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WHAT BECOMES OF GLOBAL COLOR

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Abstract

The recent demise of certain global unbroken symmetry generators in the presence of a grand unified magnetic monopole leads us to consider more carefully the notion of charges associated with gauge symmetries. It turns out that global transformations associated with the generators of the gauge group, and their charges, make sense only for extended systems which are sufficiently localized. GUT monopoles fail this criterion. Detailed consideration of the monopole-antimonopole system helps remove paradoxes related to the chromodyon excitations of a single monopole and agrees with the previous result that some, but not all, of the states naively expected do exist. The remaining states needed to fill out color multiplets are spread throughout space; they are recovered as long-lived excitations when an antimonopole is brought in from infinity.

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I. INTRODUCTION AND CONCLUSION

Four years ago, Tomaras and Dokos showed that the simplest grand unified theory, the SU(5) model of Georgi and Glashow, admitted magnetic monopoles [1]. Grand unified monopoles are time-independent solutions of the field equations, much like the monopole solutions found earlier in simpler field theories by 't Hooft and Polyakov. They differ from their predecessors in having about them not just ordinary electromagnetic magnetic fields but also color SU(3) magnetic fields; they are simultaneously electromagnetic and chromomagnetic monopoles.

't Hooft-Polyakov monopoles are known (in semiclassical quantization) to possess a spectrum of excitations called dyons. These are states that transform according to definite representations of the unbroken symmetry group of the theory, electromagnetic U(1); that is to say, they are states of definite charge. Thus one would expect grand unified monopoles to be accompanied by chromodyons, states of definite charge which also transform according to definite representations of color SU(3) [1,2].

It has become clear in the last year that this is not the case [3,4]. The excitations of grand unified monopoles do not fall into representations of the color group. The reason for this is deep and surprising. In the presence of a chromomagnetic monopole, it is impossible to *define* the color group [5,6]. There is no such thing as a global SU(3) transformation.[f1]

Our aim in this paper is to understand this phenomenon.

Our principal investigative method will be the study of a system consisting of a widely separated monopole and antimonopole. As we shall see, for this system the color group is well defined and the excitations of the system transform according to definite representations of color SU(3). We will investigate how all this structure manages to evaporate

when the separation between the monopole and the antimonopole goes to infinity.

We have found this subject to be full of booby traps, and we have been able to thread our way among them only by being very careful in our reasoning. Unfortunately, very careful reasoning makes for very long reading. Thus, out of mercy for the reader, we give in this section an overview of our results, full of handwaving and unproved assertions. The reader who only wants an idea of our results may read only this section; the reader who wants to be convinced that our answers are correct must go on to the body of the paper.

The Overview

We will work exclusively in a semiclassical approximation; indeed, for much of the time, we will restrict ourselves to classical field theory. Thus we will say nothing about the effects of confinement. This is appropriate [2]. There are fifteen orders of magnitude between the grand unification scale, which determines the size of monopoles, and the confinement scale. Confinement is as irrelevant to the physics we are discussing as the large-scale curvature of the universe is to the theory of the solar system. We will also restrict ourselves to theories of gauge fields and scalar fields only, ignoring fermions. Unlike confinement, fermions can have a profound effect on monopole structure (e.g., the Callan-Rubakov effect) but we do not believe it is an essential one for our task of sorting out issues pertaining to global color. In the same spirit, we take the vacuum angle θ to be zero. (See Ref. [4].)

Before we can describe our results we must define our notation. A gauge field, A_μ, is a Lie-Algebra valued vector field. The fieldstrength tensor, $F_{\mu\nu}$, is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$$

where g is the gauge coupling constant. In the absence of other fields, this obeys the Yang-Mills equations,

$$\partial^\mu F_{\mu\nu} + g[A^\mu, F_{\mu\nu}] \equiv D^\mu F_{\mu\nu} = 0.$$

If the group is simple, these equations are derived from the Lagrange density

$$\mathscr{L} = -\frac{1}{2}\,\mathrm{Tr}(F_{\mu\nu} \cdot F^{\mu\nu}).$$

If the group is a product, there is one such term for each factor. The electric and magnetic fields are defined in the usual way

$$F_{0i} = E_i, \quad F_{ij} = \varepsilon_{ijk} B_k.$$

In the case at hand, the gauge algebra is that of SU(5), but at distances large compared with the grand unification scale, the only gauge fields that survive are those associated with the low energy gauge algebra, that of $SU(3)_{\mathrm{color}} \times U(2)_{\mathrm{electroweak}}$. At distances large compared to the electroweak scale, this is reduced further, to the algebra of $SU(3)_{\mathrm{color}} \times U(1)_{\mathrm{electromagnetism}}$. At large distances from the grand unified monopole, in an appropriately chosen gauge [2],

$$B^i = Q\, r^i/gr^3$$

where

$$Q = \frac{1}{2}\,(Q_{em} + Y_{\mathrm{color}}).$$

Of course, we can replace the color hypercharge by any equivalent member of the color algebra; this is just one possible standard form among many.

We can now explain what we mean when we say that in the presence of a monopole, global color transformations do not exist. It will turn out

to be convenient to work in temporal gauge, $A_0 = 0$. In this gauge, we have a very large algebra of color gauge transformations: we can make independent infinitesimal color transformations at each point in space. We wish to select out of this infinite algebra an eight dimensional algebra of global color transformations, transformations that act everywhere in the same way.

What does this mean? The natural way to connect infinitesimal transformations at two separated points is by parallel transport. (As the word "natural" signals, the handwaving has begun. The real stuff, which is independent of the possibility of parallel transport, is in Ref. [5].) One starts with the element of the algebra at $\vec{r}_1$, $\Omega(\vec{r}_1)$, and defines it at $\vec{r}_2$ by picking a path, $\vec{r}(s)$, going from $\vec{r}_1$ to $\vec{r}_2$, and integrating along the path the equation

$$\frac{D\Omega}{DS} = \frac{d\vec{r}}{ds} \cdot \vec{D}\,\Omega = \frac{dr^i}{ds}\,(\partial_i \Omega + g[A_i, \Omega]) = 0.$$

The covariant derivative in this equation ensures gauge invariance; unfortunately, it also ensures path dependence. If we transport Ω about an infinitesimal planar loop, it does not rturn to its original value:

$$\delta\Omega = dA[g\hat{n}\cdot\vec{B}, \Omega],$$

where dA is the area of the loop and $\hat{n}$ is the unit normal to the plane.

However, all is not lost. We can hope to surround the system under study by a large sphere on which all the gauge fields are negligible. We can then define global color transformations on this sphere; this will enable us to study the color transformation properties of the electric flux passing through the sphere, which we might hope to link (through Gauss's Law) with those of the system inside the sphere. (This is reminiscent of the way in which we study the total energy of an isolated system in general relativity.)

But even this does not work if the system contains a magnetic monopole. In this case, no matter how large our surrounding sphere, we cannot make parallel transport path-independent. If we consider a loop on the sphere subtending fixed solid angle, as we enlarge the sphere, the area of the loop increases like r^2, exactly cancelling the $1/r^2$ fall off of the magnetic field. We can only define global transformations for that subalgebra for which

$$[Q,\Omega] = 0.$$

In the case at hand, this is the algebra of electromagnetic U(1) plus the algebra of a U(2) subgroup of color SU(3).

This is the sense in which global color does not exist in the presence of a magnetic monopole. We stress that this does not mean that a monopole on the moon prevents us from discussing the total color of an isolated system of quarks in the laboratory; the problem arises only when the system under study has net color magnetic charge. Monopoles do not "break" color symmetry.

This example suggests that we gain further insight by studying a particular system with no net color magnetic charge, a widely separated monopole-antimonopole pair. Here there is no problem in defining global color transformations on a large sphere surrounding the system, and, as we shall see, the excitations of this system transform according to definite representations of color SU(3).

We begin our analysis by showing how to continue $\Omega(\vec{r})$ in from the sphere at infinity.

Every infinitesimal symmetry transformation of a physical system can be thought of as an actual physical motion, an infinitesimal evolution of the system in time. For example, consider a molecule, a

system of N particles interacting through two-body potentials,

$$L = \sum_{a=1}^{N} \frac{1}{2} m_a |\dot{\vec{r}}_a|^2 - \sum_{a>b} V_{ab}(|\vec{r}_a - \vec{r}_b|).$$

This system is invariant under an infinitesimal rotation,

$$\delta\vec{r}_a = \vec{e} \times \vec{r}_a,$$

where $\vec{e}$ is a unit vector. This can be thought of as defining an evolution in time, steady rotation about the $\vec{e}$ axis,

$$d\vec{r}_a/dt = \omega\, \vec{e} \times \vec{r}_a,$$

where ω is the angular velocity.

Likewise, the change of a gauge field under an infinitesimal gauge transformation,

$$\delta\vec{A} = -g^{-1}\vec{\nabla}\Omega - [\vec{A},\Omega] = -g^{-1}\vec{D}\,\Omega,$$

can be thought of as defining an evolution in time,

$$\partial_0\vec{A} = -g^{-1}\omega\vec{D}\,\Omega \tag{1.1}$$

where ω is an internal-symmetry angular velocity.

But in temporal gauge, $F_{0i} = \partial_0 A_i$, and this motion is consistent with Gauss's Law if and only if $D_i F_{0i} = 0$, i.e. if

$$D_i D_i \Omega = 0. \tag{1.2}$$

It is this equation, of Laplacian type, that defines Ω throughout all space given its value at infinity.[f2] (We are here ignoring the scalar fields present in the monopole core. These make a small correction to Eq. (1.2), which we'll take proper account of in section three.)

One advantage of thinking of things in this way is that it leads immediately to the computation of the dyon spectrum, for the dyons are nothing else but the result of semiclassical quantization of the motion defined by Eq. (1.1).

To show how this occurs, let us begin with the system of N particles discussed above, and let us suppose that the system possesses a time-independent solution of the equations of motion, a local minimum of V. Then the steady rotations will be solutions of the time-dependent equations of motion, to first order in ω. (In higher orders, the rotation will begin to stretch the system.) The Lagrangian restricted to these motions is

$$L = \frac{1}{2} I \omega^2 - \text{const.},$$

where I is the usual moment of inertia,

$$I = \sum_{a=1}^{N} |\vec{e} \times \vec{r}_a|^2 .$$

The angular momentum is given by

$$J = I\omega,$$

and the Hamiltonian by

$$H = J^2/2I + \text{const.}$$

In semiclassical quantization, J is restricted to integral values, and we obtain the usual molecular rotational spectrum.

All of this carries through without alteration to a gauge theory. If we are applying our gauge transformation to a time-independent solution of the equations of motion, the Lagrangian becomes

$$L = \frac{1}{2} I \omega^2 - \text{const.},$$

where

$$I = -2g^{-2} \int d^3r \, \mathrm{Tr}(\vec{D}\Omega \cdot \vec{D}\Omega)$$

The conjugate is called not angular momentum but charge,[f3]

$$Q = I\omega,$$

and

$$H = \frac{1}{2I} Q^2 + \text{const.}$$

Semiclassical quantization leads to the dyonic spectrum.

In sections three and four we apply this analysis to a monopole-antimonopole pair separated by a large distance, R.[f4] Our main results are:

(1) For the generators of the color algebra that commute with Q, everything is normal. $\vec{D}\Omega$ is small except near the monopole and antimonopole cores, and the moments of inertia are independent of R for large R. The dyonic excitations are localized on the cores. They are the states of definite color hypercharge described in Ref. [4]; for SU(5) they fall into representations of U(1).

(2) For the orthogonal generators, everything is strange. $\vec{D}\Omega$ is non-negligible over a region of size R, and the moments of inertia are proportional to g^{-2} R. The dyonic splitting is thus proportional to g^2/R, vanishing as R goes to infinity. Furthermore, in the vicinity of the cores, Ω itself vanishes like R^{-a}, where $a = \sqrt{3/4} - 1/2$.

These dyonic excitations have nothing to do with either the monopole or the antimonopole. They are rather twistings (in color space) of the lines of magnetic force that extend from the monopole to the antimonopole. That the aether between two objects should *itself* be an object capable of supporting excitations, emitting radiation, etc. is no surprise in a nonlinear field theory. After all, two orbiting stars will have a region between them of time-varying gravitational stress, which will itself contribute to the system's gravitational radiation.

As R grows, these twistings extend over larger distances, but take place farther from the cores; this is how they are able to disappear from the spectrum in the limit of infinite R.

We have described this phenomenon in terms of dyonic excitations. These, of course, have an equivalent description in terms of the classical motions from which the excitations are derived. Suppose some external force sets the monopole-antimonopole system rotating in color space with some fixed angular velocity. If the rotation is in a direction that commutes with Q, an observer near one of the cores sees something is happening no matter how large R is. (In particular, she sees a nonzero color electric field.) But if the rotation is in an orthogonal direction, she sees nothing at all, because the associated Ω vanishes in her vicinity.

We can now see what happens in a more complex system, composed of many widely separated monopoles and antimonopoles, but still arranged such that the net magnetic charge is zero. In this case, the field near each particle is a magnetic Coulomb field, but Q might be a different element of the algebra for different particles. To find the effect of a given infinitesimal color rotation on a given monopole, we must integrate Eq. (1.2) down from infinity to the neighborhood of the monopole. Those infinitesimal transformations that commute with Q when they arrive will be allowed to penetrate the magnetic Coulomb field and transform the core; the orthogonal ones will be stopped on the lines of force and never reach the core.

The dyonic excitations we have been discussing involve distortions of color magnetic fields extending over large distances (on the order of R), and one might think that such modes would be extremely unstable to the emission of soft gluons. (In classical language, one might think that they would quickly lose their energy to non-Abelian radiation.)

In section four, we carefully study such decays. The calculation is fraught with technical complexities, but our fundamental results can be understood in terms of simple physical ideas.

Suppose we have some motion of a classical system, of angular frequency ω, that radiates energy at some rate $\dot{E}_{cl}$. The semiclassical estimate of the decay width of the associated states is obtained by computing the time it takes the system to radiate one quantum of energy,

$$\Gamma \sim \omega^{-1} \dot{E}_{cl}.$$

For the monopole-antimonopole system, by our previous analysis, the low-lying excitations have

$$\omega \sim g^2/R.$$

Thus, for small g^2, the wave length of the emitted radiation is much larger than the spatial extent of the system, and a dipole approximation is justified in computing the rate of radiation.[f5] The color dipole moment of the system is proportional to R; thus

$$\dot{E}_{cl} \propto R^2.$$

This is multiplied by powers of g and ω, which we can easily work out by dimensional analysis,

$$\dot{E}_{cl} \sim R^2\omega^4/g^2.$$

(The reader who wants to reproduce the dimensional analysis should remember that $\dot{E}_{cl}$ is a classical quantity; no powers of $\hbar$ are allowed in this formula.)

Putting all this together, we find

$$\Gamma \sim g^2\omega. \tag{1.3}$$

The dyonic excitations are metastable; in the limit of small coupling, the width of an excitation is $O(g^2)$ times the spacing between successive excitations.

We have found what becomes of global color. A magnetic monopole can only support excitations associated with color rotations that leave its long range magnetic field unchanged. The unwanted excitations are expelled along the lines of color magnetic force. If the lines of force close (that is to say, if the monopole is part of an assembly with zero total magnetic charge) the unwanted excitations hang on in a diffuse cloud between the monopoles. If these lines extend to infinity, the unwanted excitations disappear altogether.

The remainder of this paper gives the detailed arguments that lead to the picture we have just described. Section two discusses the notion of a nonabelian gauge charge, including some thought experiments to determine the color of a single chromodyon. Section three describes the monopole-antimonopole system and section four deals with the decays of its excitations. Section five contains further discussion and disposes of the thought experiments.

II. COLOR CHARGES

Gauge symmetries pose conceptual problems because they are not, strictly speaking, symmetries at all. The pure Yang-Mills lagrangian defines a nonsingular classical dynamics only when its vector potentials are interpreted as redundant, the true configuration space being the quotient [10] $\mathcal{M} = \mathscr{C}^{(3)}/\mathscr{G}^{(3)}$ of three-dimensional gauge potentials on space modulo three-dimensional gauge transformations.[f6] We can realize systems defined on this complicated space in terms of vector potentials in $\mathscr{C}^{(3)}$ if we insist that dynamics on the latter be gauge invariant and remember that two configurations differing by an element of $\mathscr{G}^{(3)}$ (henceforth called "charge rotations") are not really distinct. This viewpoint is already familiar [11] as the method used to treat the motion of a particle on the manifold $\mathbb{C}P_n$: We avoid working with the projective space by introducing a redundant degree of freedom to make it into the sphere S^{2n+1}, then demand that the action be singular with a local invariance under the one-dimensional group $U(1)$. Motion along this $U(1)$ is of course not observable.

Technically this formulation of gauge theory can be well defined only if space is taken to be compact, say S^3. Equivalently we can demand that all field configurations fall off more rapidly than r^{-1}, thus eliminating the pathologies of section one from the beginning. This requirement is no more restrictive than the fact that on compact space the total electric charge must be zero, and it seems to be necessary for quantization of the full theory to introduce a volume cutoff of this sort [10].

Formulating our theory in this way seems rather a drastic step. After all, the very motions we wish to investigate are themselves charge rotations. While gauge invariance *per se* has vanished from the theory, however, its vestiges remain in the detailed shape of the space and the level sets of the energy functional on it. $\mathscr{M}$ is a complicated space, but we will need only some of its gross features, and they are easily deduced. In place of the notion of global gauge transformations as ordinary global symmetries remaining after gauge fixing, we can substitute another: that special features of $\mathscr{M}$ imply collective coordinates which describe the relation of a system to the rest of the world and whose effects resemble global gauge transformations. These special features will in general exist only for sufficiently local systems.

We begin with an analogy to the space $\mathscr{C}^{(4)}/\mathscr{G}^{(4)}$, where exact results about the moduli space of the action functional are known [12]. In particular, the exact two-instanton solutions in the SU(2) theory are known to form a manifold of 13 dimensions, of which three can be interpreted as a relative group orientation; similarly, n instantons have 3(n-1) such coordinates. Thus even though true gauge transformations are missing, each of the two instantons has a full SU(2) of overall information relative to its partner. This hard mathematical result has an easy physical interpretation for the case when the instantons are distantly separated, as follows: There exists a gauge choice for a single instanton in which its fields fall as r^{-2}, the singular gauge. Using this choice we can find a good approximate two-instanton solution simply by linearly superposing two arbitrarily-oriented single instantons, since in the field equations the non-Abelian commutator term in the intermediate region falls with the separation like R^{-4} and so is eventually negligible compared to the derivative term, which falls like R^{-3}. While

the overall orientation of the two-instanton configuration can be changed by a gauge transformation, and so does not correspond to anything on the quotient space, the *relative* orientation cannot be changed in this way without altering the fast fall-off in the intermediate region. It therefore corresponds to a true collective coordinate of the system.

Hence two-lump systems with fast field fall-off, for which linear superposition suffices to produce approximate solutions, are guaranteed to have collective coordinates corresponding to the full gauge group, and moreover the associated motions will resemble ordinary global charge rotations in the vicinity of each lump. We can extend the analysis to the case of instantons in theories with spontaneous symmetry breaking if we demand that the asymptotic values of the Higgs fields match; again due to fast fall-off of A_μ this yields approximate solutions and we have relative coordinates corresponding now to the entire unbroken group.

When we attempt to imitate this analysis for three-dimensional gauge theories, however, we immediately run into the difficulty that the gauge fields of a single monopole do *not* fall off rapidly enough to permit linear superposition of arbitrary pairs. If however the monopoles in question have generators of the long-range fields which commute, then we get a configuration corresponding to each superposed one, distinct from others with different relative orientations. In the limit of heavy monopole cores these configurations are static approximate solutions. This is consistent with the rigorous proof by Taubes [13] that such configurations are close to exact solutions in the Bogomol'nyi-Prasad-Sommerfield limit, where no infinite mass limit is needed to render the multimonopoles stable. We expect the approximate superposition method to work as well for the $M\bar{M}$ as for the MM. If we want the $M\bar{M}$ pair to have vanishing net magnetic charge (as we do), we must choose the long-range fields not just to commute, but to cancel. This is a well-defined demand if we work in a gauge where each monopole has all gauge fields either falling faster than r^{-1} or proportional to a constant Lie algebra element, which we will call τ^1. Henceforth our classical field configurations will always be in this gauge unless otherwise stated, with string singularity chosen to lie on the line joining the cores. (In passing, we remark that this superposition trivially meets the non-Abelian stability condition discussed in Ref. [2].)

Each member of the pair will with respect to its partner have a set of collective coordinates smaller than the number of generators of the unbroken group. This is more palatable, and more accurate, than the picture of the monopole as breaking or spoiling some ordinary global symmetry. The motion under rotations of these coordinates can be quantized, giving excitations whose quantum numbers we can call "charges," but they will not resemble the weights of any representation of the unbroken group. This is not too surprising, since even in the usual interpretation of gauge theories the non-Abelian charges of an individual constituent had no gauge-invariant meaning. Nevertheless we are accustomed to saying that the individual quarks inside a hadron are in definite color representations. We now need to examine such statements to see whether they can apply to individual monopoles.

First of all, on very short distance scales QCD becomes weakly coupled, so that there are point particles corresponding to the fundamental fields of

the theory. While it makes no sense to speak of a given quark as being red, say, still we can via deep inelastic scattering count its possible states and deduce the Casimir of its representation. When we attempt to carry this reasoning over to monopoles, though, we run into trouble on two different points. The monopole is by definition a collective excitation, which has a large indeterminate number of gluons present on any scale large enough for the effective theory to make sense at all. Still it might seem that the Casimir could be well defined, whatever its value, since certainly a full basis of the unbroken algebra can be defined at every point in space. If we reconsider the argument of Ref. [5], replacing the group Aut $\mathfrak{g}$ by the larger $O(n)$ of orthonormal bases of the Lie algebra $\mathfrak{g}$, we find no obstruction to finding a set of n charge rotation generators orthonormal at each point. Implementing these operators in the quantum theory and taking the sum of their squares, we should get a candidate for a generalized Casimir operator. Unfortunately, even if this definition were sensible, no such set of operators could be implemented in the semiclassical quantization. In order to be smoothly defined and satisfy the matching condition of Ref. [5], at least some of them must at spatial infinity depend on the polar angle θ, and so correspond to classical motions of strictly infinite kinetic energy.

Next, idealizing a quark as a heavy fixed point change we can deduce its color by considering the binding of heavy scalars in some known representation of color, and in particular those bound states for which the complete system has no non-Abelian electric fields at large distances. That is, we find the color by asking how many scalars it takes to screen it. We will return later to this point.

Finally, we can examine a collection of quarks as their separation is increased to infinity. Arguments of locality then lead us to expect that then those quantum states of the whole system whose energies have a definite limit should resemble the tensor product of the possible states of a single quark, and so if for some reason the system is easier to discuss than the isolated quark, we should nevertheless be able in this way to deduce the desired properties of the latter from the former. Since indeed we do expect the $M\bar{M}$ to be simpler than the single monopole, it is to this system that we now turn. Again the problem before us is to reconcile the spectrum of single-monopole states described in section one with the fact that the $M\bar{M}$ system is expected to possess overall color excitations and hence not look like the tensor product of two representations of U(1). Again the resolution will be that the extra color states are delocalized, not associated with either monopole.

III. THE $M\bar{M}$ SYSTEM

We wish to discuss the monopole-antimonopole system in the lowest semiclassical approximation. Again it makes sense to treat the system as if it were stable, ignoring the motions of the lumps toward each other. A more serious issue is the applicability of the semiclassical expansion itself. To lowest order it amounts to the assertion that the gauge coupling g is chosen so small that every deformation of the classical solution is either perfectly flat in potential energy or infinitely steep; then quantum fluctuations off the absolute energy minimum will lie far from the low-energy spectrum we wish to investigate. When some other limit is also being taken, however, such as the separation R in our case, we must worry about the possibility that no matter what small fixed g we choose, as R increases eventually some non-flat perturbations will become shallow enough for their excitations to creep down into the low spectrum. In this case we will not be allowed to ignore them. We will return to this point in section four.

For convenience let us work in a toy theory, where SU(3) breaks via an adjoint Higgs field to H = SU(2) x U(1) divided by a discrete subgroup. We take the Higgs everywhere proportional to the matrix

$$\Phi = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}.$$

This model preserves the important features of the usual SU(5) GUT, giving qualitatively the same final result. (Analysis of the realistic case proceeds similarly, but certain coset spaces arise in place of Lie groups, due to the fact that some non-Abelian generators of the unbroken group act trivially on the monopole.)

The discussion in section two now implies that the most general minimum-energy $M\bar{M}$ solution, with a separation R much larger than the size of the cores, will be specified by an approximate solution consisting of two spherically-symmetric monopoles. The latter is in turn specified by two homomorphisms of SU(2) into the full gauge group, i.e. two ordered sets of three matrices $\vec{\tau}$, $\vec{\tau}'$ such that

i) Each set obeys the su(2) algebra,

ii) $\tau^1 = \tau^{1\prime}$, so that the monopole long-range fields match, and

iii) Only τ^1 commutes with the constant matrix Φ, and it satisfies the Dirac quantization condition [2].

Such a pair of homomorphisms determines a point of our semiclassical configuration space. Since such a description is cumbersome, we now seek a simpler parametrization of this space.

The key is the realization that the dynamical system under consideration is already familiar from mechanics. Just as in section one a single monopole was quantized by analogy with rigid body dynamics, so the $M\bar{M}$ system can be viewed as two rigid wheels partially constrained to move together by an axle. Corresponding to the two body framesinscribed on the two wheels we have the two frames $\vec{\tau}$, $\vec{\tau}'$ attached to the two monopoles and free to move subject to (i) - (iii) above. Just as in the case of the two wheels, we will be able to separate the dynamics into an overall motion and relative motion, and then use standard techniques [14] to quantize the system.

For the rigid rotator, we know that the most useful description of the configuration space is not in terms of frames, but rather in terms of the group SO(3) taking these frames into each other. Similarly, in our case we begin with a "space-fixed" frame

$$\vec{\tau}_0 = \begin{bmatrix} 0 & \\ & i\vec{\sigma}/2 \end{bmatrix},$$

where $\vec{\sigma}$ are the Pauli matrices, and the configuration $\vec{\tau} = \vec{\tau}' = \vec{\tau}_0$. Any motion of the system with finite kinetic energy will leave Φ fixed. Call the generators of these unbroken charge rotations

$$\vec{T} = \begin{bmatrix} i\vec{\sigma}/2 & \\ & 0 \end{bmatrix}$$

and $T_{u(1)} = \tau_0^3$. Of these, the latter generates the same motion as T_3, so we drop it, remembering that T_3 generates the action of ordinary electric rotations on the lump as well as one of the su(2) motions. All three generators $\vec{T}$ then act nontrivially and independently on $\vec{\tau}_0$, and in fact all choices for $\vec{\tau}$ which are connected to $\vec{\tau}_0$ by smooth paths following the bottom of the potential well can be obtained in this way.[f7]

To get $\vec{\tau}$, $\vec{\tau}'$ satisfying (i) - (iii) above, we now define a point in configuration space C as follows: Imbed the group SU(2) in the unbroken gauge group in such a way that its generator T_1 leaves τ^1 fixed. Given $(U, e^{i2\theta})$ where $U \in SU(2)$, let

$$\vec{\tau} = [\mathrm{Ad}\ U e^{\theta T_1}]\vec{\tau}_0$$
$$\vec{\tau}' = [\mathrm{Ad}\ U e^{-\theta T_1}]\vec{\tau}_0 . \qquad (3.1)$$

Here $(\mathrm{Ad}\ U)X = UXU^{-1}$, the adjoint action by the image of U under the embedding. The definition corresponds to starting with the wheels aligned, their axle along the x axis, then twisting them relative to each other by 2θ, and finally rotating the whole system by U. While the four generators of $SU(2) \times U(1)$ all act nontrivially, however, there is a global redundancy since $(U, e^{2i\theta}) = (-U, e^{i(2\theta+\pi)})$, so that finally our configuration space is $C' = C/\mathbb{Z}_2$. (The fact that C' resembles the unbroken group U(2) is a coincidence.) All told, our system has a configuration space of 4 dimensions describing motions along its minimum of energy. Moreover this space has a Lie group structure.

At this point the reader probably feels cheated. Does not Ref. [5] argue that motions such as those where U changes in time are sometimes unacceptable, having infinite energy along the strings? No, that argument took place on the sphere at infinity, which in that case contained one monopole. The proof that in our case the U motions really do have finite moments of inertia, as expected, is relegated to the appendix. There we also show that the moments of inertia for those U motions which change the long-range gauge fields are proportional to R, the monopole separation. As $R\to\infty$ we recover the infinite moments of the single monopole [1-4].

The virtue of describing configuration space as a Lie group lies in the fact that C' now has two physically meaningful actions on itself, corresponding to "space-centered" and "body-centered" motions of the wheels. By acting on the frames with $(V,e^{2i\alpha})$ either before or after the operations in Eq. (3.1) we get respectively the motions

$$(U,\theta) \to (UV,\theta+\alpha), \quad \text{space centered}$$
$$(U,\theta) \to (VU,\theta+\alpha), \quad \text{body-centered}.$$

These transformations factor through the $\mathbb{Z}_2$ above, and so define actions of C' on C'. Since the infinitesimal action of u(1) is in each case the same, we will not distinguish its two types of motion. Clearly all three of the transformations so defined commute.

A trajectory of the system is a path $(U(t), \theta(t))$ on the group. Its velocity is a tangent vector to the group. By symmetry, though, the dynamically relevant velocity must be defined in a way independent of the overall orientation of the system, so we define the body-centered angular velocity in the Lie algebra of C' as the right-invariant $(\omega(t), \dot\theta(t))$,

where $\omega(t) = \dot U(t)\, U^{-1}(t)$. Let its components be $\vec\omega = -2\,\mathrm{Tr}(\vec T\omega)$. Then we know that $\omega_{2,3}$ have large equal moments of inertia I_3 proportional to R. We will see momentarily that the motion corresponding to ω_1 resembles a global charge rotation near each monopole. Just as in the wheel analogy, therefore, the two monopoles rotate in the direction of their long-range generator T^1 with angular frequencies $\omega_1 \pm \dot\theta$, the total energy approaching the sum of two contributions with fixed moment of inertia I_1 as $R\to\infty$. The total energy is thus $H = \frac{1}{2} I_3(\omega_2^2+\omega_2^2) + \frac{1}{2} I_1((\omega_1+\dot\theta)^2 + (\omega_1-\dot\theta)^2)$.

H exhibits the promised splitting into terms involving only ω and $\dot\theta$. Immediately [14] we have that quantum states are labeled by j, m, m, q, the eigenvalues of L^2, L_1^{space}, L_1^{body}, and M, where $\vec L$ and M are the quantum generators of infinitesimal motions. j must be at least as large as $\max(|m|,|m'|)$; furthermore because of the $\mathbb{Z}_2$ factored from C we must have q even iff j is integral, since otherwise the system wavefunction would be double-valued. The energies of these states are

$$E = j(j+1)/2I_3 + \frac{1}{2}(m')^2(I_1^{-1} - I_3^{-1}) + q^2/2I_1.$$

Here $I_3 \sim g^{-2}R$, $I_1 \sim g^{-2}M_{GUT}^{-1}$. As the distance between the monopoles grows, the spectrum of the theory thus approaches exactly that predicted by Dokos and Tomaras, but the total number of states is not that of two isolated representations of color. Nor is it the same as the number expected for two isolated representations of U(1) of the type described in section one.

We can see why the T_3 rotations represent large diffuse excitations, with large moments of inertia, by examining Eq. (1.2). First, however, we must justify the approximation of ignoring the monopole cores and working in the unbroken theory. Let us circumscribe one monopole by a sphere of radius ρ, with $M_{GUT}^{-1} \ll \rho \ll R$. Outside the sphere the full Gauss's law condition indeed reduces to Eq. (1.2). This is a well-defined Schrödinger equation on all of space for a scalar adjoint-representation particle outside a monopole, and it contains but one scale: R. If Ω at infinity has a T_1 component, the equation is just Laplace's and $\Omega \neq 0$ all the way into the core, invalidating the approximation which neglects the Higgs fields. If however Ω has no T_1 component, then under the adjoint representation it splits into eigenvectors of Ad T_1 with eigenvalues ± 1. Near the chosen pole $\Omega(\vec{r})$ can be expanded in monopole harmonics, the lowest of which will have $\ell = \frac{1}{2}$. This gives [2] an effective ℓ' via $\ell'(\ell'+1) = \ell(\ell+1) - (1)^2(1/2)^2$, or $\ell' = \sqrt{3/4} - \frac{1}{2}$, so via dimensional analysis the regular solution behaves like $(r/R)^{\ell'}$ near the pole.

The equation for $r < \rho$ is complicated, and it involves a new scale M_{GUT}^{-1}. Nevertheless, it is linear, and so its solution will be homogeneous in its boundary data. Thus Ω inside the core is proportional to $(\rho/R)^{\ell'}$, and so vanishes as $R \to \infty$. As promised, *nothing* happens to the cores when we rotate the system in this limit, and we can therefore work in the unbroken theory, where the τ's disappear altogether. In the wheel analogy this corresponds to replacing the wheels by point masses at the ends of the axle, since small moments of inertia are being neglected.

Now the rotating system has an electric field $\vec{E}(t=0) = \dot{\vec{A}}(t=0) = -g^{-1}\vec{D}\Omega$. Part of this comes directly from the motion of the lump, while part is due to the spatial variation of Ω. If as in the appendix we now split Ω into a constant plus a function vanishing at infinity, we see that the latter obeys an equation with a source spread throughout a region the

size of R. In this way R can (and for dimensional reasons *must*) enter I_3 with a positive power. Therefore, supposing the total charge $\omega_3 I_3$ is held fixed as R increases, then the source inside any fixed box containing one monopole decreases, making it harder and harder for a local observer to detect whether $\omega_3 I_3$ is nonzero at all. Thus in the classical theory the qualitative behavior of one monopole is the same whether or not another exists far away.

In the quantum theory some of the states will have $j = |m'|$, corresponding to the classical motions with $\omega_2 = \omega_3 = 0$. These states contain two of the objects with definite color hypercharge described in section one, and like their classical counterparts they can be thought of as two point sources of electric and magnetic color glowing against a dark background. Also for arbitrary j our states look like two point sources of abelian electric and magnetic charge; since body T_1 rotations are also U(1) rotations, the corresponding electric charges are proportional to $m' \pm q$. By analogy with the classical case, for large R the quantum states with nonminimal j will be hard to distinguish from those with $j = |m'|$.

IV. DECAYS OF THE $M\bar{M}$

The interpretation of the T_3 motions as almost in the continuum brings us back to the issue at the beginning of section three: what right do we have to ignore the almost-flat continuum modes in the first place? We have attempted to describe stationary states by charges, with no regard to the existence of free charged particles. Indeed, just as the excited states of the hydrogen atom are not true stationary states, so interaction with the gluon continuum may make our nonminimal-j states unstable to decay. This process is controlled by the matrix element of the perturbation to our system due to a gluon plane wave, which transforms as a color triplet in the "space-centered" frame, so we get [14] that $\langle j_2,m_2,m_2',q_2|\mathcal{O}^{\text{space}}|j_1,m_1,m_1',q_1\rangle$ is the product of group-theoretic factors times internal matrix elements $\langle j_2,m_2',q_2|\mathcal{O}|j_1,m_1',q_1\rangle$. If these are nonzero at all, they will at least be diagonal in m' and q, since no field in the unbroken theory carries *abelian* charge. Also, j_1-j_2 must be an integer since the gluons are a color triplet, consistent with the relation between U(1) charge and evenness of 2j noted in section three.

A given $M\bar{M}$ state may thus decay to its minimum-j state, shaking off excess charge to minimize its color-electrostatic energy. The lowest state is still degenerate in m, which clearly just reflects the fact that while $\vec{L}^{\text{body}}$ is strictly along the long-range generator $\hat{e}_1^{\text{body}}$, still the whole system can point any direction in "space-centered" coordinates.

Since the monopole fields are very strong as $g\to 0$, it may even be possible for gluon emission to proceed so rapidly as to obliterate some of the states we have found by making their widths greater than their energy splittings from the ground state. While such a phenomenon is not

required for consistency, still it is an issue which must be resolved before the enumeration of $M\bar{M}$ states is to be trusted. In this section we therefore consider the decay of isorotational lump excitations by emission of quanta of the very same non-Abelian fields which constitute the lump. We will only be interested in the qualitative issues of whether and how such radiation can occur at all, and if so whether as $g\to 0$ its rate can exceed the energy splitting $\Delta E = \omega \sim g^2R^{-1}$. The reader who is willing to accept that this does not occur may wish to skip the anfractuous details and proceed to the end of this section. For simplicity we will continue to work with SU(2) color. The same sort of treatment could also be of use in investigating the decay $\Delta\to\pi N$ in the Skyrme model of baryons [15].

To treat the decay problem quantum-mechanically we could make use of a modified collective coordinates method along the lines of Gervais and Sakita [16]. In this language the gluons correspond to "vibrations", to be expanded in a basis of normal modes about the static lump solution, then quantized along with the "rotations". Typically in problems such as arise in molecular physics we can ignore the vibrations altogether for the lowest-order analysis of the lowest-energy states. This is because the quantized vibration states have energies much larger than the rotation states; indeed in field theory the former are $\mathcal{O}(1)$ in g while the latter are $\mathcal{O}(g^2)$ [17]. Nevertheless, when the vibrations belong to a massless continuum, their energies can be arbitrarily small, vitiating the above power-counting. The rotational excitations may then mix with the continuum, if the latter can carry the former's charge.

In the problem at hand, the emitted gluons should have an energy equal to the isorotational level splittings ΔE. The actual mechanism responsible for their emission is found by examining the collective-coordinates Lagrangian for terms involving the charge ("angular momentum") on the lump and one power of the vibration coordinates. Terms with more vibration coordinates correspond to simultaneous emission of two or more gluons, processes suppressed by phase-space factors of g due to the smallness of ΔE. The appropriate term is thus the lowest-order "stretch" term, so called because in the theory of diatomic spectra it gives the effect of the modified moment of inertia due to the molecule's deformation. If J is the angular momentum, I the moment of inertia, and δI a linear function of the vibration coordinate, this term is contained in

$$\frac{1}{2} J^2 (I+\delta I)^{-1} \simeq \frac{1}{2} J^2 I^{-1} - \frac{1}{2} J^2 I^{-2} \delta I. \qquad (4.1)$$

In our case the term of interest is thus suppressed by two powers of the frequency $\omega = J/I$, since the quantized values of J are integers of order one.

While the quantum derivation along the above lines is straightforward in principle, in practice it is difficult. For our purposes, though, we can get an adequate estimate of the emission rate by considering instead the analogous *classical* problem, as explained in section one. The rate of energy emission dE/dt for a classical system rotating at frequency ω, divided by ω, should be a good guide to the order of magnitude of the emission rate Γ, provided that ω^{-1} dE/dt turns out much smaller than ΔE. Once again we will see how stretching effects *à la* Eq. (4.1) provide the emission mechanism.

The classical problem resembles that of a dipole antenna with periodically changing polarity in electrodynamics, but without any external source of charge density. Rather, the source is the gauge field itself, so that a better analog to our problem is the gravitational radiation of two slowly orbiting black holes, a system consisting of pure

gravity everywhere on a space with two points removed. An exact solution of our problem would begin at time zero with no radiation, treat the monopoles dynamically, and solve the time-dependent field equations through time T >> R. At time T the energy flux through a shell of radius T would then give the initial rate of emission. Instead of this hyperbolic, evolution problem, though, we will solve a much easier elliptic, steady-state problem as follows: the monopoles are driven externally by linearizing the problem about a given time-dependent configuration. An ansatz for the steady-state time dependence of the perturbation is then made such that when inserted into the full field equations it yields time-independent equations for the perturbation. These equations define a well-posed elliptic boundary value problem when supplemented by gauge-fixing and outgoing-wave conditions. The solution far from the radiating system describes an energy flux which gives a good account of the initial rate of radiation in the realistic system, much as in electromagnetic theory.

Specifically, we begin with the superposed $M\bar{M}$ configuration $\tilde{A}(\vec{x}) \propto T_1$. As discussed in the preceding section, we may work in the effective unbroken theory. $\tilde{A}$ is a static solution to the Yang-Mills equations subject to the boundary condition that monopoles be present at $\pm \frac{1}{2} R\hat{z}$. We take $\tilde{A}_0 \equiv 0$. Next consider the uniform charge rotation $\bar{A} = u\tilde{A}$ where $U(t) = \exp[\Omega t]$, $u = \mathrm{Ad}\, U$ and Ω is an antihermitian generator of infinitesimal charge rotations. Since T_3 is a principal axis of the isorotator, the uniform motion $\Omega \equiv \omega T_3$ is a solution of its lowest-order Euler equations of motion. It corresponds to the quantum state with $m' \pm q = 0$ but $j=1$.

When the continuum modes are added to the system, the above motion is only an approximate solution. Accordingly we write

$$\vec{A}(\vec{x},t) = \bar{A}(\vec{x},t) + u(t)\ \vec{a}(\vec{x},t)$$

$$A_0(\vec{x},t) = u(t)\ a_0(\vec{x},t) \tag{4.2}$$

and linearize the field equations to get new equations for a_μ. The latter enjoy a gauge invariance which must be fixed before they can determine a_μ, so we choose the background field gauge $\bar{D}_\mu(ua^\mu) = 0$, where $\bar{D}_\mu$ is with respect to $\bar{A}$. Equivalently, we have $\tilde{D}\cdot\vec{a} = [\Omega, a_0] + \dot{a}_0$, where $\tilde{D}$ is with respect to $\tilde{A}$. This gauge has the advantage of being well-defined by the slice theorem [18].[f8] Solving the new field equations, we look for radiation in the part of a_μ which is $\mathcal{O}(r^{-1})$. a_μ will indeed satisfy a set of time-independent equations, and so can itself be taken independent of time in the "corotating" basis of Eq. (4.2). We will make this ansatz now for simplicity, showing later that such a solution does exist.

To get started, we digress for a moment to a simplified problem having many of the same features as ours. Consider a theory with some collection of boson fields $\phi(\vec{x})$ and invariant under an action of SU(2). The theory is characterized by a dimensionless coupling g such that $L(\phi;g) = g^{-2}L(g\phi;1)$. Suppose the theory admits a static classical soliton $\phi_0(\vec{x})$ not invariant under SU(2), e.g. $T_3\phi_0 \neq 0$. ϕ_0 will be proportional to g^{-1}. Consider $\phi(\vec{x},t) = e^{\omega t T_3}(\phi_0+\delta)$ and insert into the canonical Lagrangian $L = \frac{1}{2}\dot{\phi}^2 - V[\phi]$. Let $G = \delta^2 V/\delta\phi\delta\phi|_{\phi_0}$ be the small-oscillations operator. By hypothesis $0 = V[\phi_0] = \delta V/\delta\phi|_{\phi_0}$. Then since V is gauge-invariant we have $V[\phi] = \frac{1}{2}\delta^T G\delta + \mathcal{O}(\delta^3)$. Assuming $\dot{\delta}=0$, $\ddot{\phi} = e^{\omega t T_3}[\omega^2 T_3^2(\phi_0+\delta)]$, and the linearized equation of motion is

$\omega^2 T_3^2(\phi_0+\delta) = -G\delta$. The δ on the lhs is clearly a stretching term again, describing the change in the rotational kinetic energy due to a distortion of the lump. By linearity we would naively expect the solution to be proportional to $\omega^2 g^{-1}$, just like the source.

To find δ we need a Green function inverting the operator $G+\omega^2(T_3)^2$. Let us now specify to the simple case where T_i act pointwise on the fields, which are in the adjoint representation, ω is independent of $\vec{x}$, G is minus the Laplacian, and $\phi_0 \propto \begin{pmatrix}1\\0\\0\end{pmatrix}$ in the basis spanned by the generators T_i of su(2). We seek a solution δ also in this subspace, which we simply identify with $\mathbb{R}$. So $(\nabla^2+\omega^2)\delta = -\omega^2\phi_0 \equiv f(\vec{x})$. This is the reduced Helmholtz equation, whose solutions we obtain by Green function methods as convex linear combinations of the two functions

$$\delta(\vec{x}) = \frac{1}{4\pi}\int d^3y\ |\vec{x}-\vec{y}|^{-1}\ f(\vec{y})\ e^{\mp i\omega|\vec{x}-\vec{y}|}.$$

For $|\vec{x}| \gg \omega^{-1} \gg R$, this is

$$\simeq (4\pi|\vec{x}|)^{-1}\ e^{\mp i\omega|\vec{x}|}\int d^3y\ e^{\pm i\omega\hat{r}\cdot\vec{y}}\ f(\vec{y})$$

$$= (4\pi r)^{-1}\ e^{\mp i\omega r}\ F(\pm\omega\hat{r}), \tag{4.3}$$

where $\hat{r} = \vec{x}/r$, $r = |\vec{x}|$, and F is the Fourier transform of f. Were f a source of compact support of diameter ~R, we would approximate F by expanding $e^{i\omega\hat{r}\cdot\vec{y}}$, obtaining the usual multipole expansion for δ.

Instead, let us take ϕ_0 to be one spatial component of the magnetic dipole vector potential, say $\phi_0 = \hat{\phi}\cdot\vec{A} = (R/gr^2)\sin\theta$. This source has such slow fall-off that its dipole moment diverges quadratically. For a rough estimate we can repair the multipole expansion by continuing to expand the exponential but cutting off the space integral at the scale ω^{-1} of oscillations, giving

$$\delta(x) \sim -(4\pi r)^{-1} e^{-i\omega r} \int d^3y\, i\omega(\hat{r}\cdot\vec{y})\omega^2 \sin\theta \frac{R}{g|\vec{y}|^2} + \text{(higher multipoles)}$$

$$\sim -(4\pi r)^{-1} e^{-i\omega r} (i\omega^3 R/g)\cdot\omega^{-2}.$$

All told, δ turns out to be unexpectedly *enhanced by a power of* ω. Of course this can also be seen explicitly using the exact Fourier transform of the dipole field, but the principle is more general: if a radiation field has a contribution which is enhanced in this way, then all contributions from more rapidly-falling terms of the source can be neglected to lowest order in ω. In particular, that part of the source lying within a radius ~R can be neglected completely, since the magnitude of its contribution will obey the naive expectation above. More general linear differential operators than G will still have this property if they reduce asymptotically to $-\nabla^2$.

We should pause for a moment to note that the source term in the equation for δ comes from a term proportional to one power of δ in the Lagrangian. As noted before, a radiation process caused by this source thus corresponds to single-meson emission and proceeds steadily in the driven classical theory. Had we found that $\delta(\vec{x})$ contained no radiation part, we would have been obliged to abandon our ansatz $\dot{\delta} \equiv 0$ and to seek a solution displaying parametric resonance [19], characterized by an exponentially-increasing radiation part. Such a solution would correspond to two-meson emission, which would have been relevant had the one-meson rate vanished.

While our steady-state solution has a radiation part, however, it still is not quite correct. Calculating the Poynting vector T_{0i} at infinity, we find it to be zero. It is not difficult to trace the problem back to the ansatz $\delta \propto T_1$; transforming such a solution back to the non-

rotating frame gives $\delta(\vec{x})[T_1\cos\omega t + T_2\sin\omega t]$, which at infinity has only standing waves. We therefore relax the ansatz to allow δ a T_2 component and seek to formulate an outgoing-wave boundary condition. This is easy in the new basis T_+, T_-, T_3 for (complexified) su(2), where $T_\pm = T_1 \pm iT_2$. We simply require that at infinity the solution behave as $\delta_\pm \sim r^{-1}e^{\pm i\omega r}$. This can be rendered more precise by putting the problem in a large spherical box B of radius $L \gg \omega^{-1}$ and replacing the outgoing-wave condition by a totally-absorbing condition on the boundary: $(\partial/\partial r \mp i\omega)\delta_\pm = 0$ on ∂B. To complete the specification we can add to this $\partial\delta_3/\partial r = 0$, which will make δ_3 vanish everywhere. We will ignore it. Later it will be technically useful to clip off f just inside ∂B by replacing it with $f(\vec{x})\chi(r)$, where the smooth function $\chi = 0$ for $r > L-\varepsilon$, $\chi = 1$ for $r < L-2\varepsilon$, some small ε. Again it is clear physically that this modification does not change the radiation flux. Returning to the original T_i basis of su(2), we thus require $(\partial/\partial r + \omega M)\,\delta = 0$, where

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This condition is real, as befits a problem which was purely real to begin with. We will continue to use complex numbers whenever convenient.

The equation for δ then becomes

$$(\nabla^2 + \omega^2)\delta = \begin{pmatrix} 1 & 0 \end{pmatrix} f, \tag{4.4}$$

the two pieces coupled only by the boundary condition. Rather than attempt a solution to systems such as this one, we will later approximate their asymptotic behavior using the fact that some solution does exist. Let us sketch a proof of physically obvious assertion for the above equation, for practice. It will make the problem look more familiar if we first make yet another change of variables, letting $\alpha(\vec{x}) = e^{\omega p(r)M}\delta(\vec{x})$, M as before. The smooth function $p(r) = r$ for $r > L-\varepsilon/4$, while it vanishes for $r < L-\varepsilon/2$. The boundary condition then becomes $\partial\alpha/\partial r|_{\partial B} = 0$. Let $\mathscr{H} = L_2^N(B)$, the real Hilbert space of square-integrable functions on B with Neumann boundary conditions. We then wish to solve $\mathscr{D}\alpha = \hat{f}$ on this space, where now $\hat{f}(\vec{x}) = e^{\omega p(r)M} f(\vec{x})\chi(r)$, χ clips off f as before, and $\mathscr{D} = \nabla^2 + \omega^2(1-(\vec{\nabla}p)^2) - 2\omega M(\vec{\nabla}p)\cdot\vec{\nabla} - \omega M(\nabla^2 p)$. $\mathscr{D}$ is self-adjoint on $\mathscr{H}$ and moreover has no continuum spectrum since B is bounded.

The above transformation of variables can be symbolically replaced by a modification of the differential operator on δ from $\nabla^2 + \omega^2$ to $\mathscr{D}' = \nabla^2 + \omega^2 + 2\delta(r-L)\omega M$, where the delta function is understood in the sense of

$$\int_B 2\delta(r-L)\psi(\vec{x}) = \int_{\partial B} \psi(\vec{x}).$$

Self-adjointness of $\mathscr{D}$ on $\mathscr{H}$ corresponds to formal self-adjointness of $\mathscr{D}'$ on $\mathscr{H}'$, the space satisfying the outgoing-wave condition. In any case, to show invertibility it now suffices to show that $\mathscr{D}'$ has no null eigenvalues, i.e. that the homogeneous problem $\mathscr{D}'\delta = 0$ has no nonzero solution on $\mathscr{H}'$.

This is trivial. Setting the rhs of Eq. (4.4) zero we get a pair of equations of Schrödinger type for a "particle" of "energy" ω^2. In the case at hand the particle is free, and so all its scattering phase shifts vanish. But by the outgoing-wave condition, for large L we have $\delta_{1,2} \propto C_+ e^{i\omega r} \pm C_- e^{-i\omega r}$ respectively near r=L. This implies that $\delta_{1,2}$ must have all phase shifts differing by π. So no nonzero solution exists on $\mathscr{H}'$, and the original inhomogeneous problem has exactly one solution. This of course agrees with our physical intuition, which says that the outgoing-wave problem corresponds to the initial behavior of a well-posed Cauchy problem.

The application of the foregoing analysis to gauge theories is straightforward once gauge invariance is dealt with. In electrodynamics, we can render the static field equation $\nabla^2\vec{A} - \vec{\nabla}\vec{\nabla}\cdot\vec{A} = \vec{j}$, where $\vec{\nabla}\cdot\vec{j} = 0$, elliptic by imposing the gauge $\vec{\nabla}\cdot\vec{A} = 0$. A solution to the resulting simplified equation $\nabla^2\vec{A} = \vec{j}$ will then solve the original equation only if it also satisfies the gauge condition. Fortunately, however, with appropriate boundary condtions the simplified equation actually *contains* the gauge condition, as we see by taking its divergence. The same happy situation prevails in linearized Yang-Mills theory, so that once we have used $\tilde{D}\cdot\vec{a} = [\Omega, a_0]$ to arrive at simplified equations, we can then forget all about it, demanding simply that there be only *transverse* outgoing waves at infinity. The proof is simplest is we let $A_\mu(\vec{x},t) = \bar{A}_\mu(\vec{x},t) + b_\mu(\vec{x},t)$, where again $\bar{A}_i = u(t)\tilde{A}_i$, $\bar{A}_0=0$, b_μ is small, and $\bar{D}_i\bar{F}_{ij}=0$. We want to simplify the Yang-Mills equations using $\bar{D}\cdot\vec{b} = \dot{b}_0$. Linearizing in b_μ gives

$$D_iF_{ij} = g[b_i,\bar{F}_{ij}] + \bar{D}^2 b_j - \bar{D}_i\bar{D}_j b_i$$
$$= 2g[b_i,\bar{F}_{ij}] + \bar{D}^2 b_j - \bar{D}_j\bar{D}_i b_i.$$

Using the gauge condition the simplified equations are

$$D_iF_{0i} = 0$$
$$D_0F_{0j} - 2g[b_i,\bar{F}_{ij}] - \bar{D}^2 b_j + \bar{D}_j\dot{b}_0 = 0 \qquad (4.5)$$

We wish to recover the gauge condition from Eqs. (4.5). Taking the covariant divergence of the second and using the first,

$$[D_j,D_0]F_{0j} + D_0D_jF_{0j} - 2g[\bar{D}_jb_i,\bar{F}_{ij}] - [\bar{D}_j,\bar{D}_i](\bar{D}_ib_j) - \bar{D}_i([\bar{D}_j,\bar{D}_i]b_j)$$
$$- \bar{D}^2\bar{D}\cdot\vec{b} + \bar{D}^2\dot{b}_0 = 0$$

and

$$\bar{D}^2(\bar{D}\cdot\vec{b} - \dot{b}_0) = 0.$$

The equation $\tilde{D}^2 f = 0$ has the unique solution $f \equiv 0$ if we demand that f vanish at ∂B, following an argument similar to that in the appendix. Thus Eq. (4.5) contains the gauge condition.

The time has come to begin approximating. Returning to the variables $\vec{a}$ and a_0 of Eq. (4.2) we see that they satisfy coupled wave equations, to wit the Ampère and Gauss laws. We will show that to our approximation these equations decouple and we can estimate $\vec{a}$ and a_0 separately. First of all, we can gauge away the u in Eq. (4.2), giving $\vec{A} = \tilde{\vec{A}}+\vec{a}$, $A_0 = a_0 + g^{-1}\Omega$. Consider now the linearized Gauss law:

$$F_{0j} = ([\Omega,\tilde{A}+a]_j - \tilde{D}_j a_0)$$

$$0 = D_j F_{0j} = \tilde{D}_j([\Omega,\tilde{A}+a]_j) - \tilde{D}^2 a_0 + g[a_j,[\Omega,\tilde{A}_j]].$$

By the above analysis, the solution to this and the remaining simplified Yang-Mills equations will satisfy the gauge condition, so to estimate a_0 we can freely use the latter. We also introduce the ansatz that at large distances the solution satisfies $\vec{a} \sim g^{-1}\omega r^{-1} e^{\pm i\omega r}$, as in the earlier toy model, and that the leading term of $\hat{\phi}\cdot\vec{a}$ is independent of the azimuthal angle ϕ. Here the tilde means that we include the dependence on g, ω, and r^{-1}; enough powers of R are implicitly present to make up the right dimensions. We have then a wave equation for a_0:

$$0 = g|\tilde{A}|^2\Omega - 2g[[\Omega,\tilde{A}_j],a_j] + [\Omega,[\Omega,a_0]] - \tilde{D}^2 a_0,$$

where we have used $\nabla\cdot\tilde{A} = 0$. Since the entire problem is azimuthally symmetric, we can take $\partial_\phi a_0 = 0$, reducing the last term to $-\nabla^2 a_0 - g^2[\tilde{A}_i,[\tilde{A}_i,a_0]]$, which is diagonal in color. Then a_{03} satisfies a Poisson equation with source dominated by the first term, and so $a_{03} \sim g^{-1}\omega r^{-1}$. a_{01}, a_{02} satisfy wave equations with source $\sim g^{-1}\omega^2 r^{-3}$. This falls off too rapidly for the ω-enhancement effect to work, so $a_{01}, a_{02} \sim g^{-1}\omega^2 r^{-1} e^{\pm i\omega r}$.

Turning now to the other three linearized Yang-Mills equations, a_0 will enter via F_{0j}. We get

$$\begin{aligned} D_0 F_{0j} &= [\Omega + g a_0, [\Omega,\tilde{A}+a]_j - \tilde{D}_j a_0] \\ &= [\Omega,[\Omega,\tilde{A}+a]_j] - [\Omega,\tilde{D}_j a_0] + [g a_0,[\Omega,\tilde{A}]]. \end{aligned}$$

The sources for $\vec{a}$ are then the first term $\sim g^{-1}\omega^2 r^{-2}$, plus others involving a_0 which are all suppressed relative to it by powers of ω or r^{-1}. Since the first term falls slowly and so is enhanced by one power of ω (as in the toy example), we see that we can ignore a_0 altogether and consider the three spatial Yang-Mills equations with $a_0 = 0$.

Next we need $D_i F_{ij}$, which we are to simplify using the gauge condition. We get $D_i F_{ij} = \tilde{D}^2 a_j + 2[a_i,\tilde{F}_{ij}]$, plus an a_0 term from the gauge condition which we can drop by an argument similar to the one above. All told,

$$0 = \omega^2 A_j - [\Omega,[\Omega,a_j]] + \tilde{D}^2 a_j + 2[a_i,\tilde{F}_{ij}], \tag{4.6}$$

with boundary conditions of transversality, $\vec{\nabla}\cdot a = 0$, and outgoing transverse waves, $(\partial/\partial r + \omega M)\vec{a}_\perp = 0$. We have again moved to column-vector notation for the adjoint representation, with

$$M = \begin{bmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix}$$

and $\vec{a}_\perp = \overleftrightarrow{P}\cdot\vec{a} \equiv \vec{a} - \hat{r}\hat{r}\cdot\vec{a}$. Since we are always interested in $\vec{a}$ only to lead order in r^{-1}, we will rephrase the boundary conditions as $(\partial/\partial r + \omega M\overleftrightarrow{P})\vec{a} = 0$. This again defines a self-adjoint boundary-value problem if now we formally subtract $2\delta(r-L)\omega M\overleftrightarrow{P}$ from the differential operator, and so to show that a solution exists we turn to the homogeneous problem.

Eq. (4.6) suffers from a string singularity on the line joining the two monopoles. Since there is no genuine physical singularity anywhere in the full broken gauge theory, though, we can now invoke the ω-enhancement argument and modify the equation, setting $\tilde{A}=0$ everywhere inside $r=2R$ without disturbing the leading radiation fields. Also, outside 2R we can take $\tilde{A}$ to be pure dipole.

Equation (4.6) splits into two pieces. If we denote $T_2\vec{a}_2+T_3\vec{a}_3$ by $\vec{a}'$, then we get

$$0=\omega^2\tilde{A}+\omega^2\vec{a}_1+\nabla^2\vec{a}_1$$

$$0=-[\Omega,[\Omega,a_j']]+\tilde{D}^2a_j'+2[a_i',\tilde{F}_{ij}].$$

In spherical coordinates these equations and the boundary condition joining them are independent of ϕ, so one ansatz on $\vec{a}$ is consistent. The first equation is exactly that of the toy model, and its homogeneous version is the free Schrödinger equation. The second is not free, but becomes so for small g, as we see by rescaling $x'=\omega x$. In terms of x', we have

$$0=-\omega^2[T_3,[T_3,a_j']]+\omega^2\nabla'^2a_j'$$
$$-\omega g[\tilde{A}(\omega^{-1}x')\cdot\vec{\nabla}'+\vec{\nabla}'\cdot\tilde{A}(\omega^{-1}x')]a_j'+2[a_i',\tilde{F}_{ij}(\omega^{-1}x')].$$

But $\tilde{A}$ and $\tilde{F}$ are bounded by powers of r^{-1}, so the second line is suppressed relative to the first by one power of g. For small enough g, then, it will again be impossible for this system to have the phase shifts required by the boundary condition, and the trivial solution will be unique.

Using Eq. (4.3), we have $\vec{a}_1$ asymptotically as some convex linear combination of the functions

$$\pm\left(\frac{R\omega}{4\pi rg}\right)e^{\mp i\omega r}\,\hat{\phi}+\mathcal{O}(r^{-2},\omega^2).$$

We have used the Fourier transformation $\vec{F}(\vec{k})=-\omega^2\vec{k}\times\vec{R}/gk^2$ of the dipole vector potential. The $\vec{a}'$ equation can be shown to have a conserved "probability density" $\mathrm{Tr}(a'^{\dagger})\cdot\vec{a}'$ whose conservation law requires that $\vec{a}_2$ have equal amounts of $e^{\pm i\omega r}$ at infinity. Combining this with the boundary condition and the form of $\vec{a}_1$ shows that <u>all</u> components of $\vec{a}$ have radiation parts with magnitude bounded by $R\omega/gr$. This has the appropriate units, and it verifies the remaining ansatz on $\vec{a}$.

Now we can estimate the Poynting vector $S=\mathrm{Tr}\,\vec{E}\times\vec{B}$ as

$$\vec{S}\sim\left(\frac{R\omega^2}{gr}\right)^2\hat{r}.$$

The rate of energy loss divided by ω corresponds to the quantum emission rate and equals $\Gamma\sim R^2\omega^3/g^2$. Using $\Delta E=\omega$ and $\omega\sim g^2/R$ we at last obtain

$$\Gamma/\Delta E\sim g^2$$

in agreement with Eq. (1.3). *In the weak-coupling limit, then, the isorotator states found in section three are practically stable*; our prediction for the spectrum is not invalidated by inclusion of the continuum.

The quantum $M\bar{M}$ system for any finite separation R thus has discrete metastable color states not reflected in the tensor product of the states available to a single (anti-)monopole. This is no paradox. After all, the same is true of two hydrogen atoms, which can have relative orbital angular momentum states in addition to their internal states. In each case the extra discrete states come not from thin air but from the continuum of gluon states (respectively momentum eigenstates) available. In each case for large R localized measurements are unable to detect whether the extra states are excited, and so as $R \to \infty$ they effectively disappear, removing any contradiction with locality and the spectrum of a single system. In no sense can the extra states be thought of as new excitations of the individual constituents.

Had the analysis uncovered extra metastable states whose splittings and lifetimes did *not* depend on R, we would have been in trouble. These could have been excited by an incident gluon wavepacket much smaller than the entire system, and so we would have been forced to interpret them as unexpected new excitations of *one* of the monopoles.

Finally, we must consider the possibility of normalizable near-zero modes for large R. In the wheel analogy these correspond to flexing modes of the axle as it becomes very long and flimsy. A closer look at the analogy shows, however, that at the same time the two wheels are *separately* picking up infinite moments of inertia perpendicular to $\hat{e}_1^{\text{body}}$. This is because the only exact zero modes of a single monopole not already accounted for are non-normalizable (at least in the BPS limit [7]). Thus these motions are classically no more observable close to one monopole than were the $\omega_{2,3}$ motions. In the quantum theory this corresponds to the statement that a measurement taken on any fixed time scale can as $R \to \infty$ use a set of approximate energy eigenstates in which τ^1 is fixed, even for nonminimal j or excited near-zero modes. The error incurred in this way will be smaller than the uncertainty error of a finite-time measurement.

V. DYON SCREENING

Now that we are at last sure of what a chromodyon looks like, we can return to the remaining color test of section two, the binding of nonrelativistic test charges to a chromodyon. This is a direct generalization of the Abelian monopole problem [6,2,20]. We take the dyon to be a fixed classical background field, the test charge to be in the 5 representation of SU(5). Since a bosonic particle will not reach into the dyon core, we can work in a two-patch unitary gauge with no reference whatever to the underlying GUT, save for the relation between triality and electric charge. If we let the antihermitian matrix

$$Q' = -\frac{i}{2}\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

be the SU(3) piece of the matrix representing τ^3, then the particle wavefunction solves

$$\epsilon\psi = \left\{-\frac{1}{2m}\vec{D}^2 + iQ'\,\frac{ng^2}{r}\right\}\psi.$$

Evidently the scalar potential does not affect the invariance of this system under rotations generated by $\vec{L} = -i\vec{r}\times\vec{D} - iQ'\vec{r}/r$, and so we can diagonalize the problem in L^2, L_3, and Q'. In a sector with $iQ' = \lambda$ we get the usual radial equation with centrifugal term $[\ell(\ell+1)-\lambda^2]/2mr^2$ and potential $\lambda ng^2/r$. Test particles with $\lambda=0$ will therefore not see the long-range part of the dyon at all, while those with $\lambda\neq 0$ can bind if n has the right sign. The possibility of such binding is an important qualitative difference from the case of ordinary monopoles. For such a bound state the total electric field of the dyon-plus-particle at long distances can be obtained by linear superposition; since dyon and particle have Q' charges n and $\frac{1}{2}$ respectively, 2n particles can screen the electric field completely. The particles' mutual color interactions can be ignored due to the overwhelming magnetic field forcing them to line up.

We can repeat the argument with test charges in other color representations. Again all that matters are the Q' charges. Colored particles can screen chromodyons, but not in any way which let us infer that the latter have definite color. Fully-screened states with no long-range electric fields exist, though they still cannot be called color singlets.

This exhausts the proposed color tests. The chromodyons cannot be said to have definite color.

Before concluding there are some remarks about confinement to be made. An $M\bar{M}$ system with $j=0$ can exist by itself in the full theory so long as R is less than the confinement scale. As R is increased, however, if $q\neq 0$ the energy of the system will increase in the customary linear way. Eventually a superheavy charged vector boson will be exchanged, reducing q. In pure gauge theory, then, isolated chromodyons are not expected to exist. If colored matter particles are present, however, we have seen that they can screen the color electric field. Single chromodyons can then exist and engage with strong cross sections in a full variety of charge-exchange processes unavailable to ordinary monopoles, much as do ordinary neutral atoms.

Life without global color is not so bad after all. In the absence of monopoles the view presented here of the role of global gauge symmetries reduces to the usual one, while in their presence it gives us a consistent picture of their low-lying states. When heavy scalar quarks are present the chromodyons bind them in just the expected way. The case of light spinor quarks is of course another matter.

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APPENDIX

We are to show that the moment of inertia I_3 for the $M\bar{M}$ system at finite separation R is finite and bounded below by a function proportional to $g^{-2}R$. As explained in section one, questions about this moment of inertia are questions about solutions $\Omega(\vec{x})$ to Eq. (1.2) which at infinity approach T_3, or more properly to the equation

$$(\vec{D}^2 + M^2)\Omega = 0 \qquad (A.1)$$

in the full broken Yang-Mills-Higgs theory [3]. Here $\vec{D}$ is the adjoint representation covariant derivative with respect to the monopole gauge field $\vec{A}(\vec{x})$, and $M\Omega = [\phi,\Omega]$ where $\phi(\vec{x})$ is the adjoint Higgs field. If the static temporal-gauge configuration $(\vec{A},\phi)$ is set into uniform rotation using the generator $\Omega(\vec{x})$, the result will satisfy Gauss's law. The other field equations are of no concern here; $(\vec{A},\phi)$ need not satisfy them. The kinetic energy of the rotating system is then

$$E[\Omega] = \frac{1}{2}\int d^3x\, [\dot{\phi}(x,t)^2 + \dot{\vec{A}}(x,t)^2] = \frac{1}{2}\left\|\vec{D}\Omega\right\|_2^2 + \frac{1}{2}\left\|M\Omega\right\|_2^2, \qquad (A.2)$$

where the subscripts refer to the L_2 norm using the invariant inner product on the Lie algebra $\mathfrak{g}$.

As in the text we choose $\vec{A}(\vec{x})$, $\phi(\vec{x})$ to be the linear superposition of two fundamental monopoles in a theory where SU(3) breaks to U(2). We can write this solution with no singularities if we divide space into five regions as in Figure (1a). In the shaded regions we use the gauge where ϕ points radially in internal space, while in I, I', II we use the Wu-Yang gauges [20], where ϕ points in a constant direction. By a gauge transformation we can cover space differently as in Figure (1b). Now a single patch takes care of the entire region at infinity. Outside the cores ϕ approaches the constant $v\Phi$ exponentially, where $v \sim M_{GUT}$ and

$$\Phi = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}.$$

Splitting $\mathfrak{g}$ orthogonally into stab $\Phi \oplus$ (the rest), we have that the components of $\vec{A}$ in the second factor vanish exponentially rapidly away from the cores. Examining (A.1) we see the same is true for Ω. (Incidentally, Eqs. (A.1), (A.2) are gauge-covariant and completely insensitive to our chopping-up of space.)

Since the lower bound is easier to obtain than the existence of a finite-energy solution, we begin by showing the former given the latter. Let $\Omega(\vec{x}) = \bar{\Omega}(\vec{x}) + \eta(\vec{x})$ where $\bar{\Omega} = T_3$ outside $r = 2R$. In this region Ω essentially satisfies Eq. (1.2), i.e.

$$\vec{D}^2\eta = -\vec{D}^2\bar{\Omega}, \quad \eta \to 0 \quad \text{as } r \to \infty. \qquad (A.3)$$

Let us momentarily take $\eta \equiv 0$, so (A.3) is not satisfied. Then

$$E[\bar{\Omega}] \geq \frac{1}{2}\left\|\vec{D}\bar{\Omega}\right\|_2^2 \qquad (A.4)$$

$$\geq (\text{const.})\int_{\alpha R}^{\infty}\left(\frac{R}{gr^2}\right)^2 [1 + \mathcal{O}(R/r)]d^3r$$

$$\geq (\text{const.})\left(\frac{R}{g^2}\right)[\alpha^{-1} + \mathcal{O}(\alpha^{-2})],$$

for any $\alpha > 2$. Taking large enough α yields the desired bound.

Now we correct for nonzero η. First we show that η falls at least as fast as r^{-1}, a property of the Green function $\vec{D}^{-2}$ in (A.3) which is almost obvious, since $\vec{D}$ "approaches" $\vec{\nabla}$ at infinity. Rewrite (A.3) as

$$\nabla^2\eta = -2g[\vec{A},\vec{\nabla}\eta] - g^2[\vec{A},[\vec{A},\eta+\bar{\Omega}]]$$

and solve on $\{r \geq 2R\}$ with prescribed boundary data at $r = 2R$, zero data at infinity. This gives terms of the form,

$$\eta \sim \int \frac{d^3y}{|\vec{x}-\vec{y}|}\ \frac{1}{|\vec{y}|^4}\ (\eta+\bar{\Omega}) + \int \frac{d^3y}{|\vec{y}|^2}\ \eta\hat{\phi}\cdot\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) + (\text{surface term}).$$

Each term converges, even after one power of $|\vec{x}|^{-1}$ has been taken outside the integrals. Thus we have $\eta \sim r^{-1}$. Equation (A.4) now involves $\vec{D}\bar{\Omega} + \vec{D}\eta$. Both terms are $\mathcal{O}(r^{-2})$, but they cannot cancel, since the first has a curl to this order while the second does not. So $E[\Omega]$ is still bounded as above. Since the boundary data on $r = 2M$ were irrelevant to the proof, we see that indeed the bound on I_3 is completely due to the excitations of the long-range fields.

We must now prove existence for the exact Eq. (A.1). We will be fairly careful here since the result is central to our argument, and since it may not be clear at first that there really is no difficulty with strings and other singularities. Our strategy is to let $\Omega = \bar{\Omega}+\eta$ again and show that the differential operator acting on η is invertible on an appropriate space of functions. The trick will be to turn the problem into another in the calculus of variations, then use standard methods [21, 22]. The program succeeds because (A.1) is a smooth elliptic problem.

We begin with the remark that polarizing the quadratic form $E[\Omega]$ yields an inner product on the space Γ of functions Ω, namely

$$\langle u,v\rangle_M = \langle\vec{D}u,\vec{D}v\rangle_2 + \langle Mu,Mv\rangle_2,$$

with associated seminorm $\|u\|_M^2 = 2E[u]$. The Kato and Sobolev inequalities [21] give

$$\|u\|_M \geq \|\vec{\nabla}|u|\|_2 \geq (\text{const.})\ \|u\|_6$$

respectively, showing that $\|\cdot\|_M$ really is a norm. Next let $\mathscr{H}_M$ be the norm completion of Γ_0, the functions of compact support. $\mathscr{H}_M$ is a Hilbert space; it is here that we will pose our variational problem.

We note that $\|\bar{\Omega}\|_M < \infty$, regardless of whether $\bar{\Omega}$ is actually in $\mathscr{H}_M$. This is the step that fails for a single monopole.

Let $F[\eta] = E[\bar{\Omega}+\eta]$. We must show that F is defined on $\mathscr{H}_M$, where it is differentiable, strictly convex, and subject to a coercive lower bound. That is, F looks like a smooth convex bowl which turns up at its edges. By the calculus of variations F will achieve a unique local minimum at $\bar{\eta}$, which will be a critical point. The precise theorems are in ref. [21]. Setting the directional derivatives $\delta_\zeta F[\bar{\eta}] = 0$, all ζ, and using the nondegeneracy of the metric then shows that $\bar{\Omega}+\bar{\eta}$ satisfies the differential equation (A.1). Now

$$F[\eta] = \frac{1}{2}\|\bar{\Omega}\|_M^2 + \frac{1}{2}\|\eta\|_M^2 + \langle\bar{\Omega},\eta\rangle_M$$
$$\leq \frac{1}{2}(\|\bar{\Omega}\|_M + \|\eta\|_M)^2$$

by the Schwarz inequality. F is therefore defined on $\mathscr{H}_M$. Since F is just a (shifted) norm, it is strictly convex and satisfies a coercive bound.

Finally we need differentiability of F. This means that

$$\delta_\zeta F[\eta] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,[F(\eta+\epsilon\zeta) - F(\eta)] \in \mathbb{R}$$

is defined for all $\zeta,\eta \in \mathscr{H}_M$, and for every fixed η gives a bounded linear functional. We have $\delta_\zeta F[\eta] = \langle\bar{\Omega}+\eta,\zeta\rangle_M$, which is finite and defines a bounded linear functional by the Schwarz inequality.

By the calculus of variations, then, F has a unique local minimum $\bar{\eta} \in \mathscr{H}_M$, which is a critical point. We saw above that $\bar{\eta}$ vanishes at infinity, so $\Omega = \bar{\Omega}+\bar{\eta}$ satisfies the boundary condition. The smoothness of $\bar{\eta}$ is a question of elliptic regularity which we will not consider here.

Footnotes

1. A partial history of these statements goes as follows. For monopoles in the Bogomol'nyi-Prasad-Sommerfield limit, E. Weinberg computed the dimension of the moduli space [7]. He found that not enough normalizable zero modes exist about the fundamental SU(5) monopole solution to correspond to all the color generators naively expected to act non-trivially on it. Later Abouelsaood, in the course of an investigation into the collective-coordinates quantization of the monopole's dyonic degrees of freedom, discovered that under very general circumstances certain motions along the minimum of the system's potential well were unacceptable as collective coordinates [3]. Both of these analyses clearly show that the source of the ~~problem is~~ the slow fall-off of the monopole's non-Abelian fields, as r^{-1}, with distance. Abouelsaood concluded that there were no chromodyons whatever.
   Subsequently Nelson concluded that indeed some chromodyons exist, but not enough to form complete color multiplets [4]. This result was rendered intelligible when Nelson and Manohar proved that no complete set of smooth global color transformations could be defined on a sphere containing a monopole [5], a fact independently noted by Balachandran *et al.* [6]. Since monopole number can always be represented by a surface integral over the sphere at infinity [8], the problem is again a pathology of a soliton sector with slow field fall-off. Finally, Abouelsaood [3] discussed the $M\bar{M}$ system.
2. Abouelsaood [3] showed that this condition must be imposed on Ω in order to calculate correctly the moment of inertia in temporal gauge.

3. Throughout this paper "charge" will always refer to such angular momenta, not to the integral of any sort of Noether charge density. The latter notion is well known to be tricky [9].
4. In doing this, we ignore the attractive force between the monopole and antimonopole, which prevents the pair from being a time-independent solution of the equation of motion. It is reasonable to neglect this force for very large R, or very heavy monopoles; however, if one is worried about it, it is fairly easy to fiddle the theory to introduce new interactions or external forces that cancel it out altogether.
5. One of the reasons why this analysis is unconvincing by itself is that the relevant multipole moments of the system all diverge. We will see that nevertheless only the dipole properties of the system contribute to the radiation to lowest order in g.
6. Actually, the irreducible connections modulo all but the center of $\mathcal{G}$. We will not use these fine points.
7. Actually, for more complicated groups we must be more careful. See Ref. [7]. The problem does not arise for our case, nor for SU(5).
8. It also enables us to split cleanly the interesting effects in $\vec{a}$ from the suppressed effects in a_0, which is why we risk confusing the reader by adopting it for the balance of this section. All the effects of nontrivial solutions to Eq. (1.2) are now pushed into a_0, which will be seen to be irrelevant to the radiation problem.

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Figure Caption

Figures 1(a,b): Two singularity-free atlases for the $M\bar{M}$ system. The Higgs points in a constant direction outside the shaded regions. See the second of Refs. [3].