relates the quantifier complexity (over T) of a formula θ to the *arithmetical* complexity of the collection of texts which sustain $T \cup \{\theta\}$. The proof of this proposition is a variant of well-known demonstrations that that the satisfaction relation for first-order formulas in countable structures is Δ_1^1 (see, *e.g.*, Moschovakis (1980), p. 464); Kelly (1990), Proposition 4.1, pp. 19-20, proves a weaker result than Proposition (4) with $\mathbf{\Sigma}_n(E(T))$ and $\mathbf{\Pi}_n(E(T))$ in place of $\Sigma_n(E(T))$ and $\Pi_n(E(T))$.

(4) PROPOSITION: For all $n \geq 1$ and all $\theta \in \mathcal{L}_{form}$:

(a) if $\theta \in \Sigma_n^T$ then $E(\theta) \bigcap E(T) \in \Sigma_n(E(T))$.

(b) if $\theta \in \Pi_n^T$ then $E(\theta) \bigcap E(T) \in \Pi_n(E(T))$.

Proof: We illustrate the proof for the case where $\theta \in \Sigma_2^T$, using the special case of the normal form theorem cited under (1). The general case follows from the full normal form theorem for the arithmetical hierarchy.

Suppose $\theta \in \Sigma_2^T$. Without loss of generality we may suppose $\theta = \exists y \forall x \phi$, where ϕ is quantifier free (the case for longer quantifier strings of like kind may be handled by the standard technique of contraction of quantifiers using the recursive encoding of finite sequences of natural numbers by natural numbers). By (1), it suffices to exhibit a recursive relation $R \subseteq \mathbf{O}^{<\omega} \times \omega^2$ such that for all environments e,

(5) e sustains θ iff $\exists m \forall n R(\bar{e}(n), n, m))$.

Toward the definition of such an R note first that

(6) for all environments e and for all quantifier free ψ

(a) e sustains ψ, iff,

(b) $\exists n \in \omega(\bigwedge \bar{e}(n) \models \psi)$, iff,

(c) $\forall n \in \omega(\bigwedge \bar{e}(n) \not\models \neg\psi)$.

(5) follows from (6) if we define recursive $R \subseteq \mathbf{O}^{<\omega} \times \omega^2$ as follows:

(7) $R(\sigma, n, m) \leftrightarrow \forall i \leq n(\bigwedge \sigma \not\models \neg\phi[(x \mid v_i), (y \mid v_m)])$. ■

Proposition (4) and Proposition (2) yield the next corollary, which is proved differently in Kelly (1990), Proposition 4.3, p. 22.

(8) COROLLARY: Let $T \subseteq \mathcal{L}_{sen}$ and $\theta \in \mathcal{L}_{sen}$ be given. If $\theta \in \Sigma_2^T$, then $E(\theta)$ is effectively semi-detectable in $E(T)$.

The following is an immediate consequence of Osherson, Stob, & Weinstein (1990), Corollary 42(a); it also appears as Kelly (1990), Proposition 4.2, p. 21.

(9) COROLLARY: Let $T \subseteq \mathcal{L}_{sen}$ and $\theta \in \mathcal{L}_{sen}$ be given. If $E(\theta)$ is semi-detectable in $E(T)$, then $\theta \in \Sigma_2^T$.

Corollaries (8) and (9) yield the following characterization of semi-detectability for first-order sentences in first-order theories; it is also a corollary of Kelly (1990), Theorems 2.2, 3, Corollary 4.2.a, and Proposition 4.3.

(10) COROLLARY: Let $T \subseteq \mathcal{L}_{sen}$ and $\theta \in \mathcal{L}_{sen}$ be given. The following conditions are equivalent.

(a) $E(\theta)$ is semi-detectable in $E(T)$.

(b) $E(\theta)$ is effectively semi-detectable in $E(T)$.

(c) $\theta \in \Sigma_2^T$.

(d) $E(\theta) \bigcap E(T) \in \mathbf{\Sigma}_2(E(T))$.

(e) $E(\theta) \bigcap E(T) \in \Sigma_2(E(T))$.

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