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Scaling of negative moments of the growth probability of diffusion-limited aggregates

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The qth moment $M(q)$ of the growth probability of diffusion-limited aggregates is studied for $q<0$ in terms of the value $[M(q,N)]_{\rm av}$ obtained by averaging $M(q)$ over the ensemble of all aggregates of a given number of particles N. For a range of structures that are susceptible to precise analysis, we verify that *all moments, even those for* $q<0$, obey asymptotic power-law scaling in N. Since we cannot analyze completely arbitrary structures, our analysis is not definitive. However, it does suggest the validity of a recent proposal by one of us that there is no Lifshitz-like anomaly (similar to that found for the distribution of currents in the random resistor network) leading to non-power-law scaling of the negative moments of the growth probability.

I. INTRODUCTION

Since the model was first proposed by Witten and Sander[1] some time ago, the aggregation of particles via diffusion initiated from a source at "infinity" has been the object of intense study.[2–6] Much of this work has been aimed at determining the fractal dimension D of the aggregate which occurs in d spatial dimensions. At present numerical simulations have treated extremely large clusters.[6] However, there are a number of conceptual issues which still remain unclear, in spite of the continuing numerical assault on this problem. For instance, the relation of this problem to standard critical phenomena remains obscure. In particular, various analytic treatments seem to show behavior different from that of standard critical phenomena. For example, there does not seem to be a limit in which D becomes independent of d, as one would have in conventional mean-field theory. Instead an early result[7] was that $D=(d^2+1)/(d+1)$ for large d. Also, in low dimensions there have been questions as to whether D might be different for different lattice structures as found analytically by Turkevitch and Scher,[8] but not supported by simulations.[3] Likewise the application of renormalization group techniques[9] to describe diffusion-limited aggregates (DLA) has not led to entirely clear results. Furthermore, the connection to fractal and multifractal[10–13] distributions is not totally clear. Finally, while the fractal dimension of the aggregate has been studied in detail, there are still unanswered questions concerning the growth probability.

The growth probability $p(i)$ is defined as the probability that the aggregate grows by the attachment of the next particle at site i. Note that $p(i)$ depends on the existing configuration of the aggregate and we sometimes indicate this dependence by writing $p(i,\Gamma)$, where Γ denotes an aggregate. For a nominally spherical surface $p(i)$ is essentially site independent. However, the aggregate is unstable to formation of "arms." In the fjords between these arms the growth probability will be heavily screened and can become exceptionally small. In this paper we will study how these small growth probabilities scale with the size of the aggregate. Although the exact nature of these small growth probabilities does not influence significantly many of the gross properties of the aggregates, their behavior can affect rare growth processes and also has theoretical significance, as we shall see in more detail. For instance, in view of the multifractal formalism proposed by Halsey *et al.*,[11] one would like to know whether this growth probability distribution is really multifractal. The simplest way to phrase this question is to consider the scaling of the moments,[10–14] $M(q)=\sum_i p(i)^q$, of the distribution for q in the entire interval $-\infty<q<\infty$. Since the quantity $M(q)$ is a stochastic variable, it is convenient to consider the quantity[15]

$$[M(q,N)]_{\rm av}\equiv\sum_{\Gamma_N}P(\Gamma_N)\sum_{i\in\Gamma_N}{}' \, p(i,\Gamma_N)^q \,, \tag{1}$$

where $P(\Gamma_N)$ is the occurrence probability of the aggregate Γ_N containing N particles and the prime indicates the omission of terms, if any, for which $p(i)=0$. The occurrence probability is the probability that the cluster Γ_N be formed when the cluster has grown to a size of N sites. This is the correct weighting of the cluster Γ_N within the ensemble of all N-particle aggregates. To determine $P(\Gamma_N)$ one must consider all possible growth sequences leading to a cluster of N sites. If $W(\Gamma_{N-1},\Gamma_N)$ denotes the probability that growth will occur to form the cluster Γ_N, given the existence of the cluster Γ_{N-1}, then we have the recursive relation

$$P(\Gamma_N)=\sum_{\Gamma_{N-1}}P(\Gamma_{N-1})W(\Gamma_{N-1},\Gamma_N)\,. \tag{2}$$

[We point out that W is defined so that it is properly normalized: $\sum_{\Gamma_N}W(\Gamma_{N-1},\Gamma_N)=1$.] For the distribution to be completely multifractal it is necessary that for all q the average moments obey power-law scaling in the asymptotic limit $N\to\infty$, i.e., that

$$[M(q,N)]_{\rm av}\sim N^{\psi_q}\,. \tag{3}$$

Whether or not Eq. (3) is valid is not obvious. There is ample evidence that this power-law dependence on N

holds for positive q. In the present paper we present additional evidence that this relation also holds for negative q. As an introduction, consider the related question concerning the moments of the current distribution[16,17] for the randomly diluted resistor network in which a unit current flows between two terminals separated by an asymptotically large distance L. It has been established[18] that there are anomalous configurations, shown in Fig. 1, in which the minimum nonzero current $i_{\min}$ flowing in any resistor is of order $\exp(-KN)$, where N is the number of "rungs" in the ladder and throughout this paper K denotes a constant of order unity (not necessarily the same in all occurrences). Since these configurations occur with a probability of order $\exp[-\alpha(p)N]$, where $\alpha(p)$ is a weak function of p, it is clear that for strongly negative q, the contribution from ladders with arbitrarily large N to the qth moment of the current distribution diverge if $|q|$ is large enough even for $p < p_c$, where p_c is the critical percolation concentration. In that case for large negative q there exists a function $p_c(q) < p_c$, at which the qth moment diverges. The obvious question now is, does this type of Lifshitz phenomenon occur also for DLA? In the first paper[15] submitted which considered this question, one of us argued by considering "tubes" of length L that this anomaly did *not* occur for DLA. We found that the occurrence probability $P(\Gamma)$ of the structures (of linear dimension L) with growth probability of order $\exp(-KL)$ was too small: namely it was of order

$$P(\Gamma) \sim \exp(-KL \ln L) \ . \tag{4}$$

Two objections to this argument could be raised: first of all, the growth probabilities were not correctly estimated. Here we will show that this error is not essential to the conclusion and that the argument presented in Ref. 15 is

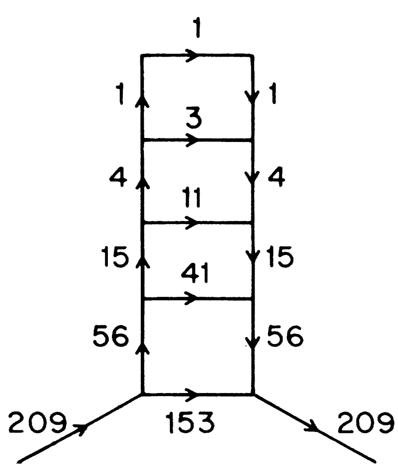

FIG. 1. Anomalous structure in the randomly diluted resistor network which leads to a minimal current whose magnitude is of order e^{-KN}, where N is a measure of the linear dimension of the structure. For percolation the occurrence probability of this structure (together with those topologically equivalent to it) is also exponentially small in N. We indicate the currents when the minimum current is normalized to have unit magnitude.

generically correct. The second objection is that the important anomalous configurations[19] may not be as simple as those considered for the random resistor network.[18] This objection is difficult to assess. In contrast, there have been suggestions[19,20] that, to the contrary, there *is* a breakdown in power-law scaling analogous to that in the resistor network. Accordingly, we consider here a number of additional possible growth sequences, some of which were suggested[21] as being responsible for the violation of power-law scaling. What we find in those cases which are amenable to analysis is in precise agreement with the original estimates: namely that structures which have growth probabilities which are exponentially small in N require a precise growth sequence. It is this requirement of a precise growth sequence which leads to an occurrence probability of order $N!^{-x}$, which for $x>0$ leads to a result of the form of Eq. (4). Note that in our discussion L denotes the length of the anomalous tube. More generally, L is defined by $p_{\min} \sim \exp(-L)$, where $p_{\min}$ is the minimum (with respect to all surface sites) of the growth probability for a given cluster Γ_N. Clearly, the whole question is, what are the occurrence probabilities for structures with a given value of L? Here we argue that these occurrence probabilities are of the form of Eq. (4) and hence that these structures are not statistically important.

What are the consequences of our conclusion, if in fact it is true? First of all, it implies that for negative moments, i.e., for $q<0$, the true asymptotic regime only occurs for extremely large L, viz., $L \sim \exp(K|q|)$, where $L \ln L$ dominates qL. Of course, even if one does not get into the true asymptotic regime, one will obtain results which have a weak dependence on L. But, putting aside numerical questions, the important conceptual result of our work would be that the growth probability of DLA in the asymptotic limit may be described completely in terms of the multifractal description.

Briefly this paper is organized as follows. In Sec. II we describe the model we use for DLA, and in particular we discuss the role of the short distance cutoff for the latticized problem. Here we correct our previous argument[15] that a linear tunnel structure does not give rise to non-power-law scaling of $[M(q,N)]_{\mathrm{av}}$. In Sec. III we estimate the minimum growth probabilities for various structures by solving the analogous electrostatic problem. In general for structures which are not one dimensional, we find that the minimum growth probability is given by a power of the length scale. In Sec. IV we construct a bound for the occurrence probability of a convoluted tunnel structure in the form of a "maze." This bound indicates that, contrary to Ref. 21, the maze does not give rise to a breakdown of power-law scaling. Some brief concluding remarks are contained in Sec. V.

II. DLA STATISTICS

There are various versions of DLA. We will focus mainly on DLA in two spatial dimensions. However, many of our results are in fact simpler and easier to establish in higher spatial dimension. Imagine growing a two-dimensional cluster starting from a single seed particle by allowing each additional particle to diffuse from a random release point on a large circle. The growth probability in this case is not strictly the harmonic measure as obtained from the solution to Laplace's equation, since there must be some coarse graining to take account of the finite size of the diffusing particle. (Viscous fingering[22,23] and dielectric breakdown[24] are probably phenomena analogous to DLA, but for which the lattice cutoff is zero.) The version of DLA which is the simplest both from a numerical and from a conceptual point of view is the latticized version in which the diffusing particle moves from one site on a lattice to a randomly chosen site in the shell of nearest neighbors. In two dimensions we will treat a square lattice and in higher dimensions a hypercubic lattice. If the diffusing particle attempts to move onto an already occupied site, it is fixed in the location it assumed before the abortive attempt. The probability that the added particle becomes fixed at site i is the growth probability. Strictly speaking, $p(i)$ is determined from the latticized diffusion equation with absorbing boundary conditions. However, in the continuum approximation the boundary value problem is equivalent to the associated electrostatic problem in which the cluster is considered to be a conductor carrying unit charge. To be precise, the particles are taken to be squares (or hypercubes in higher dimension, d) of volume b_0^d, whose edges are oriented along the lattice directions, as shown in Fig. 2. Then the growth probability of DLA at site i is equal to the total charge on the surface(s) adjacent to site i in the electrostatic problem. In this context one should note that although the charge density can be anomalously large (or small) at corners, the growth probability is equal to the total charge integrated over a region of surface

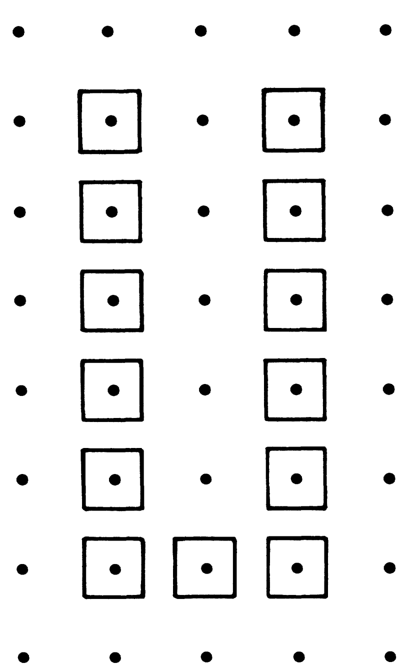

FIG. 2. "Tube" configuration, the analog for DLA of Fig. 1. Here particles are represented by squares shown schematically here.

having size of order the lattice constant. Thus, effects due to the roughness or corners in a structure which occur on a scale of length less than the lattice constant are not relevant to our discussion. Alternatively, we could confine our attention to structures whose boundaries are smooth on the scale of the lattice constant b_0.

We will treat the above described latticized version of DLA using the continuum electrostatic approach. As discussed in Refs. 15 and 18–20, a possible anomaly in the negative moments of the growth probability requires the existence of structures which (a) have a region of extremely small growth probability and (b) occur with a sufficiently large probability. In analogy with the structures having very small currents in the randomly diluted random resistor network, we previously[15] considered the growth probability in a "tube" of length L, which in the version of DLA considered here has the form shown in Fig. 2. There we noted that if site i is at the end of the tube, then

$$p(i) \sim e^{-KL} , \tag{5}$$

where L is the number of sites in the tube. We also estimated the occurrence probability $P(\Gamma)$ for this cluster to be of the form of Eq. (4), but since this discussion was not quite correct, we present a reformulation here. Although only a straight tunnel was considered in Ref. 15, it is easy to show that the entropic factor associated with a winding tunnel (which does not intersect itself) does not affect the estimate of Eq. (4).

The occurrence probability that a cluster Γ_N of N particles be formed from an initial seed Γ_1 is given by

$$P(\Gamma_N) = \sum_{\Gamma_2, \Gamma_3, \ldots, \Gamma_{N-1}} \prod_{i=2}^{N} W(\Gamma_{i-1}, \Gamma_i) . \tag{6}$$

Observe that the tube structure is in the form of a chain bent into the shape of the letter "U." Thus starting from a given seed, there are at most 2^N growth sequences to create a specified linear structure of N particles. We can obtain an upper bound to $P(\Gamma)$ by taking the maximal growth probability for adding particle $k+1$ to a chain of k particles. This maximal probability occurs for adding the particle at either end of an existing straight line of occupied sites. We denote this probability as p_k. Then for any one-dimensional structure (whether bent or straight) we have the bound

$$P(\Gamma_N) \le 2^N \prod_{k=1}^{N-1} p_k , \tag{7a}$$

with equality for a straight chain of N particles. To determine p_k, we need to find the charge distribution on a rectangular conductor which carries unit charge and has length kb and width b. Then p_k is the total charge on the end of the conductor when the conductor carries unit charge. Since the scaling properties of this electrostatics problem are not usually explicitly discussed, we will analyze this problem in some detail in the Sec. III. There we will find that $p_k \sim Kk^{-1/2}$ for large k, so that Eq. (7a) becomes

$$P(\Gamma_N) \le (2K)^N (N!)^{-1/2} \sim \exp\left[\gamma N - \tfrac{1}{2} N \ln N\right] , \quad (7b)$$

where $\gamma = \ln(2K) + \frac{1}{2}$.

The important conclusion is that the occurrence probability of a chain obeys the bound of Eq. (4) (with $K = \frac{1}{2}$). The smaller-than-exponential probability can be traced to the fact that the structure must be built up in a more or less prescribed way. The entropic factor 2^N is not sufficient to modify the smallness caused by the factor $(N!)^{-x}$, where $x = \frac{1}{2}$, here, but more generally need only be nonzero to ensure the validity of Eq. (4). In this sense the discussion given in Ref. 15 is wrong in that it incorrectly took $x = 1$. (For spatial dimension 3 or greater, $x = 1$ is correct.) However, the dominating effect is that a linear structure must be built up in a definite prescribed sequence.

III. ELECTROSTATIC ANALYSIS

A. The tube configuration

A simple way to obtain the explicit solution for the charge distribution on a rectangular conductor subject to the boundary condition that far from the conductor the field is that of a (two-dimensional) unit point charge is to use conformal mapping.[25] In this approach one maps the known solution to Laplace's equation for the potential outside an infinitely thin conducting strip of unit length centered on the origin, as shown in Fig. 3(a), into the potential *outside* a rectangular conductor, as shown in Fig. 3(b).

Therefore we start by considering the conducting strip shown in Fig. 3(a) in the z_1 plane. The electrostatic problem we wish to solve is the following: The electrostatic potential $\Phi(x_1, y_1)$ has to obey

$$\nabla^2 \Phi = 0 , \quad (8)$$

subject to the boundary conditions that Φ is a constant on the surface of the conducting strip shown in Fig. 3(a), and that far from the conductor the electric field is that of a point charge:

$$\Phi(\mathbf{r}) \sim -2Q \ln r , \quad r \to \infty . \quad (9)$$

(We will use the terminology of two dimensions, so that the solution for a "point charge" is that for a line charge in three dimensions. Also when we refer to "charge" it has the dimensions in cgs units of esu/cm. In any event, to obtain growth probabilities we set $Q = 1$.) We start from the trivial solution for the potential in the z plane outside a conductor whose boundary is the unit circle. For this purpose we consider the complex potential $W(z)$ (of which Φ is the real part)

$$W(z) = -2Q \ln z , \quad (10)$$

where $z = x + iy$. Consider the mapping $z_1 = (z + z^{-1})/2$. This transformation maps the exterior of the unit circle in the z plane into the entire z_1 plane with a branch cut along the real axis from $z_1 = -1$ to $z_1 = 1$. Applying this mapping to Eq. (10) we see that the complex potential in the presence of the charged conducting strip of Fig. 3(a) is determined by

$$z_1 = \cosh\left[\frac{W}{2Q}\right] \quad (11)$$

and the condition that $\mathrm{Re}(W/Q)$ is negative. Note that the magnitude of the charge density σ, on one face of the strip is given by

$$|\sigma| = \frac{1}{4\pi} \left| \frac{dW}{dz_1} \right| . \quad (12)$$

For $z_1 = x_1 + i0^+$, we find that

$$\sigma(x_1) = \frac{1}{4\pi i} \frac{dW}{dz_1} = \frac{Q}{2\pi} (1 - x_1^2)^{-1/2} , \quad |x_1| < 1 . \quad (13)$$

Now we wish to find the electrostatic potential outside the conducting *rectangle* covering the region in the z_2 plane $|x_2| < a$ and $|y_2| < b$, and we will assume a large aspect ratio: $a/b \gg 1$. To solve this problem we need to map the upper half of the z_1 plane into the unshaded region of the z_2 plane shown in Fig. 3. The Schwarz-Christoffel mapping which relates z_1 to z_2 is determined by the differential equation

$$\frac{dz_2}{dz_1} = \mathrm{const} \times (z_1 - a_1)^{-\alpha_1/\pi} (z_1 - a_2)^{-\alpha_2/\pi} \times (z_1 - a_3)^{-\alpha_3/\pi} \cdots . \quad (14)$$

The effect of this mapping is shown in Fig. 4. The real axis in the z_1-plane is mapped into a polygon having exterior angles $\alpha_1, \alpha_2, \ldots$. For our purpose we take $-a_1 = a_4 = 1$ and $-a_2 = a_3 = \beta$, where β is to be determined, and $\alpha_1 = -\alpha_2 = -\alpha_3 = \alpha_4 = \pi/2$. To orient the rectangle correctly the constant in Eq. (14), which we call B, must be real. Then

$$\frac{dz_2}{dz_1} = B (z_1^2 - 1)^{-1/2} (z_1^2 - \beta^2)^{1/2} . \quad (15)$$

The relation between the size of the rectangle and the constants in the mapping is obtained by integrating Eq. (15) over the two sides of the rectangle. In that way we obtain

$$B \int_0^\beta dx_1 (1 - x_1^2)^{-1/2} (\beta^2 - x_1^2)^{1/2} = a , \quad (16a)$$

$$B \int_\beta^1 dx_1 (1 - x_1^2)^{-1/2} (x_1^2 - \beta^2)^{1/2} = b . \quad (16b)$$

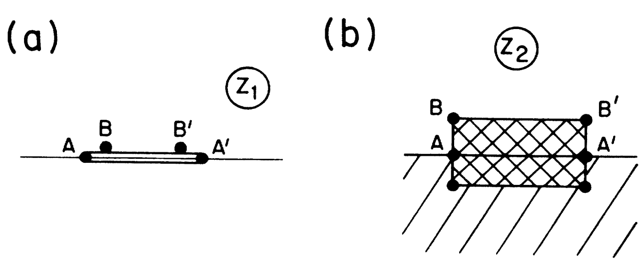

FIG. 3. Mapping of the upper half plane of the z_1 plane in panel (a), left, into the unshaded region above both the x_2 axis and the upper boundary of the rectangular conductor which occupies the cross hatched region, shown in panel (b), right. Points related by the mapping are indicated by A and A', etc.

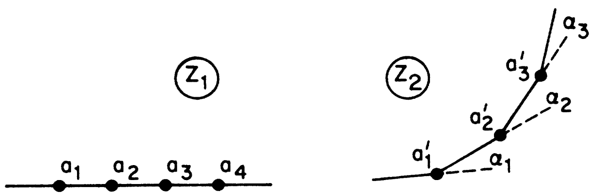

FIG. 4. Effect of the Schwartz-Christoffel mapping which takes the real axis of the z_1 plane (left) into a polygon, in the z_2 plane, some of whose vertices are shown at right. The points a_j are mapped into the vertices a_j', where the exterior angle is α_j.

From these we find that

$$\frac{\int_{\beta}^{1} ds(1-s^2)^{-1/2}(s^2-\beta^2)^{1/2}}{\int_0^{\beta} ds(1-s^2)^{-1/2}(\beta^2-s^2)^{1/2}} = \frac{b}{a} . \tag{17}$$

For small b/a, β is close to unity. In this limit the denominator can be replaced by unity. In the numerator we may approximate $(1-s^2)$ by $2(1-s)$ and $s^2-\beta^2$ by $2(s-\beta)$. Then

$$\int_{\beta}^{1}(1-s)^{-1/2}(s-\beta)^{1/2}ds = \frac{b}{a} . \tag{18}$$

Evaluating this integral we find that

$$\beta \approx 1 - \frac{2b}{\pi a} . \tag{19}$$

To find the charge on the end of the rectangle, we proceed as follows. The charge between two points $z_2^{(a)}$ and $z_2^{(b)}$ on the surface (in the z_2 plane) is given by

$$\Delta Q = -\frac{1}{4\pi}\int_{z_2^{(a)}}^{z_2^{(b)}} \frac{\partial \mathrm{Re} W}{\partial n} dl , \tag{20a}$$

where ∂n lies along the outward normal to the surface and dl is an element of length along the surface. By the Cauchy-Riemann condition (assuming now that dl describes a counterclockwise traversal of the surface) this gives

$$\Delta Q = -\frac{1}{4\pi}\int_{z_2^{(a)}}^{z_2^{(b)}} \frac{\partial \mathrm{Im} W}{\partial l} dl = \frac{1}{4\pi}[\mathrm{Im} W(z_2^{(a)}) - \mathrm{Im} W(z_2^{(b)})] . \tag{20b}$$

But since the potential itself is constant on the surface of the conductor, we have

$$\Delta Q = \frac{i}{4\pi}[W(z_1 = z_1^{(b)}) - W(z_1 = z_1^{(a)})] , \tag{20c}$$

where we wrote the result in terms of the corresponding points in the z_1 plane, in which case Eq. (11) is directly applicable. Using Eq. (20c) we write the total charge on one end of the rectangle as

$$\Delta Q = \frac{i}{2\pi}[W(z_1 = \beta + i0^+) - W(z_1 = 1)] . \tag{20d}$$

Evaluating Eq. (20d) we obtain the exact result

$$\Delta Q = (Q/\pi)\cos^{-1}\beta . \tag{21}$$

For $a \gg b$ this gives

$$\Delta Q = (2Q/\pi)\sqrt{b/(\pi a)} . \tag{22}$$

Thus the growth probability for adding a particle at the end of a chain of k sites for $k \gg 1$ is

$$p_k = 2\pi^{-3/2}k^{-1/2} . \tag{23}$$

We used this result to obtain the bound in Eq. (7), above.

B. Scaling considerations

One can easily generalize the above solution to the case of a one-dimensional structure on the x axis of length $2a$ and width $2b$, whose center is at the origin $x=0$, but whose end can have some irregular shape on the scale of length $2b$. Specifically, we are interested in the charge distribution on a conducting structure carrying total charge Q whose aspect ratio a/b is very large. We will assume that at distances much greater than $2b$ from either end, the object can be considered to be a uniform rod. In that case, as we will argue, the charge distribution is given to leading order in b/a by

$$\sigma(s) = (Q/a)(a/b)^{\eta} f(s/b), \quad |s|/b \le M \tag{24a}$$

$$\sigma(x) = (Q/a)g(x/a) , \quad ||x|-a| \gg b \tag{24b}$$

where s measures the distance along the surface from the end of the object and M is a large finite number. Thus the charge distribution near the end is characterized by a *local* shape function f which is sensitive to the detailed shape of the object near its end. The global aspect ratio a/b only appears insofar as it affects the overall amplitude factor. Equation (24b) expresses the fact that far from the end, the details of the shape of the end are irrelevant. Comparison with Eq. (13) indicates that

$$g(y) = (2\pi)^{-1}(1-y^2)^{-1/2} . \tag{25}$$

Thus, when $x = a - Mb$, Eq. (24b) gives

$$\sigma(x) \approx [Q/(2\pi a)][a/(2bM)]^{1/2} . \tag{26}$$

Requiring consistency with Eq. (24a) with $s = Mb$ indicates that $\eta = \frac{1}{2}$ and that

$$f(x) \sim (2\pi)^{-1}(2x)^{-1/2} , \quad x \to \infty . \tag{25'}$$

To motivate Eq. (24a) we note that the surface charge density σ is determined by the condition that the tangential component of the electric field vanish at all points on the surface. We write the tangential field as that, E_{far}, due to charges far from the end and that, E_{near}, due to charges near the end. For the far field we assert that the charge distribution far from the end does not depend on the detailed shape of the end. So to leading order in b/a we have

$$E_{\mathrm{far}} = \frac{Q}{2\pi a}\int_{-a}^{a-Mb} \frac{dx}{[1-(x/a)^2]^{1/2}} \frac{1}{a-x} = \mathrm{const} \times \frac{Q}{a}\left[\frac{a}{Mb}\right]^{1/2} , \tag{27a}$$

where M is a large number. Also E_{near} is the tangential component of

$$\mathbf{E}_{\rm near}=\int_{s_-(Mb)}^{s_+(Mb)}\frac{[\mathbf{r}-\mathbf{r}(s)]\sigma(s)ds}{|\mathbf{r}-\mathbf{r}(s)|^2}\ , \tag{27b}$$

where $\mathbf{r}$ is a point on the surface near the end. Here $s_+(t)$ $[s_-(t)]$ is the value of s at $x=a-t$ for the lower (upper) surface and later $\hat{\mathbf{n}}(\mathbf{r})$ is the unit tangent vector at $\mathbf{r}$. Thus the total tangential field (which must vanish) is

$$\text{const}\times\frac{Q}{a}\left[\frac{a}{Mb}\right]^{1/2}\hat{\mathbf{n}}(r)\cdot\hat{\mathbf{x}}+\int_{s_-(Mb)}^{s_+(Mb)}\frac{\hat{\mathbf{n}}(\mathbf{r})\cdot[\mathbf{r}-\mathbf{r}(s)]\sigma(s)ds}{|\mathbf{r}-\mathbf{r}(s)|^2}=0. \tag{28}$$

Thus $\sigma(s)=(Q/a)(a/b)^{1/2}h(s,b)$. But dimensionality considerations indicate that the lengths s and b can only enter in the ratio (s/b). Hence we obtain Eq. (24a). In the Appendix we explicitly verify that the exact solution for the rectangle is of the scaling form of Eq. (24) with $f(x)$ given by Eq. (25′).

In three (or more dimensions) the scaling of Eq. (24) becomes trivial. The charge density is constant far from either end so that $g(x/a)$ is a constant. Furthermore, $\eta=0$, reflecting the fact that the field due to very distant charge is negligible.

C. Comparison with ellipse result

It is interesting to compare this solution with that one obtains for a long thin ellipse which one might try to identify with a rectangle. In this connection we have to take proper account of the lattice cutoff. The ellipse can not be so thin that it passes through several lattice points when its thickness is much less than b_0. This point is illustrated in Fig. 5. Suppose the equation of the ellipse is written as

$$(x/X)^2+(y/Y)^2=1\ , \tag{29a}$$

with $Y\ll X$, so that the ratio of minor to major axes is $Y/X\ll 1$. If this ellipsoid is to have a curvature on the scale of a cluster of lattice points, then for small x near X we should have

$$x=X-(K/b_0)y^2\ , \tag{29b}$$

so that when $(X-x)/b_0$ is of order unity, (y/b_0) will also be of order unity. For convenience we will take the largest permissible value of K to be $\frac{1}{2}$. But from Eq. (29a) we actually get

$$x=X-\tfrac{1}{2}X(y/Y)^2\ . \tag{29c}$$

In view of Eq. (29b) we see that for a fixed value of Y the largest value of X consistent with the lattice cutoff is given by

$$b_0X=Y^2\ . \tag{30}$$

Now suppose the ellipsoid has a minor axis of length M lattice constants: $Y=Mb_0$. Then Eq. (30) gives the largest allowed value of X for this Y to be

$$X=M^2b_0\ . \tag{31}$$

Accordingly, the smallest allowed ratio of axes is

$$\frac{Y}{X}=\frac{Mb_0}{M^2b_0}=M^{-1}=\left[\frac{b_0}{X}\right]^{1/2}\ . \tag{32}$$

Thus, to identify an ellipse with a cluster on a lattice, the maximum value of X/Y consistent with the lattice cutoff is equal to the square root of the length (measured in lattice constants.) Note that without considering the effect of this cutoff one cannot identify the result for the ellipse with a lattice structure. The charge density at the end of the ellipse, $\sigma_{\rm end}$, is of order $\sigma_{\rm end}\sim(Q/a)(a/b)$. From this form one might have guessed that $\sigma_{\rm end}\sim Q/b$, rather than $\sigma_{\rm end}\sim Q/(ab)^{1/2}$ as given by Eq. (24a).

FIG. 5. Ellipsoids which are (left) and which are not (right) consistent with the lattice cut-off to serve as models for aggregates.

D. Lacunae

Here we estimate the growth probabilities for lacunae. The motivation for considering such structures is as follows. One believes that DLA gives rise to objects that look like snowflakes: the structure has large branches which may possibly get quite close to one another. When this happens, they will enclose a large "gallery," which we will model as a circular hole. What we will show here is that the minimum growth probability inside a circular gallery of radius R is of order AR^{-x}, where A is the probability that the diffusing particle enter the gallery. Of course, if the gallery has an aspect ratio that is far from unity, then it should be classified as a tube, as we shall see. With respect to an array of galleries in series, one can say the probability that a diffusing particle find its way through such a sequence of galleries is given by $\prod_i AR_i^{-x}$. For a large number of galleries in series, one is led back to the tube, except that here we operate on a different length scale. But a series of obstacles probably has an occurrence probability similar to that of a tube, in that in all likelihood it must be built by a prescribed growth sequence. Since this argument rests on power-law scaling for the minimum growth probability for a circular cavity, we examine that case now.

As a start, let us consider a simple electrostatics problem in which one has a conductor on the x axis from which is removed an interval of width $2d$ centered about the origin, as shown in Fig. 6. The electric field for large positive y goes to a constant and for large negative y it goes to zero. This simulates DLA with a particle source at large positive y and a seed on the x axis with a slot. The boundary condition at large positive y can be mimicked by putting a uniform line of charge density σ at very large y. One can then ask what is the y component of electric field in the slot. Clearly the charge density on the x axis does not contribute to this field. Thus, in the slot the y component of the electric field is $2\pi\sigma$. That is, the electric field is *exactly* half what it is just above the conductor far from the slot. This means that the exact result for the *total* charge Q_-, on the underside of the plate is

$$Q_- = \tfrac{1}{2} A\sigma \ , \tag{33}$$

where A is the "area," $2d$ of the slot. The factor $\frac{1}{2}$ has an immediate interpretation in terms of the diffusion problem with absorbing boundary conditions. Clearly, the particle has about the same chance to reach any point on the x axis *for the first time*. However, if it hits the seed (the conductor) it sticks and the process ends. In contrast, if it hits the slot, it can then either go through the slot or be reflected at the slot. The probability for each outcome is $\frac{1}{2}$, of course, and this explains the $\frac{1}{2}$ in Eq. (33). In fact, this symmetry is exact: each random walk, after reaching the slot, has a partner obtained by reflection about the x axis.

It is useful to solve the above problem exactly, since we will use the solution as a basis for an approximate treatment of galleries with small apertures. We start with a uniform field in the z_1 plane, where the entire x_1 axis is a conductor [See Fig. 6(a)], so that

$$W(z_1) = iAz_1 \ . \tag{34}$$

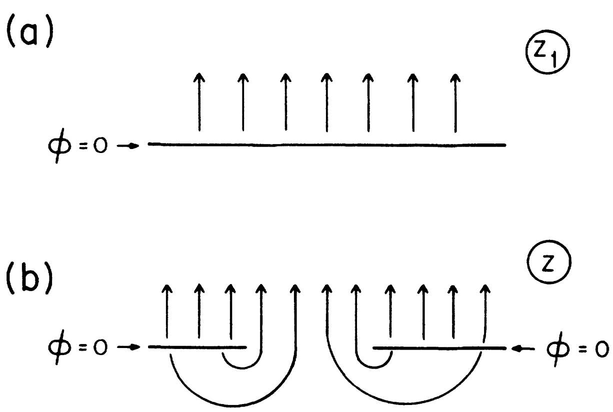

FIG. 6. (a) Top: A conductor on the x_1 axis of the z_1 plane. (b) Bottom: A conductor on the x-axis with a small slot of width $2d$ at the origin. The boundary condition at infinity is that the electric field is a constant at large positive y and tends to zero at large negative y. Some field lines are shown qualitatively. The analogous diffusion problem is one in which particles are released at infinite positive y and are absorbed at the surface of the conductor. A small fraction of particles diffuse through the slot.

Now we use the mapping

$$z = \tfrac{1}{2} d(z_1 + z_1^{-1}) \ , \tag{35}$$

which maps the upper half of the z_1 plane into the entire z plane. The region in the upper half of the z_1 plane which is exterior (interior) to the unit circle is mapped into the upper (lower) half of the z plane. For purposes of visualization, we can imagine a trap door on the x_1 axis with hinges at $x_1 = 1$ and $x_1 = -1$, the two doors meeting at $x = 0$. Both panels are opened downward completely until they hit the underside of the x_1 axis, and the doors are simultaneously stretched to infinite length. Thus, as indicated in Fig. 6(b), the right trap door ($z_1 = x_1 + i0^+$, $0 < x_1 < 1$) is mapped into ($z = x - i0^+$, $1 < x < \infty$) and similarly for the left door. Thus the simple problem in the z_1 plane is mapped into the desired problem of a conductor with a missing interval in the z plane. The charge density in the z plane (for z on the surface of the conductor) is given by

$$\sigma(z) = \frac{1}{4\pi i}\frac{dW}{dz} = \frac{Az_1}{2\pi d(z_1 - z_1^{-1})} \ . \tag{36}$$

We fix the value of A by requiring that $\sigma = \sigma_0$ far from the slot. There, where z_1 is large and real, we obtain $\sigma_0 = A/(2\pi d)$, so that $A = 2\pi\sigma_0 d$.

By inverting Eq. (35) to get z in terms of z_1, and noting that the two choices of sign in this solution correspond to points on opposite sites of the conductor, we find the charge densities on the top, σ_+, and bottom, σ_-, of the conductor to be

$$\sigma_\pm = \tfrac{1}{2}\sigma_0 \left[\pm 1 + \frac{|x/d|}{[(x/d)^2 - 1]^{1/2}} \right] , \tag{37}$$

where $|x| > d$, of course.

Since the slot corresponds to the upper half of the unit circle, it corresponds to $z_1 = e^{i\theta}$. Now use Eq. (35):

$$z = d\cos\theta \Longrightarrow x = d\cos\theta \ . \tag{38}$$

Thus the electrostatic potential in the slot is given by

$$\Phi = -Ay_1 = -A\sin\theta = -A[1-(x/d)^2]^{1/2} \ , \quad |x| \le d \ . \tag{39}$$

Now consider the relevant problem of the circle with a small opening in it, shown in Fig. 7. This problem can be solved exactly,[25] for arbitrary opening angle, by means of a conformal mapping. Instead, we use a simple approximate method which is correct when the opening angle is small and is easily generalized to rectangular galleries as well as to simple three-dimensional galleries. We will take the solution in Eq. (39) as giving the potential in the opening in the limit when the size of the opening $2d$ is much smaller than the radius R of the circle. Using the continuity of the potential to match the inside and outside solution to Laplace's equation, we write

$$\Phi = -2Q\ln(r/R) + \sum_{m=-\infty}^{\infty} a_m (R/r)^{|m|} e^{im\theta} \ , \quad r > R \tag{40a}$$

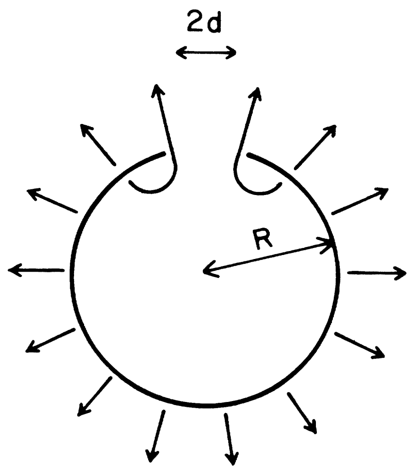

FIG. 7. Similar to Fig. 6, except that the conductor is a circle of radius R with a slot of length $2d$, with $d \ll R$.

$$\Phi = \sum_{m=-\infty}^{\infty} a_m (r/R)^{|m|} e^{im\theta} , \quad r<R . \tag{40b}$$

Knowing the potential for $r=R$ [it is zero on the conductor and is given by Eq. (39) in the opening], we can find the a_m's:

$$a_m = \frac{1}{2\pi}\int_{-d/R}^{d/R} e^{-im\theta}(-2\pi\sigma_0 d)[1-(R\theta/d)^2]^{1/2} d\theta , \tag{41}$$

where we used the previously determined value of A and for $d \ll R$ we set $\sigma_0 = Q/(2\pi R)$. Using the expressions (40a) and (40b) for Φ it is easy to show that at any angle, the charge density on the outside of the circle is larger than that on the inside by exactly σ_0. It then follows that the total charge on the inner surface of the conductor is given by $Q_{\text{in}} = Q\theta_0/(4\pi)$, where θ_0 is the total angle subtended by the slot. For small θ this reduces to $Q_{\text{in}} = \sigma_0 d$ in agreement with Eq. (33). We calculate $\partial\Phi/\partial r$ at $r=R$ inside the circle and find the charge density there to be

$$\frac{\sigma(\theta)}{\sigma_0} = -\frac{d^2}{4\pi R^2}\sum_{m=-\infty}^{\infty} |m| e^{im\theta}\int_{-1}^{1}(1-x^2)^{1/2} e^{imdx/R} dx \tag{42a}$$

$$= \frac{d^2}{8\pi R^2}\int_{-1}^{1}\frac{(1-x^2)^{1/2}}{\sin^2[\frac{1}{2}(\theta + xd/R)]} dx . \tag{42b}$$

One can write this solution in a scaling form. When the distance s from the edge of the slot is comparable to d, we obtain

$$\sigma(\theta)/\sigma_0 = F(s/d) , \tag{43a}$$

where the local function F is given by

$$F(y) = \frac{1}{2\pi}\int_{-1}^{1}\frac{(1-x^2)^{1/2}}{(1+x+y)^2} dx . \tag{43b}$$

This result agrees with that of Eq. (37) for the charge density on the underside of a slotted conducting plane. For finite angles (so that the distance to the opening is large compared to the size of the opening), we have

$$\sigma(\theta)/\sigma_0 = (d/R)^{\eta} f(\theta) , \tag{44a}$$

where $\eta = 2$, and

$$f(\theta) = [4\sin(\tfrac{1}{2}\theta)]^{-2} . \tag{44b}$$

Again, note that these two solutions fit together smoothly: as $\theta \to d/R$, $(d/R)^{\eta} f \to \text{const}$, to agree with Eq. (43a). Thus we conclude that in a cavity of aspect ratio of order unity (in which case it is generically a circle) the minimum charge density (or minimum growth probability for DLA), which occurs at $\theta = \pi$, is of order $A(b_0/R)^2$, where A is the probability of entering the cavity and we have set d equal to the minimum size of opening, i.e., the lattice constant.

Finally, we extend the above result to a cavity of arbitrary aspect ratio, i.e., a rectangular cavity of height $2a$ and width $2b$ in which there is a small hole of width $2d$ in the top. The potential $\Phi(x,y)$ is required to satisfy the boundary conditions that it vanish on the surface of the conductor and should be of the form of Eq. (9) at large distances from the conductor. To implement these boundary conditions it is convenient to choose the axes as shown in Fig. 8. Then we incorporate the boundary conditions except those on the top surface by an inside solution of the form

$$\Phi(x,y) = \sum_{n \text{ odd}} c_n \cos[n\pi x/(2b)]\sinh[n\pi y/(2b)]/\sinh(n\pi a/b) . \tag{45}$$

This will be a correct solution providing Φ is the correct potential for $y=2a$. But $\Phi=0$ except in the gap, where we know from our previous solutions that it is given (for $d/b \ll 1$) by

$$\Phi_{\text{gap}}(x) = -2\pi\sigma_0 d[1-(x/d)^2]^{1/2} , \tag{46}$$

where σ_0 is the charge density at the center of the top in the absence of a gap. Thus we determine the c_n in Eq. (45) to be

$$c_n = \frac{1}{b}\int_{-d}^{d}\Phi_{\text{gap}}(x)\cos[n\pi x/(2b)] dx . \tag{47}$$

In calculations for the bottom surface we neglect terms in Eq. (45) with $n>1$ in view of the factor $\sinh(n\pi a/b) \gg 1$. This approximation is reasonable even when $a \approx b$, and becomes rigorous for $a/b \gg 1$. Thus the charge density on the bottom *inside* surface is

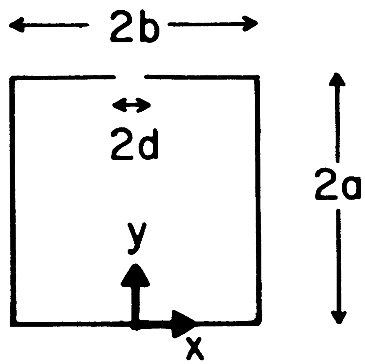

FIG. 8. Choice of axes for the rectangular cavity. The origin is taken at the center of the bottom side and the axes are oriented as shown by heavy lines.

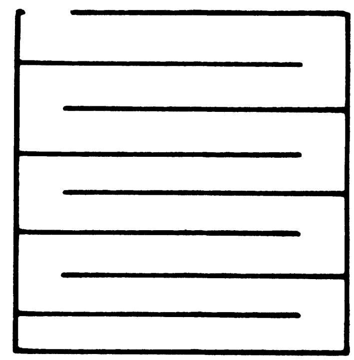

FIG. 9. The "maze" structure proposed by Lee *et al.* (Ref. 21).

$$\sigma(x,0)=-(4\pi)^{-1}\partial\Phi(x,0)/\partial y \approx \sigma(0,0)\cos[\pi x/(2b)] \ , \tag{48a}$$

where

$$\sigma(0,0)=-(4\pi)^{-1}\partial\Phi(0,0)/\partial y \approx -[c_1/(8b)]e^{-\pi a/b} \ . \tag{48b}$$

If furthermore $d/b \ll 1$, then Eqs. (46) and (47) yield:

$$c_1 \approx -2\pi\sigma_0(d/b)\int_{-d}^{d}[1-(x/d)^2]^{1/2}dx = -\pi^2\sigma_0 d^2/b \ . \tag{49}$$

Thus

$$\sigma(0,0)\approx \tfrac{1}{8}\sigma_0\pi^2(d/b)^2 e^{-\pi a/b} \ . \tag{50}$$

We see that the exponential dependence correctly crosses over to the result similar to that for the tunnel when a/b becomes large. From Eq. (48a) one sees that the charge density vanishes exactly in the inside corners. As we have said, however, one should integrate the charge density over an interval of length b_0 to take proper account of the lattice cutoff. When this is done one finds that the charge on an interval of length b_0 at the corner is of order

$$\Delta Q \sim (b_0^2/b)\sigma(0,0) \tag{51}$$

so that only the power-law prefactors are modified near the corner. The general conclusion from the rectangular cavity is that exponentially small growth only occurs in the limit when the cavity becomes a tunnel.

IV. OCCURRENCE PROBABILITY OF SPECIAL STRUCTURES

A. The "maze" structure

In this section we calculate the occurrence probability of the "maze" structure, shown in Fig. 9, which has been suggested by Lee *et al.*[21] as being responsible for a breakdown of power-law scaling. We characterize the maze structure as one in which there are N_v tubes, each of length N_h sites. For simplicity we will assume that $N_v = N_h = L$, so that the total number of sites in the maze is approximately $N = L^2$. The total linear distance (measured in units of lattice constants) of the path leading from the entrance of the maze to the bottom site in the maze is also L^2. Here we consider the asymptotic limit where N_v and N_h are large and consequently we ignore end effects. That is, we do not worry about the distinction between, say, N_h and N_h+1. We will then show that the occurrence probability P_{maze} of the maze structure is of order

$$P_{\text{maze}} \sim \exp(-KN\ln N) \ . \tag{52}$$

We now consider possible growth sequences of the maze. First of all, suppose the maze is built up "in order," i.e., sequentially from the bottom to the top. We construct an upper bound for the contribution δP_{maze}, to the occurrence probability from such a sequence. To do that note that the maze is formed by $N^{1/2}$ arms, each of which has length $N^{1/2}$ lattice constants. If we calculate the probability of forming each arm by neglecting the shielding caused by the previously constructed arms, we will clearly have an overestimate of the growth probability. Thus the probability to form a single arm of length $N^{1/2}$ obeys the bound of Eq. (7b) with N there replaced now by $N^{1/2}$. Thus for the entire structure of $N^{1/2}$ such arms we have the bound

$$\delta P_{\text{maze}} = [(2K)^{\sqrt{N}}(\sqrt{N}!)^{-1/2}]^{\sqrt{N}} = \exp(\gamma N - \tfrac{1}{4}N\ln N) \ . \tag{53}$$

Now let us consider the effect of including other growth sequences. If all growth sequences were equally likely, the occurrence probability would be of order $\exp(-KN)$ rather than of the order in Eq. (53), as we shall see in Sec. IV B. Consider a growth sequence which has as an intermediate state the one shown in Fig. 10(b). As we shall see, there is a price to pay for building the cluster "out of order," i.e., by building walls nearer the mouth of the maze before ones at the bottom of the maze. We now obtain an upper bound [which is smaller than

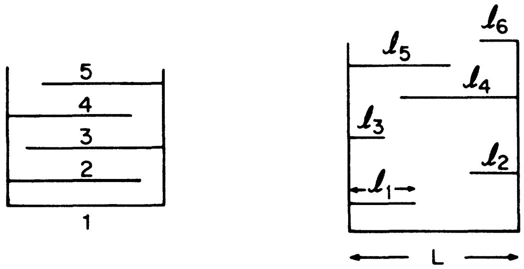

FIG. 10. Left: build-up of the maze structure "in order." The numbers on each segment indicate the order in which it is assembled. Right: Generic intermediate state of aggregation of the maze structure in which it is built up "out of order."

that of Eq. (52)] for the probability of obtaining the maze starting from the intermediate state shown in Fig. 10(b). This bound will be used to show that such growth sequences can be ignored. For this bound, we assume an optimal growth sequence from the starting point of Fig. 10(b). Let l_i be the number of sites already present in row i, where the rows are labeled with $i=1$ at the bottom and $i=L$ at the mouth of the maze. One sees that if $l_i+l_{i-1}<L$, as is shown in Fig. 11(a), then one will have to fill in l_i-1 sites under the overhang caused by row i. This will involve particles moving down a tube, first of length 1, then of lengths 2, 3 ,..., l_i-1. Since the probability of reaching the end of a tube of length k is of order $\exp(-\alpha k)$, the combined probability to form the row under the overhang of row i is of order $\exp(-\frac{1}{2}\alpha l_i^2)$. The other case occurs if $l_i+l_{i-1}>L$, as is illustrated in Fig. 11(b). Then $L-l_{i-1}$ particles have to diffuse to the end of tubes whose length varies from $l_i+l_{i-1}-L$ for the first added particle to l_i for the last added particle. In this case the combined probability for filling out row $i-1$ is therefore of order $\exp[-\frac{1}{2}(L-l_{i-1})(2l_i+l_{i-1}-L)]$. Of course, if $l_i+l_{i-1}>L$, then particles which are added to rows below $i-1$ have to pass through a constriction, but we neglect that fact, since we will already find a strong enough upper bound for the growth probability. Thus in terms of the l_i's we see that an upper bound for the probability $P_{i\to f}$ of growing from the initial configuration to the finally complete maze is given by

$$P_{i\to f}<\exp\left[-\sum_{i=2}^{L-1}F_i\right], \tag{54}$$

where

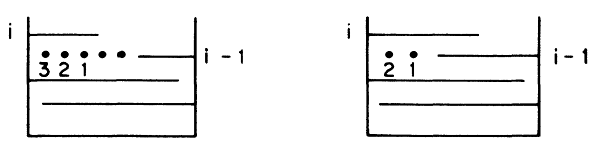

FIG. 11. Buildup of a lower layer under an overhang. Left: the case when row i and row $i-1$ are small. Right: the case when rows i and $i-1$ already overlap.

$$F_i=\tfrac{1}{2}\alpha l_i^2\ ,\quad l_i+l_{i-1}<L\ , \tag{55a}$$

$$F_i=\tfrac{1}{2}(L-l_{i-1})(2l_i+l_{i-1}-L)\ ,\quad l_i+l_{i-1}>L\ . \tag{55b}$$

Now let us use this bound when a fraction δ of the L rows satisfy

$$\epsilon L<l_i<(1-\epsilon)L\ ,\quad \epsilon L<l_{i-1}<(1-\epsilon)L\ ,\qquad i=i_1,i_2,\ldots,i_n\ , \tag{56}$$

where $n=\delta L$. For rows with l_i or l_{i-1} outside the bound of Eq. (56), we replace F_i by zero. (To obtain an upper bound for $P_{i\to f}$ we want a lower bound for the F_i's.) For the l_i's which obey the bound of Eq. (52) we have that F_i is at least of order $(\epsilon L)^2$. Since this condition holds for at least δL rows, we have that

$$P_{i\to f}<\exp(-K\delta\epsilon^2L^3)=\exp(-K\delta\epsilon^2N^{3/2})\ , \tag{57}$$

where K is a constant of order unity. Furthermore, we may take account of the "entropy" associated with all such growth sequences by multiplying the result in Eq. (57) by $N!$ which is obviously an upper bound for the number of growth sequences which form the desired final state from the intermediate state under consideration. Then if we call P^{tot} the contribution to the occurrence probability from the entire class of growth sequences involving a single intermediate state satisfying the condition of Eq. (56), we have

$$P_{i\to f}^{\text{tot}}<\exp(KN\ln N-K\delta\epsilon^2N^{3/2})\ . \tag{58}$$

Now we take $\epsilon=\delta=N^{-1/10}$. The conclusion is that in total all intermediate states in which $L^{9/10}$ rows (i.e., a fraction $L^{-1/10}$ of the rows) have l_i's in the interval $L^{9/10}<l_i<L-L^{9/10}$ (this condition means that l_i/L can not be infinitesimally close to either 0 or 1) make contributions to the occurrence probability which is small in comparison to that from the "in order" sequence. Note that the prohibited sequences of growth are those in which the maze is built up even slightly out of order. Slightly here means a growth sequence is out of order if condition (56) is obeyed.

To obtain a bound on the occurrence probability $P(\Gamma)$ we note that the occurrence probability of state i referred to in Eq. (58) is at most unity. Therefore we can interpret Eq. (58) as giving a bound on the contribution to the occurrence probability due to the family of growth sequences involving state i. The total number of such growth sequences [defined by satisfying Eq. (56)] is at most of order $N!$. Even multiplying the result of Eq. (58) by such a large factor does not lead to a contribution which is comparable to that in Eq. (53) from the "in order" growth sequence. Consequently we can calculate the occurrence probability as being due exclusively to growth sequences in which Eq. (56) is *never* fulfilled. That is, to find a bound for the occurrence probability of the maze we multiply the result in Eq. (53) by the number of sequences which *do not* pass through an intermediate state of the type of Eq. (56). In essence therefore, we can not permit growth in such a way as to have more than a small number of l_i's in the interval written in Eq. (56). One way of formulating this restriction is to treat almost all rows as consisting of $2\epsilon L+1$ elements. The first ϵL of these are sites which can be filled in any order (at least as far as our bound is concerned), the middle $L-2\epsilon L$ sites form a single element because they should all be filled at once, in order to avoid the state to which our bound applies, and finally the last ϵL sites can be filled in in any order. So in essence we need to know how many sequences of such types there are. Clearly there are at most $(2\epsilon N)!$ growth sequences of this type. But since ϵ is of order $N^{-1/10}$, the number of such sequences is smaller than $\exp(N)$. Since according to Eq. (53) any specific growth sequence gives a contribution to the occurrence probability which is bounded by $\exp(-KN\ln N)$, we see that even multiplying this by $\exp(N)$ has no effect on the result. The conclusion, although not rigorous, is nevertheless compelling: the occurrence probability of the maze is similar to that of the tube and is bounded by the expression in Eq. (52).

B. The sphere

One might ask, "Do all structures have occurrence probabilities which obey the bound of Eq. (52)?" Obviously this is not true, because the total number of clusters grows only exponentially with their size[26] (in number of particles) N. In percolation we know that the occurrence probability of a single specific large cluster of N sites is of order $\exp(-KN)$, where $K=-\ln p-\alpha\ln(1-p)$, where α is the fraction of sites on the perimeter of the cluster. Clearly, most structures in DLA will similarly have exponential occurrence probabilities. It is only the exceptional ones, such as those considered here, whose occurrence probabilities obey the bound of Eq. (52). To see this, consider a sphere. Let us ask what the probability $P_{i\to f}$ is for growing from an initial radius to a final radius which is one lattice constant larger. If the surface consists of M sites, then roughly, the probability of hitting any one of them is $1/M$. Also note that there are $M!$ sequences which lead from the initial to the desired final state. Thus

$$P_{i\to f}=M!(1/M)^M\sim e^{-M}\ , \tag{59}$$

which is exponential in the number of added particles. For a large sphere of N particles built up this way (like unpeeling an onion), one obtains a lower bound on the growth probability which is of order $\exp(-KN)$. Of course, to really calculate (rather than bound) the occurrence probability, entropy effects must be taken into account more completely. However, this argument does prove that the occurrence probability has a lower bound which is exponential in the size. Thus our arguments do not always result in the bound of Eq. (52).

V. DISCUSSION

From our examples we suggest the following conclusion. To obtain a growth probability of order $\exp(-\alpha L)$, one requires a structure generically equivalent to a tunnel whose path length (i.e., chemical length) is of order L. The one dimensionality of the tunnel implies a specific growth sequence which by Eq. (7) gives a smaller-than-exponential occurrence probability. Thus for $q<0$ these structures give a contribution to $[M(q,N)]_{\rm av}$ for $q<0$ of order

$$\delta[M(q,N)]_{\rm av}\sim e^{|q|L-KL\ln L}\ . \tag{60}$$

As $L\to\infty$ this contribution is dominated by the "regular" or power-law contribution. It is instructive to see that the "maze" structure considered in Sec. III conforms to this reasoning. While we cannot rule out the possible existence of structures which would contradict the behavior as in Eq. (60), we nevertheless conclude that at present there is no evidence in favor of the "phase transition" suggested in Refs. 19 and 20 involving non-power-law scaling of the negative moments of the growth probability in DLA.

ACKNOWLEDGMENTS

Helpful conversations with A. Aharony and R. Blumenfeld in which they pointed out the error in the previous estimate for the growth probability are gratefully acknowledged. This work was supported in part by the National Science Foundation under Grant Nos. DMR 85-19509 and DMR 88-15469.

APPENDIX: SCALING FOR THE RECTANGLE

In this appendix we verify that the exact solution for the rectangle has the scaling behavior of Eqs. (24). For this discussion we need the solution, given in Eq. (13), for the potential near an infinitely thin conducting strip and the mapping of Eq. (15) which takes the strip into the rectangle. In that mapping for large aspect ratio we have $B=a$ and $\beta=1-\epsilon$, with $\epsilon=2b/(\pi a)$. Thus

$$|\sigma|=\frac{1}{4\pi}\left|\frac{dW}{dz_2}\right|=\frac{1}{4\pi}\frac{|dW/dz_1|}{|dz_2/dz_1|}=\frac{Q}{2\pi a}|(z_1^2-\beta^2)^{-1/2}|\ . \tag{A1}$$

To get σ as a function of $z_2=x+iy$, we must integrate Eq. (15) to get $z_2(z_1)$.

We will study two cases: first when z_2 is a point on the end of the rectangle, and later when z_2 is a point near the end but on the long side of the rectangle. For the first case, set $z_1=(1-\zeta)$ and $z_2=a+iy$. As ζ increases from 0 to ϵ, y increases from 0 to b. Using the smallness of ζ we have

$$\frac{dy}{d\zeta}=a\zeta^{-1/2}(\epsilon-\zeta)^{1/2}\ . \tag{A2}$$

Thus

$$y(\zeta)=a\epsilon\int_0^{\zeta/\epsilon}dx[(1-x)/x]^{1/2}\ . \tag{A3}$$

Since $a\epsilon=2b/\pi$, this equation gives ζ/ϵ as a function of y/b:

$$y/b=F(\zeta/\epsilon)\ . \tag{A4}$$

Although it is not needed, we can give $F(x)$ explicitly

$$F(x)=\frac{2}{\pi}[\sin^{-1}(x^{1/2})+x^{1/2}\sqrt{1-x}\ ]\ . \tag{A5}$$

For z_1 and β near unity, we have $|z_1^2-\beta^2|=2|\epsilon-\zeta|$. Substituting this into Eq. (A1) we find that

$$\sigma(x=a,y)=\frac{Q}{2\pi a}(2\epsilon)^{-1/2}\left[1-\frac{\zeta}{\epsilon}\right]^{-1/2} . \tag{A6}$$

But from Eq. (A4) $\zeta/\epsilon=F^{-1}(y/b)$, so that Eq. (A6) is of the scaling form of Eq. (24a).

For the second case set $z_1=1-\epsilon-\zeta=\beta-\zeta$ and $z_2=a-s+ib$. Then as ζ increases from 0 to β, s increases from 0 to a. For small ζ Eq. (15) is

$$\frac{ds}{d\zeta}=a\left[\frac{\zeta}{\zeta+\epsilon}\right]^{1/2} . \tag{A7}$$

From this equation we deduce that $s/b=G(\zeta/\epsilon)$, where

$$G(x)=\frac{2}{\pi}\int_0^x\left[\frac{u}{1+u}\right]^{1/2}du . \tag{A8}$$

Near the end we may write $|x_1^2-\beta^2|\sim(2\zeta)$, so that

$$\sigma=\frac{Q}{2\pi a}(2\epsilon)^{-1/2}[G^{-1}(s/b)]^{-1/2} , \tag{A9}$$

which is of the scaling form of Eq. (24a). For large s, this result joins smoothly onto the form of Eq. (24b) valid far from the end.

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