Family Algebras and the Isotypic Components of $g \otimes g$

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Family Algebras and the Isotypic Components of g tensor g

Abstract
Given a complex simple Lie algebra $g$ with adjoint group $G$, the space $S(g)$ of polynomials on $mg$ is isomorphic as a graded $mg$-module to $(I(g) \otimes \mathcal{H}(g))$ where $I(g) = (S(g))^G$ is the space of $G$-invariant polynomials and $\mathcal{H}(g)$ is the space of $G$-harmonic polynomials. For a representation $V$ of $g$, the generalized exponents of $V$ are given by $\sum_{k \geq 0} \dim(Hom(g,V,H_k(g)))q^k$. We define an algebra $C_V(g) = Hom_g(End(V),S(g))$ and for the case of $V = g$ we determine the structure of $C_V(g)$ using a combination of diagrammatic methods and information about representations of the Weyl-group of $g$. We find an almost uniform description of $C_g(g)$ as an $I(g)$-algebra and as an $I(g)$-module and from there determine the generalized exponents of the irreducible components of $End(g)$. The results support conjectures about $(T(g))^G$, the $G$-invariant part of the tensor algebra, and about a relation between generalized exponents and Lusztig’s fake degrees.

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FAMILY ALGEBRAS AND THE ISOTYPIC COMPONENTS OF $\mathfrak{g} \otimes \mathfrak{g}$

Matthew Tai

A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania
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Partial Fulfillment of the Requirements for the
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ABSTRACT

FAMILY ALGEBRAS AND THE ISOTYPIC COMPONENTS OF $g \otimes g$

Matthew Tai
Alexandre Kirillov

Given a complex simple Lie algebra $g$ with adjoint group $G$, the space $S(g)$ of polynomials on $g$ is isomorphic as a graded $g$-module to $(I(g) \otimes \mathcal{H}(g))$ where $I(g) = (S(g))^G$ is the space of $G$-invariant polynomials and $\mathcal{H}(g)$ is the space of $G$-harmonic polynomials. For a representation $V$ of $g$, the generalized exponents of $V$ are given by $\sum_{k \geq 0} \text{dim}(\text{Hom}_g(V, H_k(g))) q^k$. We define an algebra $C_V(g) = \text{Hom}_g(\text{End}(V), S(g))$ and for the case of $V = g$ we determine the structure of $g$ using a combination of diagrammatic methods and information about representations of the Weyl-group of $g$. We find an almost uniform description of $C_g(g)$ as an $I(g)$-algebra and as an $I(g)$-module and from there determine the generalized exponents of the irreducible components of $\text{End}(g)$. The results support conjectures about $(T(g))^G$, the $G$-invariant part of the tensor algebra, and about a relation between generalized exponents and Lusztig’s fake degrees.
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Chapter 1

Introduction

1.1 Simple Lie Algebras and Exponents

We consider the simple Lie algebras, these being the four classical series $A_r, B_r, C_r, D_r$ and the five exceptional algebras $G_2, F_4, E_6, E_7, E_8$. Associated to each of these algebras is a list of numbers called their exponents, which appear in a number of ways. The name comes from the exponents of the hyperplane arrangement corresponding to the simple reflection planes of the Weyl Group of the Lie algebra. The exponents can also be considered topologically: for the compact group $G$ associated to $\mathfrak{g}$, the Poincare polynomial of $G$ is

$$P_G(q) = \sum_{k=0}^r r k(H^k(G, \mathbb{Z})) q^k = \prod_{k=1}^r (1 + q^{2\nu_k+1})$$

The simple Lie algebras are summarized in the following table, where the descriptions of the exceptional Lie groups are based on the Rosenfeld projective planes [Ro97].
Table 1.1: Exponents for the simple Lie algebras

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>dim $\mathfrak{g}$</th>
<th>Exponents</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_r$</td>
<td>$r^2 + 2r$</td>
<td>1, 2, 3\ldots, $r$</td>
<td>$\mathfrak{sl}(r + 1)$</td>
</tr>
<tr>
<td>$B_r$</td>
<td>$r(2r + 1)$</td>
<td>1, 3, 5\ldots, $2r - 1$</td>
<td>$\mathfrak{so}(2r + 1)$</td>
</tr>
<tr>
<td>$C_r$</td>
<td>$r(2r + 1)$</td>
<td>1, 3, 5\ldots, $2r - 1$</td>
<td>$\mathfrak{sp}(r)$</td>
</tr>
<tr>
<td>$D_r$</td>
<td>$r(2r - 1)$</td>
<td>1, 3, 5\ldots, $2r - 1$, $r - 1$</td>
<td>$\mathfrak{so}(2r)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>14</td>
<td>1, 5</td>
<td>“$\mathfrak{sl}(1, \mathbb{O})$”</td>
</tr>
<tr>
<td>$F_4$</td>
<td>52</td>
<td>1, 5, 7, 11</td>
<td>“$\mathfrak{sl}(2, \mathbb{O})$”</td>
</tr>
<tr>
<td>$E_6$</td>
<td>78</td>
<td>1, 4, 5, 7, 8, 11</td>
<td>“$\mathfrak{sl}(2, \mathbb{C} \otimes \mathbb{O})$”</td>
</tr>
<tr>
<td>$E_7$</td>
<td>133</td>
<td>1, 5, 7, 9, 11, 13, 17</td>
<td>“$\mathfrak{sl}(2, \mathbb{H} \otimes \mathbb{O})$”</td>
</tr>
<tr>
<td>$E_8$</td>
<td>248</td>
<td>1, 7, 11, 13, 17, 19, 23, 29</td>
<td>“$\mathfrak{sl}(2, \mathbb{O} \otimes \mathbb{O})$”</td>
</tr>
</tbody>
</table>
1.2 Casimir Invariants

The exponents of $\mathfrak{g}$ also have representation-theoretic interpretations. $G$ acts on $\mathfrak{g}$ via the conjugation action, and hence on $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$ considered as a vector space. We denote by $I(\mathfrak{g})$ the $G$-invariant subspace $(S(\mathfrak{g}))^G$. In 1963, Kostant showed that $I(\mathfrak{g})$ for simple $\mathfrak{g}$ is a polynomial algebra where the number of generators is equal to the rank of $\mathfrak{g}$, and that furthermore the degrees of these generators are each one more than an exponent of $\mathfrak{g}$. For later use we establish a particular choice of generators of $I(\mathfrak{g})$, which we will call primitive Casimir operators.

For each Lie algebra, we pick a representation $(V, \pi)$ with which to define the primitive Casimir operators. For $\mathfrak{sl}(r+1)$ we pick one of the two $r+1$-dimensional representations. For $\mathfrak{so}(n)$ we pick the $n$-dimensional representation, and for $\mathfrak{sp}(r)$ we pick the $2r$-dimensional representation. For $G_2, F_4, E_6, E_7,$ and $E_8$ we pick the $7-, 26-, 27-, 56-, \text{and } 248-$dimensional representations respectively. These are often also called the “standard” representations, and with a few exceptions are the nontrivial representations of minimal dimension. From here on out, unless otherwise specified, $V$ and $\pi$ refer to this defining representation.

Letting $\{x_\alpha\}$ be a basis of $\mathfrak{g}$ and $K$ be the Killing form, define

$$M_d = \pi(x_\alpha) \otimes K^{\alpha\beta} x_\beta$$

regarded as an $S(\mathfrak{g})$-valued square matrix of dimension $\dim V$.

Except in the case of $D_r$, we can order the exponents of $\mathfrak{g}$ in increasing order, so that
we can denote the exponents by

\[ e_1 < e_2 < \ldots < e_r \]

We define the Casimir operator

\[ c_k = tr(M^{e_k+1}_d) \]

where the trace is taken in the minimal representation \( V \). The set \( \{c_k\} \) are the primitive Casimir operators for \( g \).

In the case of \( D_r \), the exponents are \( 1, 3, \ldots, 2r-3, r-1 \). We order the exponents in increasing order, getting that \( e_{\left\lceil \frac{r}{2} \right\rceil} = r-1 \). For \( k \neq \left\lceil \frac{r}{2} \right\rceil \), we write

\[ c_k = tr(M^{e_k+1}_d) \]

and for \( k = \left\lceil \frac{r}{2} \right\rceil \) we define

\[ c_k = Pf = \sqrt{\det(M_d)} \]

picking the sign of \( Pf \) arbitrarily.

For \( sl(r+1) \), the exponents are \( e_i = i \) for \( 1 \leq i \leq r \). For \( k > r+1 \), we have the reduction,

\[ 0 = \sum_{n_j m_j = k} \frac{1}{m_j!} \left( -\frac{tr(M^n_d)}{n_j} \right)^{m_j} \]

where the notation \( n_j m_j = k \) indicates a partition of \( k \) where \( n_j \) appears with multiplicity \( m_j \). In particular, the relation for \( k = r+2 \) is the trace of the Cayley-Hamilton identity for \( M_d \).

For the other simple Lie algebras, there are similar formulas for reducing \( tr(M^k_d) \) for
$k$ not an exponent plus 1, with varying complexity. For the exceptional Lie algebras, \( \dim(V) \) is much larger than \( e_r + 1 \), so the Cayley-Hamilton identity doesn't yield much information about the how traces of low powers of \( M_d \) reduce. See [RSV99] for details.

### 1.3 Generalized Exponents

In 1963, Kostant [Ko63] proved that for a representation \( V \) of \( g \) and hence of \( G \), \( (V \otimes S(g))^G = \text{Hom}_G(V^\vee, S(g)) \) is a free \( I(g) \) module. Thus we can find a basis for \( (V \otimes S(g))^G \) over \( I(g) \); Kostant calls the degrees of the polynomial components of this basis the generalized exponents of \( V \) (with multiplicity), usually expressed as a polynomial \( P_V(q) \) for a variable \( q \). Note that for \( V = g \), the generalized exponents for \( g \) match the classical notion of the exponents of \( g \).

There is another description of the generalized exponents in terms of a space \( \mathcal{H}(g) \) of \( G \)-harmonic polynomials. Let \( D(g) \) be the space of \( G \)-invariant differential operators on \( S(g) \) with constant coefficients, and let \( D_+(g) \) be the subspace of \( D(g) \) with vanishing constant term. Then \( \mathcal{H}(g) \) is defined by

\[
\mathcal{H}(g) = \{f \in S(g) | d(f) = 0 \ \forall \ d \in D_+(g)\}
\]

The condition \( d(f) = 0 \) generalizes the usual harmonic condition of \( \Delta(f) = 0 \), and thus the \( G \)-harmonic polynomials allow for studying functions defined on Lie groups using methods from harmonic analysis.

Kostant showed that \( S(g) \cong I(g) \otimes \mathcal{H}(g) \) as graded \( g \)-modules. Since the elements of
\( I(g) \) are invariant, all of the interesting behavior is contained in \( \mathcal{H}(g) \). Thus we can write the generalized exponents of a representation \( V \) as

\[
P_V(q) = \sum_k \dim(\text{Hom}_g(V, \mathcal{H}(g))) q^k
\]

Hesselink [He80] gives a formula for computing the generalized exponents of an irreducible representation of a simple Lie algebra using a \( q \)-analogue of Kostant’s multiplicity formula, but using this formula is computationally infeasible, involving computing the \( q \)-analogue of the partition function, which unlike the normal partition function doesn’t vanish for negative weights, and then summing the partition function over the associated Weyl orbit. There are also combinatorial approaches such as the Kostka-Foulkes polynomials for \( \text{sl}(n) \) [DLT94]. For representations where none of the weights are twice a root (called small representations), Broer [Br95] showed that the generalized exponents of \( V \) are equal to what Lusztig calls the fake degrees [Lu77] of \( V^T \) as a representation of \( W \).

In general, however, there are no known closed-form expressions for the generalized exponents of arbitrary representations that don’t require summation over the Weyl group.
Chapter 2

Introduction to Family Algebras

2.1 Definition of Family Algebras

In [Ki00], Kirillov introduced what he calls Family Algebras in the hopes of providing a new method for determining generalized exponents that doesn't involve summing over the Weyl group.

We fix a representation $V$ of $\mathfrak{g}$ and consider $\text{End}(V)$, with the conjugation action on it induced from the action on $V$. We define the classical family algebra

$$C_V(\mathfrak{g}) = (\text{End}(V) \otimes S(\mathfrak{g}))^G$$

where $G$ is the adjoint group of $\mathfrak{g}$ and acts by the action induced from $\mathfrak{g}$. This is an algebra with multiplication $\circ \otimes m$ inherited from

$$\circ : \text{End}(V) \otimes \text{End}(V) \to \text{End}(V)$$
via composition and
\[ m : S(g) \otimes S(g) \to S(g) \]
via polynomial multiplication.

If we pick a basis \( \{ \nu_a \} \) for \( V \) and let \( E^b_a \) be defined by \( E^b_a \nu_b = \nu_a \), then we can write an element of the family algebra as
\[ E^b_a \otimes P^a_b \]
where \( P^a_b \in S(g) \). We call the set \( \{ P^a_b \} \) the polynomial component of \( E^b_a \otimes P^a_b \). Note that for two elements \( E^b_a \otimes P^a_b \) and \( E^b_a \otimes Q^a_b \), the multiplication looks like
\[ (E^b_a \otimes P^a_b) \times (E^b_a \otimes Q^a_b) = E^b_a \otimes P^a_b Q^a_b \]
So the multiplication respects the natural grading on the polynomial components, and hence we say that an element \( E^b_a \otimes P^a_b \) is homogeneous of degree \( k \) if all of the \( \{ P^a_b \} \) are homogeneous of degree \( k \).

The phrase “family algebra” comes from the decomposition of
\[ \text{End}(V) = \bigoplus_i V_i \]
for irreducible \( V_i \), which Kirillov calls the children of \( V \). The family algebra \( C_V(g) \) decomposes similarly into
\[ C_V(g) = \bigoplus_i (V_i \otimes S(g))^G \]
Thus a family algebra gives us an \( I(g) \) module that is closed under multiplication and is built from a finite set of isotypic components. Note that if \( V_i \) is a component of
\( End(V) \), then so is \( V_i^\vee \); hence

\[
C_V(g) = \bigoplus_i (V_i \otimes S(g))^G \cong \bigoplus_i Hom_g(V_i, S(g))
\]

There is a natural quantization to what Kirillov calls the quantum family algebra,

\[
Q_V(g) = (End(V) \otimes U(g))^G
\]

\( U(g) \) is isomorphic to \( S(g) \) as \( G \)-modules, so there is a map that sends \( Q_V(g) \) to \( C_V(g) \), but the classical and quantum family algebras for a given \( V \) differ in their multiplicative structures. This dissertation will only consider classical family algebras.

### 2.2 Relation to the Generalized Exponents

For an irreducible representation \( V_i \), there is an \( I(g) \)-linear basis of \( Hom_g(V_i, S(g)) \) where there is a bijection between generalized exponents \( e_{ij} \) (with multiplicity) and basis elements \( A_{ij} \) such that

\[
A_{ij} \in Hom_g(V_i, \mathcal{H}_{e_{ij}}(g))
\]

Hence, using the decomposition of \( End(V) \) into irreducible representations, there is an \( I(g) \)-linear basis of \( C_V(g) \) where each basis element is in \( Hom_g(V_i, \mathcal{H}_{e_{ij}}(g)) \) for some \( V_i \) in the decomposition of \( End(V) \) and some \( e_{ij} \) a generalized exponent of \( V_i \).

Thus the general strategy of family algebras is to determine the algebraic structure of a given family algebra, use that to determine an \( I(g) \)-linear basis, turn that basis into a harmonic basis and from there compute the generalized exponents.
2.3 Restriction to the Cartan subalgebra

Given a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with corresponding torus $T \subset G$, we can look at $S(\mathfrak{h})$, and in particular the restriction $res : S(\mathfrak{g}) \to S(\mathfrak{h})$ given by viewing the two algebras as $Pol[\mathfrak{g}^\vee]$ and $Pol[\mathfrak{h}^\vee]$ respectively, and sending an element $f \in S(\mathfrak{g})$ to $f|_{\mathfrak{h}^\vee}$. While this map is generally not an injection, there are some useful aspects. Chevalley’s restriction theorem [Br95] says that

$$res|_{I(\mathfrak{g})} : I(\mathfrak{g}) \to I(\mathfrak{h}) = S(\mathfrak{h})^W$$

is an isomorphism. We get a map

$$Res : (V \otimes S(\mathfrak{g}))^G \to (V^T \otimes S(\mathfrak{h}))^W$$

induced by restricting from $V$ to $V^T$ and from $S(\mathfrak{g})$ to $S(\mathfrak{h})$, which Kostant shows is an injection. The result by Broer mentioned in the first chapter is a necessary and sufficient condition for $Res$ to be an isomorphism.

Writing $B_V(\mathfrak{h})$ for $End(V)^T \otimes S(\mathfrak{h})$ we get that

$$Res : C_V(\mathfrak{g}) \to B_V(\mathfrak{h})^W$$

is an injection. We can make $Res$ into a surjection by localizing with respect to the non-zero part of $I(\mathfrak{g})$. In particular, let $K_0$ be the fraction field of $I(\mathfrak{g}) \cong S(\mathfrak{h})^W$. Then by [Ki01] we have that

$$C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} K_0 \cong B_V(\mathfrak{h})^W \otimes_{I(\mathfrak{h})} K_0$$
This then tells us that the dimension of $C_V(g)$ over $I(g)$ is equal to the dimension of $B_V(h)^W$ over $I(h)$.

$$End(V)^T = \bigoplus_{\mu \in Wt(V)} Mat_{m_V(\mu)}(C)$$

and so $B_V(h)^W$ is the $W$-invariant subalgebra of the sum of matrix algebras

$$\bigoplus_{\mu \in Wt(V)} Mat_{m_V(\mu)}(S(h))$$

Since the multiplicity over $I(h)$ of a representation $\phi$ of $W$ in $S(h)$ is $\dim(\phi)$, the dimension of $B_V(h)^W$ over $I(h)$, i.e. the dimension of $C_V(g)$, is given by the sum of the dimensions of the matrix algebras:

$$\sum_{\mu \in Wt(V)} m_V(\mu)^2$$

Given a weight $\lambda \in Wt(V)$, we can consider an element of $B_V(h)^W$ that is the identity on the matrix algebras corresponding to weights in $W.\lambda$ and vanish elsewhere. Such an element lifts to an element of $C_V(g)$ which, for some $P \in I(h)$, restricts to $P$ times the identity on the matrix algebras corresponding to weights in $W.\lambda$ and vanish elsewhere.

We can use these elements of $C_V(g)$ as analogues to projection operators.
Chapter 3

Results for $V = \mathfrak{g}$

This dissertation will focus on the particular case of the adjoint representation, i.e. setting $V = \mathfrak{g}$. The weights in question are then the roots of $\mathfrak{g}$ as well as 0 with multiplicity $r$, where $r$ is the rank of $\mathfrak{g}$. We denote the image of $(End(\mathfrak{g})^T \otimes S(\mathfrak{h}))^W$ in $Mat_r(S(\mathfrak{h}))$ by the torus part of the algebra, and everything else by the vector part, as it is composed of 1-dimensional and hence scalar algebras. Note that the vector part is commutative, since $S(\mathfrak{h})$ is commutative, so any non-commutativity in the family algebra appears only in the torus part.

3.1 Algebraic Structure

The decomposition of $End(\mathfrak{g})$ into irreducible components depends on $\mathfrak{g}$, but is uniform for all of the $A_n$, uniform for $B_n, C_n$ and $D_n$, and is uniform for the five exceptional Lie algebras [CV08]. The main result of this dissertation is that though the form
of $C\mathfrak{g}(\mathfrak{g})$ as a $\mathfrak{g}$-module ends up quite different, the algebraic structures of $C\mathfrak{g}(\mathfrak{g})$ are very similar for all of the simple Lie algebras. There are two generators common to all of family algebras in question, denoted $M$ and $S$, and then a set of $r$ other generators, labelled $R_1$ through $R_r$, that depend on the structure of $\mathfrak{g}$. A set of $I(\mathfrak{g})$-linearly independent basis elements of $C\mathfrak{g}(\mathfrak{g})$ is then

$$M^m R_k \text{ for } 0 \leq m \leq e_r + 1, 1 \leq k \leq r$$

$$R_m S R_n + R_n S R_m \text{ for } 1 \leq m \leq n \leq r - 1$$

$$R_m S R_n - R_n S R_m \text{ for } 1 \leq m < n \leq r$$

Here $R_1$ is a scalar, left in for uniformity of expression.

The $R_k$ can themselves be generated by either $M, S$ and $R_2$ in the cases of $A_r, B_r, C_r$ and $G_2$, by $M, S, R_2$ and $R_r$ for $D_r$, or by $M, S, R_2$ and $R_3$ in the cases of $F_4, E_6, E_7$ and $E_8$.

There are several relations common to all of the cases. The terms $M^m R_k$ for $m \geq 1$ vanish on the torus part, and any term involving $S$ vanishes on the vector part. Hence $M$ is central and $M S = S M = 0$. The $R_k$ commute with each other and with $M$, but not with $S$. $S R_k S = P_k S$ for some $P_k \in I(\mathfrak{g})$, although the form of $P_k$ depends on $\mathfrak{g}$.

The relations describing the products of the $R_k$ also depend on $\mathfrak{g}$, in particular the existence or absence of primitive Casimir elements in particular degrees.
3.2 Fake Degrees

For the Weyl group $W$ acting on the Cartan subalgebra $\mathfrak{h}$, there is a notion called “fake degrees” analogous to that of the generalized exponents, in that there is a polynomial $P_U(q)$ describing the maps from a $W$-representation $U$ into a space $\mathcal{M}(\mathfrak{h})$ of $W$-harmonic polynomials. For the representations relevant to $g \otimes g$, we have the following statement: if $V^T = \oplus_i U_i$ as $W$-modules, then

$$P_V(q) = \sum_i q^{k_i} P_{U_i}(q)$$

for some set of exponents $k_i$, although the $k_i$ are not uniquely determined.

For each classical families there are uniform expressions for the $q_i$ in terms of $r$, as well as for the $E_r$ family.
Chapter 4

Diagrams

We can write elements of the family algebra diagrammatically using the Feynman-Penrose-Cvitanović “birdtrack” notation [Cv08]. We consider graphs with two types of edges, called reference and adjoint edges. An adjoint edge is marked here by a thin line, a reference edge by a thick line with an arrow on it. All edges that end in a univalent vertex must be adjoint edges, and for every diagram one of these univalent vertices is labelled with an “I”, one with an “O”, and the rest with a white dot. A diagram with \(k\) dotted vertices is considered to have degree \(k\). The other types of allowed vertices depend on the Lie algebra in question. For example,

In the usual particle interpretation of Feynman diagrams, the adjoint edges are bosons and the reference edges are fermions carrying the corresponding charge, and vertices
of valence higher than 1 being interactions, with \( G \) being the gauge group of the interactions. Momentum constraints are ignored here.

In the Lie algebra interpretation, the reference edges correspond to copies of the reference representation, the adjoint edges are copies of the adjoint representation, and vertices of valence higher than 1 are invariants. In particular, the vertices with one reference edge pointing in, one reference edge pointing out and one adjoint edge attached are Clebsches for \( V \otimes V^\vee \rightarrow Adj \). In the standard index notation for tensors, each vertex is an invariant tensor with an upper reference index for each arrow going in, a lower reference index for each arrow going out, and an adjoint index for each adjoint edge attached; two indices are contracted if they are connected by an edge. The ability to turn upper adjoint indices into lower adjoint indices via the Killing form allows us to not require arrows on the adjoint edges.

We consider the dotted vertices as indistinguishable, so that if two diagrams differ only by which of a pair of adjoint edges connect to which of a pair of dotted vertices, we consider the diagrams equivalent.

The dotted vertices correspond to our polynomial part \( \{ P^\alpha_\beta \} \). The I and O vertices correspond to our coordinate indices \( E^\beta_\alpha \). A component that is not connected to either of the I or O vertices is contained entirely within \( S(\mathfrak{g}) \), and hence in \( I(\mathfrak{g}) \), so a component with only dotted vertices acts as a coefficient. A diagram is considered as the tensor product of its connected components.
These diagrams are all naturally $G$-invariant, being built out of $G$-invariant objects, and hence all diagrams are naturally in $(End(g) \otimes S(g))^G$. Thus any diagram as defined above automatically gives a family algebra element, as opposed to the initial setup of defining $End(g) \otimes S(g)$ and then imposing $g$-invariance as an additional property. By Cvitanović, all elements of the family algebra are formal $\mathbb{C}$-linear combinations of such diagrams, so we can consider the algebra in terms of these diagrams.

The family algebra product of two diagrams is the diagram created by removing the I vertex of one diagram and the O vertex of the other and identifying the adjoint edges those vertices were attached to, which is the equivalent of contracting the adjoint indices that the two edges corresponded to.

$$G \times H = \begin{array}{c} G \times H \\ I \ O \ I \ O \\ G \ H \ I \ O \end{array}$$

We can also define the trace of a family algebra element similarly, by removing both the $I$ and the $O$ vertices of a family algebra element and identifying the adjoint edges those vertices were attached to.

### 4.1 Casimir Operators and Structure Constants

Given a reference loop going through $n$ Clebsche vertices, we have $n$ adjoint edges coming off of the loop, and the loop corresponds to

$$tr(\pi(X_1)\pi(X_2)\cdots\pi(X_n))$$
where the $X_i$ are the adjoint edges, i.e. elements of $\mathfrak{g}$. As such, a loop of reference edges going through $k$ Clebsche vertices will be called a “trace” of order $k$ from now on. A trace of degree $e_k + 1$ whose adjoint edges all end in dotted vertices evaluates to the primitive Casimir operator $c_k$, except in the case of the $e_r = r - 1$ exponent of $D_r$, which will be handled in the section on $D_r$. A trace of order 0 evaluates to $\dim(V)$. We normalize so that traces of degree 2 are equivalent to just adjoint lines.

$$\overset{e_k + 1}{\bigcirc} = c_k \quad \bigcirc = \dim(V) \quad -\bigcirc =$$

The structure constant $f_{\beta \gamma}^\alpha$ can be written as a diagram $F$ as the difference of two traces each with three adjoint edges coming off, differing only in the direction of the reference edges. We abbreviate it using Cvitanović’s notation of a big black dot. Given two Clebsche vertices connected to single a reference edge, swapping the ends of the adjoint edges can be written using an $F$ node. This is just the Lie algebra relation $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X, Y])$ applied to the reference representation:

$$\overset{F}{-\bigcirc} = \bigcirc = \bigcirc =$$

We say a diagram is simple if the connected components containing the $I$ and $O$ vertices are each a primitive Casimir operator attached to some number of trees built out of structure constants.
4.2 Projections

When looking for generalized exponents, we want objects not in \((\mathfrak{g} \otimes \mathfrak{g} \otimes S(\mathfrak{g}))^G\), where these diagrams naturally live, but in \((V_i \otimes S(\mathfrak{g}))^G\) for a given irreducible component \(V_i\).

We denote by \(P_{ri}\) the projection operator that sends \(\mathfrak{g} \otimes \mathfrak{g}\) into the subspace isomorphic to \(V_i\). Diagrammatically, such a projector looks like a diagram with two \(I\) vertices, two \(O\) vertices and no dotted vertices. Similar to multiplication, an \(O\) vertex of the projector connects to the \(I\) vertex of the diagram being projected, but now also an \(I\) vertex of the projector connects to the \(O\) vertex of the diagram being projected, yielding a new diagram with a single \(I\) and a single \(O\) vertex:

\[
P_{ri} = \begin{array}{c}
  O \\
  I \\
  O \\
  I
\end{array} \quad P_{ri}(F) = \begin{array}{c}
  F \\
  I \\
  O
\end{array}
\]

The projection operators, like the diagrams themselves, can be expressed entirely in terms of traces of reference edges connected to adjoint edges, so the adjoint edges between the projector and the diagram being projected can be expanded out, allowing for diagrammatic evaluation of the projected diagram. See [Cv08] for details.

4.3 Symmetrization

For a generic diagram \(D\), we consider the diagram \(\overline{D}\) created by replacing the \(I\) vertex in \(D\) by a dotted vertex. We also consider the half-symmetrization \(\tilde{D}\), which is the sum
over all diagrams derived from $D$ by swapping the $I$ and one of the dotted vertices, plus $D$ itself. $\widehat{D}$ is the sum over all diagrams created by replacing one of the dotted vertices in $\overline{D}$ by the $I$ vertex.

\[
\begin{align*}
D & = \begin{array}{c} I \end{array} \quad \overline{D} = \begin{array}{c} O \end{array} \\
\widehat{D} & = \begin{array}{c} I \end{array} + \begin{array}{c} O \end{array} + \begin{array}{c} I \end{array} + \begin{array}{c} O \end{array} + \cdots \\
& \quad \begin{array}{c} D \end{array} \\
& \quad \begin{array}{c} D \end{array} \\
& \quad \begin{array}{c} D \end{array} \\
& \quad \begin{array}{c} D \end{array}
\end{align*}
\]

$\overline{D}$ belongs to $(\mathfrak{g} \otimes S(\mathfrak{g}))^G$, and thus decomposes into $\sum a_k D_k$ where $a_k \in I(\mathfrak{g})$ and $D_k$ is the diagram created by taking the diagram corresponding to the primitive Casimir element $c_k$ and replacing one of the dotted vertices with the $O$ vertex. $D_k$ is a simple diagram, and the $a_k$ is not connected to anything in $D_k$, so $\overline{D}$ can be written in terms of a finite set of simple diagrams, and thus $\widehat{D}$ can also be written in terms of a finite set of simple diagrams. Thus if the other terms in $\widehat{D}$ can be written in terms of simple diagrams, so can $D$. 

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Chapter 5

Invariant Tensors

5.1 General Statement for most Classical Lie Algebras

For $A_r$, $B_r$ and $C_r$ there is a particularly elegant expression for all the elements of the invariant tensors $\left(T(g)\right)^G$ coming from the reference representations.

**Theorem 5.1.1** (Invariant Tensors for $A_r$, $B_r$ and $C_r$). For $g = A_r, B_r$ or $C_r$ with the corresponding reference representation $(V, \pi)$, the elements of $\left(T(g)\right)^G$ can be expressed as tensor products of

$$
\text{tr}_V(\pi(X_{\alpha_1})\pi(X_{\alpha_2})\cdots\pi(X_{\alpha_k}))X^{a_1} \otimes X^{a_2} \otimes \cdots \otimes X^{a_k} \in T(g)
$$

along with permutations of the indices.

Diagrammatically, this corresponds to the statement that all diagrams with only adjoint edges leaving the diagram are expressible as loops over the reference repre-
representation with adjoint edges attached, where no two loops are connected by an adjoint edge.

5.2 $A_r$

Any invariant tensor in $\otimes A_r$ can be written in terms of representations of $A_r$, invariants of those representations, and Clebsches between representations. In turn, any representation of $A_r$ can be written in terms of $V$ and $V^\vee$, symmetrized and antisymmetrized. Thus we can write any tensor in $\otimes A_r$ in terms of $V$ and the adjoint representation. Diagrammatically, this corresponds to diagrams with only reference and adjoint edges, with all the internal edges written as reference edges and all of the edges leading out of the diagrams being adjoint edges. By the first fundamental theorem of the invariant theory of $SL(r+1)$ acting on the $r+1$-dimensional representation [FH04], the possible vertices are the Clebsches converting between the adjoint representation and $V \otimes V^\vee$, and the two forms of the Levi-Civita tensor, one with $r+1$ reference edges in, the other with $r+1$ reference edges out, corresponding to tensor that takes $r+1$ vectors and returns a scalar, and the dual of that tensor. We write the Levi-Civita tensor not as a vertex but as a black bar, following [Cv08]:

$$\epsilon_{a_1,a_2,...,a_{r+1}} = \begin{array}{c}
\text{\textbf{reference edges}}
\end{array}$$

$$\epsilon^\vee = \begin{array}{c}
\text{\textbf{adjoint edges}}
\end{array}$$
Since the only edges that can lead out of the diagram have to be adjoint edges, corresponding to the fact that all of our tensors are in $T(A_r)$, any instance of the Levi-Civita tensor in the tensor must be matched by an instance of the dual of the Levi-Civita tensor, as those are the only possible sources and sinks for reference edges. Furthermore, given a Levi-Civita tensor and a dual of the Levi-Civita tensor, we can combine them to yield reference edges without source or sink:

\[ \text{\ldots} = \text{\ldots} \]

where the black bar across the reference edges on the right side of the previous equation means a full antisymmetrization of the corresponding vectors.

Hence since every Levi-Civita tensor is matched by a dual of the Levi-Civita tensor, we can expand them into reference edges without Levi-Civita tensors. Since these reference edges cannot lead out of the diagram, they must close up. Hence we end up with loops of reference edges with Clebsche vertices attaching these loops to adjoint edges that lead out of the diagram. These are all traces of powers of the adjoint representation over $V$, as claimed.

### 5.3 $B_r$

For $B_r$, we use the $2r + 1$-dimensional representation as the reference representation; we have a symmetric form generally called the metric, which we denote by a white circle:
The invariance of the metric is given by

\[
\begin{align*}
  g_{ab} &= \frac{a}{b} = \frac{b}{a} \\
  \delta^b_a &= \frac{a}{b}
\end{align*}
\]

Although \( B_r \) has spinor representations, the group that acts on \( g \) is \( SO(2r + 1) \) rather than \( Spin(2r + 1) \) and hence the invariants must be expressible in terms of representations of \( SO(2r + 1) \), which in turn can be written in terms of the reference representation \( V \). Hence we can write all tensors in \( (T(B_r))^{SO(2r+1)} \) as graphs with reference and adjoint edges. By the first fundamental theorem of the invariant theory of \( SO(2r + 1) \) acting on the \( 2r + 1 \)-dimensional representation \([FH04]\), the relevant vertices are Clebsch between the adjoint and \( V \otimes V^\vee \), as well as the bilinear form and Levi-Civita tensor for \( V \). We will use the bilinear form on \( V \) to identify \( V \) with \( V^\vee \) and remove the arrows from the reference edges.

Since the tensors have no reference indices, every reference edge must either form a loop or end in a Levi-Civita tensor. Since the Levi-Civita tensors have odd degree, they must appear in pairs, and so again we can cancel them to leave only possible metric forms and dual metric forms. Since the metric form has two edges coming out and no edges going in, for each instance of the metric in the diagram there must be a copy of the dual of the metric form connected to it by a reference edge. The metric form can be moved past an attached adjoint edge at the cost of a sign change, so the metric form
and its dual can be placed next to each other and thus cancelled. Hence all instances of the metric forms and its dual can be removed from a diagram in $\langle T(B_r) \rangle^{SO(2r+1)}$, leaving only loops in the reference representation attached to adjoint edges.

5.4 $C_r$

For $C_r$, all representations of $Sp(r)$ can be written in terms of the $2r$-dimensional representation. By the first fundamental theorem of the invariant theory of $SO(2r+1)$ acting on the $2r+1$-dimensional representation [FH04], the relevant vertices for the $2r$-dimensional representation are the Clebsches and the symplectic form and its dual. Here we denote the symplectic form by a triangle:

$$\omega_{ab} = \begin{array}{c} a \\ b \end{array} = -\begin{array}{c} a \\ b \end{array} = -\begin{array}{c} b \\ a \end{array}$$

The inverse $\omega^{ab}$ is denoted by a triangle with the arrows pointing away, with the convention:

$$\delta^b_a = \begin{array}{c} a \\ b \end{array} = -\begin{array}{c} b \\ a \end{array}$$

The invariance of the symplectic form is given by

$$\begin{array}{c} a \\ b \\ \ldots \\ a \\ b \\ \ldots \\ a \end{array} = -\begin{array}{c} a \\ b \\ \ldots \\ a \\ b \\ \ldots \\ a \end{array}$$

The Levi-Civita tensor can itself be replaced by a fully antisymmetrized multiple of $(\omega)^r$

$$a_1 a_2 a_3 \cdots a_1 a_2 a_3 \cdots = a_1 a_2 a_3 \cdots a_1 a_2 a_3 \cdots$$
where again the black bar on the right indicates full antisymmetrization. Similarly, the
dual of the Levi-Civita tensor can be replaced by copies of the dual of the symplectic
form. Hence the only relevant invariant is the symplectic form.

Since the symplectic form has two edges coming out and no edges going in, for each
instance of the symplectic in the diagram there must be a copy of the dual of the sym-
plectic form connected to it by a reference edge. The symplectic form can be moved
past an attached adjoint edge at the cost of a sign change, so the symplectic form and
its dual can be placed next to each other and thus cancelled. Hence all instances of the
symplectic forms and its dual can be removed from a diagram in \((T(C_r))^{Sp(r)}\), leaving
only loops in the reference representation attached to adjoint edges.

5.5 Other simple Lie algebras

For \(D_r\), the Levi-Civita tensor does not need to appear in pairs since it has an even
number of reference edges attached to it. Hence there are invariants of \(D_r\) that are not
traces over the reference representation, including one of the primitive Casimir opera-
tors. While for \(r = 2k + 1\) the set of primitive Casimir operators for \(D_r\) can be expressed
as traces in one of the spins representations, for \(r\) even there are two degree \(r\) primi-
tive Casimir operators, and since there is up to scaling only one possible degree \(r\) fully
symmetrized trace in any single representation, there cannot be a single representa-
tion for which all of the invariant tensors in \((T(D_r))^{SO(2r)}\) can be expressed via traces.
The invariant tensors of \(D_r\) can be described similarly to those of the other classical
Lie algebras, but due to these complications will be handled in the chapter on $D_r$.

For the exceptional Lie algebras, the author conjectures that the statement given above does hold for them, but the above methods for showing such do not work due to the existence of higher-order invariants in their reference representations that do not vanish as simply as the ones for $A_r$, $B_r$ and $C_r$. 
Chapter 6

The $A_r$ case

As an example, we will use the case of $A_3$, as $A_1$ and $A_2$ have been fully worked out in [Ro01].

6.1 Diagrams for $A_r$

The reference representation $V$ of $A_r$ we take to be the $r + 1$-dimensional representation. $A_r$ has exponents $e_i = i$ and primitive Casimir operators all of the form $c_i = tr_V(M_d^{e_i+1})$:

In our example, $A_3$ has exponents 1, 2 and 3, and has primitive Casimir elements of degree 2, 3 and 4.
The projection from \( V \otimes V^\vee \) to the adjoint representation can be represented diagrammatically as

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{align*}
\]

which corresponds to removing the trace from a tensor in \( V \otimes V^\vee \).

6.2 Structure of the Family Algebra

The family algebra is generated as an algebra over \( I(\mathfrak{g}) \) by the following pieces, written in diagrammatic notation:

**Theorem 6.2.1 (Generators’).** The family algebra \( C_g(A_r) \) is generated over \( I(A_r) \) by the following:

\[
\begin{align*}
M &= \frac{1}{2} \begin{array}{c}
\text{Diagram 2}
\end{array} - \frac{1}{2} \begin{array}{c}
\text{Diagram 3}
\end{array} \\
R_2 &= \begin{array}{c}
\text{Diagram 4}
\end{array} + \begin{array}{c}
\text{Diagram 5}
\end{array} \\
S &= \begin{array}{c}
\text{Diagram 6}
\end{array}
\end{align*}
\]

However, the relations in terms of these generators are fairly ugly. In particular, the relations for powers of \( M \) and \( R_2 \) are complicated. Instead, we replace \( M \) and \( R_2 \) with the following:

**Theorem 6.2.2 (Generators).** The family algebra \( C_g(A_r) \) is generated over \( I(A_r) \) by the following:
Note that $K = \frac{R_2}{2} + M$ and $L = \frac{R_2}{2} - M$, so the algebra generated by $M, R_2$ and $S$ is isomorphic to the algebra generated by $K, L$ and $S$.

For the relations, first we define the following elements:

$$K_k = \begin{array}{c}
\text{I} \\
\text{O}
\end{array} \quad L_k = \begin{array}{c}
\text{I} \\
\text{O}
\end{array}$$

As will be shown, $K_k$ and $L_k$ are expressible in terms of $K, L$ and $S$ in a uniform manner.

Then for all $r$, we get the following relations:

**Theorem 6.2.3 (Relations).** The following relations are sufficient for defining the family algebra with the generators above

$$KL = LK, \ KS = LS, \ SK = SL$$

$$SK_m L_n S = (c_{m+n+1} - \frac{1}{r+1} c_m c_n) S$$

$$K_{r+1} = \sum_{k=0}^{r-1} d_{r-k+1} K_k, \ L_{r+1} = \sum_{k=0}^{r-1} d_{r-k} L_k$$

$$\sum_{l=0}^{r} K^{r-l} L^l = \sum_{k=0}^{r-2} d_{r-k} \sum_{l=0}^{k} K^{k-l} L^l$$

Note that these relations are not independent. The $SK_m L_n S$ relations become redundant when $m + n > r$. The $K_{r+1}$ relation minus the $L_{r+1}$ relation gives the $K^k L^l$ relation times $K - L$, and hence the three relations are not independent. We present
both the $K_{r+1}$ and the $L_{r+1}$ relation because each of them is easier to prove individually than any linear combination of them that isn't a multiple of the $K^k L^l$ relation.

In our example, the $r$-dependent relations become

$$SK_m R_l n S = (c_{m+n-1} - \frac{c_m c_n}{4}) S$$

$$L_k = d_2 K + d_3 K + d_4, L_4 = d_2 L_2 + d_3 L + d_4$$

$$K^3 + K^2 L + KL^2 + L^3 = d_2 (K + L) + d_3$$

Here $d_2 = c_1 / 2$, $d_3 = c_2 / 3$ and $d_4 = \frac{c_3}{4} - \frac{c_1^2}{8}$.

### 6.3 Sufficiency of the Generators

As shown in the previous chapter, all of our elements of $(\otimes A_r)^{A_r}$ are tensor products of traces, so our family algebra elements are thus all tensor products of traces. We now consider our three types of univalent vertices, the $I$, $O$ and dotted vertices. A trace with only dotted vertices on the ends of the attached adjoint edges is an element of $I(A_r)$, so we only have to generate the connected components of the diagram with the $I$ or $O$ vertices. But as we saw, the only diagrams we need are those whose connected components are traces. Thus we need to generate all diagrams where both the $I$ and the $O$ vertices are connected to the same trace, and all diagrams where they're connected to different traces.

First we show that, given $K_k$ and $L_k$ for all $k$, we can generate an element where the
$I$ vertex is connected to a trace of degree $m$ and the $O$ vertex is connected to another trace of degree $n$, the two traces being distinct connected components:

$$= K_{m-1}SK_{n-1}$$

Note that for a trace connected to the $I$ vertex but not the $O$ vertex, the direction of the arrow is irrelevant, since all of the dotted vertices are symmetrized over. Similarly, for a trace connected to the $O$ vertex but not the $I$ vertex, the direction of the arrow is irrelevant. Hence $K_{m-1}S = L_{m-1}S$ and $SK_{n-1} = SL_{n-1}$.

Now we show that $K_k$ and $L_k$ can be generated via $K$, $L$ and $S$. We shall show the derivation for $K_k$; the $L_k$ case is analogous.

**Lemma 6.3.1.** $K_k$ can be generated over $I(A_r)$ by $K$ and $S$

We first note that $K_1 = K$, and then proceed by induction.

We assume that we can generate $K_m$, $K_n$, $K_{m-1}$ and $K_{n-1}$, and now we show that we can generate $K_{m+n}$:

$$= K_{m+n} - \frac{1}{r+1} K_{m-1} SK_{n-1}$$
The second line uses the projection from $V \otimes V^\vee$ to the adjoint representation to remove the internal adjoint edge. Thus we can write

$$K_{m+n} = K_m K_n + \frac{1}{r+1} K_{m-1} S K_{n-1}$$

So thus we can generate $K_k$ for all $k$.

Finally, we just have to generate all of the other diagrams where both the $I$ and $O$ vertices are connected to the same trace. These are all traces where the reference edge attaches to the $I$ vertex, then to $m$ dotted vertices, then to the $O$ vertex, and then to $n$ vertices.

$$K_m L_n = \begin{array}{c}
\includegraphics[width=2cm]{diagram1.png}
\end{array}$$

The first term in the last line is precisely what we want, and both $K_m L_n$ and the last term, $K_{m-1} S L_{n-1}$, can be generated from $K, L$ and $S$ by assumption. Hence our generators are sufficient to generate the whole family algebra.

### 6.4 Proof of the Relations

We have already seen that the relations $K S = L S$ and $S K = S L$ hold, as special cases of $K_m S = L_m S$ and $S K_n = S L_n$. Now we can prove the other relations. We start with
where the traces in the second term of the last expression have only two adjoint edges attached, and hence by our normalization become just adjoint edges.

$LK$ differs only in the direction of the arrows on in the first term, so we end up with a trace attached to the $I$ vertex, then a dotted vertex, then the $O$ vertex, and then another dotted vertex. But the dotted vertices are interchangeable, so which of the two dotted vertices we pass through first doesn't matter. Hence the direction of the arrow doesn't matter and so $KL = LK$.  

The $SK_m L_n S$ relation follows from the expression for the product of $K_m L_n$ computed above:

\[
SK_m L_n S = \left( \begin{array}{c}
\begin{array}{cc}
&
\
&\end{array}
\end{array} \right) - \frac{1}{r+1} \left( \begin{array}{c}
\begin{array}{cc}
&
\
&\end{array}
\end{array} \right)
\]

The contents of the parentheses, being unconnected to the $I$ and $O$ vertices, is an element of $I(A_r)$, and counting the dotted vertices coming off each trace gives a factor of $c_{m+n-1} = \frac{1}{r+1} c_m c_n$.

The other relations mentioned follow from variations of the Cayley-Hamilton identity. Since the reference representation is $r+1$-dimensional, for any $X \in A_r$ we have a
relation

\[ [\pi(X)]^{r+1} = \sum_{k=0}^{r} d_{r+1-k}(X) [\pi(X)]^{k} \]

where \( d_k \) is a degree \( k \) polynomial of the entries of \( \pi(X) \). Thus for the matrix \( M_d = \pi(X) \otimes X^\alpha \), we get a relation

\[ M_d^{r+1} = \sum_{k=0}^{r} d_{r+1-k} M_d^{k} \]

where now \( d_k \) is an element of \( I(A_r) \).

Diagrammatically, this translates as

.. .

Now we note that \( K_k \) contains a reference line attached to \( k \) dotted vectors, and so for \( K_{r+1} \) we can make the above replacement. This yields the relation

\[ K_{r+1} = \sum_{k=0}^{r} d_{r+1-k} K_k \]

And similarly for \( L_{r+1} \).

The final relation comes from the decomposition of \( tr(M_d^{r+2}) \) into primitive Casimir operators. We have the following relation, mentioned in the section on Casimir operators:

\[ 0 = \sum_{n_j m_j = r+2} \prod_j \frac{1}{m_j!} \left( -\frac{tr(M_d^{n_j})}{m_j} \right)^{m_j} \]

The coefficient of \( tr(M_d^{r+2}) \) on the right side is \(-1\), so this gives an expression for \( tr(M_d^{r+2}) \) in terms of the primitive Casimir operators, recalling that \( tr(M_d^{k+1}) = c_k \) for
$1 \leq k \leq r$.

We can translate this fact into one about the family algebra by writing all of the traces as diagrammatic traces, connected only to dotted vertices, and then for each diagram writing out all the ways to replace a dotted vertex by the $I$ vertex and another dotted vertex by the $O$ vertex. This is equivalent to taking derivatives with respect to the vectors corresponding to the edges connected to the $I$ and $O$ vertices.

Given a trace of degree $d$ with only dotted vertices, there are $d$ ways to replace one dotted vertex by the $I$ vertex, and all the ways yield the same diagram. Given a product of traces with only dotted vertices, the number of ways to replace a dotted vertex by the $I$ vertex is equal to the total degree of the product, with each trace of degree $d_i$ yielding $d_i$ identical diagrams.

Given a trace with the $I$ vertex and $d - 1$ dotted vertices, there are now $d - 1$ ways to replace a dotted vertex by the $O$ vertex. Given a product of traces with one vertex being the $I$ vertex and the rest dotted, the number of ways to replace a dotted vertex by the $O$ vertex is the degree of the product (which only counts the dotted vertices). Note that we have two possibilities here: the $O$ vertex could be on the same or on a different trace as the $I$ vertex.

Using the fact that

$$d_k = \sum_{m_i n_i = k} \prod_i \frac{1}{m_i!} \left( \frac{\text{tr}(M_d)^{n_i}}{n_i!} \right)^{m_i}$$

for $k \leq r + 1$, we get that the sum is thus

$$0 = \sum_{i,j} d_{r-i-j-2} K_i SL_j + \sum_{i,j} d_{r-i-j} Q_{i,j}$$
where \( Q_{i,j} \) is a trace connected to the \( I \) vertex, then \( i \) dotted vertices, then the \( O \) vertex, and then \( j \) dotted vertices, which we saw above can be written as

\[
K_i L_j + \frac{1}{r+1} K_{i-1} S L_{j-1}
\]

Writing out \( K_m \) and \( L_n \) in terms of \( K, L \) and \( S \), we get that all of the terms involving \( S \) automatically cancel out, leaving the relation

\[
\sum_{k=0}^{r} K^k L^{r-k} = \sum_{j=0}^{r-2} d_{r-j} \sum_{k=0}^{j} K^k L^{j-k}
\]

### 6.5 The Sufficiency of the Relations

Here we show that the relations listed above are sufficient to determine the algebra, i.e. that any further relations on the algebra can be derived from the relations already given.

**Lemma 6.5.1.** No monic polynomial in \( K + L \) with coefficients in \( I(g) \) and degree less than \( r \) can vanish.

**Proof.** If we consider \( K^k L^{m-k} \), lower the raised coordinate using the Killing form, and then symmetrize over all of the indices, coordinate or otherwise, we end up with a polynomial in Casimir elements of degree \( m+2 \) including a term of \( c_{m+2} \) and all other terms products of Casimir elements of lower degree.

Now suppose that we have a monic polynomial in \( K + L \) with coefficients in \( I(g) \) and degree \( m \) less than \( r \). Then the leading terms, i.e. the terms involving no nontrivial
Casimir elements, has positive coefficients for all terms of the form $K^k L^{m-k}$ and thus the symmetrization of this polynomial then yields a polynomial in Casimir elements with nonvanishing $c_{m+2}$ coefficient.

Since $m < r$, we have that $m+2 < r+2$ and hence $c_{m+2}$ is algebraically independent of the Casimir elements of lower degree; hence the symmetrization cannot vanish, and hence the polynomial in $(L + R)$ cannot vanish. □

Consider now the $K^k L^l$ relation. The leading term is

$$\sum_k K^k L^{r-k}$$

and hence multiplying this leading term by $(K + L)^m$ yields a polynomial in $K$ and $L$ that only has positive coefficients. In particular, the term $K^r L^m$ has positive coefficient in this polynomial. For $m \leq r$, neither $K^r$ nor $L^m$ can be reduced by the $K_{r+1}$ or $L_{r+1}$ relations.

Hence we get that for $m < r$, $(K + L)^m$ times the $K^k L^l$ relation yields a relation in each degree greater than $r - 1$ that cannot be deduced from the other relations. Since no polynomial in $K + L$ vanishes for degree less than $r$, we get that the relations of the form $(K + L)^m$ times the $K^k L^l$ relation are themselves linearly independent from one another over $I(g)$. We also get a relation in degree $2r$ by squaring the $K^k L^l$ relation, and this one is also linearly independent from the other relations since the $K^k L^l$ relation is itself a polynomial in $K$ and $L$ that is linearly independent from all polynomials in $K + L$, just by comparing leading coefficients.

Now we show that there cannot be any other relations that are not in the ideal gener-
ated by the ones listed. We do so by counting the number of \( I(g) \)-linearly independent monomials.

Note that since \( K S = LS \), we can write \( K^r S = \frac{1}{r+1} \sum_{k} K^k L^{r-k} S \). Hence, using the \( K^k L^l \) relation, we can reduce \( L^r S \) to terms involving nontrivial Casimir elements. Hence we get that \( K^r SL^l \) is not linearly independent over \( I(g) \) from terms of lower degree. Similarly, \( K^k SL^r \) cannot be linearly independent.

Thus we get that our linearly independent monomials are \( K^k SL^l \) for \( 0 \leq k, l \leq r - 1 \) and \( K^k L^l \) for \( 0 \leq k, l \leq r \), minus one in each degree between \( r \) and \( 2r \) since \( (K + L)^m \) times the \( K^k L^l \) relation gives \( K^r L^m \) in terms of other monomials.

This yields a total of \( 2r^2 + r \) terms not known to be linearly dependent. If there are more relations, then there will be fewer linearly independent terms.

The dimension formula for family algebras tells us that we should be getting

\[
\dim_{I(g)} C_{I(g)}(g) = \sum_{\lambda \in W_{I(g)}} m_{I(g)}(\lambda)^2 
\]

For the adjoint representation, the weights with non-zero multiplicity are the roots, each with multiplicity 1, and 0, with multiplicity equal to the rank of the algebra. This gives us \( r(r + 1) + r^2 = 2r^2 + r \). Hence, since the relations given above limit us to at most \( 2r^2 + r \) linearly independent elements and any further relations would reduce that number, there cannot be any more relations.

Thus we can determine an \( I(g) \)-linear basis for the family algebra in terms of \( K, L \) and \( S \). Using the original basis \( M, R_2 \) and \( S \), we rewrite the set as

\[
M^m R_2^n \text{ for } m \leq e_r + 1, n \leq r - 1 
\]
\[ R^m_2 SR^n_2 + R^n_2 SR^m_2 \text{ for } m \leq n \leq r - 2 \]

\[ R^m_2 SR^n_2 - R^n_2 SR^m_2 \text{ for } m < n \leq r - 1 \]

Note that we can define an element \( R_k \) for \( k \leq r \), where \( R_k = K_r + L_k \), which in turn can be written as \( R^k_2 \) plus other terms. Hence we can write our basis as

\[ M^m R_k \text{ for } m \leq e_r + 1, k \leq r - 1 \]

\[ R_m SR_n + R_n SR_m \text{ for } m \leq n \leq r - 2 \]

\[ R_m SR_n - R_n SR_m \text{ for } m < n \leq r - 1 \]

In our example of \( A_3 \), we have the following basis

\[ 1, M, R_2, M^2, MR_2, R^2_2, S, M^3, M^2 R_2, MR^2_2, R_2 S + SR_2, \]

\[ R_2 S - SR_2, M^4, M^3 R_2, M^2 R^2_2, R_2 SR_2, R^2_2 S - SR^2_2 \]

\[ M^4 R_2, M^3 R^2_2, R^2_2 SR_2 - R_2 SR^2_2, M^4 R^2_2 \]

### 6.6 Generalized Exponents

For the irreducible component of \( \mathfrak{g} \otimes \mathfrak{g}^\vee \) with highest weight \( \lambda \), there is a projection operator \( P_\lambda \) that projects from \( \mathfrak{g} \otimes \mathfrak{g}^\vee \) to the component of type \( V_\lambda \). See [Cv08] for details.

Using the Killing form, we identify \( \mathfrak{g}^\vee \) with \( \mathfrak{g} \) and consider \( \mathfrak{g} \otimes \mathfrak{g} \). As a \( \mathfrak{g} \)-module this decomposes into \( \wedge^2 \mathfrak{g} \) and \( S^2 \mathfrak{g} \), the alternating and symmetric tensor square respectively, which then further decompose into irreducible representations.
For \( r = 1 \), \( \wedge^2 g \) is isomorphic to \( g \) itself, and hence is the adjoint representation, with generalized exponent 1. \( S^2 g \) decomposes into a trivial representation and a representation of dimension 5; these representations have generalized exponents 0 and 2 respectively.

For \( r = 2 \), \( \wedge^2 g \) decomposes into a copy \( g \), with generalized exponents 1 and 2, and two dual 10-dimensional representations with weights \( 3\omega_1 \) and \( 3\omega_2 \) respectively and each with generalized exponent 3. \( S^2 g \) decomposes into the trivial representation with generalized exponent 0, another copy of \( g \), again with generalized exponents 1 and 2, and a 27-dimensional representation with generalized exponents 2, 3 and 4. See [Ro01] for details. Note that Rozhkovskaya uses a different basis, generated by harmonic elements. Her \( M_1 \) is proportional to \( M \), her \( N_1 \) is proportional to \( R_2 \), and her \( N_2 \) is proportional to \( 3R_2^2 + 3M^2 + S + c_1 \).

For \( r \geq 3 \), the decomposition of \( g \otimes g \) is uniform. \( \wedge^2 g \) decomposes into a copy of \( g \) and two dual representations with highest weights \( 2\omega_1 + \omega_{r-1} \) and \( \omega_2 + 2\omega_r \) respectively, while \( S^2 g \) decomposes into the trivial representation, another copy of \( g \), and two representations with highest weights \( \omega_2 + \omega_{r-1} \) and \( 2\omega_1 + 2\omega_r \) respectively. In an orthonormal basis for \( g \), the corresponding elements of the family algebra are actually symmetric or antisymmetric as matrices.

The \( L^k R^l \) reduction relation gives us a relation \( \sim \) on elements in \( V_{\omega_2 + \omega_{r-1}} \); applying the differential operator \( D = \left( \frac{\partial}{\partial \alpha} c_2 \right) \frac{\partial}{\partial \alpha} \) gives a relation equivalent to the multiples of the \( L^k R^l \) relation times \( L + R \), modulo the \( L_{r+1} \) and \( R_{r+1} \) relations. Since \( D \) transforms as
the trivial representation, $D$ applied to both sides of $\sim$ again gives a relation between elements of the $V_{\omega_2 + \omega_{r-1}}$ representation; hence we get that the generalized exponents of $V_{\omega_2 + \omega_{r-1}}$ plus a copy of $\{r, \ldots, 2r\}$ gives the generalized exponents of $V_{2\omega_1 + 2\omega_r}$.

Along with the fact that $V_{\omega_1 + \omega_r}$ has generalized exponents $1, \ldots, r$ gives us enough information to get the full set of generalized exponents for the representations in question, given in table 1. Note that the two copies of $V_{\omega_1 + \omega_r}$ each give an independent set of harmonic basis elements, one symmetric, one antisymmetric. We get that $P_V(q)$ is equal to the Kostka polynomial for $V$, which are computable from Young Tableaux [DLT94]. Hence we can easily check the results given.

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>1</td>
</tr>
<tr>
<td>$V_{\omega_1 + \omega_r}$</td>
<td>$q[r]q$</td>
</tr>
<tr>
<td>$V_{\omega_2 + \omega_{r-1}}$</td>
<td>$q^2 \frac{[r+1]_q [r-2]_q}{[2]_q}$</td>
</tr>
<tr>
<td>$V_{2\omega_1 + 2\omega_r}$</td>
<td>$q^2 \binom{r+1}{2} q$</td>
</tr>
<tr>
<td>$V_{2\omega_1 + \omega_{r-1}}$</td>
<td>$q^3 \binom{r}{2} q$</td>
</tr>
<tr>
<td>$V_{\omega_2 + 2\omega_r}$</td>
<td>$q^3 \binom{r}{2} q$</td>
</tr>
</tbody>
</table>

Table 6.1: Generalized Exponents in $C_{\omega_1 + \omega_r}(A_r)$
Table 6.2: Generalized Exponents in $C_{\omega_1 + \omega_3}(A_3)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>1</td>
</tr>
<tr>
<td>$V_{\omega_1 + \omega_r}$</td>
<td>$q + q^2 + q^3$</td>
</tr>
<tr>
<td>$V_{2\omega_2}$</td>
<td>$q^2 + q^4$</td>
</tr>
<tr>
<td>$V_{2\omega_1 + 2\omega_3}$</td>
<td>$q^2 + q^3 + 2q^4 + q^5 + q^6$</td>
</tr>
<tr>
<td>$V_{2\omega_1 + \omega_2}$</td>
<td>$q^3 + q^4 + q^5$</td>
</tr>
<tr>
<td>$V_{\omega_2 + 2\omega_3}$</td>
<td>$q^3 + q^4 + q^5$</td>
</tr>
</tbody>
</table>
Chapter 7

The $B_r, C_r$ case

The cases of $B_r$ and $C_r$ end up very similar, so we treat them both here. We start with $C_r$ since it is somewhat simpler. As with the previous chapter, we use the case of $r = 3$ as an example.

7.1 Diagrams

As in the $A_r$ case, we can write the primitive Casimir operators as traces

$$c_k = \frac{e_k + 1}{e_k}$$

Because the invariant changes sign every time it passes an adjoint edge, we get that the odd-degree traces vanish, matching the fact that $C_r$ only has odd-degree exponents and hence even degree primitive Casimir elements. For $C_r$, the exponents are $e_i = 2i - 1$, so that $c_k = tr_V(M_d^{2i})$. 

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Similar to the $A_r$ case, we have a Cayley-Hamilton identity on our matrices in the reference representation. Defining
\[
d_k = \sum_{2n_i m_i = k} \frac{1}{m_i!} \left( -\frac{c_{n_i}}{2n_i} \right)^{m_i}
\]
where $m_i n_i$ indicates a sum over distinct $n_i$, we get that
\[
\sum_k d_{2r-k} Q^k = 0
\]
where
\[Q^k = \begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array}
\]
We call this the matrix Cayley-Hamilton identity, to distinguish it from the Casimir Cayley-Hamilton identity
\[
\sum n_i m_i = 2r + 2 \frac{1}{m_i!} \left( -\frac{c_{n_i}}{2n_i} \right)^{m_i} = 0
\]
which we get by multiplying the matrix Cayley-Hamilton identity by $M^2$ and then taking traces in the $2r$-dimensional representation.

The adjoint projection for getting rid of internal adjoint edges is also different:
\[
\begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array} = \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array} + \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array}
\]
Note the directions of the symplectic forms; the first term on the right-hand side has both symplectic forms attached to the top edge, where they cancel.

Now we wish to show that the tensor invariants in $(T(C_r))^{Sp(r)}$ are generated by tensor products of traces over the reference representation.
All finite dimensional representations of $C_r$ can be written in terms of the reference representation, so we only have to worry about tensors with adjoint and reference edges. The vertices are Clebsches between the adjoint and $V \otimes V^\vee$, and the Levi-Civita tensor on $V$.

Note that the reference representation, being of dimension $2r$, has a Levi-Civita tensor with $2r$ vectors coming out of it. Moreover, taking $r$ copies of the symplectic form and antisymmetrizing all of the edges yields a multiple of the Levi-Civita tensor. Hence the Levi-Civita tensor can be replaced by the symplectic form. Thus we only have loops of the reference edges with adjoint edges attached, i.e. traces over the reference representation.

### 7.2 Generators

The main result about the generators for $C_r$ is that there are again three of them:

**Theorem 7.2.1 (Generators).** *The family algebra for the adjoint representation of $C_r$ is generated by*

\[
M = \begin{array}{c}
\text{I} \\
\text{O}
\end{array}, \quad R_2 = \begin{array}{c}
\text{O}
\end{array}, \quad S = \begin{array}{c}
\text{I} \\
\text{O}
\end{array}
\]

Because of the symplectic form, we get that $L$ and $R$ are no longer independent
elements of the family algebra; in particular, $M = L = -R$ and there is no other independent degree 1 family algebra element.

**Proof.** For ease of calculation, again we use an alternate object instead of $R_2$. We write $T_{k,l}$ to be a trace attached to the $I$ vertex, $k$ dotted vertices, the $O$ vertex, and then $l$ dotted vertices, in that order. Thus $M$ is $T_{1,0}$. In this notation, $R_2 = 2T_{2,0}$.

Using the symplectic form to swap the direction of the reference tensor, we get that

$$T_{k,l} = (-1)^{k+l} T_{l,k}$$

and via the adjoint projector we get

$$T_{i,j} T_{k,l} = \frac{1}{2}(T_{i+k,j+l} + (-1)^{k+l} T_{i+l,j+k})$$

In particular,

$$MT_{k,l} = \frac{1}{2}(T_{k+1,l} + (-1)^{k+l} T_{l+1,k})$$

We see the sufficiency of the generators as given by noting that

$$M^2 = \frac{1}{2}(T_{2,0} - T_{1,1})$$

so $T_{2,0}$ and $T_{1,1}$ can be generated from $M$ and $R_2$, and then that

$$T_{k+1,l} = MT_{k,l} - MT_{l-1,k+1} + T_{2,0} T_{k,l-1}$$

so we can generate any $T_{k,l}$ via inducting from our base cases.

The only remaining possible elements are those where the $I$ and $O$ vertices are connected to unconnected traces. These we can achieve by $S$. In particular, we can realize
an element where the $I$ vertex is attached to a trace of degree $k$ and the $O$ vertex to a trace of degree $l$ by $T_{k-1,0}ST_{l-1,0}$.

7.3 Relations

Several relations are familiar from the $A_r$ case:

**Theorem 7.3.1.** The following relations hold for $C_r$:

\[
SM = MS = 0
\]

\[
R_2 M = MR_2
\]

\[
\sum_{k=0}^{r} d_{2r-2k}T_{2k,0} = 0
\]

Define

\[
Q_{2k} = \sum_{l=0}^{2k} T_{l,2k-l} - \sum_{l=0}^{k-1} T_{2l,0} ST_{2k-2l-2,0}
\]

Then

\[
\sum_{k=0}^{r} d_{2r-2k}Q_{2k} = 0
\]

In our example, the $r$-dependent relations become

\[- T_{6,0} + d_2 T_{4,0} + d_4 T_{2,0} + d_6 = 0
\]

\[2T_{6,0} + 2T_{5,1} + 2T_{4,2} + T_{3,3} - T_{4,0}S - T_{2,0}ST_{2,0} - ST_{4,0}
\]

\[= d_2(2T_{4,0} + 2T_{3,1} + T_{2,2} - T_{2,0}S - ST_{2,0}) + d_4(2T_{2,0} + T_{1,1} - S) + d_6
\]
The first two relations can be seen by expanding out the relevant diagrams, or in the case of the second relation by expanding out the $T_{k,l}$ relations.

For the third relation, we have the matrix Cayley-Hamilton relation, which, as in the $A_r$ case, gives us a relation on reference edges connected by adjoint edges to only dotted vertices. For $C_r$, the identity only involves even powers of the matrix, which translates to an even number of dotted vertices. $T_{2k,0}$ involves a reference edge connected to $2k$ dotted vertices, and thus we get the third relation.

The fourth relation comes from taking the decomposition of $tr(M_d^{2r+2})$ into primitive Casimir operators, interpreting it as diagrams, and replacing dotted vertices with $I$ and $O$ vertices, analogous to the $L^m R^n$ relation for $A_r$.

Note that the third relation has no mentions of $S$, and the fourth relation has both $ST_{2k,0}$ and $T_{2k,0}S$. Thus if $P$ times the third relation yields $R^I_2$ of the fourth relation, we get that $l = r$ and thus we get that at the very least the fourth relation yields $r$ relations that are independent of the third relation.

We now use a counting argument. We can form objects of the forms $T_{i,j}$ and $T_{k,0}ST_{l,0}$.

We note that if $k$ or $l$ is odd, then $T_{k,0}ST_{l,0}$ vanishes, due to the symplectic form. So we really only have $T_{i,j}$ and $T_{2k,0}ST_{2l,0}$. By the third relation, we can limit $i + j$ to be less than $2r$, and we can limit $i \leq j$ since $T_{i,j}$ and $T_{j,i}$ are not independent. We can similarly limit $k$ and $l$ to be less than $r$. So we have $T_{i,j}$ for $0 \leq i \leq j \leq 2r - 1$ and $T_{2k,0}ST_{2l,0}$ for $0 \leq k, l \leq r - 1$. This yields a total of $3r^2 + r$ elements, from which the fourth relation removes another $r$ elements, to yield $3r^2$ linearly independent elements. By the
dimension formula for family algebras, we should be getting $r^2 + 2r^2 = 3r^2$ elements.

Since there are no more elements to remove, there are no more relations.

We will thus take as our basis $T_{i,j}$ for $0 \leq i \leq j \leq 2r - 1$, $T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0}$ for $k,l \leq r - 2$ and $T_{2k,0}ST_{2l,0} - T_{2l,0}ST_{2k,0}$ for $k,l \leq r - 1$. To make it more in line with the results for other Lie algebras, we write this as

$$M^m R^n_2 \text{ for } m \leq e_r + 1, n \leq r - 1$$

$$R^m_2 SR^n_2 + R^n_2 SR^m_2 \text{ for } m \leq n \leq r - 2$$

$$R^m_2 SR^n_2 - R^n_2 SR^m_2 \text{ for } m < n \leq r - 1$$

Note that we can define an element $R^k_2$ for $k \leq r$ as the trace that attaches to the $I$ vertex, $e_k - 2$ dotted vertices, and then to the $O$ vertex which in turn can be written as $R^k_2$ plus other terms. Hence we can write write our basis as

$$M^m R^k_2 \text{ for } m \leq e_r + 1, k \leq r - 1$$

$$R_m SR_n + R_n SR_m \text{ for } m \leq n \leq r - 2$$

$$R_m SR_n - R_n SR_m \text{ for } m < n \leq r - 1$$

So for $C_3$, the basis is

$$1, M, M^2, R_2, S, M^3, MR_2, M^4, M^2 R_2, R^2_2, R_2 S + SR_2, R_2 S - SR_2,$$

$$M^5, M^3 R_2, MR^2_2, M^6, M^4 R_2, M^2 R^2_2, R_2 SR_2, R^2_2 S - SR^2_2, M^5 R_2,$$

$$M^3 R^2_2, M^6 R_2, M^4 R^2_2, R^2_2 SR_2 - R_2 SR^2_2, M^5 R^2_2, M^6 R^2_2$$

Note that the only difference, at least in the labelling, between this and the basis for the family algebra for $A_3$ is the maximum power of $M$ allowed.
7.4 $B_r$

Now we address the $B_r$ case. We have a change in the adjoint projector:

$$
\begin{align*}
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{align*}
= \frac{1}{2} \begin{array}{c}
\hline
\hline
\hline
\end{array} - \frac{1}{2} \begin{array}{c}
\hline
\hline
\hline
\end{array}
\end{align*}
$$

We still get a sign change whenever we move the bilinear form past an adjoint edge and hence the primitive Casimir operators are all of even degree and follow the same Casimir Cayley-Hamilton identity. The change in the adjoint projector and the symmetry of the bilinear form make the $T_{k,l}$ objects for the family algebra for $B_r$ follow the same rules as the ones for the $C_r$ family algebra. So we get that the family algebra for $C_r$ and the family algebra for $B_r$ are almost isomorphic. There is one slight difference in the relation from the matrix Cayley-Hamilton identity.

Since the reference representation for $B_r$ is $2r + 1$-dimensional, we expect that the matrix Cayley-Hamilton relation has degree $2r + 1$, and indeed it does, with no relation of lower degree working for all elements of $B_r$. Hence we get a relation

$$
\sum_k d_{2r+1-(2k+1)} T_{2k+1,0} = 0
$$

However, this relation can itself be reduced to a relation in lower degree. First we note that $d_{2r+1-(2k+1)} = d_{2r-2k}$. Secondly, we note that

$$
T_{2k+1,0} = M \sum_{l=0}^{2k} T_{2k-l,l}
$$

We further note that

$$
\sum_{l=1}^{2k} T_{2k-l,l} = 2M \sum_{j=1}^{k} T_{2k-2j,2j-1}
$$

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Hence we look at

\[ Q = \sum_{k=0}^{r} d_{2r-2k} \left( T_{2k,0} + 2M \sum_{j=1}^{k} T_{2k-2j,2j-1} \right) \]

Multiplying this by \( M \) yields the matrix Cayley-Hamilton relation, i.e. this expression times \( M \) vanishes. Now we look at this object restricted to the Cartan subalgebra.

Since \( M \) is invertible on the vector part, since the entry in the \( x_{\alpha}, x^\alpha \) position is \( \alpha^\vee \), we get that since \( MQ \) vanishes on the vector part, \( Q \) must vanish on the vector part. We also note that \( M \) vanishes on the torus part, so the torus part of \( Q \) is

\[ \sum_{k=0}^{r} d_{2r-2k} T_{2k,0} \]

Since the primitive Casimir operators for \( B_r \) restricted to the Cartan subalgebra are identical to the primitive Casimir operators for \( C_r \) restricted to the Cartan subalgebra, we get that since \( \sum_{k=0}^{r} d_{2r-2k} T_{2k,0} \) vanishes on the torus for \( C_r \), it also must vanish for \( B_r \).

So \( Q \) restricted to the Cartan subalgebra must vanish on both the vector and torus parts, and hence vanishes everywhere. Since the restriction map is an injection, \( Q \) itself must vanish. Hence we have a relation in degree \( 2r \) rather than \( 2r + 1 \).

Again, this relation involves no terms of \( S \), so again the Casimir Cayley-Hamilton identity yields \( r \) separate relations, and the counting argument above still holds. Hence we get that the relations for \( B_r \) are as follows:

**Theorem 7.4.1.** The following relations hold for \( B_r \): 

\[ SM = MS = 0 \]
Define

\[ Q_{2k} = \sum_{l=0}^{2k} T_{l,2k-l} - \sum_{l=0}^{k-1} T_{2l,0} ST_{2k-2l-2,0} \]

Then

\[ \sum_{k=0}^{r} d_{2r-2k} Q_{2k} = 0 \]

Hence the family algebra for \( B_r \) has the same \( I(g) \)-linear basis as the family algebra for \( C_r \), with the two family algebras differing only in the algebraic relations.

In terms of the example of \( B_3 \), we have the first \( r \)-dependent relation being

\[-T_{6,0} - 2M(T_{4,1} + T_{2,3} + T_{0,5}) + d_2 T_{4,0} + 2d_2 M(T_{2,1} + T_{0,3}) + d_4 T_{2,0} + 2d_4 M T_{0,1} + d_6 = 0\]

and the other \( r \)-dependent relation being identical to the case for \( C_3 \). Similarly, the basis elements are identical to those for \( C_3 \).

### 7.5 Generalized Exponents

The decomposition of the tensor square of the adjoint representation into irreducible components is very similar for \( B_r \) and \( C_r \). Both decompose into a symmetric part and an antisymmetric part, with the symmetric part decomposing into four irreducible components and the antisymmetric part decomposing into two. See [Cv08] for details and explicit projection operators.
Applying the projection operators to the basis elements computed above, we get that $P_2(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0})$ depends only on $k + l$, and that $P_2(T_{i,j})$ is a linear combination of the $P_2(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0})$ terms. So we get that the component corresponding to $V_2$ is spanned by $P_2(T_{2k,0}S + ST_{2l,0})$. Note that while there is a relation involving $T_{2r-2,0}S + ST_{2r-2,0}S$, that relation occurs in $V_2$ and thus we get that $P_2(T_{2r-2,0}S + ST_{2r-2,0})$ is linearly-independent from the set of $P_2(T_{2k,0}S + ST_{2k,0})$ for $k < r - 1$.

Since $V_2$ corresponds to $S^2 V$, we get that $V_2$ should have highest generalized exponent $2r$ and have $r$ generalized exponents. Hence we get that the generalized exponents for $V_2$ are thus $q^2[r]_{q^2}$.

$$P_4(T_{2k+1,2l+1}) = -P_4(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0})$$
$$= -\frac{1}{6}(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0}) - \frac{2}{3}T_{2k+1,2l+1}$$

and $P_4(T_{2k,2l}) = 0$. So $V_4$ has a generalized exponent for each pair $k, l$ such that $k, l \leq r - 2$. Hence $P_{V_4}(q) = q^2\binom{r}{2}_{q^2}$.

$V_1$ is the trivial representation, and thus has a single generalized exponent of degree 0. So all the rest of the degrees of the symmetric basis elements give generalized exponents for $V_3$. Thus we have that the generalized exponents of $V_3$ are $q^2[r + 1]_{q^2}[r - 1]_{q^2}$. The antisymmetric elements yield degrees $q[r]_{q^2}^2 + q^4\binom{r}{2}_{q^2}$. The adjoint representation has exponents $q[r]_{q^2}$, so we are left with $V_6$ having exponents $q^3[3]_{q^2}\binom{r}{2}_{q^2}$.
Table 7.1: Generalized Exponents in $C_g(B_r/C_r)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>1</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$q^2[r]q^2$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2[r+1]q^2[r-1]q^2$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$q^2(r)q^2$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$q[r]q^2$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$q^3[3]q^r(r)q^2$</td>
</tr>
</tbody>
</table>

Table 7.2: Generalized Exponents in $C_g(B_3/C_3)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>1</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$q^2 + q^4 + q^6$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2 + 2q^4 + 2q^6 + 2q^8 + q^{10}$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$q^2 + q^4 + q^6$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$q + q^3 + q^5$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$q^3 + q^4 + 2q^5 + q^6 + 2q^7 + q^8 + q^9$</td>
</tr>
</tbody>
</table>
Chapter 8

The $D_r$ case

Here we use both $r = 3$ and $r = 4$ as examples, as they have somewhat different behavior, and also because $r = 3$ has already been discussed, via the $A_r \cong D_r$ isomorphism.

8.1 Diagrams

The first fundamental theorem for $SO(2r)$ acting on the $2r$-dimensional representation tells us that the only invariants are a symmetric bilinear form and a Levi-Civita tensor. Thus the projection from $V \otimes V^\vee$ to the adjoint representation is the same as that for $B_r$. Unlike in the $B_r$ case, the Levi-Civita tensor has an even number of edges coming out of it, and so we can have an element of $(T(D_r))^{SO(2r)}$ with a single Levi-Civita tensor in it, and unlike in the $C_r$ case, the Levi-Civita tensor cannot be reduced to the bilinear form. As a result, the set of primitive Casimir operators for $D_r$ cannot be expressed solely as traces over the reference representation. Instead, we have traces of
2k adjoint edges for $1 \leq k \leq r - 1$, and a degree $r$ Casimir, called the Pfaffian, built from the Levi-Civita tensor.

Note that since the $2r$ reference edges going into the Levi-Civita tensor are fully anti-symmetrized, the adjoint edges attached to the Pfaffian are automatically fully symmetrized as a tensor, unlike the traces which have to be symmetrized separately.

We now have a decomposition of $tr(M_d^{2r})$ into lower order Casimir operators. We get

$$\sum_{n_i m_i = 2r} \prod_i \frac{1}{m_i!} \left( - \frac{tr(M_d)^{n_i}}{n_i} \right)^{m_i} = 4(-1)^r \left( \frac{P f}{r!} \right)^2$$

The rest of the elements of $(T(D_r))^{SO(2r)}$ are either tensor products of traces, or tensor products of traces with tensors that like the Pfaffian are built from a single Levi-Civita tensor, only with multiple adjoint edges attached to each reference edge instead of just one. Note that since the Levi-Civita tensor is fully antisymmetric, the ends of the reference edges are antisymmetrized, so there must be an odd number of adjoint edges attached to each one.

### 8.2 Generators

The generators of $B_r$ can be used to generate all of the elements of the family algebra that do not involve the Pfaffian, but always yield diagrams that only have traces. To get the diagrams that involve the Pfaffian, we add a fourth generator, $P$: 57
Since the Pfaffian is independent of the trace Casimir operators, we cannot hope to build $P$ out of the elements written as $M, R_2$ and $S$. So we have to have it as a fourth generator.

In the case of $r = 3$, the Pfaffian looks like

$$P = \begin{bmatrix} r-2 \\ I & O \end{bmatrix}$$

In the $r = 4$ case, it looks like

$$P = \begin{bmatrix} \vdots \\ I & O \end{bmatrix}$$

### 8.3 Sufficiency of the Generators

The same reasoning as used for $B_3$ shows that any element of the family algebra built only from traces in the reference representation can be generated by $M, R_2$ and $S$. Hence we consider elements that contain a Levi-Civita tensor.

Suppose our element $A$ has a single Levi-Civita tensor, and assume that for all elements $B$ such that $\deg(B) < \deg(A)$, $B$ can be generated by $M, R_2, S$ and $P$. All the reference edges going out of the Levi-Civita tensor must go back in, and must have
an odd number of adjoint edges attached. The reference edges are all fully antisymmetrized, so for a given loop we can insert an adjoint edge projector.

Suppose we have such a loop $L$ with $2k + 1$ adjoint edges attached to it. If the $I$ vertex is attached to the loop, then we can insert an adjoint projector and write the family algebra element as $A = T_{m,2k-m}B$ where $B$ has one copy of the Levi-Civita tensor. By construction, $\text{deg}(B) = \text{deg}(A) - 2k$ and furthermore has only one adjoint edge connected to the loop corresponding to the loop $L$ on our original element. Similarly, if the $O$ vertex is attached to the loop, then we can construct a $B$ such that $A = BT_{m,2k-m}$ for some $m$. Since $\text{deg}(Q) < \text{deg}(A)$ in both cases, $Q$ must be generated by $M, R_2, S$ and $P$, and thus $A$ must also be generated by them.

Now consider the case where both the $I$ and $O$ vertices are attached to loops that only have one adjoint edge attached to them. Using the symmetrization process outlined in chapter 3, we can construct $\tilde{A}$ and $\bar{A}$. $\bar{A}$ can be written in terms of primitive Casimir operators, i.e. in terms of $M, R_2, S$ and $P$. The terms in $\bar{A}$ either have the $I$ vertex attached to a loop with only one adjoint edge attached or attached to a loop with multiple adjoint edges attached. In the latter case, we can rewrite the term as $T_{m,2k-m}B$ for some $B$ of lower degree, while in the former case we get another copy of $A$, since the reference edges connected to the Pfaffian are fully antisymmetrized, so the loops are distinguishable only by how many adjoint edges are attached and to what the other ends of those adjoint edges attach. So we have a nonzero multiple of $A$ being equal
to something expressible in terms of $M, R_2, S$ and $P$, plus terms of the form $T_{m,2k-m}B$ where $B$ has degree less than $A$. Since $\deg(B) < \deg(A)$ for all of the $B$, they must be generated by $M, R_2, S$ and $P$. Thus $A$ must also be generated by them.

Hence $M, R_2, S$ and $P$ generate the entire family algebra.

### 8.4 Relations

Several of the relations for $M, R_2$ and $S$ are identical to those from $B_r$:

\[
MR_2 = R_2M
\]

\[
MS = SM = 0
\]

We have a relation coming from the Cayley-Hamilton identity for Casimir operators:

\[
2PSP + 2P f P + \sum_{k=0}^{r-1} d_{2r-2k-2} Q_{2k} = 0
\]

Expanding out $Q_{2k}$ gives us that we can read this relation as expressing $R_2^{r-2} S + SR_2^{r-2}$ in terms of other elements.

The relation also gives us that $R_2^{r-1} S$ decomposes, since the resulting diagram has a trace of degree $2r$ attached to the $I$ vertex but not the $O$ vertex, and thus is it symmetrizes to $tr(M_d^{2r})$, which decomposes, and all of the diagrams in the symmetrization are the equal. Similarly, $SR_2^{r-1}$ also decomposes.

We also have additional relations involving $P$:

\[
PS + PT_{1,1} = 2P f
\]
Defining $P_{2k} = Q_{2k} - \sum_{l=0}^{k-1} T_{2l+1,2k-2l-1}$ for $k \geq 1$ and $P_0 = 1$, we get

$$P^2 + \sum_{k=0}^{r-1} (-1)^{r-k} d_{2r-4-2k} P_{2k} = 0$$

So our possibly linearly independent elements are

$$M^m R_n^e \text{ for } m \leq 2r-2, n \leq r-2$$

$$M^m P \text{ for } m \leq 2r-2$$

$$R^m e SR^e_n + R^m e SR^e_n \text{ for } m \leq n \leq r-3$$

$$PSR^m e + R^m e SP \text{ for } m \leq r-3$$

$$R^m e SR^e_n - R^m e SR^e_n \text{ for } m < n \leq r-2$$

$$PSR^m e - R^m e SP \text{ for } m \leq r-2$$

$$PSP$$

Similarly to $A_r$, $B_r$ and $C_r$, we can define $R_k$ to be an element related to the primitive Casimir operators, although here we have to be careful about indexing. For $k \neq \left\lceil \frac{r}{2} \right\rceil$, we define $R_k$ to be the trace that attaches to the $I$ vertex, then to the $e_{k-1}$ dotted vertices, and then to the $O$ vertex, while for $R_{\left\lceil \frac{r}{2} \right\rceil}$ we use $P$. Then we can rewrite the basis as

$$M^m R_k \text{ for } m \leq e_r + 1, k \leq r-1$$

$$R_m e SR_n + R_n e SR_m \text{ for } m \leq n \leq r-2$$

$$R_m e SR_n - R_n e SR_m \text{ for } m < n \leq r-1$$
This gives a total of $3r^2 - 2r$ elements, which matches the number of elements given by the dimension formula. Note that we could expand $PSP$ by expanding the two Levi-Civita tensors into antisymmetrized reference edges, but including $PSP$ makes the full set easier to describe, as well as better matching the list of linearly independent elements of $A_3 = D_3$. In particular, for $D_3$ we have

$$1, M, P, M^2, MP, R_2, S, M^3, M^2P, MR_2, PS + SP,$$

$$PS - SP, M^4, M^3P, M^2R_2, PSP, R_2S - SR_2,$$

$$M^5, M^4P, M^3R_2, PSR_2 - R_2SP, M^4R_2$$

The $P$ here corresponds to $R^2$ for $A_3$, with $R_2$ here taking the place of $R^2_2$ for $A_3$. For $r = 4$, we have

$$1, M, M^2, R_2, P, S, M^3, MR_2, MP;$$

$$M^4, M^2R_2, M^2P, R^2_2, R_2S + SR_2, PS + SP;$$

$$R_2S - SR_2, PS - SP, M^5, M^3R_2, M^3P, MR^2_2;$$

$$M^6, M^4R_2, M^4P, M^2R^2_2, R_2SR_2, PSR_2 + R_2SP, PSP;$$

$$R^2_2S - SR^2_2, PSR_2 - R_2SP, M^5R_2, M^5P, M^3R^2_2;$$

$$M^6R_2, M^6P, M^4R^2_2, R_2^2SR_2 - R_2SR^2_2, PSR^2_2 - R^2_2SP;$$

$$M^5R^2_2, M^6R^2_2$$

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8.5 Generalized Exponents

With a few low-dimensional exceptions, for $D_r$ the $\mathfrak{g} \otimes \mathfrak{g}$ representation decomposes identically to the $B_r$ case: the symmetric subspace of $\mathfrak{g} \otimes \mathfrak{g}$ decomposes into four irreducible representations, one being the trivial representation, and the antisymmetric subspace decomposes into two representations, one being the adjoint representation.

The projection operators are identical, except instances of $2r + 1$ are replaced by $2r$. So we label the six representations $P_1$ through $P_6$, matching the notation from $B_r$. For $D_3$, the antisymmetric subspace decomposes into three irreducible representations due to the Pfaffian. This matches the decomposition of $A_3$. For $r > 3$, the Pfaffian does not appear in any projection operators, so there six components listed are all irreducible.

The projections of the elements involving only $M, R_2$ and $S$ are the same as in the $B_r$ case, so we can just examine the elements involving $P$.

$P_2(PM^{2k})$ and $P_2(PSR_2^k)$ both decompose into lower order terms. $P_2(PSP)$ can be written in terms of $P_2$ applied to elements generated by $M, R_2$ and $S$. Hence $V_2$ has generalized exponents $P_{V_2}(q) = q^2[r-1]_q^2$, matching the $B_r$ case but with $[r-1]_q^2$ instead of $r$ due to the decomposition of the degree $2r$ trace.

$P_4(PM^{2k})$ reduces to $PSR_2^k$ for $k$ positive, as does $P_4(PSP)$. $P_4(PSR_2^k)$ expands as $PSR_2^k$ plus lower order terms. Hence we get that $V_4$ has generalized exponents $P_{V_4}(q) = q^2\left(\frac{r-1}{2}\right) + q^{r-2}[r-1]_q^2$. 

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$V_3$ then has the rest of the positive degree symmetric elements, giving it generalized exponents

$$P_{V_3}(q) = q^2[r]q^2[r - 2]q^2 + q^r[r - 1]q^2 + q^{2r - 2}$$

The adjoint representation has exponents $P_{V_5}(q) = q[r - 1]q^2 + q^{r - 1}$ and therefore $V_6$ has the remaining antisymmetric degrees, giving it generalized exponents

$$P_{V_6}(q) = q^3[3]q \binom{r - 1}{2} + q^{r-2}[2r - 1]q$$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>1</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$q^2[r - 1]q^2$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2[r]q^2[r - 2]q^2 + q^r[r - 1]q^2 + q^{2r - 2}$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$q^2 \binom{r - 1}{2} + q^{r-2}[r - 1]q^2$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$q[r - 1]q^2 + q^{r - 1}$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$q^3[3]q \binom{r - 1}{2}q^2 + q^{r-2}[2r - 3]q$</td>
</tr>
</tbody>
</table>
Table 8.2: Generalized Exponents in $C_6(D_3)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
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<tbody>
<tr>
<td>$V_1$</td>
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<tr>
<td>$V_2$</td>
<td>$q^2 + q^4$</td>
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<tr>
<td>$V_3$</td>
<td>$q^2 + q^3 + 2q^4 + q^5 + q^6$</td>
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<tr>
<td>$V_4$</td>
<td>$q + q^2 + q^3$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$q + q^2 + q^3$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$2q^3 + 2q^4 + 2q^5$</td>
</tr>
</tbody>
</table>

In the $r = 3$ case, $P_4$ is a copy of the adjoint representation, and $P_6$ decomposes into two antisymmetric representations that are dual to each other, giving the results for $A_3$. 
Table 8.3: Generalized Exponents in $C_3(D_4)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
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</tr>
<tr>
<td>$V_2$</td>
<td>$q^2 + q^4 + q^6$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2 + 3q^4 + 4q^6 + 3q^8 + q^{10}$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$2q^2 + 2q^4 + 2q^6$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$q + 2q^3 + q^5$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$q^3 + 2q^4 + 3q^5 + 2q^6 + 3q^7 + 2q^8 + q^9$</td>
</tr>
</tbody>
</table>
Chapter 9

Restrictions to Subalgebras

9.1 The Weyl Group Action

The torus part is made of $r \times r$ matrices, and as a Weyl-group module is $\mathfrak{h} \otimes \mathfrak{h}$. So it decomposes as $1 \oplus S^2 \mathfrak{h} \oplus \lambda^2 \mathfrak{h}$, where $S^2 \mathfrak{h}$ is the traceless symmetric square of $\mathfrak{h}$ and $\lambda^2 \mathfrak{h}$ is the second alternating power. For the non-simply laced algebras $B_r, C_r, G_2$ and $F_4$ we further divide the vector part into a long part, where the coordinates are long roots, and a short part, where the coordinates are short roots.

Each part of an $\mathfrak{h}$-restricted family algebra element is $W$-invariant, where $W$ acts on $S(\mathfrak{h})$ in the natural way. Since the coordinates of each part form a representation of $W$, we can find the $W$-irreducible components of the subspace of $\mathfrak{g}$ generated by those coordinates. There is a module $\mathcal{M}(\mathfrak{h})$ such that $S(\mathfrak{h}) \cong (S(\mathfrak{h}))^W \otimes \mathcal{M}(\mathfrak{h})$ and $\mathcal{M}(\mathfrak{h})$ affords the regular representation; for an irreducible $W$-representation $U$, the fake degrees
are the exponents with multiplicity of the polynomial
\[ P_U(q) = \sum_k \dim(Hom_W(U, M_k(h))) q^k \]
and are analogous to generalized exponents for a Lie algebra. They describe the
degrees of a basis for \((B_V(h))^W\), although they do not explicitly give a way to find a basis.
Usually generic degrees are the objects calculated, but fake degrees are computable
from generic degrees \[Lu77\].
Elements of \((B_V(h))^W\) can be turned into elements of \(C_g(g)\) by either adding elements
on different parts together and trying to adjust for the lack of \(G\)-invariance or by mul-
tiplying an element by a \(G\)-invariant object whose Cartan restriction vanishes on all
but one part and then extending from \(h\) to \(g\).
Here we show how \(P_V(q)\) for the representations \(V\) calculated above can be expressed
in terms of the fake degrees of \(V^T\). We use \(P\) to denote both generalized exponents of
\(g\) representations and fake degrees of \(W\) representations. The decomposition of \(V^T\)
into representations of \(W\) can be extracted by setting \(q = 1\), with the labels of \(P\) denot-
ing the labels of the representation. For \(A_r\), the subscripts indicate partitions. For \(BC_r\)
and \(D_r\), the subscripts indicate pairs of partitions (allowing empty partitions or entries
of 0) \(\alpha\) and \(\beta\) such that \(|\alpha| + |\beta| = r\); for \(BC_r\) \(l(\alpha) = l(\beta) + 1\) and
for \(D_r\), \(l(\alpha) = l(\beta)\). The
\(-\) in \(\phi_{r,-}\) indicates the empty partition.
The decomposition for the exceptional Lie algebras will be discussed in the next chap-
ter.
Table 9.1: Generalized exponents as fake degrees for $A_r$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>$P_{(r+1)}(q)$</td>
</tr>
<tr>
<td>$V_{\omega_1+\omega r}$</td>
<td>$P_{r1}(q)$</td>
</tr>
<tr>
<td>$V_{\omega_2+\omega_{r-1}}$</td>
<td>$P_{(r-1)2}(q)$</td>
</tr>
<tr>
<td>$V_{2\omega_1+2\omega r}$</td>
<td>$q^2P_{(r+1)}(q) + q^2P_{r1} + q^2P_{(r-1)2}(q)$</td>
</tr>
<tr>
<td>$V_{2\omega_1+\omega_{r-1}}$</td>
<td>$P_{(r-1)11}(q)$</td>
</tr>
<tr>
<td>$V_{\omega_2+2\omega_{r-1}}$</td>
<td>$P_{(r-1)11}(q)$</td>
</tr>
</tbody>
</table>

Table 9.2: Generalized exponents as fake degrees for $BC_r$

<table>
<thead>
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<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>$P_{r-1}(q)$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$q^2P_{r-1}(q) + P_{(r-1)1,0}(q)$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2P_{r-1}(q) + (2q^2 + q^{2r})P_{(r-1)1,0}(q) + q^2P_{(r-2)2,0}(q) + P_{(r-2)11,00}(q)$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$P_{(r-2)0,0,2}(q)$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$P_{(r-1)0,1}(q)$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$P_{(r-2)00,11}(q) + q^{2r-2}P_{(r-1)0,1}(q) + P_{(r-2)1,1}(q)$</td>
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</tbody>
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Table 9.3: Generalized exponents as fake degrees for $D_r$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
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<tbody>
<tr>
<td>$V_1$</td>
<td>$P_{r,0}(q)$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$P_{(r-1)1,00}(q)$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$q^2 P_{r,0}(q) + q^2 P_{(r-1)1,00}(q) + (q^2 + q^4) P_{(r-2)2,00}(q)$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$P_{(r-2),2}(q) + q^{r-1} P_{(r-1),1}(q)$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$P_{(r-1),1}(q)$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$P_{(r-2)0,11}(q) + P_{(r-2)1,10}(q)$</td>
</tr>
</tbody>
</table>
9.2 Restriction to Maximal Subalgebras

Directly computing with the exceptional Lie algebras can be difficult due to their size and the complexity of their descriptions. Thus it is often easier to consider classical maximal subalgebras of the exceptional Lie algebras, and then try to extend results on the classical cases to the exceptional cases. Note that the maximal subalgebras are not always simple, but can at least be chosen to be direct sums of classical simple Lie algebras.

For \( h \) a Cartan subalgebra of \( g \), pick a maximal subalgebra \( \mathfrak{k} \subset g \) with the same Cartan subalgebra such that the closure of the roots of \( \mathfrak{k} \) under \( W_G \) is the set of roots of \( g \). We can take a basis of \( h \) and root vectors of \( \mathfrak{k} \) and \( g \) respectively as basis elements for \( \mathfrak{k} \) and \( g \); denote by \( V \) the span of the root vectors of \( g \) that are not root vectors of \( \mathfrak{k} \). Let \( K \subset G \) be the group corresponding to \( \mathfrak{k} \).

We get a homomorphism \( res : S(g) \rightarrow S(\mathfrak{k}) \) by sending \( V \) to 0. We also get a map \( Mat(g) \rightarrow Mat(\mathfrak{k}) \) by sending \( A \in Mat(g) \) to the submatrix where the coordinates are both in \( \mathfrak{k} \). Combining these two maps gives a map

\[
Res : (Mat(g) \otimes S(g))^G \rightarrow (Mat(\mathfrak{k}) \otimes S(\mathfrak{k}))^K
\]

In a matrix in \((Mat(g) \otimes S(g))^G\), for a non-zero entry with one coordinate in \( \mathfrak{k} \) and one in \( V \), the entry must be in the kernel of \( res \). Hence we get that \( Res \) is an algebra homomorphism. For \( A \in (Mat(g) \otimes S(g))^G \), denote \( A_\mathfrak{k} = Res(A) \).

**Lemma 9.2.1.** \( Res \) is an injection.
Proof. Suppose that $A_\mathfrak{t} = B_\mathfrak{t}$. Then $A_\mathfrak{t} - B_\mathfrak{t} = 0_\mathfrak{t}$. So $A - B$ has entries in $S(V)$ on the $\mathfrak{t}$ submatrix. When restricting to $\mathfrak{h}$, this restricts to 0 on the $\mathfrak{t}$ submatrix. Since $W_G$ allows us to move roots of $\mathfrak{g}$ to roots of $\mathfrak{t}$, we get that $A - B$ restricted to $\mathfrak{h}$ must also be 0 on the $V$ submatrix. Hence, since the restriction of an element of $(Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$ to $\mathfrak{h}$ has non-zero entries only on the Cartan submatrix and the diagonal, we get that $A - B$ restricted to $\mathfrak{h}$ is 0 everywhere. But since

$$(Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G \otimes F(\mathfrak{g}) \cong (Mat(\mathfrak{g}) \otimes S(\mathfrak{h}))^{W_G} \otimes F_G(\mathfrak{h})$$

and since the base field doesn't change if an element is 0 or not, we get that restriction to $\mathfrak{h}$ is an injection. Hence $A - B = 0$.

Therefore $Res$ is an injection.  

Using this injection, we can prove the following useful lemma:

**Lemma 9.2.2 (The Vector Restriction Lemma).** For $A \in C_\mathfrak{g}(\mathfrak{g})$, if $Res(A)$ vanishes on the torus, then $A$ is a multiple of $M$.

Proof. The exact descriptions of the family algebras for the classical cases has been handled in the previous chapters. The observation to make is that the fake degrees of the $W$-representations that $\mathfrak{h} \otimes \mathfrak{h}$ decomposes into match the degrees given by the elements of the family algebra that do not vanish on the torus, and hence anything that does vanish on the torus is a multiple of $M$.

We handle the exceptional cases by reducing to the classical case. For $\mathfrak{g} \neq G_2, E_6$, we suppose that we have an element $A$ with vanishing torus part. Then the restriction of
A to maximal subalgebra \( \mathfrak{t} \) also has vanishing torus part. For \( F_4 \), we use \( \mathfrak{t} = B_4 \), for \( E_7 \) we use \( A_7 \) and for \( E_8 \) we use \( D_8 \). We look at the part of the vector component of \( A \) that correspond to roots of \( \mathfrak{t} \). By the first part of the restriction lemma, this part is a multiple of \( M \) restricted to this part and hence can be written as \( M|_\mathfrak{t} P|_\mathfrak{t} \). \( P|_\mathfrak{t} \) is \( W(\mathfrak{t}) \)-invariant, and since both \( A|_\mathfrak{t} \) and \( M|_\mathfrak{t} \) are invariant under the subgroup of \( W(\mathfrak{g}) \) that fixes \( \mathfrak{t} \), \( P|_\mathfrak{t} \) is also invariant under that subgroup of \( W(\mathfrak{g}) \) and hence can be extended to the entire root system of \( \mathfrak{g} \); we denote the extension by \( \tilde{P} \). Since the extension of \( M|_\mathfrak{t} \) by \( W(\mathfrak{g}) \) is just \( M \), we get that \( A = M \tilde{P} \).

The cases of \( G_2 \) and \( E_6 \) have to be handled separately, since \( E_6 \) has no maximal subalgebras that are simple, while \( G_2 \) has \( A_2 \) but the image of the roots of \( A_2 \) under \( W(G_2) \) misses some roots of \( G_2 \). So we consider the case of family algebras for semisimple Lie algebras.

For a semisimple algebra \( \mathfrak{g} \oplus \mathfrak{t} \), the adjoint group is \( G \times K \) where \( G \) and \( K \) are the adjoint groups for \( \mathfrak{g} \) and \( \mathfrak{t} \) respectively. For a representation of \( \mathfrak{g} \oplus \mathfrak{t} \) that decomposes as \( U \oplus V \) where \( \mathfrak{g} \) acts trivially on \( V \) and \( \mathfrak{t} \) acts trivially on \( U \), we can write the corresponding family algebra by distributing \( \text{End}(U \oplus V) \) and noting that \( G \) leaves fixed \( V \), \( V^\vee \) and \( S(\mathfrak{t}) \), and similarly for \( K \):

\[
(\text{End}(U \oplus V) \otimes S(\mathfrak{g} \oplus \mathfrak{t}))^{G \times K} = (\text{End}(U) \otimes S(\mathfrak{g}))^G \otimes I(\mathfrak{t}) \oplus (U \otimes S(\mathfrak{g}))^G \oplus (V^\vee \otimes S(\mathfrak{t}))^K \\
\oplus (V \otimes S(\mathfrak{t}))^K \oplus (U^\vee \otimes S(\mathfrak{g}))^G \oplus (\text{End}(V) \otimes S(\mathfrak{t}))^K \otimes I(\mathfrak{g})
\]

So we get that the family algebra of \( U \oplus V \) breaks into four blocks, depending on if the coordinates are in \( U \) or \( V \). So it remains a free module over \( I(\mathfrak{g} \oplus \mathfrak{t}) = I(\mathfrak{g}) \otimes I(\mathfrak{t}) \).
For $G_2$, we use the maximal subalgebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_1 \cong \mathfrak{k}_2 \cong A_1$, with one $A_1$ containing a pair of long roots and the other a pair of short roots. The family algebra decomposes into four pieces, as above. For $E_6$, we use the maximal subalgebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$, where $\mathfrak{k}_1 \cong \mathfrak{k}_2 \cong \mathfrak{k}_3 \cong A_2$, and adjoint group $K_1 \times K_2 \times K_3$. We have that the family algebra of the adjoint representation of $\mathfrak{k}$ decomposes into nine pieces, three of which are copies of the family algebra for $A_2$ tensored with two extra copies of $S(A_2)$. Note that for $X \in \mathfrak{k}_i$ and $Y \in \mathfrak{k}_j$ for $i \neq j$, the orbit of $X$ under $K_i$ spans $\mathfrak{k}_i$ and thus the orbit of $(X, Y)$ under $K_i$ spans $(\mathfrak{k}_i, Y)$.

So suppose that we have an element of $C_\mathfrak{k}(\mathfrak{k})$ which vanishes on the torus. This element vanishes on $(h_1, h_2)$ where $h_1$ and $h_2$ are $\mathfrak{k}_i$ and $\mathfrak{k}_j$ respectively. If $i \neq j$, then by the above all entries in $(\mathfrak{k}_i \otimes \mathfrak{k}_j^\vee)$ vanish. If $i = j$, then we’re in a copy of the family algebra of $mk_i$ tensored with $S(\mathfrak{t}_m)^{K_m} \otimes S(\mathfrak{t}_n)^{K_n}$ for $i, m, n$ distinct, and thus since the vector restriction lemma holds for $\mathfrak{k}_i$, we get that the $\mathfrak{k}_i$ vector part of the family algebra element thus of the form $M_iP_i$. Because $W(\mathfrak{g})$ intertwines the actions of the $\mathfrak{k}_i$, we get that the element that the action of $W(\mathfrak{g})$ sends the $P_i$ to each other, and hence $M_iP_i$ extends to a $W(\mathfrak{g})$-invariant element on the vector part of the family algebra of $\mathfrak{g}$, as was desired. □
Chapter 10

The Exceptional Lie algebras

10.1 Invariants

The exceptional Lie algebras are not uniform in many senses, so we give a table listing the data for the reference representation and the exponents:

Table 10.1: Exponents for the Exceptional Lie algebras

<table>
<thead>
<tr>
<th>g</th>
<th>dim V</th>
<th>Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>G₂</td>
<td>7</td>
<td>1, 5</td>
</tr>
<tr>
<td>F₄</td>
<td>26</td>
<td>1, 5, 7, 11</td>
</tr>
<tr>
<td>E₆</td>
<td>27</td>
<td>1, 4, 5, 7, 8, 11</td>
</tr>
<tr>
<td>E₇</td>
<td>56</td>
<td>1, 5, 7, 9, 11, 13, 17</td>
</tr>
<tr>
<td>E₈</td>
<td>248</td>
<td>1, 7, 11, 13, 17, 19, 23, 29</td>
</tr>
</tbody>
</table>

Unlike for the classical Lie algebras, the reference representations of the excep-
tional Lie algebras carry invariants of degree higher than 2. For $G_2$, the 7 dimensional representation has a degree 2 symmetric invariant and a degree 3 antisymmetric invariant. For $F_4$, the 26 dimensional representation has a degree 2 symmetric invariant and a degree 3 symmetric invariant. For $E_6$, the 27 dimensional representation has a degree 3 symmetric invariant. For $E_7$, the 56 dimensional representation has a degree 2 antisymmetric invariant and a degree 4 symmetric invariant. For $E_8$ the 248 dimensional representation has a degree 2 symmetric invariant and a degree 3 antisymmetric invariant. The existence of these higher degree invariants gives us elements of $\left( T(g) \right)^G$ that are not obviously traces over the reference representation.

10.2 The Decomposition of $g \otimes g$

The decomposition of $g \otimes g$ into irreducible representations is uniform for the five exceptional Lie algebras. There are three symmetric representations and two antisymmetric representations. Of the three symmetric representations, one is the trivial representation, and one has highest weight twice that of the adjoint representation, so we label it $S^2 Adj$. The remaining symmetric representation we label $Sym$. Of the antisymmetric representations, one is the adjoint representation; the other we label $\wedge^2 Adj$. 

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Table 10.2: Decomposition of $V^T$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$G_2$</th>
<th>$F_4$</th>
<th>$E_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Triv$</td>
<td>$\phi_{1,0}$</td>
<td>$\phi_{1,0}$</td>
<td>$\phi_{1,0}$</td>
</tr>
<tr>
<td>$Sym$</td>
<td>$\phi_{2,2} \oplus \phi_{1,0}$</td>
<td>$\phi_{9,2} \oplus \phi_{1,0} \oplus \phi_{2,4}$</td>
<td>$\phi_{r,-1,2}$</td>
</tr>
<tr>
<td>$S^2\text{Adj}$</td>
<td>$\phi_{1,0} \oplus 2\phi_{2,2}$</td>
<td>$\phi_{1,0} \oplus 2\phi_{9,2} \oplus \phi_{2,4}$</td>
<td>$\phi_{1,0} \oplus \phi_{r,-1,2} \oplus \phi_{N_{-r},4}$</td>
</tr>
<tr>
<td>$\text{Adj}$</td>
<td>$\phi_{2,1}$</td>
<td>$\phi_{4,1}$</td>
<td>$\phi_{r,1}$</td>
</tr>
<tr>
<td>$\wedge^2\text{Adj}$</td>
<td>$2\phi_{1,3} \oplus \phi_{2,1} \oplus \phi_{1,6}$</td>
<td>$2\phi_{8,3} \oplus \phi_{4,1} \oplus \phi_{6,6}$</td>
<td>$\phi_{N_{-r},3} \oplus \phi_{r,-1,2,2+1}$</td>
</tr>
</tbody>
</table>

where in the $E_r$ column, $N$ is the number of positive roots. Here the first subscript is the dimension of the corresponding representation, the second index is the lowest fake degree of the representation. The subscripts do not uniquely determine the representation, but they do uniquely determine the set of associated fake degrees.

### 10.3 General Structure

To find a linearly-independent basis, we restrict to the Cartan subalgebra, allowing us to decompose elements of the family algebra into torus and vector parts. The structure of the family algebras is first determined by determining the generators on the torus, in particular by showing that $(End(\mathfrak{h}) \otimes S(\mathfrak{h}))^W$ is generated by the torus parts of $R_i$ and $R_iSR_j$. This in turn is done by showing that $R_i$ for $i \leq r$ and $R_iSR_j + R_jSR_i$ for $i, j \leq r - 1$ are linearly independent, and that $R_iSR_j - R_jSR_i$ for $i, j \leq r$ are linearly independent.
independent. There are \( r^2 \) elements of the given form, and using the isomorphism to \( \text{Mat}_\mathfrak{h}(S(\mathfrak{h})) \) gives us that there can only be \( r^2 \) linearly independent elements, so the elements \( R_i, R_iSR_j + R_jSR_i \) and \( R_iSR_j - R_jSR_i \) would form a basis.

The linear independence of \( R_iSR_j - R_jSR_i \) is proven in [So64], theorem 2, which describes the basis of the \( \wedge^2 \mathfrak{h} \) isotypic component of \( S(\mathfrak{h})/I(W) \) as the order 2 minors of the Jacobian matrix for \( c_1, \ldots, c_r \). As noted in the section on restriction to subalgebras, this leaves the symmetric parts of \( \text{End}(\mathfrak{h}) \), which decompose as \( 1 \oplus S^2 \mathfrak{h} \). The fake degrees of both representations are known, calculated in [Ca], and match the degrees of \( R_i \) and \( R_iSR_j \). Hence we only have to check the linear independence of \( R_i \) and \( R_iSR_j + R_jSR_i \).

Once the generation of the torus parts by the torus parts of \( R_i \) and \( S \) is established, suppose that \( A \) is an element of the family algebra and for all elements \( B \) with smaller degree, \( B \) can be generated by \( M, R_i \) and \( S \). The generation of the torus parts by \( R_i \) and \( S \) shows that there is some element \( P(R_i, S) \) such that \( A - P(R_i, S) \) vanishes on the torus. The vector restriction lemma then shows that \( A - P(R_i, S) = MQ \) for some family algebra element \( Q \). \( \deg(Q) < \deg(A) \), so \( Q \) can be generated by \( M, R_i \) and \( S \), so \( A \) can also be generated. Hence we get a generating set, \( M, R_i \) and \( S \).

In terms of relations, for \( E_r \) the elements of the form \( M^kR_i \) for \( k \leq e_r - 1 \) match the fake degrees of the representations of \( W \) on the vector part, so since they are a minimal spanning set, they must be independent. Hence multiplying by \( M^2 \) gets a set of objects linearly independent from the aforementioned \( R_i \) and \( R_iSR_j \). So our \( I(\mathfrak{g}) \)-basis
of our family algebra is $R_i, R_iSR_j, M^k R_i$ for $1 \leq k \leq e_r + 1$.

For $F_4$ and $G_2$, the vector part splits in two, and so must be handled separately, but the resulting statement of the $I(g)$-basis for the family algebra is the same.

## 10.4 Larger exceptional Lie algebras

The $G_2$ case acts differently from the other exceptional Lie algebras, so we list it first.

**Theorem 10.4.1** (Generators for the family algebra of $G_2$). The family algebra $C_g(G_2)$ is generated by the following:

\[
M = \begin{array}{c}
I \\
O
\end{array}, \quad R_2 = 2 \begin{array}{c}
I \\
O
\end{array}, \quad S = \begin{array}{c}
I \\
O
\end{array}
\]

The relations come from the Cayley-Hamilton relation on the 7-dimensional reference representation, which yield a degree 6 relation, as well as higher-degree relations coming from higher degree traces as there are no primitive Casimir elements in degree higher than 6. Unlike the classical cases, here the relations coming from higher-degree traces are sometimes independent of the Cayley-Hamilton relations.

Unlike $G_2$, the larger exceptional Lie algebras cannot be generated by the three generators listed above. Instead they appear more similar to the case of $D_r$, requiring a fourth generator due to the structure of the primitive Casimir invariants.
Theorem 10.4.2 (Generators for \( C_g(g) \) for \( g = F_4, E_6, E_7, E_8 \)). For \( g = F_4, E_6, E_7 \) or \( E_8 \), \( C_g(g) \) is generated over \( I(g) \) by

\[
M = \begin{array}{c}
I \\
O
\end{array}
\quad R'_2 = \begin{array}{c}
I \\
O
\end{array} + \begin{array}{c}
I \\
O
\end{array}
\quad \frac{e_2 - 1}{I O}
\quad \frac{e_2 - 1}{I O}
\quad R'_3 = \begin{array}{c}
I \\
O
\end{array} + \begin{array}{c}
I \\
O
\end{array}
\quad \frac{e_3 - 1}{I O}
\quad \frac{e_3 - 1}{I O}
\quad S = \begin{array}{c}
I \\
O
\end{array}
\]

While for \( F_4, E_7 \) and \( E_8 \), the two terms in \( R'_2 \) are equal, just as in the \( G_2 \) case, and similarly for the two terms in \( R'_3 \), for \( E_6 \) the two terms are different, since the 27-dimensional reference representation of \( E_6 \) is not self-dual. Compare this to the \( A_r \) case, for which the reference representation is not self-dual, and contrast with the \( B_r/C_r \) and \( D_r \) cases, where it is.

For each of them, the elements \( R'_2 \) and \( R'_3 \) do not commute, but there are commuting elements \( R_2 \) and \( R_3 \) that can be written as \( R'_2 \) plus lower order terms and \( R'_3 \) plus lower order terms respectively.

As with the other cases, \( M \) is central and vanishes on the torus part, \( S \) vanishes on the vector part, and the other relations come from reductions of traces with degrees that aren’t primitive Casimir operators. In particular, the terms of the form \( R'^m_2 R'^n_3 \) are expressible in terms of other elements for \( m(e_2 - 1) + n(e_3 - 1) \) not equal to \( e_1 - 1 \).
for some \( i \), and \( R_2^n R_3^m S + SR_2^m R_3^n \) is also expressible in terms of other elements when

\[
m(e_2 - 1) + n(e_3 - 1) = e_r - 1.
\]
Chapter 11

$G_2$

The real compact group associated to $G_2$ is usually defined as the group of automorphisms of the octonions $\mathcal{O}$ as a division algebra over $\mathbb{R}$; we define $G_2$ to be the complexification of the Lie algebra of this group. Since the real part of $\mathcal{O}$ is fixed, we get an irreducible 7-dimensional representation of the automorphism group, which we complexify to get an irreducible representation of $G_2$. We use this as the reference representation.

The 7-dimensional representation has two primitive invariant forms, a symmetric one $g_{ab}$ corresponding to the norm on the octonions, and an antisymmetric trilinear form $f_{abc}$ corresponding to the imaginary component of the multiplication. Writing $e_a$ for a basis of the imaginary part of $\mathcal{O}$, we can write our invariant forms as

$$e_a e_b = -g_{ab} + f_{abc} g^{cd} e_d$$
The torus part of $G_2$ decomposes as $\phi_{1,0} \oplus \phi_{2,2} \oplus \phi_{1,6}$, while the vector parts decompose as $\phi_{1,0} \oplus \phi_{2,2} \oplus \phi_{2,1} \oplus \phi_{1,3}$. $\phi_{2,1}$ has fake degrees $q + q^5$, $\phi_{2,2}$ has fake degrees $q^2 + q^6$ and $\phi_{1,3}$ has fake degree $q^k$. Note that there are actually several inequivalent representations denoted by $\phi_{1,6}$, $\phi_{2,2}$ and $\phi_{1,3}$, but they have the same fake degrees so we do not distinguish between them.

The elements $1, S, R_2$ and $R_2 S - SR_2$ are all linearly independent on the torus, and hence the torus parts of these elements generate the torus parts of all elements of the family algebra. So now we can determine the linearly independent elements on the vector part.

By the fake degrees, there is only 1 linearly independent order 4 element when restricted to the short roots, but since we can construct $M^4$ and $R_2$, there must be some nonzero element $P$ of the family algebra of order 4 that vanishes on the short roots. Hence $M^4$ restricted to the short roots can be written in terms of $R_2$ and lower order terms. Similarly there must be some element nonzero $Q$ of order 6 that vanishes on the long roots, and hence $M^4$ restricted to the long roots can be written in terms of $R_2$ and lower order terms. $P$ is not a multiple of $Q$.

Using the fact that $R_2^2$ is not amongst the list of linearly independent torus elements above, we get that the following must be linearly independent on the short roots:

$$1, M^2, R_2$$

and the same set is also linearly independent on the long roots. Denote this set $\{A_i\}_i$.

Since the set is linearly independent on the long roots, the set $\{PA_i\}_i$ is also linearly independent.
dependent, and vanishes on the short roots. No linear combination of the $A_i$ vanishes on the short roots, so the set $\{A_i, PA_i\}$ is linearly independent. Since $M^2$ is nonzero on both the short and long roots, $\{M^2 A_i, M^2 PA_i\}$ are linearly independent, and are themselves linearly independent from the set of linearly independent torus elements above, since they all have vanishing torus part. Hence we have all of the linearly independent symmetric pieces.

An $I(G_2)$ basis of the family algebra can be written as

$$M^m R^0_2 \text{ for } 0 \leq m \leq 6, n = 0, 1$$

$$S$$

$$R_2 S - SR_2$$

Considering $R_1$ as a scalar, we get

$$M^m R_n \text{ for } 0 \leq m \leq e_2, n \leq 2$$

$$R_m SR_n + R_n SR_m \text{ for } 1 \leq m \leq n \leq 1$$

$$R_m SR_n - R_n SR_m \text{ for } 1 \leq m < n \leq 2$$

The generalized exponents for the representations in question have already been computed by Pieter Mostert [Mo12] and we can compare them to both the basis above and the fake degrees.
Table 11.1: Generalized Exponents in $C_9(G_2)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Triv$</td>
<td>$1 = P_{1,0}(q)$</td>
</tr>
<tr>
<td>$Sym$</td>
<td>$q^2[3]<em>q^2 = P</em>{2,2}(q) + q^4 P_{1,0}(q)$</td>
</tr>
<tr>
<td>$S^2 Adj$</td>
<td>$q^2[5]<em>q^2 = q^2 P</em>{1,0}(q) + q^2 + q^4 P_{2,2}(q)$</td>
</tr>
<tr>
<td>$Adj$</td>
<td>$q[2]<em>q^4 = P</em>{2,1}(q)$</td>
</tr>
<tr>
<td>$∧^2 Adj$</td>
<td>$q[4]<em>q^2 + q^6 = (1 + q^4) P</em>{1,3}(q) + q^4 P_{2,1}(q) + P_{1,6}(q)$</td>
</tr>
</tbody>
</table>
Chapter 12

$F_4$

$F_4$ can be thought of as the Lie algebra of the isometry group of the octonionic projective plane, or as the Lie algebra of the automorphism group of traceless part of the Albert algebra. It has a symmetric bilinear form and a symmetric trilinear form, both from the multiplication in the Albert algebra: using $e_1, \ldots, e_{26}$ as the basis of the traceless part of the Algebra algebra, for $v = v^a e_a$ and $w = w^b e_b$ we get $v \circ = g_{ab} v^a w^b I_3 + d_{abc} g^{cd} v^a w^b e_d$, where $I_3$ is the 3x3-identity matrix.

Alternatively, we view $F_4$ as $B_4 \oplus \Delta$, where $\Delta$ is the spinor representation of $B_4$ with Lie bracket given by $[X, s] = X.s$ for $X \in B_4$ and $s \in \Delta$, and $[s, t]$ is defined by $\langle X, [s, t] \rangle_{B_4} = -\langle X, t, s \rangle_{\Delta}$. In this setup, the 26-dimensional representation decomposes as $\mathbb{C} \oplus V_9 \oplus \Delta$ where $V_9$ is the standard 9-dimensional representation of $B_4$; here $B_4$ acts on each component via the usual representation, while for $s \in \Delta$, $s.a = a.s$ for $a \in \mathbb{C}$, $s.v = v(s)$ for $v \in V_9$ where the right-hand side is the usual action of $V_9$ on $\Delta$, and then for $t \in \Delta,$
s.t is defined by $\langle u, s.t \rangle_{V_9} = -\langle u.t, s \rangle_{\Delta}$.

To describe the trilinear form, it is easiest to view everything in terms of $so(8)$ representations, with $\Delta$ decomposing as $\Delta^+$ and $\Delta^-$. The two spinor representations of $so(8)$, and $V_9$ as $V_8 \oplus \mathbb{C}$. We now get that the 26-dimensional representation can be described as $S \oplus V_1 \oplus V_2 \oplus V_3$, where $V_1 = V_8$, $V_2 = \Delta^+$, $V_3 = \Delta^-$, and $S$ is the subspace of $\mathbb{C}^3$ such that $s_1 + s_2 + s_3 = 0$ for $(s_1, s_2, s_3) \in S$.

The bilinear form on $S$ is given by $(s, t) = \frac{1}{2}(s_1 t_1 + s_2 t_2 + s_3 t_3)$, and so the bilinear form on the 26-dimensional representation is the sum of the bilinear forms on each part.

The trilinear form becomes

$$(v_1, v_2, v_3) = f(v_1, v_2, v_3) \text{ for } v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$$

$$(s, u, v) = s_i(u, v) \text{ for } u, v \in V_i$$

$$(r, s, t) = -(r_1 s_1 t_1 + r_2 s_2 t_2 + r_3 s_3 t_3) \text{ for } r, s, t \in S$$

where $f$ is the triality map from $V_8 \otimes \Delta^+ \oplus \Delta^-$ to $\mathbb{C}$, with the scaling such that $|f(x, y, z)| \leq 1$ when $\|x\| = \|y\| = \|z\| = 1$ and the bound is attained for some triple.

with permutations of the arguments above giving the same values and all other triples of arguments giving 0.

The presence of the bilinear form gives us that all odd degree Casimir operators vanish, while the cubic invariant gives us that the quartic and degree 10 Casimir operators reduce. We are left with exponents of 1, 5, 7, and 11, giving us primitive Casimir operators of degrees 2, 6, 8 and 12.
Because $F_4$ is not simply laced, there are two root lengths, long and short, so the vector part of the family algebra decomposes into a long root part and a short root part.

Hence we have three parts. Following [Ca], we write the $W$-representations as $\phi_{a,b}$ where $a$ is the dimension and $b$ is the lowest fake degree. Writing the fake degrees as exponents of $q$-multiplicities $Q_{a,b}$, the torus decomposes as $\phi_{1,0} \oplus \phi_{3,2} \oplus \phi_{6,6}$. The vector parts each decompose as $\phi_{1,0} \oplus \phi_{3,2} \oplus \phi_{2,4} \oplus \phi_{4,1} \oplus \phi_{8,3}$. Note that the $\phi_{2,4}$ and $\phi_{8,3}$ labels actually each denote two inequivalent representations, but although the $\phi_{2,4}$ of the long roots is not equivalent to the $\phi_{2,4}$ of the short roots, the fake degrees are the same, and similarly for the $\phi_{8,3}$ representations, so the distinction is ignored here. Also note that except for the trivial and the adjoint representation, none of these $W(F_4)$ representations correspond nicely to representations of $F_4$ itself. They do provide an upper-limit for the number of linearly-independent elements we can have in each degree, though. See Table 12.1 for the fake degrees of these representations.

$R_3$ is of degree 6 and hence cannot be generated by $M, S$ and $R_2$. Hence we need four generators. $R'_2 R'_3 - R'_3 R'_2$ has degree 10, but $\phi_{6,6}$ has no fake degree of 10, so $R'_2 R'_3 - R'_3 R'_2$ must be reducible to lower order terms. Hence we can define an $R_2$ and an $R_3$ of degrees 3 and 4 respectively that commute and that, along with $M$ and $S$, generate $R'_2$ and $R'_3$.

The torus parts of the set $R_i$ for $i \leq 4$, $R_i SR_j + R_j SR_i$ for $i \leq j \leq 3$ and $R_i SR_j - R_j SR_i$ for $i < j \leq 4$ form a basis of the torus parts of the algebra, as expected in the general strategy. Now we determine the linearly independent elements on the vector part.
Table 12.1: Fake degrees in $End(F_4)^T$

<table>
<thead>
<tr>
<th>$\phi_{a,b}$</th>
<th>$P_{a,b}(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_{9,2}$</td>
<td>$q^2 + q^4 + 2q^6 + q^8 + 2q^{10} + q^{12} + q^{14}$</td>
</tr>
<tr>
<td>$\phi_{6,6}$</td>
<td>$q^6 + q^8 + q^{12} + q^{16} + q^{18}$</td>
</tr>
<tr>
<td>$\phi_{2,4}$</td>
<td>$q^4 + q^8$</td>
</tr>
<tr>
<td>$\phi_{4,1}$</td>
<td>$q + q^5 + q^7 + q^{11}$</td>
</tr>
<tr>
<td>$\phi_{8,3}$</td>
<td>$q^3 + q^5 + q^7 + 2q^9 + q^{11} + q^{13} + q^{15}$</td>
</tr>
</tbody>
</table>

By the fake degrees, there are no more than 2 linearly independent order 6 elements when restricted to the short roots, but since we can construct $M^6, R_3$ and $M^2 R_2$, there must be some nonzero element $P$ of the family algebra of order 6 that vanishes on the short roots. Hence $M^6$ restricted to the short roots can be written in terms of $R_3, M^2 R_2$ and lower order terms. Similarly there must be some element nonzero $Q$ of order 6 that vanishes on the long roots, and hence $M^6$ restricted to the long roots can be written in terms of $R_3, M^2 R_2$ and lower order terms. $P$ is not a multiple of $Q$.

Using the fact that $R_2^2$ and $R_3^2$ are not amongst the list of linearly independent torus elements above, we get that the following must be linearly independent on the short roots:

$$1, M^2, M^4, R_2, M^2 R_2, R_3, M^4 R_2, M^2 R_3, R_2 R_3, M^4 R_3, M^2 R_2 R_3, M^4 R_2 R_3$$
and the same set is also linearly independent on the long roots. Denote this set \( \{ A_i \} \).

Since the set is linearly independent on the long roots, the set \( \{ PA_i \} \) is also linearly independent, and vanishes on the short roots. No linear combination of the \( A_i \) vanishes on the short roots, so the set \( \{ A_i, PA_i \} \) is linearly independent. Since \( M^2 \) is nonzero on both the short and long roots, \( \{ M^2 A_i, M^2 PA_i \} \) are linearly independent, and are themselves linearly independent from the set of linearly independent torus elements above, since they all have vanishing torus part. Hence we have all of the linearly independent symmetric pieces.

The symmetric representations, \( \text{Triv}, \text{Sym} \) and \( S^2 \text{Adj} \), have 0-weight multiplicities 1, 12 and 21 respectively. Unfortunately, the elements corresponding to parts of the \( W(F_4) \) representation do not at all match those corresponding to the various representations of \( F_4 \) since the decomposition into torus and vector parts is not \( F_4 \)-invariant. Hence the determination of which degrees give generalized exponents for which representation is nontrivial despite knowing the decompositions of the 0-weight spaces of the \( F_4 \)-representations into \( W(F_4) \)-representations.

To determine the generalized exponents of the symmetric representations we examine the family algebra of the reference representation \( V \) of \( F_4 \). \( V \otimes V \) decomposes as \( \text{Triv} \oplus V \oplus \text{Sym} \oplus \text{Adj} \oplus \text{Sym}, \) where \( \text{Sym} \) is a 273-dimensional representation. The first three representations belong to \( S^2 V \), the latter two to \( \wedge^2 V \).

The family algebra \( C_V(F_4) \), restricted to the maximal torus, splits into a \( 2 \times 2 \) matrix corresponding to the 0-weight space of \( V \) and a set of \( 1 \times 1 \) matrix algebras corresponding
to the nonzero weights. Analogous to the case of $C_{g}(g)$, we call the matrix the torus part of the family algebra, and the $1 \times 1$ matrix algebras the vector part.

The torus part decomposes into $W(F_{4})$ representations as $\phi_{1,0} \oplus \phi_{2,4} \oplus \phi_{1,12}$. The vector part decomposes as $\phi_{1,0} \oplus \phi_{2,4} \oplus \phi_{9,2} \oplus \phi_{4,1} \oplus \phi_{8,3}$. The $\phi_{1,12}$, $\phi_{4,1}$ and $\phi_{8,3}$ are antisymmetric, the rest are symmetric.

We write the trilinear form on $V$ as a white dot with three reference edges attached. Note that the trilinear form is precisely the projector from $V \otimes V$ to $V$.

$V$ is a small representation, in the sense of Broer, so its generalized exponents are 4 and 8. Thus for the following diagram we get that

![Diagram](image)

vanishes for $k$ odd and for $k = 0$ or 2, and for all other $k$ can be written in terms of the diagrams for $k = 4$ and $k = 8$. We denote the corresponding elements of the family algebra as $P$ and $Q$. Note that the restrictions of $P$ and $Q$ to the torus are linearly independent.

We note that $M_{d}$, treated as an element of $C_{V}(F_{4})$, vanishes on the torus. Hence, by the same reasoning for generating $C_{g}(g)$ elements on the vector part from $W(g)$-invariant elements on the vector part, multiplying a set of linearly independent $W(F_{4})$-invariant elements yields linearly independent elements of $C_{V}(F_{4})$ that vanish on the torus. Since we have that $P$ and $Q$ restricted to the torus are linearly independent, they are linearly independent from the elements of $C_{V}(F_{4})$ that vanish on the torus.
Hence the generalized exponents of the symmetric parts of $C_V(F_4)$ are

$$(1 + q^2)P_{1,0} + (1 + q^2)P_{2,4} + q^2P_{9,2}$$

Since we have that $q^0$ is the generalized exponent for the trivial representation and $P_{2,4}$ gives the generalized exponents for $V$, we get that the generalized exponents for $Sym$ are $q^2P_{1,0} + q^2P_{2,4} + q^2P_{9,2}$.

Returning to the case of $C_g(F_4)$, we can now determine the generalized exponents of $S^2Adj$ by elimination. We get

$$P_{Triv}(q) = q^0$$

$$P_{Sym}(q) = q^2P_{1,0} + q^2P_{2,4} + q^2P_{9,2}$$

$$P_{S^2Adj}(q) = q^8P_{1,0} + q^8P_{2,4} + (1 + q^8)P_{9,2}$$

We can rewrite the last two expressions as

$$P_{Sym}(q) = P_{9,2} + q^8P_{1,0} + q^8P_{2,4}$$

$$P_{S^2Adj}(q) = q^2P_{1,0} + q^2P_{2,4} + (q^2 + q^8)P_{9,2}$$

to better match the results for the other exceptional Lie algebras.

The antisymmetric basis elements are the ones that come from $\wedge^2\mathfrak{h}$ or that vanish on the torus. As with the symmetric elements, we get that $\{MA_i, MPA_i\}$ are linearly independent, and this in turn gives us that the $M^{2k+1}, M^{2k+1}R_2, M^{2k+1}R_3$ and $M^{2k+1}R_4$ are linearly independent for $k \leq 5$. 

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Table 12.2: Generalized Exponents in $C_9(F_4)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Triv$</td>
<td>1</td>
</tr>
<tr>
<td>$Sym$</td>
<td>$P_{9,2}(q) + q^8 P_{1,0}(q) + q^8 P_{2,4}(q)$</td>
</tr>
<tr>
<td>$S^2 Adj$</td>
<td>$q^2 P_{1,0}(q) + q^2 P_{2,4}(q) + (q^2 + q^8) P_{9,2}(q)$</td>
</tr>
<tr>
<td>$Adj$</td>
<td>$P_{4,1}(q)$</td>
</tr>
<tr>
<td>$\wedge^2 Adj$</td>
<td>$(1 + q^6)P_{0,3}(q) + q^8 P_{4,1}(q) + P_{6,6}(q)$</td>
</tr>
</tbody>
</table>
Chapter 13

$E_6$

The reference representation of $E_6$ is 27-dimensional, bearing a symmetric trilinear form derived from the multiplication on the Albert algebra. Using $e_1, \ldots, e_{26}$ to denote the 26 traceless basis elements, we write $e_0$ for the identity element and extend $d_{abc}$ by $d_{0ab} = g_{ab}$ for $a, b \neq 0$, and $d_{000} = 0$. $E_6$ is then the group that preserves this extended $d_{abc}$ without regard to preserving $g_{ab}$.

Alternatively, we can write $E_6$ as a module over $(A_2)^3$. We take $(A_2)_1, (A_2)_2$ and $(A_2)_3$ to be three copies of $A_2$, with standard representations $U_1, U_2$ and $U_3$. Then

$$E_6 = (A_2)_1 \oplus (A_2)_2 \oplus (A_2)_3 \oplus U_1 \otimes U_2 \otimes U_3 \oplus U_1^\vee \otimes U_2^\vee \otimes U_3^\vee$$

The bracket of an element of one of the $(A_2)_i$ with something in either $U_1 \otimes U_2 \otimes U_3$ or $U_1^\vee \otimes U_2^\vee \otimes U_3^\vee$ is given by the usual action of $(A_2)_i$ on $U_i$ or $U_i^\vee$. We write

$$[u_1 \otimes u_2 \otimes u_3, v_1^\vee \otimes v_2^\vee \otimes v_3^\vee] = \sum_i (u_i \otimes v_i^\vee - tr(u_i \otimes v_i^\vee)) v_j^\vee (u_j) v_k^\vee (u_k)$$
where $i, j$ and $k$ cycle through 1, 2, 3.

The 27-dimensional representation of $E_6$ then becomes $U_1 \otimes U_2^\vee \oplus U_2 \otimes U_3^\vee \oplus U_3 \otimes U_1^\vee$. The action of the three copies of $A_2$ is the usual action. Define a pair of maps $\phi : U_i^\vee \otimes U_i \to U_i$ and $\phi^\vee : U_i \otimes U_i \to U_i^\vee$ via the determinant on $U_i$. Then the action of $U_1 \otimes U_2 \otimes U_3$ on $U_i \otimes U_j^\vee$ is given by

\[ u_1 \otimes u_2 \otimes u_3 \cdot v_i \otimes v_j^\vee = v_j^\vee(u_j)u_k \otimes \phi^\vee(u_i, v_i) \]

And similarly for $U_i^\vee \otimes U_j^\vee \otimes U_3^\vee$,

\[ u_i^\vee \otimes u_j^\vee \otimes u_3^\vee \cdot v_i \otimes v_j^\vee = u_i^\vee(v_i)\phi(u_j^\vee, v_j^\vee) \otimes u_k^\vee \]

The trilinear form now looks like

\[ (u_1 \otimes u_2^\vee, v_2 \otimes v_3^\vee, w_3 \otimes w_1^\vee) = u_2^\vee(v_2) v_3^\vee(w_3) w_1^\vee(u_1) \]

with all permutations of the three arguments giving the same value, and all other triples of arguments giving 0.

The lack of a bilinear form allows for odd-degree nonvanishing Casimir operators, and indeed the degrees of the primitive Casimir operators for $E_6$ are 2, 5, 6, 8, 9, 12, with corresponding exponents 1, 4, 5, 7, 8, 11.

Again following [Ca], we get that the torus part decomposes as $\phi_{1,0} \oplus \phi_{20,2} \oplus \phi_{15,5}$. Unlike $F_4$, $E_6$ is simply laced, so we only have one vector part, which decomposes as $\phi_{1,0} \oplus \phi_{20,2} \oplus \phi_{15,4} \oplus \phi_{6,1} \oplus \phi_{30,3}$. See Table 13.1 for the fake degrees of these representations.
Table 13.1: Fake degrees in $End(E_6)^T$

<table>
<thead>
<tr>
<th>$\phi_{a,b}$</th>
<th>$P_{a,b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>(1)</td>
</tr>
<tr>
<td>$\phi_{20,2}$</td>
<td>(q^2 + q^3 + q^4 + q^5 + 2q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + q^{11} + 2q^{12} + q^{13} + q^{14} + q^{15} + q^{16})</td>
</tr>
<tr>
<td>$\phi_{15,5}$</td>
<td>(q^5 + q^6 + q^8 + 2q^9 + q^{11} + 3q^{12} + q^{13} + 2q^{15} + q^{16} + q^{18} + q^{19})</td>
</tr>
<tr>
<td>$\phi_{15,4}$</td>
<td>(q^4 + q^6 + q^7 + q^8 + 2q^{10} + q^{11} + q^{12} + q^{13} + 2q^{14} + q^{16} + q^{17} + q^{18} + q^{20})</td>
</tr>
<tr>
<td>$\phi_{6,1}$</td>
<td>(q + q^4 + q^5 + q^7 + q^8 + q^{11})</td>
</tr>
<tr>
<td>$\phi_{30,3}$</td>
<td>(q^3 + q^5 + q^6 + 2q^7 + q^8 + 3q^9 + 2q^{10} + 3q^{11} + 2q^{12} + 3q^{13} + 2q^{14} + 3q^{15} + q^{16} + 2q^{17} + q^{18} + q^{19} + q^{21})</td>
</tr>
</tbody>
</table>

$R_3$ is of degree 4 and hence cannot be generated by $M, S$ and $R_2$. Hence we need four generators. $R'_2 R'_3 - R'_3 R'_2$ has degree 7, but $\phi_{15,5}$ doesn’t have a fake degree of 7, and hence $R'_2 R'_3 - R'_3 R'_2$ must be reducible to lower order terms. Hence we can define an $R_2$ and an $R_3$ of degrees 3 and 4 respectively that commute and that, along with $M$ and $S$, generate $R'_2$ and $R'_3$.

The torus parts of the set $R_i$ for $i \leq 6$, $R_i SR_j + R_j SR_i$ for $i \leq j \leq 5$ and $R_i SR_j - R_j SR_i$ for $i < j \leq 6$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by $M, S$ and the $R_i$, with the following elements being linearly independent:

$$M^i R_j \text{ for } i \leq 12, j \leq 6$$
\( R_i SR_j + R_j SR_i \) for \( i \leq j \leq 5 \)

\( R_i SR_j - R_j SR_i \) for \( i < j \leq 6 \)

As in previous cases, multiplying an element by \( S \) on both sides yields \( S \) times an element of \( I(E_6) \), so we can define \( d_{22}, d_{23} \) and \( d_{223} \) in \( I(E_6) \) by

\[
d_{22} S = SR_2^2 S
\]

\[
d_{23} S = SR_2 R_3 S
\]

\[
d_{223} S = SR_2^2 R_3 S
\]

\( d_{22} \) has degree 8, \( d_{23} \) has degree 9 and \( d_{223} \) has degree 12. Moreover, the set

\[
\{c_1, c_2, c_3, d_{22}, d_{23}, d_{223}\}
\]

is algebraically independent, so \( d_{22} \) must be a \( \mathbb{C}^* \)-multiple of \( c_4 \) plus products of lower degree primitive Casimir operators, \( d_{23} \) must be a \( \mathbb{C}^* \)-multiple of \( c_5 \) plus products of lower degree primitive Casimir operators, and \( d_{223} \) must be a \( \mathbb{C}^* \)-multiple of \( c_6 \) plus products of lower degree primitive Casimir operators. Hence, since we can write \( R_4, R_5 \) and \( R_6 \) in terms of \( c_4, c_5 \) and \( c_6 \), they can be generated by \( M, S, R_2 \) and \( R_3 \). Hence the entire family algebra can be generated over \( I(E_6) \) by \( M, S, R_2 \) and \( R_3 \), as stated in the theorem.
Table 13.2: Generalized Exponents in $C_g(E_6)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Triv$</td>
<td>1</td>
</tr>
<tr>
<td>$Sym$</td>
<td>$P_{20,2}$</td>
</tr>
<tr>
<td>$S^2 Adj$</td>
<td>$q^2 P_{1,0} + q^2 P_{15,4} + q^2 P_{20,2}$</td>
</tr>
<tr>
<td>$Adj$</td>
<td>$P_{6,1}$</td>
</tr>
<tr>
<td>$\wedge^2 Adj$</td>
<td>$P_{30,3} + P_{15,5}$</td>
</tr>
</tbody>
</table>
Chapter 14

$E_7$

The reference representation of $E_7$ is 56-dimensional, bearing an antisymmetric bilinear form and a symmetric quartic form. We can view it as the Grassman component of a supervector space whose even part is Euclidean $\mathbb{R}^4$, with the metric on $\mathbb{R}^4$ becoming the bilinear form and the determinant on $\mathbb{R}^4$ becoming the quartic form.

Alternatively, we can write $E_7$ as a module over $A_7$. We take $V$ to be the standard 8-dimensional representation of $A_7$, and we take an identification of $\Lambda^8 V$ with $\mathbb{C}$, along with the induced identification of $\Lambda^8 V^\vee$ with $\mathbb{C}$. This gives us identifications of $\Lambda^k V$ with $\Lambda^{8-k} V^\vee$.

$E_7$ decomposes as $E_7 = A_7 \oplus \Lambda^4 V$, with $V$ being the standard 8-dimensional representation of $A_7$. For $X \in A_7$ and $v \in \Lambda^4 V$, we define $[X, v] = X.v$, and for $u, v \in \Lambda^4 V$, we define $[u, v]$ by $\langle X, [u, v] \rangle_{A_7} = -(X.u) \wedge v$.

We can view the 56-dimensional representation as $\Lambda^2 V \oplus \Lambda^2 V^\vee$. The action of $A_7$...
is the usual action on $\wedge^2 V$ and $\wedge^2 V^\vee$, while the action of $u \in \wedge^4 V$ on $t \in \wedge^2 V$ is $u.t = u \wedge t \in \wedge^6 V \cong \wedge^2 V^\vee$ and the action of $u$ on $\wedge^2 V^\vee$ is $u.t = u \wedge t \in \wedge^6 V^\vee \cong \wedge^2 V$.

Let $a, b, c, d \in \wedge^2 V$ and $a^\vee, b^\vee c^\vee, d^\vee \in \wedge^2 V^\vee$. Then we have that the bilinear form looks like

$$(a, b) = (a^\vee, b^\vee) = 0, \quad (a, b^\vee) = -(b^\vee, a) = b^\vee(a)$$

and the quadratic form looks like

$$(a, b, c, d) = a \wedge b \wedge c \wedge d \in \wedge^8 V \cong \mathbb{C}$$

$$(a, b, c, d^\vee) = 0$$

$$(a, b, c^\vee, d^\vee) = \frac{1}{2} c^\vee(a)d^\vee(b) + \frac{1}{2} d^\vee(a)c^\vee(b) - (c^\vee \wedge d^\vee)(a \wedge b)$$

$$(a, b^\vee, c^\vee, d^\vee) = 0$$

$$(a^\vee, b^\vee, c^\vee, d^\vee) = a^\vee \wedge b^\vee \wedge c^\vee \wedge d^\vee \in \wedge^8 V^\vee \cong \mathbb{C}$$

with permutations of the arguments treated via the full symmetry of the quadratic form.

The existence of the bilinear form forces all odd-degree Casimir elements to vanish, and we are left with the degrees of the primitive Casimir elements being 2, 6, 8, 10, 12, 14 and 18, with corresponding exponents 1, 5, 7, 9, 11, 13, and 17.

The torus part of the family algebra decomposes as $\phi_{1,0} \oplus \phi_{27,2} \oplus \phi_{21,6}$, and the vector part as $\phi_{1,0} \oplus \phi_{27,2} \oplus \phi_{35,4} \oplus \phi_{7,1} \oplus \phi_{56,3}$. See Table 14.1 for the fake degrees of these representations.

$R_3$ is of degree 6 and hence cannot be generated by $M, S$ and $R_2$. Hence we need four
Table 14.1: Fake degrees in $\text{End}(E_7)^T$

<table>
<thead>
<tr>
<th>$\phi_{a,b}$</th>
<th>$P_{a,b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\phi_{27,2}$</td>
<td>$q^2 + q^4 + 2q^6 + 2q^8 + 3q^{10} + 3q^{12} + 3q^{14}$ $+ 3q^{16} + 3q^{18} + 2q^{20} + 2q^{22} + q^{24} + q^{26}$</td>
</tr>
<tr>
<td>$\phi_{21,6}$</td>
<td>$q^6 + q^8 + q^{10} + 2q^{12} + 2q^{14} + 2q^{16}$ $+ 3q^{18} + 2q^{20} + 2q^{22} + 2q^{24} + q^{26} + q^{28} + q^{30}$</td>
</tr>
<tr>
<td>$\phi_{35,4}$</td>
<td>$q^4 + q^6 + 2q^8 + 2q^{10} + 3q^{12} + 3q^{14} + 4q^{16} + 3q^{18}$ $+ 4q^{20} + 3q^{22} + 3q^{24} + 2q^{26} + 2q^{28} + q^{30} + q^{32}$</td>
</tr>
<tr>
<td>$\phi_{7,1}$</td>
<td>$q + q^5 + q^7 + q^9 + q^{11} + q^{13} + q^{17}$</td>
</tr>
<tr>
<td>$\phi_{56,3}$</td>
<td>$q^3 + q^5 + 2q^7 + 3q^9 + 4q^{11} + 5q^{13} + 6q^{15} + 6q^{17}$ $+ 6q^{19} + 6q^{21} + 5q^{23} + 4q^{25} + 3q^{27} + 2q^{29} + q^{31} + q^{33}$</td>
</tr>
</tbody>
</table>

generators. $R_2'R_3' - R_2'R_3'$ has degree 10; while $\phi_{21,6}$ has a fake degree of 10, it only has one of them, so $R_2'R_3' - R_2'R_3'$ must be reducible to $R_4S - SR_4$ and lower order terms. Hence (once we prove that the torus part of $R_4$ can be generated by $R_2', R_3'$ and $S$) we can define an $R_2$ and an $R_3$ of degrees 4 and 6 respectively that commute and that, along with $M$ and $S$, generate $R_2'$ and $R_3'$.

The torus parts of the set $R_i$ for $i \leq 7$, $R_iSR_j + R_jSR_i$ for $i \leq j \leq 6$ and $R_iSR_j - R_jSR_i$ for $i < j \leq 7$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by $M, S$ and the $R_i$, 101
with the following elements being linearly independent:

\[ M^i R_j \text{ for } i \leq 18, j \leq 7 \]

\[ R_i SR_j + R_j SR_i \text{ for } i \leq j \leq 6 \]

\[ R_i SR_j - R_j SR_i \text{ for } i < j \leq 7 \]

As in previous cases, multiplying an element by \( S \) on both sides yields \( S \) times an element of \( I(E_7) \), so we can define \( d_{22}, d_{23}, d_{33} \) and \( d_{233} \) in \( I(E_7) \) by

\[ d_{22} S = SR_2^2 S \]

\[ d_{23} S = SR_2 R_3 S \]

\[ d_{33} S = SR_3^2 S \]

\[ d_{233} S = SR_2 R_3^2 S \]

\( d_{22} \) has degree 10, \( d_{23} \) has degree 12, \( d_{33} \) has degree 14 and \( d_{233} \) has degree 18. Moreover, the set

\[ \{c_1, c_2, c_3, d_{22}, d_{23}, d_{33}, d_{233}\} \]

is algebraically independent, so \( d_{22} \) must be a \( \mathbb{C}^* \)-multiple of \( c_4 \) plus products of lower degree primitive Casimir operators, \( d_{23} \) must be a \( \mathbb{C}^* \)-multiple of \( c_5 \) plus products of lower degree primitive Casimir operators, \( d_{33} \) must be a \( \mathbb{C}^* \)-multiple of \( c_6 \) plus products of lower degree primitive Casimir operators, and \( d_{233} \) must be a \( \mathbb{C}^* \)-multiple of \( c_7 \) plus products of lower degree primitive Casimir operators. Hence, since we can write \( R_4, R_5, R_6 \) and \( R_7 \) in terms of \( c_4, c_5, c_6 \) and \( c_7 \), they can be generated by \( M, S, R_2 \) and
 Hence the entire family algebra can be generated over $I(E_7)$ by $M, S, R_2$ and $R_3$, as stated in the theorem.

Note that $R_2^3$ also has degree 12, but the $\phi_{27,2}$ representation only allows 3 degree 12 elements. Since $R_3^2, R_2 R_3 S + SR_2 R_3$ and $R_2 SR_3 + R_3 SR_2$ are all linearly-independent on the torus, $R_2^3$ must be expressible as a $\mathbb{C}$-linear combination of $R_3^2, R_2 R_3 S + SR_2 R_3$ and $R_2 SR_3 + R_3 SR_2$ plus lower order terms.

Table 14.2: Generalized Exponents in $C_9(E_7)$

<table>
<thead>
<tr>
<th>$V$</th>
<th>$P_V(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triv</td>
<td>1</td>
</tr>
<tr>
<td>Sym</td>
<td>$P_{27,2}$</td>
</tr>
<tr>
<td>$S^2 Ad$</td>
<td>$q^2 P_{1,0} + q^2 P_{35,4} + q^2 P_{27,2}$</td>
</tr>
<tr>
<td>Ad</td>
<td>$P_{7,1}$</td>
</tr>
<tr>
<td>$\wedge^2 Ad$</td>
<td>$P_{56,3} + P_{21,6}$</td>
</tr>
</tbody>
</table>
Chapter 15

$E_8$

The reference representation of $E_8$ is the adjoint representation, bearing a symmetric bilinear form (the Killing form) and an antisymmetric trilinear form (the structure constants). Because the reference representation is the adjoint representation, there is no nice description of the invariant forms other than as themselves.

Alternatively, we can write $E_8$ as a module over $D_8$. $E_8$ decomposes as $E_8 = D_8 \oplus S$, where $S$ is one of the two spinor representations of $D_8$; it doesn’t matter which one.

The Lie bracket is extended by $[X, s] = X.s$ for $X \in D_8$ and $s \in S$, and for $s, t \in S$, $[s, t]$ is defined by $\langle X, [s, t] \rangle_{D_4} = -\langle X.t, s \rangle_S$. The Lie bracket also provides the action of $E_8$ on itself.

The existence of the bilinear form forces all odd-degree Casimir elements to vanish, and we are left with the degrees of the primitive Casimir elements being 2, 8, 12, 14, 18, 20, 24, and 30, with corresponding exponents 1, 7, 11, 13, 17, 19, 23, and 29.
The torus part of the family algebra decomposes as $\phi_{1,0}^{} \oplus \phi_{35,2}^{} \oplus \phi_{28,8}^{}$. The vector part decomposes as $\phi_{1,0}^{} \oplus \phi_{35,2}^{} \oplus \phi_{84,4}^{} \oplus \phi_{8,1}^{} \oplus \phi_{112,3}^{}$. See Table 15.1 for the fake degrees of these representations.

$R_3$ is of degree 10 and hence cannot be generated by $M$, $S$ and $R_2$. Hence we need

<table>
<thead>
<tr>
<th>Table 15.1: Fake degrees in $End(E_8)^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{a,b}$</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>$\phi_{1,0}^{}$</td>
</tr>
<tr>
<td>$\phi_{35,2}^{}$</td>
</tr>
<tr>
<td>$\phi_{28,8}^{}$</td>
</tr>
<tr>
<td>$\phi_{84,4}^{}$</td>
</tr>
<tr>
<td>$\phi_{8,1}^{}$</td>
</tr>
<tr>
<td>$\phi_{112,3}^{}$</td>
</tr>
</tbody>
</table>

four generators. $R'_2R'_3^{} - R'_3^{}R'_2^{}$ has degree 16, but $\phi_{28,8}^{}$ has no fake degree of 16, so $R'_2R'_3^{} - R'_3^{}R'_2^{}$ must be reducible to lower order terms. Hence we can define an $R_2$ and
an $R_3$ of degrees 6 and 10 respectively that commute and that, along with $M$ and $S$, generate $R'_2$ and $R'_3$.

The torus parts of the set $R_i$ for $i \leq 8$, $R_iSR_j + R_jSR_i$ for $i \leq j \leq 7$ and $R_iSR_j - R_jSR_i$ for $i < j \leq 8$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by $M, S$ and the $R_i$, with the following elements being linearly independent:

$$M^i R_j \text{ for } i \leq 18, j \leq 7$$

$$R_iSR_j + R_jSR_i \text{ for } i \leq j \leq 6$$

$$R_iSR_j - R_jSR_i \text{ for } i < j \leq 7$$

As in previous cases, multiplying an element by $S$ on both sides yields $S$ times an element of $I(E_8)$, so we can define $d_{22}$, $d_{23}$, $d_{222}$, $d_{223}$ and $d_{2223}$ in $I(E_8)$ by

$$d_{22}S = SR_2^2S$$

$$d_{23}S = SR_2R_3S$$

$$d_{222}S = SR_2^3S$$

$$d_{223}S = SR_2^2R_3S$$

$$d_{2223}S = SR_2^3R_3S$$

$d_{22}$ has degree 14, $d_{23}$ has degree 18, $d_{222}$ has degree 20, $d_{223}$ has degree 24, and $d_{2223}$ has degree 30. Moreover, the set $c_1, c_2, c_3, d_{22}, d_{23}, d_{222}, d_{223}$ and $d_{2223}$ is algebraically
independent, so $d_{22}$ must be a $\mathbb{C}^*$-multiple of $c_4$ plus products of lower degree primitive Casimir operators, $d_{23}$ must be a $\mathbb{C}^*$-multiple of $c_5$ plus products of lower degree primitive Casimir operators, $d_{222}$ must be a $\mathbb{C}^*$-multiple of $c_6$ plus products of lower degree primitive Casimir operators, $d_{223}$ must be a $\mathbb{C}^*$-multiple of $c_7$ plus products of lower degree primitive Casimir operators, and $d_{2223}$ must be a $\mathbb{C}^*$-multiple of $c_8$ plus products of lower degree primitive Casimir operators. Hence, since we can write $R_4$, $R_5$, $R_6$, $R_7$ and $R_8$ in terms of $c_4$, $c_5$, $c_6$, $c_7$ and $c_8$, they can be generated by $M, S, R_2$ and $R_3$. Hence the entire family algebra can be generated over $I(E_8)$ by $M, S, R_2$ and $R_3$, as stated in the theorem.

Table 15.2: Generalized Exponents in $C_g(E_8)$

<table>
<thead>
<tr>
<th>$V$</th>
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<tr>
<td>$S^2 Adj$</td>
<td>$q^2 P_{1,0} + q^2 P_{84,4} + q^2 P_{35,2}$</td>
</tr>
<tr>
<td>$Adj$</td>
<td>$P_{8,1}$</td>
</tr>
<tr>
<td>$\wedge^2 Adj$</td>
<td>$P_{112,3} + P_{28,8}$</td>
</tr>
</tbody>
</table>
Bibliography

Be79  Benson, C. T., *The generic degrees of the irreducible characters of $E_8$*, 
Comm. Algebra 7 (1979), 1199-1209


