EFFECTIVE FIELD THEORY ON MANIFOLDS WITH BOUNDARY

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ABSTRACT

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In the monograph *Renormalization and Effective Field Theory*, Costello made two major advances in rigorous quantum field theory. Firstly, he gave an inductive position space renormalization procedure for constructing an effective field theory that is based on heat kernel regularization of the propagator. Secondly, he gave a rigorous formulation of quantum gauge theory within effective field theory that makes use of the BV formalism. In this work, we extend Costello's renormalization procedure to a class of manifolds with boundary and make preliminary steps towards also extending his formulation of gauge theory to manifolds with boundary. In addition, we reorganize the presentation of the preexisting material, filling in details and strengthening the results.

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Chapter 1

Introduction

Effective field theory, in the context of the renormalization group, was developed by Wilson [7] [8] based on earlier work of Kadanoff [4]. There are many variations, but the basic procedure involves two steps: mode elimination and rescaling [5] [3]. In this introduction, we shall present the intuitive idea of mode elimininination and how it relates to the body of the paper.

Suppose that we have an action functional $S[\Lambda_H](\phi)$ describing physics below an energy scale Λ_H. Then the action functional $S[\Lambda_L](\phi)$ describing physics at a lower energy scale should be given by "eliminating the modes" with energy between Λ_L and Λ_H. This is described by the renormalization group equation (RGE)

$$e^{S[\Lambda_L](\phi)/\hbar} = \int_{\phi' \in \mathcal{E}_{(\Lambda_L,\Lambda_H]}} e^{S[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'. \tag{1.1}$$

where the integral is over $\mathcal{E}_{(\Lambda_L,\Lambda_H]}$, the space of fields with energy between Λ_L and Λ_H. And $S[\Lambda](\phi)$ is defined on the low energy fields $\phi \in \mathcal{E}_{[0,\Lambda]}$. Equivalently, we can write

$$S[\Lambda_L](\phi) = \hbar \log \int_{\phi' \in \mathcal{E}_{(\Lambda_L, \Lambda_H]}} e^{S[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'. \tag{1.2}$$

In order to define the effective action $S[\Lambda](\phi)$, one might be tempted to let $\Lambda \to \infty$ and write

$$S[\Lambda](\phi) = \hbar \log \int_{\phi' \in \mathcal{E}_{(\Lambda, \infty)}} e^{S(\phi+\phi')/\hbar} \mathcal{D}\phi'. \tag{1.3}$$

but this limit will not exist due to ultraviolet divergences. However, the limit should exist after an appropriate renormalization of the functional integral (1.3).

The focus of the first part of this thesis will be on constructing effective field theory, albeit in a slightly different formulation, which we now begin to move towards.

For the remainder of the introduction, for expository reasons, we shall work with a scalar theory on a compact manifold M. Let D be the Laplacian on M, $\mathcal{E} = C^\infty(M)$, and the "modes" the eigenvalues of D. Assume that the action is of the form

$$S(\phi) = -\frac{1}{2}\langle \phi, D\phi \rangle + I(\phi). \tag{1.4}$$

where $\langle \phi, D\phi \rangle = \int_M \phi D\phi$ is the quadratic part of the action. Because $\phi \in \mathcal{E}_{[0,\Lambda]}$ and $\phi' \in \mathcal{E}_{(\Lambda,\infty)}$ are orthogonal,

$$S(\phi + \phi') = -\frac{1}{2}\langle \phi, D\phi \rangle - \frac{1}{2}\langle \phi', D\phi' \rangle + I(\phi + \phi'). \tag{1.5}$$

If $S[\Lambda](\phi) = -\frac{1}{2}\langle \phi, D\phi\rangle + I[\Lambda](\phi)$, then the renormalization group equation simpifies to

$$e^{I[\Lambda_L](\phi)/\hbar} = \int_{\phi' \in \mathcal{E}_{(\Lambda_L, \Lambda_H]}} e^{-\frac{1}{2}\langle \phi', D\phi'\rangle/\hbar + I[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'. \tag{1.6}$$

or equivalently

$$I[\Lambda_L](\phi) = \hbar \log \int_{\phi' \in \mathcal{E}_{(\Lambda_L, \Lambda_H]}} e^{-\frac{1}{2}\langle \phi', D\phi'\rangle/\hbar + I[\Lambda_H](\phi+\phi')/\hbar} \mathcal{D}\phi'. \tag{1.7}$$

Let $P = P(\Lambda_L, \Lambda_H)$ be the inverse of the quadratic form $\langle \phi', D\phi'\rangle$ on $\mathcal{E}_{(\Lambda_L, \Lambda_H]}$ and let ∂_P be the second order contraction operator associated to P. By Wick's theorem on the finite dimensional vector space $\mathcal{E}_{(\Lambda_L, \Lambda_H]}$, the integral (1.6) is equal to the Wick contraction

$$V(P, I[\Lambda_H]) := e^{\hbar\partial_P} e^{I[\Lambda_H]/\hbar}. \tag{1.8}$$

and (1.7) is equal to the expression

$$W(P, I[\Lambda_H]) := \hbar \log[e^{\hbar\partial_P} e^{I[\Lambda_H]/\hbar}]. \tag{1.9}$$

While the version of effective field theory with sharp energy cutoffs described above paints an intuitive physical picture, there are disadvantages to working with it, as discussed in [3]. Costello gives an alternative approach that comes from noticing the relationship between the uncut propagator and the heat kernel. Let $K_t(x, y)$ be the heat kernel for D. That is

$$\partial_t K_t(x, y) + D_x K_t(x, y) = 0 \tag{1.10}$$

and $\lim_{t\to 0^+}\int_M K_t(x,y)\phi(y)\,dy = \phi(x)$. Then if the integral

$$G(x,y) = \int_0^\infty K_t\,dt \tag{1.11}$$

exists the operator it induces provides an inverse to D on $\mathcal{E}_{(0,\infty)}$. That is, away from the energy zero fields.

Instead of cutting off the space of fields, we work with the entire space of fields $\mathcal{E}$ and introduce the regularized propagator

$$P_\epsilon^L = \int_\epsilon^L K_t\,dt \tag{1.12}$$

An effective field theory now becomes a collection of length scale regularized interactions satisfying

$$I[L] = \hbar\log[e^{\hbar\partial_{P_\epsilon^L}}e^{I[\epsilon]/\hbar}]. \tag{1.13}$$

or more compactly $I[L] = W(P_\epsilon^L, I[\epsilon])$.

We naively might try to define the scale L effective interaction as

$$I[L] = \lim_{\epsilon\to 0^+}\hbar\log\left[\exp\left(\hbar\partial_{P_\epsilon^L}\right)\exp\left(I/\hbar\right)\right] \tag{1.14}$$

However, this limit may not exist and expression then has to be renormalized. That is, an interaction functional $I(\epsilon)$ with counterterms for I is constructed such that

$$I[L] = \lim_{\epsilon\to 0^+}\hbar\log\left[\exp\left(\hbar\partial_{P_\epsilon^L}\right)\exp\left((I - I(\epsilon))/\hbar\right)\right] \tag{1.15}$$

exists.

In Chapter 2, we define the spaces to which the propagator P and the interaction functional belong. We define stable Feynman graphs which give a way of organizing the combinatorics of the contractions in $V(P,I)$ and $W(P,I)$. Theorem 1 expresses $V(P,I)$ as a summation over all stable graphs while Corollary 1 expresses $W(P,I)$ as a summation over connected stable graphs.

In Chapter 3, we state and prove several variations of Wick's theorem. In 3.1.2, we calculate the 1 dimensional Gaussian integral

$$I_{m,\alpha}(a,b) = \int_a^b x^m e^{-\alpha x^2/2}\, dx \tag{1.16}$$

in terms of $I_{0,\alpha}(a,b)$ and $J_{i,\alpha}(a,b) = x^i e^{-\alpha x^2/2}|_{x=a}^{x=b}$ for $i < m$. The formula reduces to expected results on $\mathbb{R}$ and $\mathbb{R}^+$ which are recalled in 3.1.1 and 3.1.3 respectively. In 3.1.4, we generalize the formula for $I_{m,\alpha}(a,b)$ to one for

$$I_{m,\alpha,\beta}(a,b) = \int_a^b x^m e^{-\alpha x^2/2+\beta x}\, dx. \tag{1.17}$$

The proof, which is analogous to the one in 3.1.2 is omitted. The next two sections are focused on the many variables Wick's theorem. That is, the computation of the integral

$$\int_P x_{m_1} \dots x_{m_k} e^{-Q(x)/2}\, dx \tag{1.18}$$

where $Q(x)$ is a nondegenerate quadratic form. In 3.1.5, we recall the standard statement of Wick's theorem on $P = \mathbb{R}^n$ and give a proof by diagonalizing the quadratic form and applying the result of 3.1.1. This will be used to calculate the counterterms on $\mathbb{R}^n$ in 4.1.4. In 3.1.6 it is shown that the result of 3.1.2 is sufficient to compute (1.18) inductively, when P is any polytope. Lastly, we show that as long as P is bounded the $Q(x)$ may be degenerate and even inhomogeneous. In this case, the result of 3.1.4 can be applied iteratively to compute the answer. We specialize to the case relevant for the counterterms on $\mathbb{H}^n$, the upper half space with the Euclidean metric, in 4.1.6.

Chapter 4, in particular Section 4.1, forms the body of the paper. We begin with 4.1.1, where the construction of the counterterms in general is motivated by carrying out the procedure for the Feynman weight associated to a particular 1-loop graph in the ϕ_4^4-theory. The renormalization procedure is based on the ability to cover $(0, \infty)^k$ and a fortiori $(\epsilon, 1)^k$ by sets defined by inequalities of the form $t_i \leq t_j^R$, where $R \geq 1$. In the next section, the covering lemma that was proved by Costello in [3] is strengthened and proved. Much more detail about the nature of the sets in the cover is given. Other preliminary concepts needed for the renormalization procedure like local functionals and the form of their Feynman weights are then discussed.

In 4.1.4, we formulate Costello's renormalization procedure on $\mathbb{R}^n$. We give explicit formulas whenever possible and fill in a few steps in the argument omitted by Costello, such as the introduction of what we call spanning tree coodinates. In 4.1.5, we show how to control the error and how the basic result of 4.1.4 can be used inductively to provide counterterms on each of sets in the cover of $(\epsilon, 1)^k$ where k is the number of edges in the Feynman graph whose weight we are trying to renormalize.

In 4.1.6, the renormalization is adapted to $\mathbb{H}^n$, the upper half space with the Euclidean metric. The procedure does not carry over without modification since the quadratic form in the integral computing the Feynman weight is both no longer non-degenerate and no longer homogeneous. Luckily, this difficulty can be circumvented by a clever change of coordinates in the direction normal to boundary. The counterterms have a more complicated form than those on $\mathbb{R}^n$, but we argue that the inductive procedure of 4.1.5 can be carried out with appropriate modifications.

In 4.1.7, we correct what seems to be an oversight in Costello's reasoning in [3]. On a compact manifold M, Costello uses the asymptotic expansion of the heat kernel $K_t(x,y) \sim e^{-d(x,y)^2/4t} \sum_i \phi_i(x,y) t^i$, but for each chart in a cover replaces $d(x,y)$ with the coordinate distance $\|x-y\|$. Thus, taking a partition of unity, the Feynman weight under consideration becomes a sum of integrals whose integrands will contain the exponential of a quadratic form, which allows us to apply Wick's theorem. However, it does not seem to be correct that $K_t(x,y) \sim e^{-\|x-y\|^2/4t} \sum_i \phi_i(x,y) t^i$, at least not uniformly in x and y. Again, we show how this difficulty is not fatal. While the counterterms will not simplify as they do on $\mathbb{R}^n$, through the introduction of spanning tree coordinates, one can still bound the error. The inductive step in the constuction thus remains valid.

The culmination of these results is 4.1.8 where we show the renormalization procedure can be carried out on a class of compact manifolds with boundary where the argument reduces that of 4.1.6 near the boundary and 4.1.7 away from the boundary.

In Section 4.2 we move beyond the construction of counterterms for each Feynman weight and construct the counterterms $I^{CT}(\epsilon)$ for the entire effective interaction.

Chapter 5, contains preliminary work done towards extending Costello's formulation of quantum gauge theory within effective field theory to manifolds with boundary.

In 5.1.1, we recall the required graded linear algebra and state and verify the classical master equation for generalized Chern-Simons theory. In the last part of 5.1.1, we show that the requisite algebraic assumptions on the space of fields for the classical master equation to hold can be satisfied beginning from the data of a compact smooth manifold M of dimension n and a graded Lie algebra $\mathfrak{g}$ with a symmetric bilinear pairing κ of degree $n-3$. In 5.1.2, following [1], we extend these constructions in our language to manifolds with boundary.

Lastly, in Section 5.2, we recall and generalize slightly some of the constructions that can be found in [2] and [6], as a preliminary step towards adapting them to manifolds with boundary.

Chapter 2

Feynman Diagrams

2.1 General Setup

Let $\mathcal{E}$ be a graded object in an appropriate symmetric monoidal category, which contains a field $\mathbb{K}$ as its monoidal unit. For toy examples one can work with the category of finite dimensional vector spaces. For quantum field theory one will need to work with a category of topological vector spaces like the category of nuclear spaces with the projective tensor product. The identifications $(\mathcal{E} \otimes \mathcal{F})^* \cong \mathcal{E}^* \otimes \mathcal{F}^*$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F}$ will be made throughout. We will not dwell on the issue any further.

Fix an element $P \in \mathrm{Sym}^2(\mathcal{E})$ which will be called a *propagator*. We define the algebra of formal power series on $\mathcal{E}$,

$$\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \mathrm{Hom}(\otimes^n \mathcal{E}, \mathbb{K})_{S_n} = \prod_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}^*) \tag{2.1}$$

Here Sym means taking coinvariants of the n-fold tensor product with respect to the symmetric group action. An element of $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$ is of the form $I = \sum_{i,k\geq 0} I_{i,k}\hbar^i$, where $I_{i,k} \in \mathrm{Sym}^k(\mathcal{E}, \mathbb{R})$. Let

$$\mathcal{O}(\mathcal{E})^+[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]] \tag{2.2}$$

be the functionals of the form $I = \sum_{i,k\geq 0} I_{i,k}\hbar^i$, where $I_{0,k} = 0$ for $k < 3$ and $I_{1,0} = 0$. We will see the reason for this restricted class of functionals later in the section.

We are interested in combinatorial formulas for "functional integrals" of the form

$$V(P, I) = e^{\hbar\partial_P} e^{I/\hbar} \tag{2.3}$$

and

$$W(P, I) = \hbar \log(e^{\hbar\partial_P} e^{I/\hbar}), \tag{2.4}$$

where ∂_P denotes the contraction operator $\frac{1}{2}\sum_i \partial_{P_i^{(1)}}\partial_{P_i^{(2)}}$ where $P = \sum_i P_i^{(1)} \otimes P_i^{(2)}$.

Lemma 1 (Feynman Expansion)**.**

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j C(\{n_{i,k}\}, j)\hbar^{p(\{n_{i,k}\},j)} \partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}} \tag{2.5}$$

where

$$C(\{n_{i,k}\}, j) = \frac{1}{j!} \prod_{i,k} \frac{1}{n_{i,k}!}$$

and

$$p(\{n_{i,k}\}, j) = \sum_{i,k} i\, n_{i,k} - \sum_{i,k} n_{i,k} + j.$$

In the outer summation, we sum over the collection of double sequences of non-negative integers $\{n_{i,k}\}_{i,k\geq 0}$ with the requirement that for all but finitely many i, k, $n_{i,k} = 0$.

Proof. By the multinomial formula

$$\begin{aligned}
\exp\left(\sum_{i,k} I_{i,k}\hbar^{i-1}\right) &= \sum_j \frac{(\sum_{i,k} I_{i,k}\hbar^{i-1})^j}{j!} \\
&= \sum_j \sum \prod_{i,k} \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}},
\end{aligned}$$

where the inner sum is over sequences of nonnegative numbers $\{n_{i,k}\}$ such that $\sum_{i,k} n_{i,k} = j$. We can reexpress this as a single sum over sequences of almost all zero nonnegative integers $\{n_{i,k}\}$

$$\exp\left(\sum_{i,k} I_{i,k}\hbar^{i-1}\right) = \sum_{\{n_{i,k}\}} \prod_{i,k} \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}}.$$

Thus,

$$\begin{aligned}
V(P, I) &= \sum_{\{n_{i,k}\}} \sum_j \frac{\hbar^j}{j!} \partial_P^j \prod_{i,k} \frac{\hbar^{(i-1)n_{i,k}}}{n_{i,k}!} I_{i,k}^{n_{i,k}} \\
&= \sum_{\{n_{i,k}\}} \sum_j C(\{n_{i,k}\}, j)\hbar^{p(\{n_{i,k}\},j)} \partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}}
\end{aligned}$$

□

It remains to investigate the combinatorial structure of the expression

$$\partial_P^j \prod_{i,k} I_{i,k}^{n_{i,k}}.$$

Before doing so, we shall make a definition.

Definition 1. *A* stable graph *is defined by*

$V(\gamma)$ *a set of vertices*

$E(\gamma)$ *a set of edges each connecting two vertices*

$T(\gamma)$ *a set of tails each connected to a single vertex*

and a function $g : V(\gamma) \to \mathbb{Z}^{\geq 0}$ *associating a "genus" to each vertex.*

There is a natural preorder on vertices: If v_1 has $g(v_1) = i_1$ and valency k_1, and v_2 has $g(v_2) = i_2$ and valency k_2, then $v_1 \preceq v_2$ if $i_1 < i_2$ or $i_1 = i_2$ and $k_1 \leq k_2$.

2.2 Feynman Diagram Expansion

Begin with the expression

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \left(\frac{1}{j!2^j} \prod_{i,k} \frac{1}{n_{i,k}!} \right) \hbar^{p(\{n_{i,k}\},j)} \left(\sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}} \right)^j \prod_{i,k} I_{i,k}^{n_{i,k}}.$$

.

Let $I_{i_1,k_1}, \ldots, I_{i_n,k_n}$ be the sequence of interactions for which $n_{i,k} \neq 0$. Recall that the propagator $P \in \operatorname{Sym}^2 \mathcal{E}$ with $P = \sum_l P_l^{(1)} \otimes P_l^{(2)}$, and we are assuming that $\mathcal{E}$ is ungraded. Make the substitution $I_{i,k} = S^k I_{i,k}/k!$ where $S^k I_{i,k} = \sum_{\sigma \in S_k} I_{i,k}^{\sigma} = k! I_{i,k}$. Then

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \left(\frac{1}{j!2^j} \prod_{i,k} \frac{1}{n_{i,k}!(k!)^{n_{i,k}}} \right) \hbar^{p(\{n_{i,k}\},j)} \left(\sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}} \right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.$$

Then

$$\left(\sum_l \partial_{P_l^{(1)}} \partial_{P_l^{(2)}}\right)^j \prod_{i,k} (S^k I_{i,k})^{n_{i,k}}.$$

will be a sum over contractions that can be parametrized by injections $Q : H \to V$ of the set $H = \{1^{(1)}, 1^{(2)}, \ldots, j^{(1)}, j^{(2)}\}$ into the set of inputs to the interactions $V = \{1^{(1)}, \ldots, k_1^{(1)}, \ldots, 1^{(n)}, \ldots, k_n^{(n)}\}$.

Since $I_{i,k} \in \mathrm{Sym}^\bullet \mathcal{E}^*$ and $I_{n,k}^{n_{i,k}} \in \mathrm{Sym}^\bullet \mathcal{E}^*$, we can reorder the contractions so that the images of the index (1) elements in H, $Q(1^{(1)}), \ldots, Q(j^{(1)})$ are in ascending order. There are $j!$ contractions that will be reordered to the same contraction in this way. We can also reorder so that $Q(\alpha^{(1)})$ comes before $Q(\alpha^{(2)})$. There are 2^j contractions that will be reordered to the same contraction in this way.

Injections up to these reorderings are in one-to-one correspondence with partitions of V into j subsets with two elements and 1 additional subset containing the remaining $|V| - 2j$ elements. Let $\mathcal{Q}(\{n_{i,k}\}, j)$ be the collection of such partitions and for $Q \in \mathcal{Q}(\{n_{i,k}\}, j)$ let $w_Q(P, I)$ denote the corresponding contraction.

Then

$$V(P, I) = \sum_{\{n_{i,k}\}} \sum_j \sum_{Q \in \mathcal{Q}(\{n_{i,k}\}, j)} \left(\prod_{i,k} \frac{1}{n_{i,k}!(k!)^{n_{i,k}}}\right) \hbar^{p(\{n_{i,k}\}, j)} w_Q(P, I) \tag{2.6}$$

Any partition $Q \in \mathcal{Q}(\{n_{i,k}\}, j)$ determines a stable graph γ in an obvious way. Consider $\mathcal{Q}_\gamma(\{n_{i,k}\}, j)$, the collection of partitions which determine the same stable graph γ. Let $G(\{n_{i,k}\}, j) = \prod_{i,k} (S_k^{n_{i,k}} \rtimes S_{n_{i,k}})$. Note that

$$|G(\{n_{i,k}\}, j)| = \prod_{i,k} n_{i,k}!(k!)^{n_{i,k}}$$

This acts on V by permuting the interactions of type i, k and their k inputs. As a consequence, it acts on $\mathcal{Q}(\{n_{i,k}\}, j)$. In fact, it acts transitively on $\mathcal{Q}_\gamma(\{n_{i,k}\}, j)$. The stabilizer subgroup of a given partition $Q \in \mathcal{Q}_\gamma(\{n_{i,k}\}, j)$ is equal to $\mathrm{Aut}(\gamma)$, the group of automorphisms of the stable graph γ. By the orbit-stabilizer theorem, the number of partitions which determine the same stable graph γ is given by

$$\frac{|G(\{n_{i,k}\}, j)|}{|\operatorname{Aut}(\gamma)|} = \frac{\prod_{i,k} n_{i,k}!(k!)^{n_{i,k}}}{|\operatorname{Aut}(\gamma)|} \tag{2.7}$$

Therefore,

Theorem 1 (Feynman Diagram Expansion)**.** *For a stable graph γ, we define*

$$g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(v) \tag{2.8}$$

where $b(\gamma)$ is the first Betti number of γ. Let $C(\gamma)$ be the number of connected components of γ. Then

$$V(P, I) = \sum_{\gamma} \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{g(\gamma) - C(\gamma)} w_\gamma(P, I) \tag{2.9}$$

Proof. The constant $p(\{n_{i,k}\}, j) = \sum_{i,k} i n_{i,k} - \sum_{i,k} n_{i,k} + j$ has a very simple interpretation in terms of the stable graph γ since

$$\sum_{v \in V(\gamma)} g(v) = \sum_{i,k} i\, n_{i,k},$$

$|V(\gamma)| = \sum_{i,k} n_{i,k}$ and $|E(\gamma)| = j$. Using the fact that

$$b(\gamma) = |E(\gamma)| - |V(\gamma)| + C(\gamma), \tag{2.10}$$

and the definition

$$g(\gamma) = b(\gamma) + \sum_{v \in V(\gamma)} g(\gamma) \tag{2.11}$$

we have

$$p(\{n_{i,k}\}, j) = g(\gamma) - C(\gamma). \tag{2.12}$$

Lastly define $w_\gamma(P, I)$ to be $w_Q(P, I)$ where Q is any partition that determines γ. The formula now follows from (2.6) and (2.7). □

Now we describe a combinatorial formula for $W(P, I) = \hbar \log(e^{\hbar\partial_P} e^{I/\hbar})$ or equivalently $e^{W(P,I)/\hbar} = e^{\hbar\partial_P} e^{I/\hbar}$.

Corollary 1.

$$W(P, I) = \sum_{\gamma\ conn} \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{g(\gamma)} w_\gamma(P, I) \tag{2.13}$$

Proof. If $\gamma_1 \cup \cdots \cup \gamma_k$ is the disjoint union of not necessarily distinct connected stable graphs $\gamma_1, \ldots, \gamma_k$, then it is clear that

$$g(\gamma_1 \cup \cdots \cup \gamma_k) = g(\gamma_1) + \cdots + g(\gamma_k)$$

$$C(\gamma_1 \cup \cdots \cup \gamma_k) = C(\gamma_1) + \cdots + C(\gamma_2)$$

and if $\gamma = (\cup^{k_1}\gamma_1) \cup \cdots \cup (\cup^{k_n}\gamma_n)$ where $\gamma_1, \ldots, \gamma_n$ are distinct

$$|\operatorname{Aut}(\gamma)| = k_1! \ldots k_n! |\operatorname{Aut}(\gamma_1)|^{k_1} \ldots |\operatorname{Aut}(\gamma_n)|^{k_n}$$

Thus,

$$\begin{aligned}
\exp\left(W(P,I)/\hbar\right) &= \exp\left(\sum_{\gamma \text{ conn}} \frac{1}{|\operatorname{Aut}(\gamma)|}\hbar^{g(\gamma)-1} w_\gamma(P,I)\right) \\
&= \sum_{\{k_\gamma\}} \prod_{\gamma \text{ conn}} \frac{1}{|\operatorname{Aut}(\gamma)|^{k_\gamma} k_\gamma!}\hbar^{k_\gamma(g(\gamma)-1)} w_{\cup_{k_\gamma}\gamma}(P,I) \\
&= \sum_{\gamma} \frac{1}{|\operatorname{Aut}(\gamma)|}\hbar^{g(\gamma)-C(\gamma)} w_\gamma(P,I)
\end{aligned}$$

In the second line above, for each sequence $\{k_\gamma\}_{\gamma \text{ conn}}$ in the outer summation, $k_\gamma = 0$ for all but finitely many γ, and k_γ is a nonnegative integer for all γ. $\square$

Corollary 2. *For $I \in \mathcal{O}(\mathcal{E})^+[[\hbar]]$,*

$$W(P,I) \in \mathcal{O}(\mathcal{E})^+[[\hbar]]$$

Chapter 3

Wick's Theorem

3.1 Bosonic Wick's Theorem

3.1.1 Wick's Theorem on $\mathbb{R}$

In one variable, Wick's theorem reduces to the statement

$$\int_{-\infty}^{\infty} x^m e^{-\alpha x^2/2}\, dx = \begin{cases} \sqrt{2\pi}\frac{(2k)!}{k!2^k}\frac{1}{\alpha^{(2k+1)/2}} & \text{if } m = 2k \\ 0 & \text{if } m = 2k+1. \end{cases} \tag{3.1}$$

$$= C_m \frac{1}{\alpha^{(m+1)/2}} \tag{3.2}$$

3.1.2 Wick's Theorem on (a, b)

There are several ways of proving the formula for $\mathbb{R}$ which one might try to adapt. The proof by integration by parts seems the best suited and is the one we develop here.

We wish to compute the integral

$$I_{m,\alpha}(a,b) = \int_a^b x^m e^{-\alpha x^2/2}\,dx$$

for $-\infty \le a \le b \le \infty$ and to check that the result agrees with the standard formula for $a = -\infty$ and $b = \infty$. Let

$$J_{m,\alpha}(a,b) = x^m e^{-\alpha x^2/2}\Big|_{x=a}^{x=b}.$$

By integration by parts,

$$\begin{aligned}\int_a^b x^m e^{-\alpha x^2/2}\,dx &= \int_a^b x^{m-1}(xe^{-\alpha x^2/2})\,dx \\ &= \frac{m-1}{\alpha}\int_a^b x^{m-2}e^{-\alpha x^2/2}\,dx - \frac{x^{m-1}}{\alpha}e^{-\alpha x^2/2}\Big|_a^b\end{aligned}$$

That is,

$$I_{m,\alpha}(a,b) = \frac{m-1}{\alpha} I_{m-2,\alpha}(a,b) - \frac{1}{\alpha} J_{m-1,\alpha}(a,b). \tag{3.3}$$

For m even, we can thus express $I_{m,\alpha}(a,b)$ in terms of $I_{0,\alpha}(a,b)$ and $J_{l,\alpha}(a,b)$ where l ranges over odd integers less than m. For m odd, since $I_{1,\alpha}(a,b) = -(1/\alpha)J_{0,\alpha}(a,b)$, we can express $I_{m,\alpha}(a,b)$ in terms of $J_{l,\alpha}(a,b)$, where l ranges over even integers less than m.

We can then prove a precise formula by induction:

Proposition 1.

$$I_{m,\alpha}(a,b) = \frac{C_m}{\alpha^{m/2}} I_{0,\alpha}(a,b) - \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\tilde{C}_{i,m}}{\alpha^{i+1}} J_{m-1-2i,\alpha}(a,b), \tag{3.4}$$

where $C_m = 0$ *when* m *is odd and* $C_m = (m-1)!!$ *when* m *is even and for all* m

$$\tilde{C}_{i,m} = \frac{(m-1)!!}{(m-1-2i)!!} \tag{3.5}$$

Proof. The even and odd base cases when $m = 0$ and $m = 1$ are clearly satisfied. Suppose the result is true for $I_{m,\alpha}(a,b)$. Then using (3.3),

$$\begin{aligned} I_{m+2,\alpha}(a,b) = & \frac{m+1}{\alpha}\frac{C_m}{\alpha^{m/2}} - \frac{m+1}{\alpha}\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\frac{\tilde{C}_{i,m}}{\alpha^{i+1}}J_{m-1-2i,\alpha}(a,b) \\ & - \frac{1}{\alpha}J_{m+1,\alpha}(a,b) \end{aligned}$$

and

$$\begin{aligned} (m+1)\tilde{C}_{i,m} &= \frac{(m+1)!!}{(m+1-2(i+1))!!} \\ &= \tilde{C}_{i+1,m+2} \end{aligned}$$

so

$$\begin{aligned} & \frac{m+1}{\alpha}\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\frac{\tilde{C}_{i,m}}{\alpha^{i+1}}J_{m-1-2i,\alpha}(a,b) \\ &= \sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\frac{\tilde{C}_{i+1,m+2}}{\alpha^{(i+1)+1}}J_{(m+2)-1-2(i+1),\alpha}(a,b) \\ &= \sum_{i=1}^{\lfloor\frac{(m+2)+1}{2}\rfloor}\frac{\tilde{C}_{i,m+2}}{\alpha^{i+1}}J_{(m+2)-1-2i}(a,b) \end{aligned}$$

The induction step is now completed by employing the fact that

$$\frac{(m+1)C_m}{\alpha\alpha^{m/2}} = \frac{C_{m+2}}{\alpha^{(m+1)/2}}$$

$\square$

As $a \to -\infty$ and $b \to \infty$, we have $J_{m,\alpha} \to 0$ and $I_{0,\alpha} \to \sqrt{2\pi/\alpha}$. Combining this with identity for the double factorial

$$(2k-1)!! = \frac{(2k)!}{k!2^k}$$

we recover the statement of Wick's theorem on $\mathbb{R}$.

3.1.3 Wick's theorem on $\mathbb{R}^+$

Note that if $a = 0$ and $b = \infty$, then $J_{l,\alpha} = 0$ for $l \neq 0$ and $J_{0,\alpha} = -1$. Since $(2k)!! = 2^k k!$,

$$\int_0^\infty x^m e^{-\alpha x^2/2}\, dx = \begin{cases} \frac{C_m}{\alpha^{m/2}} I_{0,\alpha}(0,\infty) & m \text{ even} \\ \frac{\tilde{C}_{m,\frac{m-1}{2}}}{\alpha^{(m+1)/2}} J_{0,\alpha}(0,\infty) & m \text{ odd} \end{cases} \tag{3.6}$$

$$= \begin{cases} \sqrt{2\pi}\frac{(2k)!}{k!2^{k+1}}\frac{1}{\alpha^{(2k+1)/2}} & \text{if } m = 2k \\ 2^k k! \frac{1}{\alpha^{k+1}} & \text{if } m = 2k+1. \end{cases} \tag{3.7}$$

3.1.4 Generalized Wick's Theorem on (a, b)

In 4.1.6 we shall encounter integrals of polynomials with respect to inhomogeneous quadratic forms. Here we establish the one dimensional result that can be used iteratively to calculate such integrals explicitly.

We wish to compute the integral

$$I_{m,\alpha,\beta}(a,b) = \int_a^b x^m e^{-\alpha x^2/2+\beta x}\, dx$$

for $-\infty \leq a \leq b \leq \infty$ and to check that the result agrees with the standard formula for $a = -\infty$ and $b = \infty$. Let

$$J_{m,\alpha,\beta}(a,b) = x^m e^{-\alpha x^2/2+\beta x}\Big|_{x=a}^{x=b}.$$

Firstly, (3.3) generalizes to

$$I_{m,\alpha,\beta}(a,b) = -\frac{1}{\alpha} J_{m-1,\alpha,\beta}(a,b) + \frac{\beta}{\alpha} I_{m-1,\alpha}(a,b) + \frac{m-1}{\alpha} I_{m-2,\alpha,\beta}(a,b). \tag{3.8}$$

The following is a generalization of Proposition 1

Proposition 2.

$$I_{m,\alpha,\beta}(a,b) = -\sum_{i=0}^{m-1} \sum_{\substack{\{a_j\} \\ \sum a_j = i}} \frac{\beta^{|a^{-1}(1)|} \prod_{k \in a^{-1}(2)} (s_k - 1 + m - i)}{\alpha^{|l(a)|+1}} J_{\alpha,\beta,m-i-1} \tag{3.9}$$

$$+ \sum_{\substack{\{a_j\} \\ \sum a_j = m}} \frac{\beta^{|a^{-1}(1)|} \prod_{k \in a^{-1}(2)} (s_k - 1)}{\alpha^{|l(a)|}} I_{\alpha,\beta,0} \tag{3.10}$$

where $\{a_j\}$ ranges over finite sequences such that $a_j \in \{1,2\}$ for all j. We use $l(a)$ to denote the length of the sequence $\{a_i\}$ and $s_i = \sum_{j=1}^{l(a)} a_j$.

We shall not give the proof which is a straightforward induction like the proof of Proposition 1. However, let us just check that it reduces to the formula of Proposition 1 in the case that $\beta = 0$. Since $0^0 = 1$ and $0^k = 0$ for $k > 0$ the only nonzero terms in the sums will come from sequences with $a^{-1}(1) = \emptyset$. But there is exactly one such sequence such that $\sum a_j = i$ for i even and it has $l(a) = i/2$ and no such sequences for i odd. It is clear that this then becomes the formula of Proposition 1.

3.1.5 Wick's theorem on $\mathbb{R}^n$

Suppose that A is an invertible symmetric $n \times n$ matrix and consider the associated quadratic form $Q(x) = \langle x, Ax \rangle = x^i A_{ij} x^j$. We wish to compute the integral

$$I_{J,A} = \int_P x_{m_1} \dots x_{m_k} e^{-Q(x)/2}\, dx$$

where $J = (j_1, \dots, j_n)$ is a multi-index such that $x_1^{j_1} \dots x_n^{j_n} = x_{m_1} \dots x_{m_k}$ and P is a polytope.

Theorem 2 (Wick's Theorem on $\mathbb{R}^n$)**.** *For k even*

$$\int_{\mathbb{R}^n} x_{m_1} \dots x_{m_k} e^{-Q(x)/2}\, dx = \frac{\sqrt{2\pi}}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} A^{-1}_{m_{\beta_j^{(1)}}, m_{\beta_j^{(2)}}} \tag{3.11}$$

where the sum is over partitions of the set $1, \dots, k$ into $k/2$ subsets of 2 elements. Here $\beta_j^{(1)}$ and $\beta_j^{(2)}$ denote respectively the first and second elements of the j-th set in the partition.

Proof. Let D denote the diagonalization of A and assume that D has diagonal entries $\alpha_1, \dots, \alpha_n$. In this new basis, using the change of basis matrix S, we have a linear combination

$$\sum_{i_1, \dots, i_k} S^{i_1}_{m_1} \dots S^{i_k}_{m_k} \int_{\mathbb{R}^n} y_{i_1} \dots y_{i_k} e^{-\alpha_1 x^2/2} \dots e^{-\alpha_n x^2/2}\, dx.$$

Apply Wick's theorem on $\mathbb{R}$ separately in each variable. For each such integral, this gives

$$\frac{1}{\sqrt{\alpha_1 \dots \alpha_n}} \prod_{i=1}^{n} \frac{C_{k_i}}{\alpha_i^{k_i}} = \frac{1}{\sqrt{\det(A)}} \prod_{i=1}^{n} \frac{C_{k_i}}{D_{ii}^{k_i}}$$
$$= \frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} D^{-1}_{i_{\beta_j^{(1)}}, i_{\beta_j^{(2)}}}$$

where the sum is over partitions of the set $1, \dots, k$ into $k/2$ subsets of 2 elements. We then switch the order of summation so that the sum over partitions is the outer sum and then

$$\frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \sum_{i_1,\dots,i_k} S^{i_1}_{m_1} \dots S^{i_k}_{m_k} \prod_{j=1}^{k/2} D^{-1}_{i_{\beta_j^{(1)}}, i_{\beta_j^{(2)}}} = \frac{(\sqrt{2\pi})^n}{\sqrt{\det(A)}} \sum_{\beta} \prod_{j=1}^{k/2} A^{-1}_{m_{\beta_j^{(1)}}, m_{\beta_j^{(2)}}}$$

□

3.1.6 Wick's Theorem for Polytopes

The purpose of this section is to show that the result of 3.1.2 can be used inductively to calculate a Wick integral over a polytope in $\mathbb{R}^n$.

By the spectral theorem, A can be diagonalized by an orthogonal transformation. This will produce a linear combination of integrals of the form

$$\int_P x_1^{k_1} \dots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_n x_n^2/2}\, dx.$$

where P is some polytope.

Decompose the integral as

$$\int_{P'} \int_{a_n}^{b_n} x_1^{k_1} \dots x_n^{k_n} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_n x_n^2/2}\, dx_n dx'.$$

where P' is the projection of P onto the hyperplane $x_n = 0$ and a_n and b_n are piecewise linear in the variables $x_1, \dots, x_{n-1}$. The subdomains where a_n is linear are the projections of the $(n-1)$-dimensional faces of P onto the hyperplane $x_n = 0$. Denote an arbitrary projection of an $(n-1)$-dimensional face by P^a. Similarly, use P^b for an arbitrary subdomain where b_n is linear.

We apply Wick's theorem in one variable to x_n to get a linear combination of elements of the form

$$\int_{P'} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} J_{m-2i-1,\alpha_n}(a_n, b_n)\, dx' \tag{3.12}$$

and an element of the form

$$\int_{P'} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2/2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} I_{0,\alpha_n}(a_n, b_n)\, dx.$$

By substituting the definition of $J_{m-2i-1,\alpha}$, terms of the first form are equal to the summation

$$\begin{aligned} &\sum_{P^b} \int_{P^b} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} b_n^{k_n - 2i - 1} e^{-\alpha_n b_n^2/2}\, dx' \\ -&\sum_{P^a} \int_{P^a} x_1^{k_1} \dots x_{n-1}^{k_{n-1}} e^{-\alpha_1 x_1^2} \dots e^{-\alpha_{n-1} x_{n-1}^2/2} a_n^{k_n - 2i - 1} e^{-\alpha_n a_n^2/2}\, dx' \end{aligned}$$

We emphasize that $b_n|_{P^b}$ is a linear function in the variables $x_1, \dots, x_{n-1}$ and similarly for $a_n|_{P_a}$.

Let us focus our attention on any term involving $b_n|_{P^b} = d_1 x_1 + \dots + d_{n-1} x_{n-1}$; that is, those in the first summation. The analysis for terms involving $a_n|_{P_a}$ in the second summation is similar.

Since P^b is a polytope, to complete the inductive step, it suffices to show that

$$\alpha_1 x_1^2 + \cdots + \alpha_{n-1} x_{n-1}^2 + \alpha_n (d_1 x_1 + \cdots + d_{n-1} x_{n-1})^2 \tag{3.13}$$

is nondegenerate. Let d be the thought of as a column vector. Let $c = \sqrt{\alpha_n} d$ and let $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_{n-1})$. Then

$$\begin{aligned}\det(A + \alpha_n d d^t) &= \det(A)\det(I + A^{-1} c c^t) \\ &= \det(A)(1 + c^t A^{-1} c) = \det(A)(1 + |\sqrt{A^{-1}} c|^2) > 0\end{aligned}$$

which implies that the quadratic form is nondegenerate.

3.1.7 Generalized Wick's Theorem for Compact Polytopes

In Section 4.1.6, the counterterms that will be introduced to renormalize the theory on $\mathbb{H}^n$ will involve integrals of the form

$$\int_{P_u} e^{-Q(z,u,\mathbf{t})} z^{K'}\, dz \tag{3.14}$$

where P_u is given by the inequalities

$$\begin{aligned}&0 \leq u + z_1 \\ &0 \leq u + z_2 - z_1 \\ &\cdots \\ &0 \leq u + z_{m-1} - z_{m-2} \\ &0 \leq u - z_{m-1}.\end{aligned}$$

and $Q(z,u,\mathbf{t}) = \sum_{i,j} a_{ij} z_i z_j + \sum_i b_i u z_i$

We reexpress the above inequalities in a form which makes it possible to calculate the integral.

$$-u \leq z_1 \leq (m-1)u$$

$$z_1 - u \leq z_2 \leq (m-2)u$$

$$\dots$$

$$z_{m-3} - u \leq z_{m-2} \leq 2u$$

$$z_{m-2} - u \leq z_{m-1} \leq u$$

So for $z^{K'} = z_1^{p_1} \dots z_{m-1}^{p_{m-1}}$, we have $\int_{P_u} e^{-Q(z,u,\mathbf{t})} z^{K'}\, dz$ is equal to

$$\int_{-u}^{(m-1)u} \cdots \int_{z_{m-2}-u}^{u} e^{-\sum_{i,j} a_{ij} z_i z_j - \sum_i b_i u z_i} z_1^{p_1} \dots z_{m-1}^{p_{m-1}}\, dz_{m-1} \dots dz_1. \tag{3.15}$$

Despite the quadratic form being inhomogeneous with homogeneous part not necessarily being nondegenerate, since the bounds are linear and finite we are able to inductively apply the result of 3.1.4.

Chapter 4

Renormalization

4.1 Heat Kernel Counter Terms

4.1.1 A Motivating Example

Because of the inherent complexity of the renormalization procedure for a general Feynman graph, it is helpful to begin with an example that can elucidate most of the structure that arises. For the sake of simplicity and concreteness, we will work in the ϕ^4 theory, i.e. the scalar field theory theory with classical interaction

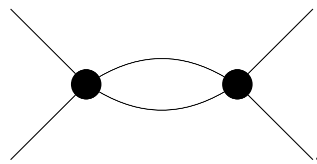
$$I = c\frac{1}{4!}\int \phi^4. \tag{4.1}$$

Nothing needs to be done at the 0-loop level, since the limit

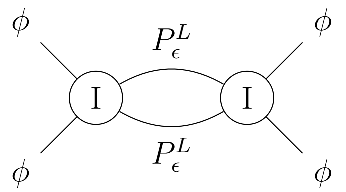
$$\lim_{\epsilon\to 0^+} w_\gamma(P_\epsilon^L, I) \tag{4.2}$$

already exists for any tree γ.

To illustrate what happens at the higher loop level, we will work with the 1-loop graph γ



The Feynman weight $w_\gamma(P_\epsilon^L, I)$ is computed by labelling the vertices by the interaction I, the edges by the propagator P_ϵ^L, and the tails by the input field ϕ



and then contracting. As defined in (1.12), the regularized propagator P_ϵ^L is the integral of the heat kernel K_t over the time interval $[\epsilon, L]$. This produces an integral

$$w_\gamma(P_\epsilon^L, I)[\phi] = \int_{[\epsilon,L]^2} f_{\gamma,I}(t_1,t_2)[\phi]\, dt_1 dt_2 \tag{4.3}$$

where

$$f_{\gamma,I}(t_1,t_2)[\phi] = \int_{M^2} K_{t_1}(x_1,x_2)K_{t_2}(x_1,x_2)\phi(x_1)^2\phi(x_2)^2. \tag{4.4}$$

Make the definition $\Phi(x_1,x_2) = \phi(x_1)^2\phi(x_2)^2$ to avoid unecessary detail in subsequent equations.

We begin with the case $M = \mathbb{R}^n$, where the heat kernel is given by

$$K_t(x_1,x_2) = (4\pi t)^{-n/2} e^{-|x_1-x_2|^2/4t}. \tag{4.5}$$

Making the substitution for K_t and the change of variables

$$w = x_1 + x_2 \tag{4.6}$$

$$y = x_1 - x_2 \tag{4.7}$$

we have

$$f_{\gamma,I}(t_1,t_2)[\phi] = C(t_1t_2)^{-n/2}\int_{(\mathbb{R}^n)^2} e^{-|y|^2\left(\frac{1}{4t_1}+\frac{1}{4t_2}\right)}\Phi(w,y). \tag{4.8}$$

for some constant C.

Let $\Phi^N(w,y)$ be the Taylor polynomial of degree N of $\Phi(w,\cdot)$ and define $f^N_\gamma(t_1,t_2)[\phi]$ by substituting $\Phi^N(w,y)$ in place of $\Phi(w,y)$ in the above formula for $f_{\gamma,I}(t_1,t_2)[\phi]$.

Roughly, we would like to define $w^N_\gamma(P^L_\epsilon, I)$ likewise by substituting $f^N_{\gamma,I}(t_1,t_2)[\phi]$ for $f_{\gamma,I}(t_1,t_2)[\phi]$ in the above formula (4.3) for $w_\gamma(P^L_\epsilon, I)$. Then we would hope that by making N sufficiently large, we can sufficiently control the error $|f_{\gamma,I}(t_1,t_2)[\phi] - f^N_{\gamma,I}(t_1,t_2)[\phi]|$ to force the limit $\lim_{\epsilon\to 0^+}[w_\gamma(P^L_\epsilon, I) - w^N_\gamma(P^L_\epsilon, I)]$ to exist.

This is the idea in spirit, but there are additional subtleties needed to ensure we can always sufficiently bound the error. Firstly, we give an ordering to the edges. Our graph is symmetric with respect to interchange of the edges, so, for this particular graph, in fact we can assume without loss of generality that $t_1 \leq t_2$.

Choose $R > 2$. If $t_2^R \leq t_1$ we will indeed be able to bound the error by

$$|f_{\gamma,I}(t_1,t_2)[\phi] - f^N_{\gamma,I}(t_1,t_2)[\phi]| \leq Ct_2^{-Rn}\int_{\mathbb{R}^n} e^{-|y|^2/2t_2}|y|^{N+1} \tag{4.9}$$

$$\leq Ct_2^{-Rn}t_2^{\frac{1}{2}(N+1)+\frac{n}{2}} \tag{4.10}$$

for some constant C and then by making N large enough we can ensure that $\frac{1}{2}(N+1)+\frac{n}{2}-Rn\geq 0$. Let N_1 be such an N.

It is worth fleshing out the structure of $f_{\gamma,I}(t_1,t_2)[\phi]$, which is of independent interest. Write $\Phi^N(w,y)=\sum_{|K|\leq N}\Psi_K(w)y^K$ so that

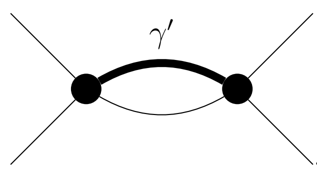
$$f_{\gamma,I}(t_1,t_2)[\phi]=C(t_1t_2)^{-n/2}\sum_{|K|\leq N}\int_{\mathbb{R}^n}e^{-|y|^2\left(\frac{1}{4t_1}+\frac{1}{4t_2}\right)}y^K\int_{\mathbb{R}^n}\Psi_K(w). \tag{4.11}$$

Applying Wick's theorem to the integral over y, we have

$$f_{\gamma,I}(t_1,t_2)[\phi]=\sum_{|K|\leq N}F_K(t_1,t_2)^{\frac{1}{2}}\int_{\mathbb{R}^n}\Psi_K(w). \tag{4.12}$$

Note that for each K, $F_K(t_1,t_2)$ is a rational function of t_1 and t_2 and $\int_{\mathbb{R}^n}\Psi_K(w)$ is a local functional of ϕ.

If $t_1\leq t_2^R$, however, we must do something different. Essentially, we begin by choosing the subgraph γ' of γ corresponding to the edge labelled by t_1



and treating the edges outside γ' as input tails. Let

$$f_{\gamma',\gamma,I}(t_1,t_2)[\phi]=f_{\gamma,I}(t_1,t_2)[\phi] \tag{4.13}$$

$$=C(t_1t_2)^{-n/2}\int_{(\mathbb{R})^2}e^{-|y|^2/4t_1}\Psi(w,y,t_2) \tag{4.14}$$

where $\Psi(w,y,t_2)=e^{-|y|^2/4t_2}\Phi(w,y)$. Now define

$$f^N_{\gamma',\gamma}(t_1,t_2)[\phi]=C(t_1t_2)^{-n/2}\int_{(\mathbb{R})^2}e^{-|y|^2/4t_1}\Psi^N(w,y,t_2) \tag{4.15}$$

where $\Psi^N(w,y,t_2)$ is the order N Taylor polynomial of $\Psi(w,y,t_2)$. Then because the $(N+1)$-st derivative in the y variable of $\Psi(w,y,t_2)$ is bounded by a constant times $t_2^{-(N+1)}$, we have the bound

$$|f_{\gamma',\gamma,I}(t_1,t_2)[\phi] - f^N_{\gamma',\gamma,I}(t_1,t_2)[\phi]| \leq C(t_1t_2)^{-n/2}t_2^{-(N+1)}t_1^{\frac{1}{2}(N+1)+\frac{n}{2}} \tag{4.16}$$

$$\leq Ct_2^{-\frac{n}{2}-(N+1)}t_2^{\frac{R}{2}(N+1)} \tag{4.17}$$

$$\leq Ct_2^{\left(\frac{R}{2}-1\right)(N+1)-\frac{n}{2}}. \tag{4.18}$$

This is why we require that $R > 2$, so that for N sufficiently large $\left(\frac{R}{2}-1\right)(N+1)-\frac{n}{2} \geq 0$. Let N_2 be such an N.

The counterterm is given by

$$w^{\mathrm{ct}}_\gamma(P^L_\epsilon, I) = \int\limits_{\substack{\epsilon\leq t_1,t_2\leq L \\ t_2^R\leq t_1}} f^{N_1}_{\gamma,I}(t_1,t_2)[\phi] + \int\limits_{\substack{\epsilon\leq t_1,t_2\leq L \\ t_1\leq t_2^R}} f^{N_2}_{\gamma',\gamma,I}(t_1,t_2)[\phi] \tag{4.19}$$

by construction, the limit

$$\lim_{\epsilon\to 0^+}[w_\gamma(P^L_\epsilon, I) - w^{\mathrm{ct}}_\gamma(P^L_\epsilon, I)] \tag{4.20}$$

exists, as desired.

In the case of the Euclidean half space $\mathbb{H}^n$, the Dirichlet heat kernel is given by

$$K_t(x_1,x_2) = (4\pi t)^{-n/2}[e^{-|x_1-x_2|^2/4t} - e^{-|x_1-x_2^*|^2/4t}] \tag{4.21}$$

where x_2 is the reflection through the boundary. We will try to follow the same procedure writing

$$w_\gamma(P^1_\epsilon, I) = \int_{[\epsilon,1]^2}\int_{(\mathbb{H}^4)^2} K_{t_1}(x_1,x_2)K_{t_2}(x_1,x_2)\Phi(x_1,x_2) \tag{4.22}$$

which is equal to

$$\int_{[\epsilon,1]^2} [f_{\gamma,I;0,0}(t_1,t_2)[\phi] + f_{\gamma,I;1,0}(t_1,t_2)[\phi] + f_{\gamma,I;0,1}(t_1,t_2)[\phi] + f_{\gamma,I;1,1}(t_1,t_2)[\phi]] \tag{4.23}$$

where

$$f_{\gamma,I;0,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_{(\mathbb{H}^4)^2} e^{-|x_1-x_2|^2(1/4t_1+1/4t_2)}\Phi(x_1,x_2) \tag{4.24}$$

$$f_{\gamma,I;0,1}(t_1,t_2)[\phi] = -C(t_1t_2)^{-2}\int_{(\mathbb{H}^4)^2} e^{-|x_1-x_2|^2/4t_1+-|x_1-x_2^*|^2/4t_2}\Phi(x_1,x_2) \tag{4.25}$$

$$f_{\gamma,I;1,0}(t_1,t_2)[\phi] = -C(t_1t_2)^{-2}\int_{(\mathbb{H}^4)^2} e^{-|x_1-x_2^*|^2/4t_1+-|x_1-x_2|^2/4t_2}\Phi(x_1,x_2) \tag{4.26}$$

$$f_{\gamma,I;1,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_{(\mathbb{H}^4)^2} e^{-|x_1-x_2^*|^2/4t_1-|x_1-x_2^*|/4t_2}\Phi(x_1,x_2). \tag{4.27}$$

where C is some constant.

Introduce the coordinates, $\overline{w} = \frac{\overline{x}_1+\overline{x}_2}{2}$ and $\overline{y} = \frac{\overline{x}_1-\overline{x}_2}{2}$ and $x_{1,n} = u + z$ and $x_{2,n} = u - z$.

When $t_2^R \leq t_1$, take the Taylor expansion to order N at 0 of $\Phi(x_1, x_2)$ in $\overline{y}$ and

z substitute it in the definition of $f_{\gamma,i,j}(t_1,t_2)[\phi]$ to get

$$f^N_{\gamma,I;0,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_0^\infty\int_{-u}^u\int_{(\mathbb{R}^3)^2} e^{-(|\overline{y}|^2+z^2)(1/t_1+1/t_2)}\Phi^N(\overline{w},u,\overline{y},z) \quad (4.28)$$

$$f^N_{\gamma,I;1,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_0^\infty\int_{-u}^u\int_{(\mathbb{R}^3)^2} e^{-(|\overline{y}|^2+u^2)/t_1-(|\overline{y}|^2+z^2)/t_2}\Phi^N(\overline{w},u,\overline{y},z) \quad (4.29)$$

$$f^N_{\gamma,I;0,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_0^\infty\int_{-u}^u\int_{(\mathbb{R}^3)^2} e^{-(|\overline{y}|^2+z^2)/t_1-(|\overline{y}|^2+u^2)/t_2}\Phi^N(\overline{w},u,\overline{y},z) \quad (4.30)$$

$$f^N_{\gamma,I;1,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\int_0^\infty\int_{-u}^u\int_{(\mathbb{R}^3)^2} e^{-(|\overline{y}|^2+u^2)(1/t_1+1/t_2)}\Phi^N(\overline{w},u,\overline{y},z) \quad (4.31)$$

where C is some new constant. One can show that we get the same bound

$$|f_{\gamma,I;i,j}(t_1,t_2)[\phi] - f^N_{\gamma,I;0,0}(t_1,t_2)[\phi]| \leq Ct_2^{\frac{1}{2}(N+1)+\frac{n}{2}-Rn}. \quad (4.32)$$

for all i,j. Details will be given in 4.1.6.

Upon examining the structure of the $f^N_{\gamma,I,i,j}(t_1,t_2)[\phi]$, we find that we no longer have a summation of local integrals, each weighted by the square root of some rational function in t_1 and t_2. For example,

$$f^0_{\gamma,I;0,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^{3/2}\int_{\mathbb{H}^4}\int_{-u}^u\left[e^{-z^2(1/t_1+1/t_2)}\right]\phi(\overline{w},u)^4 \quad (4.33)$$

$$f^0_{\gamma,I;0,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^{3/2}\int_{\mathbb{H}^4}\left[e^{-u^2/t_1}\int_{-u}^u e^{-z^2/t_2}\right]\phi(\overline{w},u)^4 \quad (4.34)$$

$$f^0_{\gamma,I;1,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^{3/2}\int_{\mathbb{H}^4}\left[e^{-u^2/t_2}\int_{-u}^u e^{-z^2/t_1}\right]\phi(\overline{w},u)^4 \quad (4.35)$$

$$f^0_{\gamma,I;1,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^{3/2}\int_{\mathbb{H}^4}\left[e^{-u^2(1/t_1+1/t_2)}\int_{-u}^u 1\right]\phi(\overline{w},u)^4 \quad (4.36)$$

In fact,

$$f^0_{\gamma,I;1,1}(t_1,t_2)[\phi] = 0 \tag{4.37}$$

and

$$f^0_{\gamma,I;i,j}(t_1,t_2)[\phi] = \int_{\mathbb{H}^4} F(t_1,t_2,u)\phi(\overline{w},u)^4 \tag{4.38}$$

for $(i,j) \neq (1,1)$, which is no longer a local integral due to the presence of $F(t_1,t_2,u)$. We shall investigate the structure of these "pseudo-local" integrals more carefully in 4.1.6.

Lastly, on the set where $t_1 \leq t_2^R$, we introduce the notation $f_{\gamma',\gamma,I,i,j}(t_1,t_2)[\phi]$ for $f_{\gamma,I,i,j}(t_1,t_2)[\phi]$ as before and construct $f^N_{\gamma',\gamma,I,i,j}(t_1,t_2)[\phi]$ analogously to the way we did on $\mathbb{R}^n$. We can bound the error similarly, but again $f^N_{\gamma',\gamma,I;i,j}(t_1,t_2)[\phi]$ will not be a sum of local integrals of ϕ each multiplied by the square root of a rational function in t_1 and t_2. The case $N=1$ makes evident the general structure

$$f^1_{\gamma',\gamma,I;0,0}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}t_1^{3/2}\int_{\mathbb{H}^4}\left[\int_{-u}^{u} e^{-z^2/t_1}\right]\phi(\overline{w},u)^4 \tag{4.39}$$

$$f^1_{\gamma',\gamma,I;0,1}(t_1,t_2)[\phi] = -C(t_1t_2)^{-2}t_1^{3/2}\int_{\mathbb{H}^4}\left[e^{-u^2/t_2}\int_{-u}^{u} e^{-z^2/t_1}\right]\phi(\overline{w},u)^4 \tag{4.40}$$

$$f^1_{\gamma',\gamma,I;1,0}(t_1,t_2)[\phi] = -C(t_1t_2)^{-2}t_1^{3/2}\int_{\mathbb{H}^4}\left[2ue^{-u^2/t_1}\right]\phi(\overline{w},u)^4 \tag{4.41}$$

$$f^1_{\gamma',\gamma,I;1,1}(t_1,t_2)[\phi] = C(t_1t_2)^{-2}t_1^{3/2}\int_{\mathbb{H}^4}\left[2ue^{-u^2(1/t_1+1/t_2)}\right]\phi(\overline{w},u)^4. \tag{4.42}$$

More details will be given in 4.1.6.

4.1.2 Covering $(0,\infty)^{|E(\gamma)|}$

Let $k = |E(\gamma)|$. We denote $\mathbf{t} = (t_1, \ldots, t_k)$. For each permutations $\sigma \in S_k$, there is a subset

$$S_\sigma = \{\mathbf{t} \in (0,\infty)^k : t_{\sigma(1)} < \cdots < t_{\sigma(k)}\}. \tag{4.43}$$

and it is clear that

$$\cup_{\sigma \in S_k} \overline{S_\sigma} = (0,\infty)^k. \tag{4.44}$$

The procedure we are about to describe is applied separately within each of the S_σ, but we work within

$$S_{\mathrm{id}} = \{\mathbf{t} \in (0,\infty)^k : t_1 < \cdots < t_k\}.$$

for notational clarity. We assume that $R > 1$.

Definition 2. *For $j \in \{1, \ldots, k-1\}$, let*

$$B_R^j = \{\mathbf{t} \in S_{id} : t_j < t_{j+1}^R\}. \tag{4.45}$$

For $i, j \in \{1, \ldots, k\}$ with $i < j$, define

$$C_R^{i,j} = \{\mathbf{t} \in S_{id} : t_j^R < t_i\}. \tag{4.46}$$

and define

$$\begin{aligned} D_R^{i,j} &= S_{id} \setminus \overline{C_R^{i,j}} \\ &= \{\mathbf{t} \in S_{id} : t_j^R > t_i\}. \end{aligned}$$

And lastly for $j \in \{2, \ldots, k-1\}$*, define*

$$A_R^j = B_R^j \cap C_R^{1,j} \tag{4.47}$$

$$= \{\mathbf{t} \in S_{id} : t_j < t_{j+1}^R \text{ and } t_j^R < t_1\} \tag{4.48}$$

and let $A_R^1 = B_R^1$ *and* $A^k = C_R^{1,k}$*.*

Note that $D_R^{j,j+1} = B_R^j$. A couple of facts about these subsets are collected in the following proposition:

Proposition 3. *For* $i_1 < i_2 < i_3$*.*

$$C_R^{i_1,i_2} \cap C_S^{i_2,i_3} \subset C_{RS}^{i_1,i_3} \tag{4.49}$$

and similarly

$$D_R^{i_1,i_2} \cap D_S^{i_2,i_3} \subset D_{RS}^{i_1,i_3} \tag{4.50}$$

Proof. If $\mathbf{t} \in C_R^{j_1,j_2} \cap C_S^{j_2,j_3}$, then $t_{i_2}^R < t_{i_1}$ and $t_{i_3}^S < t_{i_2}$. This implies that

$$t_{i_3}^{RS} < t_{i_1}.$$

The proof of the second inclusion is similar. □

The following statements are trivially true:

Proposition 4. *For* $j \in \{1, \ldots, k-1\}$*, let*

$$\tilde{B}_R^j = \{\mathbf{t} \in S_{id} : t_\alpha < t_\beta^R, \text{ for } \alpha \leq j \text{ and } j+1 \leq \beta\}. \tag{4.51}$$

For $i, j \in \{1, \ldots, k\}$ *with* $i < j$, *define*

$$\tilde{C}_R^{i,j} = \{\mathbf{t} \in S_{id} : t_\alpha^R < t_\beta, \text{ for } \alpha \leq i \text{ and } j \leq \beta\}.\}. \tag{4.52}$$

Then $\tilde{B}_R^j = B_R^j$ *and* $\tilde{C}_R^{i,j} = C_R^{i,j}$

Proposition 5. *For* $j_1 \leq j_2$, *if* $C_R^{i,j_1} \supseteq C_R^{i,j_2}$.

The next two proposions are needed to prove Theorem 4.

Proposition 6. $C_R^{i,j} \cap D_R^{l,m} = \emptyset$ *for* $i \leq l$ *and* $m \leq j$.

Proof. If $t_j^R < t_i$ and $t_l < t_m^R$. Then

$$t_i \leq t_l < t_m^R < t_j^R < t_i,$$

a contradiction. □

Proposition 7. $B_R^l \cap C_R^{i,j} = \emptyset$ *for* $i \leq l < j$.

Proof. Since $B_R^l = D_R^{l,l+1}$, we can apply the previous proposition. □

Definition 3. *We consider sequences of the form* $1 = i_0 < i_1 < \cdots < i_m \leq k$, *where* $m \leq k - 1$. *For any sequence of this form* I, *we define the sets* $E_R^I = \cap_{j=0}^m E_{R,j}^I$, *where* $E_{R,i}^I$ *is defined such that*

$$E_{R,0}^I = \begin{cases} B_{R^{s_1}}^1 & \text{if } m = 0 \\ S_{id} & \text{otherwise} \end{cases}$$

and

$$E^I_{R,j} = C^{i_{j-1},i_j}_{R^{s_j}} \cap D^{i_{j-1},i_j+1}_{R^{s_j}}$$

and for $j = m$

$$E^I_{R,m} = \begin{cases} C^{i_{m-1},i_m}_{R^{s_m}} \cap D^{i_{m-1},i_m+1}_{R^{s_m}} \cap B^{i_m}_{R^{s_{m+1}}} & \textit{if } i_m \neq k \\ C^{i_{m-1},i_m}_{R^{s_m}} \textit{if } i_m = k. \end{cases}$$

where $s_0, \ldots, s_m$ *is a fixed sequence.*

Theorem 3. *Their closures* $\overline{E}^I_R$ *form a cover of* $(0, \infty)^k$*.*

Proof. If $t_1 \leq t_2^{R^{s_1}}$, then $\mathbf{t} \in \overline{B}^1_R$. Thus let $m = 0$.

Otherwise, assume let i_1 be the largest integer such that $t^{R^{s_1}}_{i_1} \leq t_1 = t_{i_0}$. Then $i_1 \in \overline{C}^{i_0,i_1}_{R^{s_1}}$. If $i_1 = k$, let $m = 1$. If $i_1 < k$, then $\mathbf{t} \in D^{i_0,i_1+1}_{R^{s_1}}$. If $t_{i_1} \leq t^R_{i_1+1}$ then $\mathbf{t} \in \overline{B}^{i_1}_{R^{s_2}}$ and we let $m = 1$.

Otherwise, let i_2 be the largest integer such that $t^{R^{s_2}}_{i_2} \leq t_{i_1}$. Then $i_2 \in \overline{C}^{i_1,i_2}_{R^{s_2}}$. If $i_2 = k$, let $m = 2$. If $i_2 < k$, then $\mathbf{t} \in D^{i_1,i_2+1}_{R^{s_2}}$. If $t_{i_2} \leq t^{R^{s_3}}_{i_2+1}$ then $\mathbf{t} \in \overline{B}^{i_2}_{R^{s_3}}$ and we let $m = 2$.

And so on ... □

Theorem 4. *The sets* E^I_R *are disjoint.*

Proof. We prove this by induction. Consider the distinct sequences $1 = i_0 < i_1 < \cdots < i_m \leq k$ and $1 = j_0 < j_1 < \cdots < j_n \leq k$, where without loss of generality we assume that $m \leq n$.

Suppose that $i_l \neq j_l$, but $i_1 = j_1, \ldots, i_{l-1} = j_{l-1}$. Then

$$E^I_{R,l} \cap E^J_{R,l} \subseteq C^{i_{l-1},i_l}_{R^{s_l}} \cap D^{i_{l-1},i_l+1}_{R^{s_l}} \cap C^{i_{l-1},j_l}_{R^{s_l}} \cap D^{i_{l-1},j_l+1}_{R^{s_l}}$$

$$= \emptyset.$$

because $C^{i,j}_R \cap D^{i,m}_R = \emptyset$ for $m \leq j$ by Proposition 6.

It is also possible that $i_1 = j_1, \ldots, i_m = j_m$, but $m < n$. Then

$$E^I_m \cap E^J_{m+1} \subseteq B^{i_m}_{R^{s_{m+1}}} \cap C^{i_m,j_{m+1}}_{R^{s_{m+1}}}$$

$$= \emptyset.$$

by Proposition 7. $\square$

Now specialize to a specific sequence $s_0 = 1$ and $s_i = 2^{i-1}$ for $i > 0$.

Theorem 5. *Consider the sequence* $1 = i_0 < i_1 < \cdots < i_m \leq k$. *Then*

$$E^I_R \subseteq A^{i_m}_{R^{2^m}}$$

Proof. If $m = 0$, it is clear that $E^I_R \subseteq A^1_R = B^1_R$.

If $m > 0$,

$$E^I_R \subseteq \begin{cases} C^{i_0,i_1}_R \cap C^{i_1,i_2}_R \cap C^{i_2,i_3}_{R^2} \cdots \cap C^{i_{m-1},i_m}_{R^{2^{m-1}}} \cap B^{i_m}_{R^{2^m}} & \text{if } i_m < k \\ C^{i_0,i_1}_R \cap C^{i_1,i_2}_R \cap C^{i_2,i_3}_{R^2} \cdots \cap C^{i_{m-1},i_m}_{R^{2^{m-1}}} & \text{if } i_m = k \end{cases}$$

$$\subseteq \begin{cases} C^{i_0,i_m}_{R^{2^m}} \cap B^{i_m}_{R^{2^m}} & \text{if } i_m < k \\ C^{i_0,i_m}_{R^{2^m}} & \text{if } i_m = k \end{cases}$$

$$= A^{i_m}_{R^{2^m}}$$

$\square$

The construction of the counterterms in 4.1.5 will be based on a refinement of the covering $\{\overline{E}_R^I\}$. For $l < k$, given a sequence $l = i_0 < \cdots < i_m \leq k$, introduce the more general sets E_R^I which are defined by applying the definition of E_R^I, but replacing the set $\{t_1 < \cdots < t_k\}$ with the set $\{t_l < \cdots < t_k\}$. For $l = 1$, we recover E_R^I in the sense in which it was defined earlier.

The following is a corollary of Theorem 3:

Corollary 3. *Consider the collection of sequences of the form*

$$1 = i_0^{(1)} < i_1^{(1)} < \cdots < i_{m^{(1)}}^{(1)}$$

$$i_{m^{(1)}}^{(1)} = i_0^{(2)} < i_1^{(2)} < \cdots < i_{m^{(2)}}^{(2)}$$

$$\cdots$$

$$i_{m^{(p-1)}}^{(p-1)} = i_0^{(p)} < i_1^{(p)} < \cdots < i_{m^{(p)}}^{(p)} = k.$$

Then the sets

$$\overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} \cdots \cap \overline{E}_R^{I^{(p)}} \tag{4.53}$$

form a cover of S_{id}.

4.1.3 Local Functionals and Feynman Weights

Differential Operators

Let M be a smooth manifold, let E be a graded vector bundle and let $\underline{\mathbb{R}}$ be the trivial line bundle. Let $\mathcal{E} = \Gamma(E)$ and $C^\infty(M) = \Gamma(\underline{\mathbb{R}})$. A differential operator

$P : E \to \underline{\mathbb{R}}$ is an $\mathbb{R}$-linear map $\mathcal{E} \to C^\infty(M)$ which can be given locally using Einstein notation

$$s = \alpha^i e_i \mapsto a^I_j \frac{\partial \alpha^j}{\partial x^I}$$

where $e_1, \ldots, e_r$ is a local (homogeneous) frame for E on some sufficiently small coordinate neighborhood U, and $\alpha^1, \ldots, \alpha^r$ and a^I_i are functions on U.

Equivalently, there is a bundle map $\iota_P : J(E) \to \underline{\mathbb{R}}$, where $J(E)$ is the jet bundle of E. The differential operator P is determined by ι_P by composing with the jet prolongation of s, $j(s) : M \to J(E)$. That is, $P(s) = \iota_P \circ j(s)$.

Local Functionals

Definition 4. *A* local functional $I \in \mathcal{O}^k_{loc}(\mathcal{E})$ *of degree k is a functional $I \in \mathcal{O}^k(\mathcal{E})$ of the form*

$$I(s) = \sum_{\beta=1}^{m} \int_M D_{\beta,1}(s) \ldots D_{\beta,k}(s) \tag{4.54}$$

for some collection of differential operators $D_{i,j} : E \to \underline{\mathbb{R}}$.

Substituting the local formula for the differential operators

$$D_{\beta,j}(\alpha^i e_i) = (a_{\beta,j})^I_k \frac{\partial \alpha^k}{\partial x^I}$$

we get that locally

$$I(\alpha^i e_i) = \sum_{\beta=1}^{m} \int_U (a_{\beta,1})^{I_{\beta,1}}_{j_{\beta,1}} \ldots (a_{\beta,k})^{I_{\beta,k}}_{j_{\beta,k}} \frac{\partial \alpha^{j_{\beta,1}}}{\partial x^{I_{\beta,1}}} \cdots \frac{\partial \alpha^{j_{\beta,k}}}{\partial x^{I_{\beta,k}}} \tag{4.55}$$

$$= \int_U a^{I_1,\ldots,I_k}_{j_1,\cdots j_k} \frac{\partial \alpha^{j_1}}{\partial x^{I_1}} \cdots \frac{\partial \alpha^{j_k}}{\partial x^{I_k}}. \tag{4.56}$$

for a collection of functions $a^{I_1,\ldots,I_k}_{j_1,\ldots,j_k}$ on U.

Evaluation of $w_\gamma(P, I)$

We shall work in the ungraded case. For notation simplicity, we shall also assume from hereon out that $E = \underline{\mathbb{R}}$, although the method remains valid for any vector bundle.

We would like to describe the form of $w_\gamma(P_\epsilon^L, I)$ when $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[[\hbar]]$ is a power series of local functionals and

$$P_\epsilon^L = \int_\epsilon^L K_t \, dt \tag{4.57}$$

where K_t is the heat kernel of M.

For each vertex $v \in V(\gamma)$, we associate the functional $I_{g(v),k(v)}$, where $k(v)$ is the valency of the vertex v. Assume that within a given chart U,

$$S^{k(v)} I_{g(v),k(v)}(\alpha_1^i e_i, \ldots, \alpha_{k(v)}^i e_i) = \int_U a^{I^{v^1}, \ldots, I^{v^{k(v)}}} \frac{\partial \alpha_1}{\partial x^{I^{v^1}}} \cdots \frac{\partial \alpha_{k(v)}}{\partial x^{I^{v^k}}}.$$

where $I^{v^1}, \ldots, I^{v^{k(v)}}$ ranges over multi-indices with $|I^{v^1}| + \cdots + |I^{v^k}| \leq \operatorname{ord} I_{g(v),k(v)}$ Choose an ordering on the set of half edges $v^1, \ldots, v^{k(v)}$ incident on each vertex v and an orientation on each edge. Then γ determines the maps

$$Q : T(\gamma) \to \cup_{v \in V} \{v^1, \ldots, v^{k(v)}\}$$

$$Q_1 : E(\gamma) \to \cup_{v \in V} \{v^1, \ldots, v^{k(v)}\}$$

$$Q_2 : E(\gamma) \to \cup_{v \in V} \{v^1, \ldots, v^{k(v)}\}$$

where Q_1 and Q_2 map an edge to its first and second half edges respectively, and Q maps a tail to itself. Also denote by $v_1(e)$ and $v_2(e)$ the first and second vertices of the edge e. Similarly, let $v(h)$ denote the vertex of the tail h.

With these data, we can give the expression

$$w_\gamma(P_\epsilon^L, I)[\alpha] = \int_{(\epsilon,L)^{|E(\gamma)|}} f_{\gamma,I}(\mathbf{t})[\alpha]. \tag{4.58}$$

where for $M = \mathbb{R}^n$,

$$f_{\gamma,I}(\mathbf{t})[\alpha] = \int_{\mathbb{R}^{n|V(\gamma)|}} \prod_{v\in V(\gamma)} a^{I^{v^1},\ldots,I^{v^k}}(x_v) \prod_{e\in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h\in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}} \tag{4.59}$$

where k is used to stand for $k(v)$. If M is a compact manifold then choose a partition of unity subordinate to a finite cover of M (on which E is trivialized). Then $f_{\gamma,I}(\mathbf{t})[\alpha]$ is a sum of integrals of the form

$$\int_{U^{|V(\gamma)|}} \chi \prod_{v\in V(\gamma)} a^{I^{v^1},\ldots,I^{v^k}}(x_v) \prod_{e\in E(\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h\in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}} \tag{4.60}$$

where χ is the partition of unity function for the open set U in the cover and α^i are the coordinates of α in U.

Due to the symmetry of P_ϵ^L and $I_{i,k}$, the value of $w_\gamma(P_\epsilon^L, I)$ is independent of the choices of ordering and orientation.

4.1.4 Counterterms on $\mathbb{R}^n$: Preliminaries

For simplicity, in this section and subsequent sections in this chapter, we shall only consider scalar field theories.

When working with $\mathbb{R}^n$, we shall really mean locally on a flat compact manifold. Essentially, all that this means is that the input fields will be compactly-supported functions. The procedure can also be carried out on $\mathbb{R}^n$ in earnest by working with Schwartz functions instead of compactly supported functions.

On $\mathbb{R}^n$, the heat kernel has the simple form

$$K_t(x, y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}. \tag{4.61}$$

Derivatives of K_t

Proposition 8. *For a multi-index $I = (i_1, \ldots, i_n)$, $\frac{\partial K_t}{\partial x^I}$ is a polynomial in $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and $1/t$ which is multiplied by K_t. The degree in $1/t$ is $|I|$.*

The proof is a consequence of the Lemma 2 in Section 4.1.9 which also gives explicit formulas for the single variable derivatives.

Proof of Proposition 8. For a multi-index $I = (i_1, \ldots, i_n)$,

$$\frac{\partial K_t}{\partial x^I} = P_{1,i_1} \ldots P_{n,i_n} K_t. \tag{4.62}$$

□

Powers of t in $w_\gamma(P_\epsilon^L, I)$

Let $O(\gamma)$ be the sum of the orders of the local functionals $I_{g(v),k(v)}$ for all $v \in V(\gamma)$. As a consequence of Corollary 8, if we group the terms in $w_\gamma(P_\epsilon^L, I)$ by their powers of t, we see that

$$w_\gamma(P, I)[\alpha] = \int_{(\epsilon,L)^{|E(\gamma)|}} f_{\gamma,I}(\mathbf{t})[\alpha]. \tag{4.63}$$

where J is a multi-index and

$$f_{\gamma,I}(\mathbf{t})[\alpha] = \sum_{-O(\gamma)\leq|J|\leq 0} t^{J-n/2} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} \Phi_J. \tag{4.64}$$

This formula requires some explanation. The outer integral is over the time variables. Secondly, let

$$t^{J-n/2} = \prod_{e\in E(\gamma)} t_e^{j_e-n/2}.$$

In the exponential, $Q_e = \|x_{v_1(e)} - x_{v_2(e)}\|^2$.

The multi-index $I : E(\gamma) \to \mathbb{Z}$ and for each I, Φ_I a sum of terms of the form

$$\prod_{v\in V(\gamma)} D_v\alpha(x_v)$$

where for each vertex v,

$$D_v\alpha = D_{v,1}\alpha \dots D_{v,l}\alpha \tag{4.65}$$

is a product of differential operators applied to α.

Spanning Tree Coordinates

We would like to evaluate (4.64). This will require a special change of coordinates. Choose a spanning tree T of γ. For each edge in the tree we define a coordinate $y_e = x_{v_1(e)} - x_{v_2(e)}$.

Proposition 9. *Given a spanning tree T, the coordinates $y_e = x_{v_1(e)} - x_{v_2(e)}$ for $e \in E(T)$ and*

$$w = x_1 + \cdots + x_{|V(\gamma)|} \tag{4.66}$$

form a coordinate system on $\mathbb{R}^{n|V(\gamma)|}$.

Proof. This is a linear transformation from $\mathbb{R}^{|V(\gamma)|}$ to $\mathbb{R}^{|V(\gamma)|}$. It is invertible if and only if it has trivial kernel. But if $y_e = 0$ for all $e \in T(\gamma)$ then $x_i = x_j$ for all i and j. The condition that $x_1 + \cdots + x_{|V(\gamma)|} = 0$ then implies that $x_i = 0$ for all i. $\square$

The quadratic form $Q(x) = \sum_{e \in E(\gamma)} Q_e(x)/4t_e$ can be written in the spanning tree coordinates as $Q(w, y)$.

Let A be the matrix of $Q(0, y)$. Then A is an $n(|V(\gamma)| - 1)$ by $n(|V(\gamma)| - 1)$ matrix.

Proposition 10. *The quadratic form* $Q(w, y)$ *is independent of* w.

Proof. For any edge $e \in E(\gamma)$, let $f_1^e, \ldots, f_{l(e)}^e$ be the unique path of edges in T connecting $v_1(e)$ and $v_2(e)$. Then

$$x_{v_1(e)} - x_{v_2(e)} = \sum_{i=1}^{l(e)} (x_{v_1(f_i^e)} - x_{v_2(f_i^e)}) = \sum_{i=1}^{l(e)} y_{f_i^e}.$$

Therefore,

$$\begin{aligned} Q(x) &= \sum_{e \in E(\gamma)} Q_e(x)/4t_e \\ &= \sum_{e \in E(\gamma)} \left\| \sum_{i=1}^{l(e)} y_{f_i^e} \right\|^2 /4t_e \\ &= Q(w, y) \end{aligned}$$

which clearly does not depend on w. $\square$

Proposition 11. *The matrix* $B = (4\prod_{e\in E(\gamma)} t_e)A$ *has entries that are integer polynomials in* $\{t_e\}_{e\in E(\gamma)}$*. Consequently,* $P_\gamma = \det B$ *is an integer polynomial in* $\{t_e\}_{e\in E(\gamma)}$*.*

Proof. It is clear that the matrix B, which is the matrix of the quadratic form $(4\prod_{e\in E(\gamma)} t_e)Q(0,y)$, has entries which are polynomials in $\{t_e\}_{e\in E(\gamma)}$ with integer coefficients. □

Proposition 12.

$$\det A = 4^{-n(|V(\gamma)|-1)} t^{-n(|V(\gamma)|-1)} P_\gamma \tag{4.67}$$

and

$$A^{-1} = \frac{1}{P_\gamma} C, \tag{4.68}$$

where C *is a matrix with polynomial entries in* t_e*.*

Proof. To prove the second statement, use Cramer's rule

$$B^{-1} = \frac{1}{\det B}\operatorname{adj}(B) = \frac{1}{P_\gamma}\operatorname{adj}(B) \tag{4.69}$$

and that $A^{-1} = (4\prod_{e\in E(\gamma)} t_e)B^{-1}$. So, the statement follows by letting $C = (4\prod_{e\in E(\gamma)} t_e)\operatorname{adj}(B)$. □

Taylor Expansion of Φ_I

In (4.64), replace Φ_J in $f_{\gamma,I}(\mathbf{t})[\alpha]$ with its Taylor polynomial of degree N' in y, $\Phi^{N'}(w,y) = \sum_{|K|\le N'} c_{J,K} y^K$, where N' is a non-negative integer to be determined.

This gives

$$f_{\gamma,I}^{N'}(\mathbf{t})[\alpha] = \sum_{\substack{|K|\leq N' \\ -O(\gamma)\leq|J|\leq 0}} t^{J-n/2} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e(w,y)/4t_e} c_{J,K}(w) y^K \, dy dw \tag{4.70}$$

$$= \sum_{\substack{|K|\leq N', K \text{ even} \\ -O(\gamma)\leq|J|\leq 0}} t^{J-n/2} \mathcal{I}_A^K(\mathbf{t}) \int_{\mathbb{R}^n} c_{J,K}(w) \, dw \tag{4.71}$$

where

$$\mathcal{I}_A^K(\mathbf{t}) = \int_{\mathbb{R}^{n(|V(\gamma)|-1)}} e^{-\langle y, Ay\rangle} y^K \, dy. \tag{4.72}$$

and $c_{J,K}$ is a function of w only.

We can calulate $\mathcal{I}_A^K$ rather explicitly. This is the content of Theorem 9 in 4.1.9.

The structure of $c_{J,K}$

In this section we prove that $\Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} c_{J,K}(w) \, dw$ is a local functional.

Recall that $c_{J,K}(w) = \frac{\partial \Phi_J}{\partial y^K}(0, w)$ and that

$$\Phi_J = \prod_{v\in V(\gamma)} D_v \alpha(x_v) \tag{4.73}$$

where D_v is a product of differential operators. So

$$c_{J,K}(w) = \prod_{v\in V(\gamma)} \tilde{D}_v \alpha(w). \tag{4.74}$$

$\tilde{D}_v$ is a product of differential operators on $\mathbb{R}^n_{x_v}$. Finally, we see that

$$\Psi_{J,K}(\alpha) = \int_{\mathbb{R}^n} \prod_{v\in V(\gamma)} \tilde{D}_v \alpha(w) \, dw \tag{4.75}$$

is a local functional.

Alternatively, we can say that Φ_J is a sum of terms of the form

$$f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \tag{4.76}$$

where f is a compactly supported function and I^h is a collection of multi-indices, one for each tail h, satisfying the condition $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma)$. This implies that $\frac{\partial \Phi_I}{\partial y^J}$ is a sum of terms of the same form, but satisfying the condition $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) + N' + 1$.

From (4.71) and (4.122) it is now clear that

Corollary 4.

$$f_{\gamma,I}^{N'}(\mathbf{t}) = \sum_{\substack{|K| \leq N', K \ even \\ -O(\gamma) \leq |J| \leq 0}} \frac{\mathcal{P}_A^K(\mathbf{t})}{\mathcal{Q}_A^{J,K}(\mathbf{t})} \Psi_{J,K}(\alpha). \tag{4.77}$$

where $\mathcal{Q}_A^{J,K}$ is of homogeneous degree

$$-|J| + \frac{n}{2}|E(\gamma)| + n(|V(\gamma)| - 1)(|E(\gamma)| - 1)(|K| + 1)/2. \tag{4.78}$$

and the degree of $\mathcal{P}_A^K$ is given within Theorem 9 in 4.1.9.

As an aside, note that for a fixed spanning tree T, x_v and $\{y_e\}_{e \in E(T)}$ and w are related by a linear coordinate change. In order to calculate $c_{J,K}$, one would like to make this change explicit. Let $e_1, \ldots, e_l$ with $v_1(e_1) = v$ and $v_2(e_2) = w$ be the unique path in T connecting v and w. Then

$$x_v - x_w = \sum_{i=1}^{l} y_{e_i}$$

and thus we can express x_v in terms of $\{y_e\}_{e \in E(T)}$ and w using the equation

$$x_v = \frac{1}{|V(\gamma)|}\left[w + \sum_{w \neq v} (x_v - x_w)\right].$$

4.1.5 Counterterms on $\mathbb{R}^n$: Error Bounds and Iteration

Bounding the Error

Importantly, an elementary change of variables gives the following bounds

Proposition 13.

$$\int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} |y^K| \leq C t_k^{\frac{1}{2}(|K|+(|V(\gamma)|-1))} \leq C t_k^{n(|V(\gamma)|-1))} \tag{4.79}$$

for some constant $C > 0$.

The last inequality above is a consequence of the assumption $t_k \leq 1$. Thus,

Proposition 14.

$$|\mathcal{I}_A^K(\mathbf{t})| \leq C t_k^{\frac{1}{2}(|K|+n(|V(\gamma)|-1))} \leq C t_k^{n(|V(\gamma)|-1))} \tag{4.80}$$

and consequently,

$$|f_{\gamma,I}^{N'}(\mathbf{t})[\alpha]| \leq \left(\sum_p C_p \prod_{h\in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{\frac{n}{2}(|V(\gamma)|-1)-R|E(\gamma)|\frac{n}{2}-RO(\gamma)} \tag{4.81}$$

$$\leq \|\alpha\|_{O(\gamma)+N'}^{|T(\gamma)|} t_k^{\frac{n}{2}(|V(\gamma)|-1)-R|E(\gamma)|\frac{n}{2}-RO(\gamma)} \tag{4.82}$$

$$\tag{4.83}$$

where the summation is over multi-indices $p : T(\gamma) \to \mathbb{Z}^{\geq 0}$ such that $\sum_{h\in T(\gamma)} p_h \leq O(\gamma) + N'$ and $\|\alpha\|_{p_h}$ is the C^{p_h} norm of α.

Let $k = |E(\gamma)|$. Assume that we order the edges so that $t_1 \leq \cdots \leq t_k$ and $t_k^R \leq t_1$ so that $\mathbf{t} \in A_R^k$ and that $\mathbf{t} \in (0,1)^k$.

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f^{N'}_{\gamma,I}(\mathbf{t})[\alpha]| \leq \sum_{|K|=N'+1} t_k^{-RO(\gamma)} t^{-n/2} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} d_K(w)|y^K| \tag{4.84}$$

$$\leq \sum_{|K|=N'+1} t_k^{-RO(\gamma)-|E(\gamma)|n/2} \int_{\mathbb{R}^{n|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_k} d_K(w)|y^K| \tag{4.85}$$

$$\leq C t_k^{\frac{1}{2}(N'+1)+\frac{n}{2}(|V(\gamma)|-1))} t_k^{-RO(\gamma)-R|E(\gamma)|n/2} \sum_{|K|=N'+1} \int d_K(w)\, dw \tag{4.86}$$

using Proposition 13. In the formula above,

$$d_K(w) = \sum_J \sup_y \left| \frac{\partial \Phi_J}{\partial y^K}(y,w) \right|$$

But

$$\sup_y \left| f \prod_{h\in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^h}_{v(h)}} \right| \leq \sup_y |f| \cdot \sup_{y,w} \left| \prod_{h\in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^h}_{v(h)}} \right|$$
$$\leq \sup_y |f| \prod_{h\in T(\gamma)} \|\alpha\|_{p_h}$$

where $p_h = |I^h|$. Note that $\sup_y |f|$ is a compactly-supported function in the variable w. Thus,

$$\int d_K\, dw \leq \sum_p C_p \prod_{h\in T(\gamma)} \|\alpha\|_{p_h}$$

where the summation is over multi-indices $p : T(\gamma) \to \mathbb{Z}^{\geq 0}$ such that $\sum_{h\in T(\gamma)} p_h \leq O(\gamma) + N' + 1$.

In conclusion, we have shown that

Theorem 6.

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^{N'}(\mathbf{t})[\alpha]| \leq \left(\sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{\frac{1}{2}(N'+1)+\frac{n}{2}(|V(\gamma)|-1)-R(O(\gamma)+\frac{n}{2}|E(\gamma)|)} \tag{4.87}$$

where the summation is over multi-indices $p : T(\gamma) \to \mathbb{Z}^{\geq 0}$ *such that* $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$.

Inductive Construction of the Counterterms

We shall use Corollary 3, which gives a finite cover of $(0,1)^k$ given by sets of the form $\overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} \cdots \cap \overline{E}_R^{I^{(p)}}$, for $p \leq k$. The structure of the multi-indices $I^{(1)}, \ldots, I^{(p)}$ is given in 3 and the sets E_R^I are defined in Definition 3.

Also we shall need Theorem 5 which states that for E_R^I where I is a sequence of the form $1 < i_1 < \cdots < i_m \leq k$ with $m < k$, we have $E_R^I \subseteq A_{R^{2m}}^{i_m}$, where

$$A_{R^{2m}}^{i_m} = \{t_1 < t_2 < \cdots < t_k : t_{i_m} < t_{i_m+1}^{R^{2m}} \text{ and } t_{i_m}^{R^{2m}} < t_1\}. \tag{4.88}$$

Theorem 7. *For any sequence* $I^{(1)}, \ldots, I^{(p)}$ *as in Corollary 3, for nonnegative integers* $N_1', \ldots, N_p'$.

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^{N_1',\ldots,N_p'}(\mathbf{t})[\alpha]| \leq \|\alpha\|_l^{|T(\gamma)|} \sum_i C_i t^{d_i}, \tag{4.89}$$

where l *some positive integer, where* $d_i = d_i(N_1', \ldots, N_i')$ *increases linearly in* N_i' *for* $N_1', \ldots, N_{i-1}'$ *fixed and sufficiently large, and where* $f_{\gamma,I}^{N_1',\ldots,N_p'}(\mathbf{t})[\alpha]$ *is defined by iterative Taylor expansion of* $f_{\gamma,I}(\mathbf{t})[\alpha]$ *as illustrated in the proof of the theorem.*

Proof. Fix an ordering of the edge set so that we can identify $E(\gamma) = \{e_1, \dots, e_k\} = \{1, \dots, k\}$. The procedure should be carried out in

$$S_\sigma = \{\mathbf{t} \in (0, \infty)^k : t_{\sigma(1)} < \cdots < t_{\sigma(k)}\}. \tag{4.90}$$

for each permutation $\sigma \in S_k$. However, we shall work in S_{id} for notational simplicity.

For general p, we would consider the cover of S_{id} by sets of the form $\overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} \cdots \cap \overline{E}_R^{I^{(p)}}$, for each of the $k!$ possible orderings of the edges of γ.

For illustrative purposes and notational simplicity we prove the main theorem only for $p = 2$. The inductive step in the proof of the general case is similar. Let $i^{(1)} = i^{(1)}_{m^{(1)}}$ and let $R_1 = R^{s_{m^{(1)}+1}}$ and $R_2 = R^{s_{m^{(2)}+1}}$ so that we are working within

$$\begin{aligned} \overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} &\subseteq \overline{A}_{R_1}^{i^{(1)}} \cap \overline{A}_{R_2}^{k} \\ &= \{\mathbf{t} \in S_{\text{id}} : t_{i^{(1)}}^{R_1} \leq t_1 \text{ and } t_{i^{(1)}} \leq t_{i^{(1)}+1}^{R_1} \text{ and } t_k \leq t_{i^{(1)}+1}^{R_2}\} \end{aligned}$$

where the inclusion follows from Theorem 5. Note that $i^{(2)}_{m^{(2)}} = k$.

The collection of edges $e_1, \dots, e_{i^{(1)}}$ determines a subgraph of γ, which we denote by γ'. The remaining edges $e_{i^{(1)}+1}, \dots, e_k$ form the edge set of γ/γ'.

A tail $h \in T(\gamma')$ will either be in $T(\gamma)$ or will be one of the two half edges forming an edge in $E(\gamma)$. In the formula above, $T(\gamma', \gamma) = T(\gamma') \cap T(\gamma)$ and $E(\gamma', \gamma) = E(\gamma') \cup F(\gamma', \gamma)$, where $F(\gamma', \gamma)$ is the set of all edges in γ for which one half edge making up the edge is a tail in γ'. Let $V(\gamma', \gamma)$ denote the vertices not in γ' that are incident on an edge in $F(\gamma', \gamma)$.

The integral in the formula for $f_{\gamma,I}(\mathbf{t})[\alpha]$ is over $\mathbb{R}^{n|V(\gamma)|}$ and we can order the integration so that we integrate first with respect to the vertices in $V(\gamma')$. This inner integral, which is of the form

$$\int_{\mathbb{R}^{n|V(\gamma')|}} \prod_{v \in V(\gamma')} a^{I^{v^1},\ldots,I^{v^k}}(x_v) \prod_{e \in E(\gamma',\gamma)} \frac{\partial K_t(x_{v_1(e)}, x_{v_2(e)})}{\partial x^{I^{Q_1(e)}} \partial x^{I^{Q_2(e)}}} \prod_{h \in T(\gamma',\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x^{I^{Q(h)}}} \qquad (4.91)$$

Let us use $g_{\gamma',\gamma,I}(\mathbf{t})[\alpha]$ to denote the integral above. It is a function of $\mathbf{t} = \{t_e\}_{e \in E(\gamma',\gamma)}$ and x_v for $v \in V(\gamma',\gamma)$. Let $f_{\gamma/\gamma',I,g}(\mathbf{t})[\alpha]$ be as in (4.64) but with the functional $g(\mathbf{t})$ for $\mathbf{t} = \{t_e\}_{e \in E(\gamma',\gamma)}$ used for the distinguished vertex in γ/γ'.

Use the same procedure which led to Corollary 4 and Theorem 6. We have that $\left| g_{\gamma',\gamma,I}(\mathbf{t})[\alpha] - g^{N_1'}_{\gamma',\gamma,I}(\mathbf{t})[\alpha] \right|$ is less than or equal to

$$\left(\sum_p C_p \prod_{h \in T(\gamma')} \|\alpha\|_{p_h} \prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_{h(e)}} \right) t_{i^{(1)}}^{\frac{1}{2}N_1' + C(\gamma',n,R_1)}$$

where

$$C(\gamma', n, R_1) = \frac{1}{2} + (|V(\gamma')| - 1)\frac{n}{2} - R_1 \left(O(\gamma') - |E(\gamma')| \frac{n}{2} \right).$$

But

$$\|K_{t_e}\|_{p_{h(e)}} \le C t_e^{-\frac{n}{2} - p_h(e)} \le C t_{i^{(1)}+1}^{-\frac{n}{2} - p_h(e)}$$

for some constant C, and thus

$$\begin{aligned} \prod_{e \in F(\gamma',\gamma)} \|K_{t_e}\|_{p_{h(e)}} &\le C t_{i^{(1)}+1}^{-|F(\gamma',\gamma)|\frac{n}{2} - \sum_{e \in F(\gamma',\gamma)} p_{h(e)}} \\ &\le C t_{i^{(1)}+1}^{-|E(\gamma)|\frac{n}{2} - O(\gamma') - N_1' - 1} \end{aligned}$$

for some constant C.

Thus, $\left|g_{\gamma',\gamma,I}(\mathbf{t})[\alpha] - g^{N_1'}_{\gamma',\gamma,I}(\mathbf{t})[\alpha]\right|$ is less than or equal to

$$\left(\sum_p C_p \prod_{h\in T(\gamma')} \|\alpha\|_{p_h}\right) t_{i^{(1)}+1}^{\frac{R_1}{2}N_1'+R_1C(\gamma',n,R_1)-|E(\gamma)|\frac{n}{2}-O(\gamma')-N_1'}.$$

So as long as $R_1 > 2$, $\left|g_{\gamma',\gamma,I}(\mathbf{t})[\alpha] - g^{N_1'}_{\gamma',\gamma,I}(\mathbf{t})[\alpha]\right|$ and consequently

$$\left|f_{\gamma/\gamma',I,g_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]\right|$$

will be bounded by a power of t_k, can it $d_1(N_1')$ which grows linearly with N_1'.

To finish the argument, we need to show the same thing for

$$|f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]|$$

From Proposition 14,

$$|g^{N_1'}_{\gamma',\gamma,I}(\mathbf{t})[\alpha]| \leq \left(\sum_p C_p \prod_{h\in T(\gamma')} \|\alpha\|_{p_h}\right) t_{i^{(1)}}^{\frac{n}{2}(|V(\gamma)|-1)-R_1|E(\gamma)|\frac{n}{2}-R_1O(\gamma)-R_1N_1'}.$$

Let $C_1(N_1',\gamma',\gamma,n,R_1)$ be the power of $t_{i^{(1)}}$ in the inequality above.

We are able to bound $|f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]|$ by

$$C\|\alpha\|_l^{|T(\gamma)|} t_k^{\frac{1}{2}N_2'+\frac{1}{2}+(|V(\gamma/\gamma')|-1)\frac{n}{2}-R_1(O(\gamma)+N_1')+R_1C_1(N_1',\gamma',\gamma,n,R_1)}$$

for some positive integer l.

In conclusion, using the triangle inequality and that we can bound

$$|f_{\gamma/\gamma',I,g_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f^{N_2'}_{\gamma/\gamma',I,g^{N_1}_{\gamma,\gamma',I}}(\mathbf{t})[\alpha]| \leq C_1 t_k^{d_1(N_1')} + C_2 t_k^{d_2(N_1',N_2')}$$

where by $d_1(N_1')$ grows linearly in N_1' and $d_2(N_1',N_2')$ grows linearly in N_2' for N_1' fixed. $\square$

4.1.6 Counterterms on the Euclidean Upper Half Space

The Dirichlet heat kernel on the upper half space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ with the Euclidean metric is given by

$$K_t(x, y) = (4\pi t)^{-n/2}[e^{-|x-y|^2/4t} - e^{-|x-y^*|^2/4t}], \tag{4.92}$$

where y^* is the reflection through the hyperplane $y_n = 0$.

Note that K_t is solves the heat equation, for $y \in \partial\,\mathbb{H}^n_+$, $K_t(x, y) = 0$, and

$$\lim_{t \to 0^+} \int_{\mathbb{H}^n} K_t(x, y)\phi(y)\, dy = \phi(x)$$

for any $\phi \in C^\infty(\mathbb{H}^n)$.

Similarly to 4.1.4, we form

$$w_\gamma(P^L_\epsilon, I)[\alpha] = \int_{(\epsilon,L)^{|E(\gamma)|}} f_{\gamma,I}(\mathbf{t})[\alpha] \tag{4.93}$$

but now

$$f_{\gamma,I}(\mathbf{t})[\alpha] = \sum_\beta \sum_{-O(\gamma) \le |J| \le 0} t^{J-n/2} \int_{\mathbb{H}^{n|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e^{(\beta e)}/4t_e} \Phi_{J,\beta} \tag{4.94}$$

where $Q_e^{(1)} = \|x_{v_1(e)} - x_{v_2(e)}\|^2$ and $Q_e^{(-1)} = \|x_{v_1(e)} - x^*_{v_2(e)}\|^2$ and β ranges over all functions $E(\gamma) \to \{-1, 1\}$. As in 4.1.4, we wish to apply Wick's theorem after taking the Taylor expansion of $\Phi_{J,\beta}$.

Coordinate System on $\mathbb{H}^{n|V(\gamma)|}$

Using the decomposition $\mathbb{H}^n = \mathbb{R}^{(n-1)} \times \mathbb{R}^{\ge 0}$ introduce the coordinates $x_v = (\overline{x}_v, x_{v,n})$. Split the integral into an integral over $\mathbb{R}^{(n-1)|V(\gamma)|}$ followed by an integral over

$(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$. The quadratic form with these coordinates becomes

$$\sum_{e \in E(\gamma)} \|\overline{x}_{v_1(e)} - \overline{x}_{v_2(e)}\|/4t_e$$

plus the part depending on the variables $x_{v,n}$

$$\sum_{e \in \beta^{-1}(1)} |x_{v_1(e),n} - x_{v_1(e),n}|^2/4t_e + \sum_{e \in \beta^{-1}(-1)} |x_{v_1(e),n} + x_{v_1(e),n}|^2/4t_e$$

We shall only concentrate on the part of the quadratic form depending on the variables $x_{v,n}$ since the integral over the variables $\overline{x}_v$ can be treated by the methods of 4.1.4.

Choose an ordering on the set of vertices and consider the basis, $f_{|V(\gamma)|} = e_1 + \cdots + e_n$, $f_1 = e_1 - e_2, \ldots, f_{|V(\gamma)|-1} = e_{|V(\gamma)|-1} - e_{|V(\gamma)|}$. This induces the coordinate system $u, z_1, \ldots, z_{|V(\gamma)|-1}$ on $\mathbb{R}^{|V(\gamma)|}$ related to standard coordinates by

$$\begin{aligned}
x_{1,n} &= u + z_1 = u + \tilde{z}_1 \\
x_{2,n} &= u + z_2 - z_1 = u + \tilde{z}_2 \\
&\cdots \\
x_{|V(\gamma)|-1,n} &= u + z_{|V(\gamma)|-1} - z_{|V(\gamma)|-2} = u + \tilde{z}_{|V(\gamma)|-1} \\
x_{|V(\gamma)|,n} &= u - z_{|V(\gamma)|-1} = u + \tilde{z}_{|V(\gamma)|}
\end{aligned}$$

In this coordinate system the second part of the quadratic form becomes

$$\sum_{e \in \beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2/4t_e + \sum_{e \in \beta^{-1}(-1)} |2u + \tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2/4t_e.$$

Let P be the plane spanned by f_i for i between 1 and $|V(\gamma)|-1$ the subscript indicates the dependence on $\{t_e\}_{e\in E(\gamma)}$. Then for $u \geq 0$

$$u(e_1+\cdots+e_{|V(\gamma)|})+P$$

intersects $(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$ in a bounded set (in particular a simplex) whose projection onto P we denote P_u.

Taylor Expansion of Φ_J

For a fixed spanning tree of γ, choose spanning tree coordinates on $\mathbb{R}^{(n-1)||V(\gamma)|}$, $\overline{y}_e = \overline{x}_{v_1(e)} - \overline{x}_{v_2(e)}$ and $\overline{w} = \overline{x}_1 + \cdots + \overline{x}_{|V(\gamma)|}$ on $\mathbb{R}^{(n-1)||V(\gamma)|}$. As in the previous section, choose coordinates $z_1, \ldots z_{|V(\gamma)|-1}$ and u on $(\mathbb{R}^{\geq 0})^{|V(\gamma)|}$.

In these coordinates, the quadratic form $\sum_{e\in E(\gamma)} Q_e/t_e$ decomposes into a sum of three terms

$$Q(\overline{y},\mathbf{t}) + Q^{(\beta)}(z,u,\mathbf{t}) + Q^{(\beta)}(u,\mathbf{t}),$$

where as in the proof of Proposition 10,

$$Q(\overline{y},\mathbf{t}) = \sum_{e\in E(\gamma)} \left\| \sum_{i=1}^{l(e)} \overline{y}_{f_i^e} \right\|^2 /4t_e,$$

$$\begin{aligned} Q^{(\beta)}(z,u,\mathbf{t}) = &\sum_{e\in\beta^{-1}(1)} |\tilde{z}_{v_1(e)} - \tilde{z}_{v_2(e)}|^2/4t_e + \sum_{e\in\beta^{-1}(-1)} |\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)}|^2/4t_e \\ &+ \sum_{e\in\beta^{-1}(-1)} u(\tilde{z}_{v_1(e)} + \tilde{z}_{v_2(e)})/t_e \end{aligned}$$

and

$$Q^{(\beta)}(u, \mathbf{t}) = \left(\sum_{e \in \beta^{-1}(-1)} t_e^{-1} \right) u^2.$$

We shall Taylor expand Φ_J in $f_{\gamma,I}(\mathbf{t})[\alpha]$ in both $\overline{y}$ and z to order N'. For $f_{\gamma,I}^{N'}(\mathbf{t})[\alpha]$ we have a sum of integrals of the form

$$t^{J-n/2} \int_{\mathbb{R}^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-Q(\overline{y},\mathbf{t}) - Q^{(\beta)}(z,u,\mathbf{t}) - Q^{(\beta)}(u,\mathbf{t})} c_{J,K,K',\beta} \overline{y}^K z^{K'} \, d\overline{y} d\overline{w} dz du,$$

over $|K| + |K'| \leq N'$, J even, $-O(\gamma) \leq |J| \leq 0$ and β functions $E(\gamma) \to \{-1, 1\}$. Also, $c_{J,K,K',\beta}$, the Taylor coefficient is a function of w and u only.

The integral over $\overline{y}$ gives an answer like that of Theorem 9 and Corollary 4, but with the dimension n replaced by $n-1$ in all the formulas. The integral over z exists because P_u is a bounded set. Let

$$\phi_{K'}(u, \mathbf{t}) = \int_{P_u} e^{-Q^{(\beta)}(z,u,\mathbf{t})} z^{K'} \, dz \tag{4.95}$$

Let

$$\begin{aligned} \mathcal{I}_{K,K'}(u) &= \int_{P_u} \int_{\mathbb{R}^{(n-1)(|V(\gamma)|-1)}} e^{-Q(\overline{y},\mathbf{t}) - Q^{(\beta)}(z,u,\mathbf{t})} \overline{y}^K z^{K'} \, d\overline{y} dz & (4.96) \\ &= \frac{\mathcal{P}_A^K(\mathbf{t})}{\mathcal{Q}_A^K(\mathbf{t})} \phi_{K'}(u, \mathbf{t}) & (4.97) \end{aligned}$$

where $\mathcal{P}_A^K(\mathbf{t})$ and $\mathcal{Q}_A^K(\mathbf{t})$ are defined analogously to the functions in 4.

Then $f_{\gamma,I}^{N'}(\mathbf{t})[\alpha]$ becomes a sum of integrals

$$\begin{aligned} & t^{J-n/2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{\geq 0}} e^{-Q^{(\beta)}(u,\mathbf{t})} \mathcal{I}_{K,K'}(u) c_{J,K,K',\beta}(\overline{w}, u) du d\overline{w} & (4.98) \\ =& t^{J-n/2} \frac{\mathcal{P}_A^K(\mathbf{t})}{\mathcal{Q}_A^K(\mathbf{t})} \int_{\mathbb{H}} e^{-Q^{(P)}(u,\mathbf{t})} \phi_{K'}(u, \mathbf{t}) c_{J,K,K',\beta}(\overline{w}, u) du d\overline{w} & (4.99) \\ =& \frac{\mathcal{P}_A^K(\mathbf{t})}{\mathcal{Q}_A^{J,K}(\mathbf{t})} \Psi_{J,K,K'}(\mathbf{t}, \alpha) & (4.100) \end{aligned}$$

This is not quite the product of a function of $\mathbf{t}$ and a local functional because $\psi_{K'}(u,\mathbf{t}) = e^{-Q^{(P)}(u,\mathbf{t})}\phi_{K'}(u,\mathbf{t})$ depends on $\mathbf{t}$.

Recall that $c_{J,K,K'}(\overline{w},u) = \frac{\partial \Phi_J}{\partial \overline{y}^K \partial z^{K'}}(0,\overline{w},0,u)$ and that

$$\Phi_J = \prod_{v \in V(\gamma)} D_v \alpha(x_v)$$

where D_v is a product of differential operators. So

$$c_{J,K,K'}(\overline{w},u) = \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(\overline{w},u).$$

$\tilde{D}_v$ is a product of differential operators. Finally, we see that the integrand in

$$\Psi_{J,K,K'}(\mathbf{t},\alpha) = \int_{\mathbb{H}^n} \psi_{K'}(u,\mathbf{t}) \prod_{v \in V(\gamma)} \tilde{D}_v \alpha(\overline{w},u)\, du d\overline{w} \tag{4.101}$$

is almost a local functional with $\psi_{K'}(u,\mathbf{u})$ as the $\mathbf{t}$-dependent factor. For $t_1 \leq \cdots \leq t_k$, we do have control on the $\mathbf{t}$ dependence

$$|\psi_{K'}(\sqrt{t_k}u,\mathbf{t})| \leq C t_k^{\frac{1}{2}(|K'|+|V(\gamma)|-1)}. \tag{4.102}$$

We will show how to the renormalization procedure can be adapted to this situation in the next section.

Bounding the Error

Note that

Proposition 15.

$$\int_{P_{\sqrt{t_k}u}} \int_{\mathbb{R}^{(n-1)(|V(\gamma)|-1)}} e^{-Q(\overline{y},\mathbf{t})-Q^{(\beta)}(z,\sqrt{t_k}u,\mathbf{t})} |\overline{y}^K||z^{K'}|\, d\overline{y}dz \leq t_k^{\frac{1}{2}(N'+n(|V(\gamma)|-1))} \tag{4.103}$$

where $|K|+|K'| = N'$.

Assume that we have ordered the set of edges so that $t_1 \leq \cdots \leq t_k$ and $t_k^R \leq t_1$ for some $R > 1$. By Taylor's theorem $|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^{N'}(\mathbf{t})[\alpha]|$ is bounded above by a sum of terms of the form

$$t_k^{-RO(\gamma)} t^{-n/2} \int_{\mathbb{R}_u^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_e} d_{K,K',\beta}(\overline{w}, u) |\overline{y}^K| |z^{K'}| \tag{4.104}$$

over multi-indices $|K| + |K'| = N' + 1$ and $\beta : E(\gamma) \to \{-1, 1\}$. Each such term is bounded above by

$$t_k^{-RO(\gamma) - R|E(\gamma)|n/2)} \int_{\mathbb{R}_u^{\geq 0}} \int_{P_u} \int_{\mathbb{R}^{(n-1)|V(\gamma)|}} e^{-\sum_{e \in E(\gamma)} Q_e/4t_k} d_{K,K',\beta}(\overline{w}, u) |\overline{y}^K| |z^{K'}| \tag{4.105}$$

$$\leq t_k^{\frac{1}{2} + \frac{1}{2}(N'+1+n(|V(\gamma)|-1))} t_k^{-RO(\gamma) - R|E(\gamma)|n/2)} \left(\int d_{K,K',\beta}(\overline{w}, \sqrt{t_k} u) C_{K,K'}(u) \, d\overline{w} du \right) \tag{4.106}$$

$$\leq t_k^{\frac{1}{2}(N'+2) + \frac{n}{2}(|V(\gamma)|-1))} t_k^{-RO(\gamma) - R|E(\gamma)|\frac{n}{2}} \left(\int e_{K,K',\beta}(w) C_{K,K'}(u) \, d\overline{w} du \right) \tag{4.107}$$

where

$$C_{K,K'}(u) = e^{-|\beta^{-1}(-1)|u^2} \int_{P_u} |z|^{K'} \, dz \int_{\mathbb{R}^{(n-1)(|V(\gamma)|-1)}} e^{-Q(\overline{y})} \overline{y}^K \, d\overline{y} \tag{4.108}$$

is a Schwartz function in u for $\beta^{-1}(-1) \neq 0$, and

$$e_{K,K',\beta}(w) = \sup_u d_{K,K',\beta}(\overline{w}, u).$$

It remains to understand the integral

$$\int e_{K,K',\beta}(\overline{w}) C_{K,K'}(u) \, d\overline{w} du$$

in terms of the field α and its derivatives. In the formula above,

$$d_{K,K',\beta}(\overline{w}) = \sum_J \sup_{\overline{y},z} \left| \frac{\partial \Phi_J}{\partial \overline{y}^K z^{K'}} \right|$$

so

$$e_{K,K',\beta}(\overline{w}) = \sum_J \sup_{\overline{y},z,u} \left| \frac{\partial \Phi_J}{\partial \overline{y}^K z^{K'}} \right|.$$

But Φ_J is a sum of terms of the form

$$f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}}$$

where f is a compactly-supported function and I^h is a collection of multi-indices, one for each tail h satisfying the condition $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma)$. This implies that $\frac{\partial \Phi_I}{\partial \overline{y}^K z^{K'}}$ is a sum of terms of the same form, but satisfying the condition $\sum_{h \in T(\gamma)} |I^h| \leq O(\gamma) + N' + 1$.

So

$$\begin{aligned} \sup_{\overline{y},z,u} \left| f \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| &\leq \sup_{\overline{y},z,u} |f| \cdot \sup_{\overline{y},\overline{w},z,u} \left| \prod_{h \in T(\gamma)} \frac{\partial \alpha(x_{v(h)})}{\partial x_{v(h)}^{I^h}} \right| \\ &\leq \sup_{\overline{y},z,u} |f| \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \end{aligned}$$

where $p_h = |I^h|$. Note that $\sup_{\overline{y},z,u} |f|$ is a compactly-supported function in the variable $\overline{w}$. Thus,

$$\int e_{K,K',\beta}(w) C_{K,K'}(u)\, d\overline{w} du \leq \sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h}$$

where the summation is over multi-indices $p : T(\gamma) \to \mathbb{Z}^{\geq 0}$ such that $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$.

In conclusion, we have shown that

Theorem 8.

$$|f_{\gamma,I}(\mathbf{t}) - f^{N'}_{\gamma,P,I}(\mathbf{t})| \leq \left(\sum_p C_p \prod_{h \in T(\gamma)} \|\alpha\|_{p_h} \right) t_k^{\frac{1}{2}(N'+2)+(|V(\gamma)|-1)\frac{n}{2} - RO(\gamma) - R|E(\gamma)|\frac{n}{2}} \tag{4.109}$$

where the summation is over multi-indices $p : T(\gamma) \to \mathbb{Z}^{\geq 0}$ *such that* $\sum_{h \in T(\gamma)} p_h \leq O(\gamma) + N' + 1$.

Inductive Construction of the Counterterms

We will only note the differences from Section 4.1.5.

The proof of Theorem 7, which was given only in the case $p = 2$ involves two steps. In the first step, we show that $\left| g_{\gamma',\gamma,I}(\mathbf{t})[\alpha] - g^{N_1'}_{\gamma',\gamma,I}(\mathbf{t})[\alpha] \right|$ is bounded by t_k to a power that grows linearly in N_1'. This implies that

$$\left| f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma/\gamma',I,g^{N'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] \right|$$

is bounded by a power of t_k that grows linearly in N_1', where we have used that directly from the definitions, we have

$$f_{\gamma,I}(\mathbf{t})[\alpha] = f_{\gamma/\gamma',I,g_{\gamma',\gamma,I}}(\mathbf{t})[\alpha].$$

In the second step, we show that

$$\left| f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] \right|$$

is bounded by a power of t_k that grows linearly in N_2' for N_1' fixed. This will require a slight modification from the procedure of Section 4.1.5. To construct

$f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]$, we start with $f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]$, which is formed from the Feynman rules applied to the pointed graph γ/γ', where we place the functional $g^{N_1'}_{\gamma',\gamma,I}$ on the distinguished vertex.

In the case of $\mathbb{H}^n$, $g^{N_1'}_{\gamma',\gamma,I}$ is no longer a sum of functions of $t_1, \ldots, t_{i^{(1)}}$ multiplied by local functionals applied to the inputs on the tails of γ'. Now the local functional depends on $\mathbf{t}$ in the integrand.

When forming $f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]$ from $f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha]$ only take the Taylor expansion of the factors in the integrand of $g^{N_1'}_{\gamma',\gamma,I}$ that do not depend on $\mathbf{t}$. That is, in the integrand of each $\Psi_{J,K,K'}$, neglect the first factor $\psi_{K'}(u,\mathbf{t})$ and only take the Taylor expansion of the second factor. Because $|\psi_{K'}(\sqrt{t_k}u,\mathbf{t})| \leq t_{i^{(1)}}^{\frac{1}{2}(|V(\gamma)|-1)}$, this will contribute factor of $t_k^{\frac{R_1}{2}(|V(\gamma)|-1)}$ to the bound on

$$\left| f^{N_2'}_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f_{\gamma/\gamma',I,g^{N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] \right|$$

so the overall power of t_k will in fact be the same as in the case of $\mathbb{R}^n$.

4.1.7 Counterterms on a Compact Manifold

The asymptotic formula for the scalar heat kernel $K_t(x,y) \sim (4\pi t)^{-n/2} e^{-d(x,y)^2/4t} \sum_i \phi_i(x,y) t^i$ states precisely that there exists some sequence of smooth functions ϕ_i on $M \times M$ supported on a neighborhood of the diagonal such that

$$\|K_t(x,y) - (4\pi t)^{-n/2} e^{-d(x,y)^2/4t} \sum_{i=0}^{N} \phi_i(x,y) t^i \|_l = O(t^{N-n/2-l}), \qquad (4.110)$$

Let

$$K_t^N(x,y) = (4\pi t)^{-n/2} e^{-d(x,y)^2/4t} \sum_{i=0}^{N} \phi_i(x,y) t^i.$$

Beginning from (4.60), for each edge, we replace K_t with K_t^N. We let $f_{\gamma,I}^N$ denote the result of making all $|E(\gamma)|$ of such substitutions. Assume that we've ordered the edges so that $t_1 \leq \cdots \leq t_k$. Then

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^N(\mathbf{t})[\alpha]| \leq \|\alpha\|_{O(\gamma)}^{|T(\gamma)|} \sum_{i=1}^{k} C_i t_i^{N-n/2-p_i} \prod_{j\neq i} t_j^{-p_j-n/2}$$

for some nonnegative integers p_j with $\sum_j p_j \leq O(\gamma)$ and some constants C_i. Thus

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^N(\mathbf{t})[\alpha]| \leq C\|\alpha\|_{O(\gamma)}^{|T(\gamma)|} t_k^{N-O(\gamma)-|E(\gamma)|\frac{n}{2}}.$$

An analogous statement to Proposition 8 can be made for K_t^N giving that

Proposition 16. *For the heat kernel in* (4.61)

$$\frac{\partial K_t^N}{\partial x_i^k} = P_{i,k} e^{-d(x,y)^2/4t} \tag{4.111}$$

where $P_{i,k}$ is a Laurent polynomial in t of degree between $-k$ and N.

Therefore, we have a formula

$$f_{\gamma,I}(\mathbf{t})[\alpha] = \sum_{-O(\gamma)\leq|J|\leq|E(\gamma)|N} t^{J-n/2} \int_{U^{|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} \Phi_J \tag{4.112}$$

where $Q_e = d^2(x_{v_1(e)}, x_{v_2(e)})$.

Spanning Tree Coordinates

Choose a spanning tree T and vertex v_0 of γ. For any vertex v of γ, there is a unique path $e_1^v, \ldots, e_{l(v)}^v$ from v_0 to v. We can then make the inductive definition

Definition 5. *Spanning tree coordinates are defined by $w = x_{v_0}$ and $\{y_e\}_{e \in T}$ where y_e is defined inductively so that $x_{v_2(e)} = \exp_{x_{v_1(e)}(y_{e_1}, \ldots, y_{e_l})}(y_e)$ where $e_1, \ldots, e_l$ is the unique path from v_0 to $v_1(e)$.*

The reason for introducing these coordinates is that for all $e \in T$,

$$d(x_{v_1(e)}, x_{v_2(e)}) = \|y_e\|.$$

More explicitly, the spanning tree is the union of q maximal paths originating at v_0. Let us denote the i-th such path by $e_1^{(i)}, \ldots, e_{l_i}^{(i)}$. Let $V_1^{(i)}$ be a neighborhood of the zero section in TM such that $\Phi_1^{(i)} : V_1^{(i)} \to U_1^{(i)} \subseteq M \times M$ given by $(x_{v_0}, y_{e_1^{(i)}}) \mapsto (x_{v_0}, \exp_{x_{v_0}}(y_{e_1^{(i)}}))$ is a diffeomorphism. Inductively, given $\Phi_{j-1}^{(i)} : V_{j-1}^{(i)} \to U_{j-1}^{(i)} \subseteq M^j$ let $\exp_{j-1}^{(i)} = p_j \circ \Phi_{j-1}^{(i)}$, where p_j is the projection onto the last factor. Now let $V_j^{(i)}$ be a neighborhood of the zero section in $(\exp_{j-1}^{(i)})^* TM$ such that the map $\Phi_j^{(i)} : V_j^{(i)} \to U_j^{(i)} \subseteq M^{j+1}$ given by

$$\left(x_{v_0}, y_{e_1^{(i)}}, \ldots, y_{e_j^{(i)}}\right) \mapsto \left(\Phi_{j-1}^{(i)}(x_{v_0}, y_{e_1^{(i)}}, \ldots, y_{e_{j-1}^{(i)}}), \exp_{\exp_{j-1}^{(i)}\left(x_{v_0}, y_{e_1^{(i)}}, \ldots, y_{e_{j-1}^{(i)}}\right)}(y_{e_j})\right)$$

is a diffeomorphism.

We then take the fiber product over M of the maps $\Phi_{l_i}^{(i)}$ which produces the desired diffeomorphism

$$V_{l_1}^{(1)} \times_M \cdots \times_M V_{l_m}^{(q)} \to M^{|V(\gamma)|}.$$

Taylor Expanding Φ_J and Bounding the Error

Taking the Taylor expansion of Φ with respect to $\{y_e\}_{e\in E(\gamma)}$.

$$f^{N,N'}_{\gamma,I}(\mathbf{t})[\alpha] = \sum_{\substack{|K|\le N' \\ -O(\gamma)\le |J|\le N|E(\gamma)|}} t^{J-n/2}\int_{U^{|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} c_{J,K}(w) y^K \, dydw \tag{4.113}$$

$$= \sum_{\substack{|K|\le N', K \text{ even} \\ -O(\gamma)\le |J|\le N|E(\gamma)|}} t^{J-n/2}\int_{U} c_{J,K}(w)\mathcal{I}^K_A(w,\mathbf{t})\, dw \tag{4.114}$$

$$= \sum_{\substack{|K|\le N', K \text{ even} \\ -O(\gamma)\le |J|\le N|E(\gamma)|}} t^{J-n/2}\Psi_K(\mathbf{t},\alpha)\, dw \tag{4.115}$$

This differs from the case of $\mathbb{R}^n$ where $\mathcal{I}^K_A(w,\mathbf{t})$ does not depend on w. Note that $\int_U c_{J,K}(w)\mathcal{I}^K_A(w,\mathbf{t})\, dw$ is not a local functional due to the factor of $\mathcal{I}^K_A(w,\mathbf{t})$, which depends on $\mathbf{t}$. We will say below why the procedure to construct the counterterms still works.

We have the bound

Proposition 17.

$$\int_{U^{|V(\gamma)|}} e^{-\sum_{e\in E(\gamma)} Q_e/4t_e}|y|^K\, dy \le C t_k^{\frac{1}{2}|K|+\frac{n}{2}(|V(\gamma)|-1)} \tag{4.116}$$

Proof. This follows from the fact that

$$e^{-\sum_{e\in E(\gamma)} Q_e/4t_e} \le e^{-\sum_{e\in E(T)} Q_e/4t_e} = e^{-\sum_{e\in E(T)} \|y_e\|^2/4t_e}$$

□

Using Proposition 17, the bound on

$$|f^N_{\gamma,I}(\mathbf{t})[\alpha] - f^{N,N'}_{\gamma,I}(\mathbf{t})[\alpha]|$$

can be established as in the proof of Theorem 6.

As in the case of $\mathbb{H}^n$, in the proof of Theorem 7 we make the following modification. When forming $f^{N_2,N_2'}_{\gamma/\gamma',I,g^{N_1,N_1'}_{\gamma',\gamma,I}}$ from $f^{N_2}_{\gamma/\gamma',I,g^{N_1,N_1'}_{\gamma',\gamma,I}}$ only take the Taylor expansion of the factors in the integrand of $g^{N_1,N_1'}_{\gamma',\gamma,I}$ that do not depend on $\mathbf{t}$. That is, in the integrand of each term $\Psi_{J,K}$, neglect the factor $\mathcal{I}^K_A(w,\mathbf{t})$ and only take the Taylor expansion of the other factor.

Because $|\mathcal{I}^K_A(w,\mathbf{t})| \leq t_{i^{(1)}}^{\frac{n}{2}(|V(\gamma)|-1)}$, this will contribute factor of $t_k^{\frac{R}{2}(|V(\gamma)|-1)}$ to the bound on

$$\left| f^{N_2,N_2'}_{\gamma/\gamma',I,g^{N_1,N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] - f^{N_2}_{\gamma/\gamma',I,g^{N_1,N_1'}_{\gamma',\gamma,I}}(\mathbf{t})[\alpha] \right|$$

so the overall power of t_k will be the same as in the case of $\mathbb{R}^n$.

4.1.8 Counterterms on a Compact Manifold with Boundary

The renormalization procedure can also be carried out in the case of compact Riemannian manifolds with boundary M, such that there exists a neighborhood W of ∂M that is isometric to a product $\partial M \times [0,\epsilon)$.

When we have such a manifold with boundary M, the double of M which will be denoted by M' will be a smooth compact manifold without boundary equipped with an involution $p \mapsto p^*$ that sends a point p to its reflection through the boundary.

The Dirichlet heat kernel on M is

$$K_t(x,y) = K'_t(x,y) - K'_t(x,y^*) \tag{4.117}$$

where $K'_t(x,y)$ is the heat kernel on M'.

The existence of an asymptotic expansion of $K'_t(x,y)$ implies that

$$K_t(x,y) \sim e^{-d(x,y)^2/4t} \sum_i \phi_i(x,y)t^i + e^{-d(x,y^*)^2/4t} \sum_i \psi_i(x,y)t^i \tag{4.118}$$

where $\psi_i(x,y) = -\phi_i(x,y^*)$.

This can be used to define $f^N_{\gamma,I}$ and to show that $|f_{\gamma,I}(\mathbf{t})[\alpha] - f^N_{\gamma,I}(\mathbf{t})[\alpha]|$ is bounded by a power of t_k the increases linearly with N.

Choose a finite cover $U_1, \ldots, U_m$ of ∂M by coordinate neighborhoods. This induces a finite cover $U_1 \times [0,\epsilon), \ldots, U_m \times [0,\epsilon)$ of $W \cong \partial M \times [0,\epsilon)$ by coordinate neighborhoods.

Then choose a finite cover $V_1, \ldots, V_{m'}$ of the complement of $M \times [0,\epsilon)$.

On the open sets U_i that intersect the boundary, since M is a product, we can use the Pythagorean theorem and the square distance becomes

$$d^2_M(x,y) = d^2_{\partial M}(\overline{x},\overline{y}) + |x_n - y_n|^2 \tag{4.119}$$

Therefore, on these open sets we can apply the analysis of 4.1.6 for the direction normal to the boundary and the analysis of 4.1.7 to the ∂M direction. On open sets V_i whose closures do not intersect the boundary,

$$K_t(x,y) \sim e^{-d(x,y)^2/4t} \sum_i \phi_i(x,y)t^i \tag{4.120}$$

so we can apply the analysis of 4.1.7.

4.1.9 Appendix

Lemma 2. *For the heat kernel in* (4.61)

$$\frac{\partial K_t}{\partial x_i^k} = P_{i,k} K_t \tag{4.121}$$

where $P_{i,k}$ is polynomial in x_i and y_i and $1/t$. The degree of $P_{i,k}$ in $1/t$ is k.

Proof. We would like to find an explicit expression for $P_{i,k}$.

Note that

$$\frac{\partial K_t}{\partial x_i} = \frac{x_i - y_i}{2t} K_t.$$

For each sequence of the form $s_1, \ldots, s_{k'}$ where $s_j \geq 1$ for all j and $s_1 + \cdots + s_{k'} = k$, consider the functions

$$F_{s_1,\ldots,s_{k'}}(t, x_i, y_i) = \begin{cases} \partial_{x_i}^{s_1} \left[\left(\frac{x_i-y_i}{2t}\right)^{s_2} \ldots \partial_{x_i}^{s_{k'-1}} \left[\left(\frac{x_i-y_i}{2t}\right)^{s_{k'}} \right] \ldots \right] & k' \text{ even} \\ \left(\frac{x_i-y_i}{2t}\right)^{s_1} \partial_{x_i}^{s_2} \left[\left(\frac{x_i-y_i}{2t}\right)^{s_3} \ldots \partial_{x_i}^{s_{k'-1}} \left[\left(\frac{x_i-y_i}{2t}\right)^{s_{k'}} \right] \ldots \right] & k' \text{ odd.} \end{cases}$$

We argue by induction that

$$\frac{\partial^k K_t}{\partial x_i^k} = \sum_{\substack{s_1+\cdots+s_{k'}=k \\ s_j \geq 1 \text{ for all } j}} F_{s_1,\ldots,s_{k'}} K_t.$$

For $k = 1$, this is clearly true. Suppose it is true for some $k \geq 1$, then

$$\begin{aligned}\frac{\partial^{k+1} K_t}{\partial x_i^{k+1}} &= \frac{\partial}{\partial x_i} \sum_{\substack{s_1+\cdots+s_{k'}=k \\ s_j \geq 1 \text{ for all } j}} F_{s_1,\ldots,s_{k'}} K_t \\ &= \sum_{\substack{s_1+\cdots+s_{k'}=k \\ s_j \geq 1 \text{ for all } j}} \partial_{x_i} F_{s_1,\ldots,s_{k'}} K_t \\ &+ \sum_{\substack{s_1+\cdots+s_{k'}=k \\ s_j \geq 1 \text{ for all } j}} F_{s_1,\ldots,s_{k'}} \left(\frac{x_i - y_i}{2t}\right) K_t \\ &= \sum_{\substack{s_1+\cdots+s_{k'}=k+1 \\ s_j \geq 1 \text{ for all } j}} F_{s_1,\ldots,s_{k'}} K_t.\end{aligned}$$

In fact, we can be more precise. That is, for k' even

$$F_{s_1,\ldots,s_{k'}} = \frac{s_{k'}!}{(s_{k'} - s_{k'-1})!} \cdots \frac{(s_{k'} - s_{k'-1} + \cdots + s_2)!}{(s_{k'} - s_{k'-1} + \cdots - s_1)!} \frac{((x_i - y_i)/2t)^{s_{k'}+s_{k'-2}+\cdots+s_2}}{(x_i - y_i)^{s_{k'-1}+s_{k'-3}+\ldots s_1}}$$

as long as $s_{k'} - s_{k'-1} + \cdots + s_{2i} - s_{2i-1} \geq 0$ for all $i \geq 1$ such that $2i \leq k'$. Otherwise $F_{s_1,\ldots,s_{k'}} = 0$. If k' is odd then $F_{s_1,\ldots,s_{k'}} = \left(\frac{x_i - y_i}{2t}\right)^{s_1} F_{s_2,\ldots,s_{k'}}$.

The leading term of $P_{i,k}$ in $\frac{1}{t}$ is $F_k = \left(\frac{x_i - y_i}{2t}\right)^k$. $\square$

Theorem 9.

$$\mathcal{I}_A^K(\mathbf{t}) = \frac{1}{P_\gamma^{(|K|+1)/2}} \mathcal{P}_A^K. \tag{4.122}$$

where $\mathcal{P}_A^K$ *is a homogeneous polynomial in* $\mathbf{t}$ *of degree* $R(\gamma, n, K) = C_1(\gamma, n) + |K| C_2(\gamma, n)$ *for constants* $C_1(\gamma, n)$ *and* $C_2(\gamma, n)$ *which are defined in the body of the proof of the theorem.*

Proof. Writing $y^K = y_{m_1} \dots y_{m_{|K|}}$, we have

$$\begin{aligned}
\mathcal{I}_A^K &= \int_{\mathbb{R}^{n(V(\gamma)-1)}} e^{-\langle y, Ay\rangle} y_{m_1} \dots y_{m_{|K|}} \, dy \\
&= \frac{(\sqrt{\pi})^{n(V(\gamma)-1)}}{\sqrt{\det A}} \sum_{\beta} \prod_{i=1}^{|K|/2} (A^{-1})_{m_{\beta_i^{(1)}}, m_{\beta_i^{(2)}}} \\
&= \frac{(\sqrt{\pi})^{n(V(\gamma)-1)}}{P_\gamma^{1/2}} 2^{n(|V(\gamma)|-1)} t^{n(|V(\gamma)|-1)/2} \frac{1}{P_\gamma^{|K|/2}} \sum_{Q} \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}},
\end{aligned}$$

where we have used Proposition 12 and Wick's Theorem on $\mathbb{R}^n$. Let

$$\mathcal{P}_A^K = (\sqrt{\pi})^{n(V(\gamma)-1)} 2^{n(|V(\gamma)|-1)} t^{n(|V(\gamma)|-1)/2} \sum_{Q} \prod_{i=1}^{|K|/2} C_{Q_i^{(1)}, Q_i^{(2)}}. \tag{4.123}$$

Recall the definition of C which is $\left(4 \prod_{e \in E(\gamma)} t_e\right) \mathrm{adj}(B)$. But B is $n(|V(\gamma)|-1)$ by $n(|V(\gamma)|-1)$ and its entries are homogeneous of degree $|E(\gamma)|-1$ in $\{t_e\}_{e \in E(\gamma)}$. So $\mathrm{adj}(B)$ has entries of degree $(|E(\gamma)|-1)[(n|V(\gamma)|-1)-1])$. Therefore C has entries of degree

$$(|E(\gamma)|-1)[n(|V(\gamma)|-1)-1] + |E(\gamma)| = (|E(\gamma)|-1)n(|V(\gamma)|-1)+1 \tag{4.124}$$

This implies that $\mathcal{P}_A^K$ is of homogeneous degree

$$R_\gamma(n,K) = n|E(\gamma)|(|V(\gamma)|-1)/2 + \frac{|K|}{2}[(|E(\gamma)|-1)n(|V(\gamma)-1)+1] \tag{4.125}$$

With the definition of $\mathcal{P}_A^K$ in hand, the theorem is now evident.

□

4.2 Construction of an Effective Field Theory from a Local Functional

In this section, we show that an effective action can be constructed from a local functional $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$ using a procedure that is based on Theorem 7.

Do the following for each sequence $I^{(1)}, \ldots, I^{(p)}$ as in Corollary 3: Let N_1' be the smallest nonnegative integer such that $d_1(N_1') \geq 0$. Let N_2' be the smallest nonnegative integer such that $d_2(N_1', N_2') \geq 0$ and so on. Then by Theorem 7,

$$|f_{\gamma,I}(\mathbf{t})[\alpha] - f_{\gamma,I}^{N_1',\ldots,N_p'}(\mathbf{t})[\alpha]| \leq C \tag{4.126}$$

for some constant C. Let $\overline{E}_R^{I^{(1)},\ldots,I^{(p)}} = \overline{E}_R^{I^{(1)}} \cap \overline{E}_R^{I^{(2)}} \cdots \cap \overline{E}_R^{I^{(p)}}$,

Let

$$w_\gamma^{\mathrm{CT}}(P_\epsilon^1, I)[\alpha] = \sum_{p=1}^{k} \sum_{I^{(1)},\ldots,I^{(p)}} \int_{\overline{E}_R^{I^{(1)},\ldots,I^{(p)}}} f_{\gamma,I}^{N_1',\ldots,N_p'}(\mathbf{t})[\alpha]\, d\mathbf{t} \tag{4.127}$$

We can integrate this formula on $(\epsilon, 1)^k \cap \overline{E}_R^{I^{(1)},\ldots,I^{(p)}}$. This gives

$$|w_\gamma(P_\epsilon^1, I)[\alpha] - w_\gamma^{\mathrm{CT}}(P_\epsilon^1, I)[\alpha]| \leq C(1 - \epsilon^k). \tag{4.128}$$

In particular, by Lebesgue's dominated convergence theorem, we can let $\epsilon \to 0^+$.

Thus, the limit as $\epsilon \to 0^+$ of $w_\gamma(P_\epsilon^L, I)[\alpha] - w_\gamma^{\mathrm{CT}}(P_\epsilon^1, I)[\alpha]$ exists as well. We shall call this the renormalized Feynman weight.

The counterterms for the effective action are defined by

$$I_{i,k}^{\mathrm{CT}}(\epsilon) = W_{i,k}^{\mathrm{CT}}\left(P_\epsilon^1, I - \sum_{(i',k') \prec (i,k)} I_{i',k'}^{\mathrm{CT}}(\epsilon)\right), \tag{4.129}$$

where

$$W^{\mathrm{CT}}_{i,k}(P^1_\epsilon, I) = \sum_{\substack{\gamma \text{ conn} \\ g(\gamma)=i, T(\gamma)=k}} \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{g(\gamma)} w^{\mathrm{CT}}_\gamma \left(P^1_\epsilon, I - \sum_{(i',k') \prec (i,k)} I^{\mathrm{CT}}_{i',k'}(\epsilon) \right) \quad (4.130)$$

.

Then the effective action is defined by

$$I[L] = \lim_{\epsilon \to 0^+} W(P^L_\epsilon, I - I^{\mathrm{CT}}(\epsilon)). \quad (4.131)$$

This is well-defined because for all i, k,

$$\begin{aligned} I_{i,k}[L] &= \lim_{\epsilon \to 0^+} W_{i,k}(P^L_\epsilon, I - I^{\mathrm{CT}}(\epsilon)) & (4.132) \\ &= \lim_{\epsilon \to 0^+} \left[W_{i,k} \left(P^L_\epsilon, I - \sum_{(i',k') \prec (i,k)} I^{\mathrm{CT}}_{i',k'}(\epsilon) \right) - I^{\mathrm{CT}}_{(i,k)}(\epsilon) \right]. & (4.133) \end{aligned}$$

Chapter 5

Gauge Theory

5.1 Classical BV Theory

5.1.1 On Manifolds without Boundary

Graded Lie Algebras

Instead of a graded Lie algebra $\mathfrak{g}$ consider its shift $\mathfrak{g}[1]$. Then the bracket $[\cdot,\cdot]$ becomes a symmetric bilinear map of degree 1 that we denote $\ell_2(\cdot,\cdot)$. Let s be the suspension map $\mathfrak{g}[1] \to \mathfrak{g}$. Then for $X, Y \in \mathfrak{g}[1]$ the product ℓ_2 is defined by

$$\ell_2(X,Y) = (s^{-1} \circ [\cdot,\cdot] \circ s \otimes s)(X,Y) \tag{5.1}$$

$$= (-1)^{|X|} s^{-1}[sX, sY]. \tag{5.2}$$

One can readily see that

$$\ell_2(X,Y) = -(-1)^{|X|+(|X|+1)(|Y|+1)} s^{-1}[sY, sX] = (-1)^{|X||Y|}\ell_2(Y,X) \tag{5.3}$$

On $\mathfrak{g}[1]$ the Jacobi identity becomes

$$\ell_2(\ell_2(X,Y),Z) + (-1)^{|Z||Y|}\ell_2(\ell_2(X,Z),Y) + (-1)^{|X|(|Y|+|Z|)}\ell_2(\ell_2(Y,Z),X) \quad (5.4)$$

In particular if X has degree 0, then

$$\ell_2(\ell_2(X,X),Y) = -2\ell_2(\ell_2(X,Y),X) \tag{5.5}$$

and

$$\ell_2(\ell_2(X,X),X) = 0 \tag{5.6}$$

A symmetric bilinear pairing κ of degree $d-2$ on $\mathfrak{g}$ becomes a degree d symplectic pairing $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}[1]$.

Graded Lie Algebra Modules

Suppose that M is a graded module for the graded Lie algebra $\mathfrak{g}$. That means that there is an action of $\mathfrak{g}$ on M such that

$$X\cdot Y\cdot m - (-1)^{|X||Y|}Y\cdot X\cdot m = [X,Y]\cdot m. \tag{5.7}$$

The prototypical example arises in BF theory, where $\mathfrak{g}$ acts by the coadjoint representation on $\mathfrak{g}^*[d-3]$, for some integer d. One can combine $\mathfrak{g}$ and $\mathfrak{g}^*[d-3]$ into a single Lie algebra $\mathfrak{g}\oplus\mathfrak{g}^*[d-3]$ by defining $[X,m] = X\cdot m$ and $[m,X] = -(-1)^{|X||m|}[X,m]$ and extending the bracket by zero on $\mathfrak{g}^*[d-3]$.

For any module M one can constuct as above the Lie algebra $\mathfrak{g}\oplus M$ which we call the crossed product Lie algebra. The Jacobi identity of $\mathfrak{g}\oplus M$ is equivalent to the statement that $\mathfrak{g}$ is a Lie algebra and M is a module for $\mathfrak{g}$.

In what follows, it is natural to shift and consider instead $\mathfrak{g}[1] \oplus M[1]$. Once again, what the Lie bracket $[\cdot,\cdot]$ corresponds to will be denoted $\ell_2(\cdot,\cdot)$.

Maurer Cartan Elements

If $A \in \mathfrak{g}[1]$ is an element of degree 0, then one can deform a differential d to $d_A = d + \ell_2(A,\cdot)$. We compute

$$\begin{aligned}(d+\ell_2(A,\cdot))^2 B &= d(dB + \ell_2(A,B)) + \ell_2(A,dB) + \ell_2(A,\ell_2(A,B)) && (5.8)\\ &= -\ell_2(dA,B) - \ell_2(A,dB) + \ell_2(A,dB) - \frac{1}{2}\ell_2(\ell_2(A,A),B) && (5.9)\\ &= -\ell_2\left(dA + \frac{1}{2}\ell_2(A,A), B\right) && (5.10)\end{aligned}$$

Therefore $d_A^2 = 0$ if A is a Maurer-Cartan element. Another standard computation establishes the Bianchi identity

$$\begin{aligned}d_A\left(dA + \frac{1}{2}\ell(A,A)\right) &= d\left(dA + \frac{1}{2}\ell_2(A,A)\right) + \ell_2\left(A, dA + \frac{1}{2}\ell_2(A,A)\right) && (5.11)\\ &= -\frac{1}{2}\ell_2(dA,A) - \frac{1}{2}\ell_2(A,dA) + \ell_2(A,dA) = 0. && (5.12)\end{aligned}$$

In what follows, we will see the Maurer-Cartan equation, but not necessarily for a homogeneous element of degree 0. Suppose that $A = A^{(m)} + \cdots + A^{(n)}$, where $m \leq n$ and $|A_i| = i$. Then

$$dA + \frac{1}{2}\ell_2(A,A) = 0 \tag{5.13}$$

is equivalent to

$$dA^{(k)} + \sum_{i+j=k} \frac{1}{2}\ell_2(A^{(i)}, A^{(j)}) = 0 \tag{5.14}$$

for all k.

Vector Fields

We shall only assume that we have a finite dimensional vector space or nice infinite dimensional graded vector space like a nuclear Fréchet space $\mathcal{E}$ in which case we would take the appropriate dual and tensor product. Given a polynomial function F, we write $D_n F \in \operatorname{Sym}^n \mathcal{E}^*$ for its components.

We identify $\mathcal{E}$ with its tangent space; in other words, a constant vector field is an element $X \in \mathcal{E}$. More generally, a vector field X is defined to be an element of $\operatorname{Hom}(\operatorname{Sym}^\bullet \mathcal{E}, \mathcal{E})$. We write $D_n X \in \operatorname{Hom}(\operatorname{Sym}^n \mathcal{E}, \mathcal{E})$ for its components. A constant vector field X acts on a polynomial function $F = D_n F$ by

$$(XF)(A_1, \ldots, A_{n-1}) = \sum_{i=1}^{n} \pm F(A_1, \ldots, X, \ldots, A_{n-1}) \tag{5.15}$$

$$= nF(X, A_1, \ldots, A_{n-1}). \tag{5.16}$$

and $XF = D_{n-1}(XF)$.

More generally, suppose $X = D_m X$ for $m \geq 0$; i.e. X is homogeneous and is not necessarily constant. Intuitively, we should let X act like a derivation and then symmetrize. We have $XF(A_1, \ldots, A_{m+n-1})$ is equal to

$$C'_{m,n} \sum_{\sigma \in S_{m+n-1}} \pm nf(X(A_{\sigma(1)}, \ldots, A_{\sigma(m)}), A_{\sigma(m+1)}, \ldots, A_{\sigma(m+n-1)}) \tag{5.17}$$

$$= C_{m,n} \sum_{\sigma \in \operatorname{Sh}(m,n-1)} \pm f(X(A_{\sigma(1)}, \ldots, A_{\sigma(m)}), A_{\sigma(m+1)}, \ldots, A_{\sigma(m+n-1)}) \tag{5.18}$$

where $C'_{m,n} = \frac{1}{(m+n-1)!}$ and $C_{m,n} = \frac{m!n!}{(m+n-1)!}$. Here $\operatorname{Sh}(m, n-1)$ denotes the set of

$(m, n-1)$ shuffles, i.e. the permutions $\sigma \in S_{m+n-1}$ satisfying the conditions

$$\sigma(1) < \cdots < \sigma(m) \tag{5.19}$$

$$\sigma(m+1) < \cdots < \sigma(m+n-1). \tag{5.20}$$

Generalized Chern-Simons Theory

Assume the existence of a degree 1 symmetric product $\ell_2(\cdot,\cdot)$ satisfying (5.4) and a differential Q compatible with $\ell_2(\cdot,\cdot)$. Also assume the existence of a degree -1 symplectic pairing on $\mathcal{E}$, which we denote $\langle\cdot,\cdot\rangle$, that is compatible with Q and ℓ_2 in the sense that

$$\langle Q(\cdot),\cdot\rangle \in \mathrm{Sym}^2\,\mathcal{E}^* \tag{5.21}$$

$$\langle \ell_2(\cdot,\cdot),\cdot\rangle \in \mathrm{Sym}^3\,\mathcal{E}^* \tag{5.22}$$

Let $I(\cdot,\cdot,\cdot) := \frac{1}{6}\langle \ell_2(\cdot,\cdot),\cdot\rangle$ and let

$$K(\cdot,\cdot) = \frac{1}{2}\langle Q(\cdot),\cdot\rangle \tag{5.23}$$

Then K and I will also be symmetric polynomials of degree 0 as a consequence of the conditions (5.21) and (5.23). This means that $|A_1| + |A_2| = 0$ is necessary to have $K(A_1, A_2) \neq 0$. This implies that

$$K(A_1, A_2) = (-1)^{|A_1||A_2|} K(A_2, A_1) = (-1)^{|A_1|} K(A_2, A_1). \tag{5.24}$$

Similarly, $|A_1|+|A_2|+|A_3| = 0$ is necessary to have $I(A_1, A_2, A_3) \neq 0$, which implies that $I(A_1, A_2, A_3) = (-1)^{|A_1|} I(A_2, A_3, A_1)$.

Note that the condition

$$\langle QA_1, A_2\rangle + (-1)^{|A_1|}\langle A_1, QA_2\rangle = 0 \tag{5.25}$$

for all $A_1, A_2 \in \mathcal{E}$ is equivalent to (5.21)

Define the generalized Chern-Simons action

$$S(A) = K(A) + I(A). \tag{5.26}$$

Then

$$XS(A) = 2(-1)^{|X|}\cdot K(A,X) + 3(-1)^{|X|}\cdot I(A,A,X) \tag{5.27}$$

$$= (-1)^{|X|}\langle QA + \frac{1}{2}\ell_2(A,A), X\rangle \tag{5.28}$$

Let X_S be the vector field defined by $D_1X_S(A_1) = QA_1$ and $D_2X_S(A_1,A_2) = \frac{1}{2}\ell_2(A_1,A_2)$. The Chern-Simons equation of motion $X_S(A) = 0$ is the familiar Maurer-Cartan equation for A. By definition in our convention, this is the Hamiltonian vector field of S because $XS = (-1)^{|X|}\langle X_S, X\rangle$. It is sometimes also called the BRST operator.

Generalized BF Theory

We now define the generalized BF action. Assume that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and there is a symmetric product ℓ_2 on $\mathcal{E}$ with $\ell(A,B) \in \mathcal{E}_2$ for all $A \in \mathcal{E}_1$ and $B \in \mathcal{E}_2$ and $\ell_2(B_1,B_1) = 0$ for all $B_1, B_2 \in \mathcal{E}_2$. Suppose that $\langle\cdot,\cdot\rangle$ is a degree -1 symplectic pairing the such that for all $A_1, A_2 \in \mathcal{E}_1$ and $B_1, B_2 \in \mathcal{E}_2$, we have $\langle A_1, A_2\rangle = \langle B_1, B_2\rangle = 0$. Let Q be a compatible differential on $\mathcal{E}$.

The BF action is just the Chern-Simons action for this theory.

$$S(A+B) = K(A+B) + I(A+B) \tag{5.29}$$

$$= 2K(B,A) + 3I(B,A,A) \tag{5.30}$$

Using the calculation from the previous section, we have that

$$(X+Y)S(A+B) = (-1)^{|X|}\langle Q(A+B) + \frac{1}{2}\ell_2(A+B,A+B), X\rangle \tag{5.31}$$

$$+ (-1)^{|Y|}\langle Q(A+B) + \frac{1}{2}\ell_2(A+B,A+B), Y\rangle \tag{5.32}$$

$$= (-1)^{|X|}\langle QB + \ell_2(A,B), X\rangle + (-1)^{|Y|}\langle QA + \frac{1}{2}\ell_2(A,A), Y\rangle \tag{5.33}$$

Therefore the Hamiltonian vector field $X_S + Y_S$ has $\mathcal{E}_1$ component X_S with $D_1 X_S(A) = QA$ and $D_2 X_S(A) = \frac{1}{2}\ell_2(A,A)$ and it has $\mathcal{E}_2$ component Y_S with $D_1 Y_S(A+B) = QB$ and $D_2 Y_S(A+B) = \ell_2(A,B)$. The familiar (perhaps) BF equations of motions are given by $X_S(A) = 0$ and $Y_S(A+B) = 0$.

Classical Master Equation

The purpose of this section is to define and verify the classical master equation for generalized Chern-Simons theory and a fortiori generalized BF theory. For some other polynomial function F we can define the Poisson bracket with the generalized Chern Simons action S as

$$\{S, F\} := X_S F. \tag{5.34}$$

We would like to show that $\{S,S\}=0$. This is called the classical master equation. We can prove this one of two ways: by showing that $X_S S=0$ or showing that $\langle X_S, X_S\rangle = 0$.

Using the definition of the action of a vector field on a polynomial function $F = D_2F + D_3F$, we calculate the four terms of $X_S F$.

$$D_1X_S(D_2F)(A_1,A_2) = D_2F(Q(A_1),A_2) + (-1)^{|A_1||A_2|}D_3F(Q(A_2),A_1) \tag{5.35}$$

and

$$D_1X_S(D_3F)(A_1,A_2,A_3) = D_2F(Q(A_1),A_2,A_3) + (-1)^{|A_1||A_2|}D_2F(Q(A_2),A_1,A_3) \tag{5.36}$$

$$+ (-1)^{(|A_1|+|A_2|)|A_3|}D_2F(Q(A_3),A_1,A_2) \tag{5.37}$$

and

$$D_2X_S(D_2F)(A_1,A_2,A_3) = \frac{C_{2,2}}{2}\left[D_2F(\ell_2(A_1,A_2),A_3)\right. \tag{5.38}$$

$$+ (-1)^{|A_2||A_3|}D_2F(\ell_2(A_1,A_3),A_2) \tag{5.39}$$

$$\left.+(-1)^{|A|_1(|A_2|+|A_3|)}D_2F(\ell_2(A_2,A_3),A_1)\right] \tag{5.40}$$

$$\tag{5.41}$$

and lastly, $D_2X_S(D_3F)(A_1,A_2,A_3,A_4)$ will be a sum with summands given by the

$$\pm\frac{C_{2,3}}{2}D_3F(\ell_2(A_{\sigma(1)},A_{\sigma(2)}),A_{\sigma(3)},A_{\sigma(4)}) \tag{5.42}$$

for each of the $\binom{4}{2}=6$ shuffles σ in $\mathrm{Sh}(2,2)$.

Finally we turn to the case $F = S$ the generalized Chern-Simons action where we claim that each of these four terms vanish. Notice that the vanishing of (5.35) is equivalent to our assumption (5.25). The vanishing of (5.36) and (5.38) are each equivalent to our assumption that the differential Q compatible with $\ell_2(\cdot,\cdot)$ and (5.25). Lastly, the vanishing of $D_2X_S(D_3S) = D_2X_S(I)$ is is a consequence of our assumptions (5.22) and (5.4). Therefore, the classical master equation holds.

Chern-Simons and BF on a Closed Manifold

Let M be a closed 3-manifold and let $(\mathfrak{g}, [\cdot,\cdot], \kappa)$ be a quadratic Lie algebra. Define $\mathcal{E}^\bullet = \Omega^\bullet(M,\mathfrak{g})[1]$. We denote the Lie bracket after shifting by $\ell_2'(\cdot,\cdot)$. We define

$$\ell_2(\omega_1 \otimes X_1, \omega_2 \otimes X_2) = (-1)^{|X_1||\omega_2|}(\omega_1 \wedge \omega_2) \otimes \ell_2'(X_1, X_2). \tag{5.43}$$

for $\omega_i \otimes X_i \in \mathcal{E}^\bullet$. A short calculation shows that

$$\ell_2(\omega_1 \otimes X_1, \omega_2 \otimes X_2) = (-1)^{(|\omega_1|+|X_1|)(|X_2|+|\omega_2|)}\ell_2(\omega_2 \otimes X_2, \omega_1 \otimes X_1), \tag{5.44}$$

verifying that $\ell_2(\cdot,\cdot)$ is a symmetric product on $\mathcal{E}^\bullet$. Let $\langle\cdot,\cdot\rangle'$ be the degree 2 symplectic pairing induced by shifting the compatible symmetric bilinear form κ on $\mathfrak{g}$. We define

$$\langle \omega_1 \otimes X_1, \omega_2 \otimes X_2\rangle = (-1)^{|X_1||\omega_2|}\int_M \omega_1 \wedge \omega_2 \, \langle X_1, X_2\rangle' \tag{5.45}$$

Then $\langle\cdot,\cdot\rangle$ is a degree -1 symplectic pairing on $\mathcal{E}^\bullet$.

Let $Q = d \otimes 1$, where d is the de Rham differential. Then

$$\begin{aligned}
\langle Q(\omega_1 \otimes X_1), \omega_2 \otimes X_2 \rangle &= (-1)^{|X_1||\omega_2|} \int_M d\omega_1 \wedge \omega_2 \, \langle X_1, X_2 \rangle' && (5.46)\\
&= -(-1)^{|X_1||\omega_2|}(-1)^{|\omega_1|} \int_M \omega_1 \wedge d\omega_2 \, \langle X_1, X_2 \rangle' && (5.47)\\
&= -(-1)^{|\omega_1|+|X_1|} \langle \omega_1 \otimes X_1, Q(\omega_2 \otimes X_2) \rangle && (5.48)
\end{aligned}$$

which is equivalent to the statement that $\langle Q(\cdot), \cdot \rangle$ is a symmetric polynomial on $\mathcal{E}^\bullet$.

Secondly the compatibility of $[\cdot, \cdot]$ and κ is the statement that for all $X, Y, Z \in \mathfrak{g}$,

$$\kappa(X, [Y, Z]) = -(-1)^{|X||Y|} \kappa(Y, [X, Z]) \tag{5.49}$$

which on $\mathfrak{g}[1]$ implies that

$$\langle \ell_2'(\cdot, \cdot), \cdot \rangle' \in \mathrm{Sym}^3(\mathfrak{g}^*[-1]). \tag{5.50}$$

On $\mathcal{E}^\bullet$, we have the desired consequence

$$\langle \ell_2(\cdot, \cdot), \cdot \rangle \in \mathrm{Sym}^3(\mathcal{E}^\bullet)^* \tag{5.51}$$

Lastly, $\ell_2'(\cdot, \cdot)$ satisfies the shift of the Jacobi identity, which implies that ℓ_2 satisfies it too; that is,

$$\ell_2(\ell_2(X, Y), Z) + (-1)^{|Z||Y|} \ell_2(\ell_2(X, Z), Y) + (-1)^{|X|(|Y|+|Z|)} \ell_2(\ell_2(Y, Z), X) = 0. \tag{5.52}$$

In the case of BF theory, let M be a closed manifold of any dimension n. Then we consider the crossed product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*[n-3]$ we can symmetrize and extend the duality pairing by 0 to construct a symmetric bilinear pairing κ of degree

$n-3$. Then κ, by construction is compatible with $[\cdot,\cdot]$. On $\mathfrak{g}[1]\oplus\mathfrak{g}^*[n-2]$ the shift of the crossed product algebra κ becomes $\langle\cdot,\cdot\rangle'$ a symplectic form of degree $n-1$, as desired.

Let

$$\mathcal{E}^\bullet = \Omega^\bullet(M)\otimes(\mathfrak{g}[1]\oplus\mathfrak{g}^*[n-2]) \cong \Omega^\bullet(M,\mathfrak{g})[1]\oplus\Omega^\bullet(M,\mathfrak{g}^*)[n-2]. \tag{5.53}$$

The symplectic pairing $\langle\cdot,\cdot\rangle$ on $\mathcal{E}^\bullet$ coming from the integration pairing on M and $\langle\cdot,\cdot\rangle'$ is of degree -1 and satisfies the desired properties.

5.1.2 On Manifolds with Boundary

Generalized Abelian Chern-Simons

On a manifold with boundary M, integration by parts introduces an integration over ∂M. There is also a natural map $\pi:\mathcal{E}_M\to\mathcal{E}_{\partial M}$ of degree 0 restricting fields $\mathcal{E}_M$ on M to fields $\mathcal{E}_{\partial M}$ on ∂M . We shall try to account for this structure algebraically in our generalized setting.

As before all the linear objects should be taken in an appropriate category like the category of finite dimensional vector spaces or the category of nuclear Fréchet spaces. Suppose that $\mathcal{E}_M$ and $\mathcal{E}_{\partial M}$ are two such objects and there is map between them $\pi:\mathcal{E}_M\to\mathcal{E}_{\partial M}$. Let $\mathcal{E}_M$ have a degree -1 symplectic pairing $\langle\cdot,\cdot\rangle_M$ and let $\mathcal{E}_{\partial M}$ have a degree 0 pairing $\langle\cdot,\cdot\rangle_{\partial M}$. Let Q be a differential on $\mathcal{E}_M$ that satisfies the property

$$\langle QA_1, A_2\rangle_M + (-1)^{|A_1|}\langle A_1, QA_2\rangle_M = \langle \pi A_1, \pi A_2\rangle_{\partial M}. \tag{5.54}$$

We shall also assume the existence of a differential on $\mathcal{E}_{\partial M}$ which we will also denote by Q, and use the context to distinguish between the two operators. For $a_1, a_2 \in \mathcal{E}_{\partial M}$ we shall in fact assume that Q satisfies

$$\langle Qa_1, a_2\rangle_{\partial M} + (-1)^{|a_1|}\langle a_1, Qa_2\rangle_{\partial M} = 0 \tag{5.55}$$

Define the abelian generalized Chern-Simons action on $\mathcal{E}_M$ by

$$S_M(A) = K_M(A) \tag{5.56}$$

where

$$K_M = \frac{1}{2}\langle Q(\cdot), \cdot\rangle \tag{5.57}$$

for $A \in \mathcal{E}_M$.

Then (5.54) means that unfortunately, $K_M(\cdot,\cdot)$ is no longer symmetric, and its failure to be symmetric is measured by $\pi^*\langle\cdot,\cdot\rangle_{\partial M}$.

We can still act on it with a constant vector field $X \in \mathcal{E}_M$ to get

$$XK_M(A) = \frac{1}{2}[\langle QX, A\rangle_M + (-1)^{|X|}\langle QA, X\rangle_M] \tag{5.58}$$

$$= (-1)^{|X|}\langle QA, X\rangle_M + \langle \pi X, \pi A\rangle_{\partial M} \tag{5.59}$$

$$= (-1)^{|X|}\langle X_{S_M}, X\rangle_M + \langle \pi X, \pi A\rangle_{\partial M} \tag{5.60}$$

We compute

$$\langle X_{S_M}, X_{S_M}\rangle_M(A) = -X_{S_M}D_2S_M(A) + \langle \pi X_{S_M}(A), \pi A\rangle \tag{5.61}$$

$$= 2\,\langle Q(\pi A), \pi A\rangle_{\partial M} \tag{5.62}$$

$$= \pi^* S_{\partial M}(A) \tag{5.63}$$

where we define the boundary correction as $S_{\partial M}(a) = 2\langle Qa, a\rangle_{\partial M}$ for $a \in \mathcal{E}_{\partial M}$ which is a quadratic polynomial of degree 1.

Following [1], for the purpose of quantum field theory we shall want to assume the existence of additional structure. Assume the existence of a polarization on the space of boundary fields. In this generalized setting, that simply means that we can choose subspaces $\mathcal{B}_{\partial M}$ and $\mathcal{P}_{\partial M}$ such that

$$\mathcal{E}_{\partial M} = \mathcal{B}_{\partial M} \oplus \mathcal{P}_{\partial M}. \tag{5.64}$$

Also assume the existence $\mathcal{Y}_M$, a subspace of $\mathcal{E}_M$ such that $\pi(\mathcal{Y}_M) \subseteq \mathcal{P}_{\partial M}$ and that

$$\langle QA_1, A_2\rangle_M + (-1)^{|A_1|}\langle A_1, QA_2\rangle_M = 0. \tag{5.65}$$

In what follows, $\mathcal{E}_M$ will be replaced with a new space of fields $\mathcal{Y}_M \oplus \mathcal{B}_{\partial M}$. We shall try to replace S_M by defining a new action functional on $\mathcal{Y}_M \oplus \mathcal{B}_{\partial M}$. To start, we naively substitute $A + a \in \mathcal{Y}_M \oplus \mathcal{B}_{\partial M}$, assume that $\langle a, \cdot\rangle_M = \langle \cdot, a\rangle = 0$ and try to formally integrate by parts

$$K(A+a) = \frac{1}{2}\langle Q(A+a), A\rangle_M + \frac{1}{2}\langle Q(A+a), a\rangle_M \tag{5.66}$$
$$= \frac{1}{2}\langle QA, A\rangle_M + \frac{1}{2}\langle Qa, A\rangle_M \tag{5.67}$$
$$= \frac{1}{2}\langle QA, A\rangle_M + \frac{1}{2}\langle a, \pi A\rangle_{\partial M} \tag{5.68}$$

From now on K as a polynomial function on $\mathcal{Y}_M \oplus \mathcal{B}_{\partial M}$ will be defined by the expression on the last line.

For $X \in \mathcal{Y}_M$ a constant vector field, we compute

$$XK(A+a) = (-1)^{|X|}\langle X_S(A), X\rangle_M + \frac{1}{2}\langle \pi a, \pi X\rangle_{\partial M} \tag{5.69}$$

where $X_S(A) = QA$ and thus

$$X_S K(A+a) = (-1)^{|X|}\langle X_S(A), X_S(A)\rangle_M + \frac{1}{2}\langle a, \pi X_S(A)\rangle_{\partial M} \tag{5.70}$$
$$= \frac{1}{2}\langle a, \pi QA\rangle_{\partial M} \tag{5.71}$$
$$\tag{5.72}$$

Generalized abelian BF Theory

We now define the generalized abelian BF action. Assume that $\mathcal{E}_M = \mathcal{E}_{M,1} \oplus \mathcal{E}_{M,2}$. Suppose that $\langle\cdot,\cdot\rangle_M$ is a degree -1 symplectic pairing the such that for all $A_1, A_2 \in \mathcal{E}_1$ and $B_1, B_2 \in \mathcal{E}_2$, we have $\langle A_1, A_2\rangle_M = \langle B_1, B_2\rangle_M = 0$. Suppose that $\mathcal{E}_{\partial M} = \mathcal{E}_{\partial M,1} \oplus \mathcal{E}_{\partial M,2}$ and has a pairing $\langle\cdot,\cdot\rangle_{\partial M}$ with the analogous properties as $\langle\cdot,\cdot\rangle_M$. Suppose we have a map $\pi : \mathcal{E}_M \to \mathcal{E}_{\partial M}$. Let Q be a differential on $\mathcal{E}_M$

that satifies (5.54). And denote by the same letter Q the differential on $\mathcal{E}_{\partial M}$ that satisfies (5.55).

For the purposes of quantum field theory, assume that we can choose subspaces $\mathcal{B}_{\partial M,i}$ and $\mathcal{P}_{\partial M,i}$ such that

$$\mathcal{E}_{\partial M,i} = \mathcal{B}_{\partial M,i} \oplus \mathcal{P}_{\partial M,i}. \tag{5.73}$$

Also assume the existence $\mathcal{Y}_{M,i}$, a subspace of $\mathcal{E}_{M,i}$ such that $\pi(\mathcal{Y}_{M,i}) \subseteq \mathcal{P}_{\partial M,i}$ and that on $\mathcal{Y}_M = \mathcal{Y}_{M,1} \oplus \mathcal{Y}_{M,2}$, we have (5.65). Then for $A + a \in \mathcal{Y}_{M,1} \oplus \mathcal{B}_{\partial M,1}$ and $B + b \in \mathcal{Y}_{M,2} \oplus \mathcal{B}_{\partial M,2}$, we have following

$$K(A + B + a + b) = \frac{1}{2}\langle QA, B\rangle_M + \frac{1}{2}\langle QB, A\rangle_M + \frac{1}{2}\langle a, \pi B\rangle_{\partial M} + \frac{1}{2}\langle b, \pi A\rangle_{\partial M} \tag{5.74}$$

$$= \langle QA, B\rangle_M + \frac{1}{2}\langle \pi B, \pi A\rangle_{\partial M} + \frac{1}{2}\langle a, \pi B\rangle_{\partial M} + \frac{1}{2}\langle b, \pi A\rangle_{\partial M} \tag{5.75}$$

BF Theory on a Compact Manifold with Boundary

Following [1], on a compact manifold with boundary M, choose a decomposition of the boundary as $\partial M = \partial_1 M \oplus \partial_2 M$. Let $\Omega^\bullet(M)_{Di}$ be the space of differential forms on M with Dirichlet boundary conditions on $\partial_i M$. Then define

$$\mathcal{E}_{M,1} = \Omega^\bullet(M, \mathfrak{g})[1] \tag{5.76}$$

$$\mathcal{E}_{M,2} = \Omega^\bullet(M, \mathfrak{g}^*)[n-2] \tag{5.77}$$

and

$$\mathcal{E}_{\partial M,1} = \Omega^\bullet(\partial M, \mathfrak{g})[1] \tag{5.78}$$

$$\mathcal{E}_{\partial M,2} = \Omega^\bullet(\partial M, \mathfrak{g}^*)[n-2]. \tag{5.79}$$

Let $\pi : \mathcal{E}_M \to \mathcal{E}_{\partial M}$ be the map induced by pullback of forms to the boundary. and define

$$\mathcal{B}_{\partial M,1} = \Omega^\bullet(\partial_1 M, \mathfrak{g})[1] \tag{5.80}$$

$$\mathcal{B}_{\partial M,2} = \Omega^\bullet(\partial_2 M, \mathfrak{g}^*)[n-2] \tag{5.81}$$

and

$$\mathcal{Y}_{M,1} = \Omega^\bullet(M, \mathfrak{g})_{D1}[1] \tag{5.82}$$

$$\mathcal{Y}_{M,2} = \Omega^\bullet(M, \mathfrak{g}^*)_{D2}[n-2]. \tag{5.83}$$

Define $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_{\partial M}$ to be the integration pairings on M and ∂M respectively. Because for $\omega_i \in \Omega^\bullet(M)_{Di}$, we have $\omega_1 \wedge \omega_2$ always vanishes on the boundary, (5.65) is satisfied on $\mathcal{Y}_M = \mathcal{Y}_{M,1} \oplus \mathcal{Y}_{M,2}$. It is straightforward that the other properties of the previous section are satisfied, as well.

5.2 Quantum Effective BV Theory

5.2.1 On Manifolds without Boundary

Choose a "gauge fixing" Q^*, a map $\mathcal{E} \to \mathcal{E}$ of degree -1. Define

$$D = [Q, Q^*] = QQ^* + Q^*Q, \tag{5.84}$$

a map of degree 0.

Since $\mathcal{E}$ is a topological vector space it makes sense to state the heat equation for $\phi : (0, \infty) \to \mathcal{E}$

$$\partial_t \phi(t) + D\phi(t) = 0. \tag{5.85}$$

Suppose that there exists a degree 1 integral kernel $K_t : (0, \infty) \to \mathcal{E} \otimes \mathcal{E}$, satisfying

$$\partial_t K_t + (D \otimes \mathrm{id})K_t = 0 \tag{5.86}$$

and define $e^{-tD}\phi = K_t \star \phi$, where

$$K \star \phi = (\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(K_t \otimes \phi). \tag{5.87}$$

for $K \in \mathrm{Sym}^2 \mathcal{E}^*$. Then $e^{-tD}\phi$ satisfies the heat equation because

$$\partial_t e^{-tD}\phi = -(\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(D \otimes \mathrm{id})(K_t \otimes \phi) \tag{5.88}$$

$$= -D(\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(K_t \otimes \phi) = -De^{-tD}\phi \tag{5.89}$$

We claim that $(1 \otimes Q)K_t = -(Q \otimes 1)K_t$ is equivalent to the desired relation

$$Qe^{-tD} = e^{-tD}Q. \tag{5.90}$$

This is because

$$Q(K_t \star \phi) = -(\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(Q \otimes \mathrm{id} \otimes \mathrm{id})(K_t \otimes \phi) \tag{5.91}$$

$$= (\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(\mathrm{id} \otimes Q \otimes \mathrm{id})(K_t \otimes \phi) \tag{5.92}$$

$$= -(\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(\mathrm{id} \otimes \mathrm{id} \otimes Q)(K_t \otimes \phi) \tag{5.93}$$

$$= (\mathrm{id} \otimes \langle \cdot, \cdot \rangle)(K_t \otimes Q\phi) \tag{5.94}$$

$$= K_t \star (Q\phi). \tag{5.95}$$

We define the propagator $P_\epsilon^L = \int_0^L (Q^* \otimes \mathrm{id}) K_t$. The propagator satisfies the equation

$$(Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) P_\epsilon^L = \int_\epsilon^L [(Q \otimes \mathrm{id})(Q^* \otimes \mathrm{id}) - (Q^* \otimes \mathrm{id})(\mathrm{id} \otimes Q) K_t \tag{5.96}$$

$$= \int_\epsilon^L [(Q \otimes \mathrm{id})(Q^* \otimes \mathrm{id}) + (Q^* \otimes \mathrm{id})(Q \otimes \mathrm{id}) K_t \tag{5.97}$$

$$= -\int_\epsilon^L \partial_t K_t \tag{5.98}$$

$$= K_\epsilon - K_L \tag{5.99}$$

This implies that

$$[Q, \partial_{P_\epsilon^L}] = \Delta_L - \Delta_\epsilon \tag{5.100}$$

where $\Delta_L = -\partial_{K_L}$ and ∂_X is the contraction operator of a symmetric tensor $X \in \mathrm{Sym}^\bullet \mathcal{E}$ acting on functions. We call Δ_L the regularized BV operator and define the regularized Poisson bracket by

$$\{f, g\}_L = \Delta_L(fg) - \Delta_L f\, g - (-1)^{|f|} f \Delta_L g \tag{5.101}$$

We wish to construct a family of effective interactions $I[L] \in \mathrm{Sym}^\bullet \mathcal{E}^*[[\hbar]]$ satisfying Costello's renormalization group equation

$$e^{I[L]/\hbar} = e^{\hbar \partial_{P_\epsilon^L}} e^{I[\epsilon]/\hbar} \tag{5.102}$$

and the quantum master equation

$$QI[L] + \frac{1}{2}\{I[L], I[L]\} + \hbar \Delta_L I[L] = 0. \tag{5.103}$$

Because

$$\begin{aligned}(Q + \hbar\Delta_L) e^{I[L]/\hbar} &= (Q + \hbar\Delta_L) e^{I[L]/\hbar} && (5.104)\\ &= \left[\frac{QI[L]}{\hbar} + \frac{\{I[L], I[L]\}}{2\hbar} + \Delta_L I[L]\right] e^{I[L]/\hbar} && (5.105)\end{aligned}$$

The quantum master equation is equivalent to the condition $(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0$. This exponential form of the quantum master equation is often more convenient to work with. One important consequence of (5.100) is that

$$\begin{aligned}(Q + \hbar\Delta_L) e^{I[L]/\hbar} &= (Q + \hbar\Delta_L) e^{\hbar \partial_{P_\epsilon^L}} e^{I[\epsilon]/\hbar} && (5.106)\\ &= (Q + \hbar\Delta_L) e^{\hbar \partial_{P_\epsilon^L}} e^{I[\epsilon]/\hbar} && (5.107)\\ &= e^{\hbar \partial_{P_\epsilon^L}} (Q + \hbar\Delta_\epsilon) e^{I[\epsilon]/\hbar} && (5.108)\end{aligned}$$

Thus, the quantum master equation is satisfied at all length scales if it is satisfied at any particular length scale.

Suppose that for some classical interaction functional I, the naive effective quantization

$$I[L] = \lim_{\epsilon \to 0^+} \hbar \log\left[\exp\left(\hbar \partial_{P_\epsilon^L}\right) \exp\left(I/\hbar\right)\right] \tag{5.109}$$

exists. Can we calculate the obstructon (aka anomaly) to $I[L]$ satisfying the quantum master equation? Using the same manipulation as above,

$$\frac{(QI[L] + \frac{1}{2}\{I[L], I[L]\} + \Delta_L I[L])}{\hbar} e^{I[L]/\hbar} = (Q + \hbar\Delta_L)e^{I[L]/\hbar} \tag{5.110}$$

$$= \lim_{\epsilon\to 0^+} e^{\hbar\partial_{P_\epsilon^L}}(Q + \hbar\Delta_\epsilon)e^{I/\hbar} \tag{5.111}$$

$$= \lim_{\epsilon\to 0^+} e^{\hbar\partial_{P_\epsilon^L}}(Q + \hbar\Delta_\epsilon)e^{I/\hbar} \tag{5.112}$$

$$= \lim_{\epsilon\to 0^+} e^{\hbar\partial_{P_\epsilon^L}} \left(\frac{QI + \frac{1}{2}\{I, I\}_\epsilon + \hbar\Delta_\epsilon I}{\hbar}\right) e^{I/\hbar} \tag{5.113}$$

We are assuming that I satifies the classical master equation so that we can substitute $QI = -\frac{1}{2}\{I, I\}$.

Naive Quantization at 1-loop

In this section, we adapt the arguments in 2 dimensions in [2] and [6] to show that the 1-loop quantization for generalized Chern-Simons theory exists.

On $\mathbb{R}^n$, the heat kernel on differential forms has a very elegant form

$$K_t^{\mathrm{an}}(x_1, x_2) = (4\pi t)^{-n/2} e^{-|x_1-x_2|^2/4t} \prod_i (dx_2^i - dx_1^i) \tag{5.114}$$

$$\tag{5.115}$$

Expand

$$\prod_{i=1}^n (dx_2^i - dx_1^i) = \sum_{I\cup J=[n]} (-1)^{|I|+\sigma(I,J)} dx_1^I \wedge dx_2^J \tag{5.116}$$

where $I = i_1 < \cdots < i_j$ and $J = j_1 < \cdots < j_k$ are disjoint multi-indices and $\sigma(I, J)$ is the sign of the unshuffle premutation needed to put I followed by J in increasing order. Because

$$\sum_{I \cup J = [n]} (-1)^{|K| + \sigma(I,J)} dx_1^I \wedge dx_2^J \wedge dx_2^K = (-1)^{|K| + |K|(n - |K|)} dx_1^K \wedge dx_2 \tag{5.117}$$

$$= (-1)^{|K|n} dx_1^K \wedge dx_2 \tag{5.118}$$

and the integral $\int_{\mathbb{R}^n_{x_2}}$ is of degree n, we have

$$\int_{\mathbb{R}^n_{x_2}} \left[(4\pi t)^{-n/2} e^{-|x_1 - x_2|^2/4t} \prod_{i=1}^{n} (dx_2^i - dx_1^i) \right] \left[\phi(x_2) dx_2^K \right] = \phi(x_1, t) dx_1^K \tag{5.119}$$

where $\phi(x_1, t)$ is the solution of the scalar heat equation with $\phi(x_1, 0) = \phi(x_1)$.

For the purpose of Chern-Simons theory, let $\mathfrak{g}$ be a graded Lie algebra of dimension m with compatible symmetric bilinear pairing of degree $n - 3$.

Choose a $X_1, \ldots, X_k, Y_1, \ldots, Y_k, Z_1, \ldots, Z_l \in \mathfrak{g}[1]$ each linearly independent so that

$$\langle X_i, Y_j \rangle' = \delta_{i,j} \tag{5.120}$$

and

$$\langle X_i, Y_j \rangle' = \delta_{i,j} \tag{5.121}$$

This implies that $|X_i| + |Y_i| = 1 - n$ and $2|Z_i| = 1 - n$. Clearly, $l \neq 0$ only if n is odd.

We define the Casimir element

$$C_{\mathfrak{g}} = \sum_i (a_i X^i \otimes Y^i + b_i Y^i \otimes X^i) + \sum_i c_i Z^i \otimes Z^i \tag{5.122}$$

$$= C_{\mathfrak{g},1} + C_{\mathfrak{g},2} \tag{5.123}$$

We would like to study when $K_t = K_t^{an} \otimes C_{\mathfrak{g}}$ is a heat kernel on $\mathcal{E}^\bullet$.

Let

$$K_t \star (\omega \otimes X) = (-1)^{(n-1)|\omega|} \left(\mathrm{id} \otimes \int_{M_2}\right) (\mathrm{id} \otimes \langle \cdot, \cdot \rangle') K_t^{\mathrm{an}} \otimes \omega \otimes C_{\mathfrak{g}} \otimes X \tag{5.124}$$

$$= \left(\mathrm{id} \otimes \int_{M_2}\right) (K_t^{\mathrm{an}} \otimes \omega) \otimes (\mathrm{id} \otimes \langle \cdot, \cdot \rangle') C_{\mathfrak{g}} \otimes X \tag{5.125}$$

Therefore in order to have

$$\lim_{\epsilon \to 0^+} K_t^{\mathrm{an}} \star (\omega \otimes X) = \omega \otimes X \tag{5.126}$$

for all $\omega \otimes X \in \mathcal{E}^\bullet$ it is necessary and sufficient that

$$X = \sum_i [(-1)^{(n-1)|X_i|} a_i X^i \langle Y^i, X \rangle + \sum_i (-1)^{(n-1)|Y^i|} b_i Y^i \langle X^i, X \rangle \tag{5.127}$$

$$+ \sum_i (-1)^{(n-1)|Z_i|} c_i Z^i \langle Z^i, X \rangle] \tag{5.128}$$

Therefore $a_i = (-1)^{|X_i|+1}$, $b_i = (-1)^{(n-1)|Y_i|}$ and $c_i = (-1)^{(n-1)|Z^i|}$.

So, how does $C_{\mathfrak{g}}$ behave under transposition of factors?

$$\tau C_{\mathfrak{g}} = \sum_i [(-1)^{|Y^i||X^i|} a_i Y^i \otimes X^i + (-1)^{|Y^i||X^i|} b_i X^i \otimes Y^i] \tag{5.129}$$

$$+ \sum_i (-1)^{|Z^i|} c_i Z^i \otimes Z^i \tag{5.130}$$

But

$$(-1)^{|Y^i||X^i|}b_i = (-1)^{|Y^i|} = (-1)^n a_i \tag{5.131}$$

$$(-1)^{|Y^i||X^i|}a_i = (-1)^{(n-1)|X^i|+1} = (-1)^n b_i \tag{5.132}$$

$$\tag{5.133}$$

So if $l = 0$,

$$\tau C_{\mathfrak{g}} = (-1)^n C_{\mathfrak{g},1} \tag{5.134}$$

and if $l \neq 0$ and therefore n is odd we have

$$\tau C_{\mathfrak{g}} = -C_{\mathfrak{g},1} + (-1)^{\frac{n-1}{2}} C_{\mathfrak{g},2}. \tag{5.135}$$

This means that in the case $l \neq 0$ we require that $n-1$ is congruent to 2 modulo 4.

For $l = 0$,

$$\tau K_t = \tau K_t^{\mathrm{an}} \otimes \tau C_{\mathfrak{g}} \tag{5.136}$$

$$= (-1)^n(-1)^n K_t^{\mathrm{an}} \otimes C_{\mathfrak{g}} = K_t \tag{5.137}$$

That is, K_t is symmetric, as desired.

We calculate the contraction of $C_{\mathfrak{g}}$ with $\ell_2'(\cdot,\cdot)$

$$n\ell_2'(\cdot,\cdot)(C_{\mathfrak{g}}) = \sum_i [(-1)^{|X_i|+1}\ell_2'(X^i,Y^i) + (-1)^{(n-1)|Y_i|}\ell_2'(Y^i,X^i)] \tag{5.138}$$

$$+ \sum_i (-1)^{(n-1)|Z^i|}\ell_2'(Z^i,Z^i) \tag{5.139}$$

$$= \sum_i [(-1)^{(n-1)|X_i|}\ell_2'(Y^i,X^i) + (-1)^{(n-1)|Y_i|}\ell_2'(Y^i,X^i) \tag{5.140}$$

$$+ \sum_{\substack{i \\ |Z^i| \text{ even}}} \ell_2'(Z^i,Z^i) \tag{5.141}$$

$$\tag{5.142}$$

For this to be 0 we must require $|Z^i|$ is odd for all i. We must also requrie that $k = 0$ or $n-1$ is odd.

But $\Delta_\epsilon I = 0$ on $\mathbb{R}^n$ for all n without needing such conditions on $C_{\mathfrak{g}}$ because K_t^{an} pulled back to the diagonal in $M \times M$ is 0.

Let $Q^* = d^* \otimes \text{id}$. In coordinates, on $\mathbb{R}^n$

$$d_x^* = -\sum_i \iota_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^i}. \tag{5.143}$$

We compute

$$P_\epsilon^L = \int_\epsilon^L Q^* K_t = \int_\epsilon^L d^* K_t^{\text{an}} \otimes C_{\mathfrak{g}} = (P^{\text{an}})_\epsilon^L \otimes C_{\mathfrak{g}} \tag{5.144}$$

where

$$d^* K_t^{\text{an}} = (4\pi t)^{-n/2} e^{-|x_1-x_2|^2/4t} \sum_i \frac{(x_2^i - x_1^i)}{2t} \prod_{i\neq j} (dx_2^j - dx_1^j) \tag{5.145}$$

For any tree γ, $\lim_{\epsilon\to 0+} w_\gamma(P_\epsilon^L, I)$ exists. This is by a simple inductive procedure. If γ_1 and γ_2 are graphs for which $\lim_{\epsilon\to 0+} w_{\gamma_i}(P_\epsilon^L, I)$ exists, then if γ is γ_1 and γ_2 with a tail of γ_1 connected to a tail of γ_2, it follows that $\lim_{\epsilon\to 0+} w_\gamma(P_\epsilon^L, I)$ exists.

The 1-loop quantization exists for generalized Chern-Simons, but the argument relies on the structure of the classical interaction I. It is clear from the previous parragraph that that it suffices to show the existence of the limit when γ is a wheel of n edges for all $n \geq 0$.

Because $w_\gamma(P_\epsilon^L, I)(\alpha_1, \ldots, \alpha_{|T(\gamma)|})$ is equal to

$$\pm w_\gamma^{\mathrm{an}}((P_\epsilon^L)^{\mathrm{an}}, I^{\mathrm{an}})(\omega_1, \ldots, \omega_{|T(\gamma)|}) w_\gamma^{\mathrm{alg}}(C_{\mathfrak{g}}, I^{\mathrm{alg}})(X_1, \ldots, X_{|T(\gamma)|}) \tag{5.146}$$

where $\alpha_i = \omega_i \otimes X_i$, it suffices to show that

$$\lim_{\epsilon\to 0^+} w_\gamma^{\mathrm{an}}((P_\epsilon^L)^{\mathrm{an}}, I^{\mathrm{an}}) \tag{5.147}$$

exists.

Let $x_0, \ldots, x_m$ be coordinates on $(\mathbb{R}^n)^{m+1}$ and identify $x_0 = x_{m+1}$. Make the change of variables $y_\alpha = x_\alpha - x_{\alpha-1}$ for $\alpha = 1, \ldots, m$ and $y_0 = x_0$. Note that

$$\prod_{\alpha=1}^{m+1} \sum_i (x_\alpha^i - x_{\alpha-1}^i) \prod_{i\neq j} (dx_{\alpha+1}^j - dx_\alpha^j) \tag{5.148}$$

is a form of degree $(m+1)(n-1) = mn+n-m-1$ on $\mathbb{R}^{nm}$. Therefore for $n > m+1$ it must vanish.

Since $dy_m^i = -\sum_{\alpha=1}^m dy_\alpha^i$, it is in the subspace generated by the 1-forms dy_α^i for $\alpha = 1, \ldots, m$. For $n = m+1$, it is also at each point in the $\wedge^{nm}\mathbb{R}^{nm}$ so it must again vanish. These statements imply that for $n \geq m-1$

$$w^{\mathrm{an}}_{\Gamma_{m+1}}((P^L_\epsilon)^{\mathrm{an}}, I^{\mathrm{an}}) = 0 \tag{5.149}$$

Before proceding, let

$$M(\mathbf{t},\epsilon) = \begin{pmatrix} \frac{1}{t_1}+\frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & \frac{1}{t_2}+\frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{t_m}+\frac{1}{\epsilon} \end{pmatrix} \tag{5.150}$$

Lemma 3.

$$\det(M(\mathbf{t},\epsilon)) = (t_1 \dots t_m)^{-1}\left(1+\epsilon^{-1}\sum_{i=1}^m t_i\right) \tag{5.151}$$

$$= \frac{t_1+\cdots+t_m+\epsilon}{t_1 \dots t_m \epsilon} \tag{5.152}$$

and

$$M^{-1}(\mathbf{t},\epsilon) = \begin{pmatrix} t_1+\frac{t_1^2}{\epsilon+t_1+\cdots+t_m} & \frac{t_1t_2}{\epsilon+t_1+\cdots+t_m} & \cdots & \frac{t_1t_m}{\epsilon+t_1+\cdots+t_m} \\ \frac{t_2t_1}{\epsilon+t_1+\cdots+t_m} & t_2+\frac{t_2^2}{\epsilon+t_1+\cdots+t_m} & \cdots & \frac{t_2t_m}{\epsilon+t_1+\cdots+t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t_mt_1}{\epsilon+t_1+\cdots+t_m} & \frac{t_mt_2}{\epsilon+t_1+\cdots+t_m} & \cdots & t_m+\frac{t_m^2}{\epsilon+t_1+\cdots+t_m} \end{pmatrix} \tag{5.153}$$

For $m+1>n$, $\lim_{\epsilon\to 0+} w^{\mathrm{an}}_{\Gamma_{m+1}}((P^L_\epsilon)^{\mathrm{an}}, I^{\mathrm{an}})$ exists. This is a consequence of the fact that $w^{\mathrm{an}}_{\Gamma_{m+1}}((P^L_\epsilon)^{\mathrm{an}}, I^{\mathrm{an}})(\omega_0,\dots,\omega_m)$ is bounded by

$$\int_{[0,L]}\int_{(\mathbb{R}^n)^m}\prod_{\alpha=0}^m \frac{dt_\alpha}{(4\pi t_\alpha)^{n/2}} e^{-\sum_{\alpha=1}^m |y_\alpha|^2/4t_\alpha - |\sum_{a=1}^m y_\alpha|/4t_{m+1}} \tag{5.154}$$

which is finite because upon performing the Gaussian integral and is equal up to a constant to

$$\int_{[0,L]} \prod_{\alpha=1}^{m+1} \frac{dt_\alpha}{t_\alpha^{n/2}} \frac{1}{\det(M(\mathbf{t},\epsilon))^{n/2}} = \int_{[0,L]} \frac{1}{(t_1 + \cdots + t_{m+1})^{n/2}} \tag{5.155}$$

$$\leq \int_{[0,L]} \frac{1}{(t_1 \ldots t_{m+1})^{\frac{n}{2(m+1)}}} < \infty \tag{5.156}$$

This implies that for $2(m+1) > n$,

$$\lim_{\epsilon \to 0^+} w^{\mathrm{an}}_{\Gamma_{m+1}}((P^L_\epsilon)^{\mathrm{an}}, I^{\mathrm{an}}) \tag{5.157}$$

exists, so in particular, the limit exists for $m+1 > n$.

We conclude that

$$\lim_{\epsilon \to 0^+} w_\gamma(P^L_\epsilon, I) \tag{5.158}$$

exists in generalized Chern-Simons theory on $\mathbb{R}^n$, for all graphs γ with $b(\gamma) = 1$.

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