Shifted Symplectic Structures on Spaces of Framed Maps

Theodore Spaide
University of Pennsylvania, tspaide@gmail.com

Follow this and additional works at: http://repository.upenn.edu/edissertations

Part of the Mathematics Commons

Recommended Citation
http://repository.upenn.edu/edissertations/1143

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/edissertations/1143
For more information, please contact libraryrepository@pobox.upenn.edu.
Shifted Symplectic Structures on Spaces of Framed Maps

Abstract
This work examines the existence of shifted symplectic and Poisson structures on certain spaces of framed maps.

We define n-shifted Poisson structures and coisotropic structures in terms of shifted symplectic structures and Lagrangian structures. Shifted Poisson structures are shown to have properties analogous to those of shifted symplectic structures, and reduce to ordinary Poisson structures in the classical case.

Next, we examine the space Map(X,D,Y) of maps from X to Y, framed along some divisor D. These are shown to inherit a shifted symplectic or Poisson structure from Y in certain conditions. This construction is used to rederive the existence of symplectic and Poisson structures in classical examples.

Degree Type
Dissertation

Degree Name
Doctor of Philosophy (PhD)

Graduate Group
Mathematics

First Advisor
Tony Pantev

Keywords
Derived, Framed, Mapping, Poisson, Shifted, Symplectic

Subject Categories
Mathematics

This dissertation is available at ScholarlyCommons: http://repository.upenn.edu/edissertations/1143
SHIFTED SYMPLECTIC STRUCTURES ON SPACES OF FRAMED MAPS
Theodore Spaide
A DISSERTATION
in
Mathematics
Presented to the Faculties of the University of Pennsylvania
in
Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy
2015

Supervisor of Dissertation

Tony Pantev, Professor of Mathematics

Graduate Group Chairperson

David Harbater, Christopher H. Browne Distinguished Professor
in the School of Arts and Sciences

Dissertation Committee
Tony Pantev, Professor of Mathematics
Jonathan Block, Professor of Mathematics
Ron Donagi, Professor of Mathematics
ACKNOWLEDGEMENT

I would like to offer my deepest thanks to my advisor, Tony Pantev. His patience with my foolishness and personal foibles is saint-like; his advice is always useful and true.

I give my heartfelt gratitude to the administrative staff of the math department, particularly Janet Burns and Reshma Tanna, for their invaluable assistance.

I would like to thank my parents, my brother, and my sister, who have supported me throughout the long process of my education.

Finally, I would like to thank Kinoko Nasu and Gen Urobuchi, whose works have provided me with the resolve I needed in difficult times.
This work examines the existence of shifted symplectic and Poisson structures on certain spaces of framed maps.

We define n-shifted Poisson structures and coisotropic structures in terms of shifted symplectic structures and Lagrangian structures. Shifted Poisson structures are shown to have properties analogous to those of shifted symplectic structures, and reduce to ordinary Poisson structures in the classical case.

Next, we examine the space Map(X,D,Y) of maps from X to Y, framed along some divisor D. These are shown to inherit a shifted symplectic or Poisson structure from Y in certain conditions. This construction is used to rederive the existence of symplectic and Poisson structures in classical examples.
# TABLE OF CONTENTS

## ACKNOWLEDGEMENT

## ABSTRACT

### 1 Introduction

1.1 Shifted Symplectic Structures .............................................. 1  
1.2 Maps and Framed Maps ..................................................... 2  
1.3 Shifted Poisson Structures ................................................. 3  
1.4 Organization ............................................................... 4  

### 2 Shifted Symplectic Structures

2.1 Lagrangian Structures ....................................................... 7  
2.2 Symplectic Structures on Mapping Stacks ................................ 9  

### 3 Shifted Poisson Structures

### 4 Framed Mapping Spaces

4.1 Framed Vector Bundles on Surfaces ..................................... 38  

### 5 Monopoles

5.1 Classical Construction of the Symplectic Structure .................... 39  
5.2 Construction via Shifted Poisson Structures ............................ 41  
5.3 Other Fibers .............................................................. 45  

iv
Chapter 1

Introduction

This thesis is an investigation into shifted symplectic structures and shifted Poisson structures, and their relation to certain spaces of framed maps. Shifted symplectic structures could pithily be described via a pullback square of “symplectic geometry” and “derived algebraic geometry” over “algebraic geometry”, although maybe not if you want anyone to understand what you’re saying.

1.1 Shifted Symplectic Structures

Shifted symplectic structures, first described in [PTVV], are the natural extension of ordinary (algebraic) symplectic structures to the land of derived algebraic geometry. Promoting everything to the level of derived stacks has the notable effect of replacing the cotangent sheaf $L_X$ with the cotangent complex $L_X$, and dually the tangent sheaf with the tangent complex. This has the expected “homotopic” effect of replacing isomorphisms with quasi-isomorphisms, equalities with equalities up to homotopy, et cetera. More interestingly, the usual nondegeneracy requirement that the induced map $T_X \to L_X$ is an isomorphism seems to become a requirement that $T_X \to L_X$ is a quasi-isomorphism. This is generally impossible, unless $L_X$ is concentrated in cohomological degrees $[-a, a]$ for some $a$.

More generally, we might look for a map $T_X \to L_X[n]$, which would correspond to a 2-form of degree $n$, and which would ultimately correspond to an $n$-shifted symplectic structure. Assuming $L_X$ is concentrated in degrees $[a, b]$ (with $|a|, |b| < \infty$), an $n$-shifted symplectic structure may be possible for $n = a + b$.

This is a wide generalization of ordinary symplectic structures. First, it really is a generalization:

\footnote{This is somewhat misleading; in the classical case, the form is equivalent to the map $T \to L$, but in the derived case more structure is needed.}
0-shifted symplectic structures on smooth varieties are symplectic structures in the ordinary sense. Second, there are constructions which will provide new shifted symplectic spaces from old ones, often with different shifts. Thus, someone uninterested in derived algebraic geometry might still use this machinery to end up with a 0-shifted structure on a smooth variety. Finally, a lot of derived stacks—like the classifying stack $BG$—have shifted symplectic structures.

1.2 Maps and Framed Maps

One particular example, discussed in [PTVV], is as follows. Let $Y$ have an $n$-shifted symplectic structure, and let $X$ be $\mathcal{O}$-compact oriented in dimension $d$. This latter condition is nontrivial; it is, for example, satisfied for $X$ a smooth compact Calabi-Yau variety. Then the mapping stack $\text{Map}(X,Y)$ has an $(n-d)$-shifted symplectic structure.

This is a powerful theorem that also recreates some examples of known classical symplectic structures. For example, if $X$ is a K3 surface and $G$ a semisimple group, then the symplectic structure on the stable locus of $\text{Map}(X,BG)$ was described by Mukai.

However, there are also a number of cases this does not cover. Let $G$ be a reductive group. Let $\text{Map}(\mathbb{P}^2, L, BG)$ be the space of stable principal $G$-bundles on $\mathbb{P}^2$ with a trivial framing along a line $L$. This space has a symplectic structure as described in [Bo]. For another example, let $\text{Map}(\mathbb{P}^1, p, G/B)$ be the space of maps from $\mathbb{P}^1$ to a flag variety $G/B$ sending a marked point in the source to a marked point in the target. This space also has a symplectic structure, as described in [FKMM].

Both of these examples detail spaces of framed maps. Specifically, fix maps $i : D \to X$ and $f : D \to Y$, and look at the homotopy fiber of $\text{Map}(X,Y) \to \text{Map}(D,Y)$ above $f$. The resulting space, $\text{Map}(X,D,Y)$ parametrizes maps $g : X \to Y$ with homotopies $g \circ i \sim f$ on $D$. As the above examples show, these spaces will have shifted symplectic structures under certain circumstances. Looking at things from a different perspective, in the above cases the source $X$ does not have a $d$-orientation, and $\text{Map}(X,Y)$ does not a have a symplectic structure; $\text{Map}(X,D,Y)$ is the “correct” space for symplectic structures.

The main result for this is

**Theorem 1.1.** Let $X$ be a $d$-dimensional proper smooth scheme and $D$ an effective divisor. Suppose $E$ is an effective divisor of $X$ such that $\tilde{D} = 2D + E$ is anticanonical. Let $Y$ be a derived Artin stack such that $\text{Map}(X,Y)$, $\text{Map}(\tilde{D},Y)$, $\text{Map}(D,Y)$, and $\text{Map}(D+E,Y)$ are themselves derived...
Artin stacks of locally finite presentation over $k$. Fix a base map $f : D \to Y$. Suppose $Y$ is $n$-shifted symplectic and the projection $\text{Map}(D + E, Y) \to \text{Map}(D, Y)$ is etale over $f$. Then $\text{Map}(X, D, Y)$ has an $(n - d)$-shifted symplectic structure.

This theorem will provide, for example, the symplectic structure on $\text{Map}(\mathbb{P}^2, L, BG)$. On the other hand, it is a bit fragile; varieties with effective anticanonical divisor are common enough, but the cohomological condition that $\text{Map}(D + E, Y) \to \text{Map}(D, Y)$ is etale over $f$ is not guaranteed. On a more conceptual level, we would like to know why $\text{Map}(X, Y)$ isn’t symplectic in this scenario (and $\text{Map}(X, D, Y)$ whenever the etaleness condition is not satisfied).

1.3 Shifted Poisson Structures

To justify this sudden change of topics, we note that in the above examples, even when the mapping space doesn’t have a symplectic structure, it still has a Poisson structure. To give two examples, if $G$ is a semisimple group and $P$ is a parabolic subgroup, $\text{Map}(\mathbb{P}^1, p, G/P)$ will have a Poisson structure [FKMM]. The space $\text{Map}(\mathbb{P}^2, L, BG)$ will not have a symplectic structure if we choose a nontrivial framing on $L$, but it will still have a Poisson structure [Bo].

To motivate the definition of a shifted Poisson structure, we note two things. First, if we let $\bullet_{n+1}$ denote a point with the trivial $(n + 1)$-shifted symplectic structure, then an $n$-shifted symplectic structure on a stack $X$ is exactly the same as a Lagrangian structure on $X \to \bullet_{n+1}$. Second, if $X$ is a smooth underived scheme, then a Poisson structure on $X$ can be used to construct a 1-shifted symplectic space $Y$ and a morphism $X \to Y$ with Lagrangian structure; conversely, given such a map to a 1-shifted symplectic $Y$, we can construct a Poisson structure on $X$.

With this in mind, we take an $n$-shifted Poisson structure on $X$ to be a formal derived stack $Y$ with an $(n + 1)$-shifted symplectic structure, and morphism $X \to Y$ with Lagrangian structure. The preceding paragraph tells us that any $n$-shifted symplectic structure is $n$-shifted Poisson, and that a 0-shifted Poisson structure on a smooth scheme is a Poisson structure in the usual sense.

With this definition, we prove a number of results about Poisson structures generalizing those about symplectic structures. For framed mapping spaces we have the following theorem:

**Theorem 1.2.** Let $X$ be a $d$-dimensional proper smooth scheme and $D$ an effective divisor. Suppose $E$ is an effective divisor of $X$ such that $\bar{D} = 2D + E$ is anticanonical. Let $Y$ be a derived Artin stack such that $\text{Map}(X, Y)$, $\text{Map}(\bar{D}, Y)$, $\text{Map}(D, Y)$, and $\text{Map}(D + E, Y)$ are themselves derived Artin stacks of locally finite presentation over $k$. Fix a base map $f : D \to Y$. Suppose $Y$ is $n$-shifted
Poisson. Then $\text{Map}(X, D, Y)$ has an $(n - d)$-shifted Poisson structure.

We have, notably, discarded the etaleness assumption. This theorem is pretty broadly applicable. For example, we obtain the Poisson structure on the remaining cases of $\text{Map}(\mathbb{P}^2, L, BG)$. The case of $\text{Map}(\mathbb{P}^1, p, G/P)$ requires a more roundabout approach, but ultimately is understood via these tools.

Even in the case that $Y$ is symplectic, this theorem is illuminating. For example, it tells us “why”, if $X$ has nonzero effective anticanonical divisor, the space $\text{Map}(X, Y)$ does not have a symplectic structure; it has a (nonsymplectic) Poisson structure. In fact, the same is true for any $\text{Map}(X, D, Y)$ for which the etaleness condition of Theorem 1.1 does not hold.

1.4 Organization

Chapter 1 is an overview of shifted symplectic structures. It collects some definitions and results but is not a detailed reference.

Chapter 2 is about shifted Poisson structures. Poisson structures and coisotropic morphisms are defined. 0-shifted Poisson structures on smooth schemes are shown to be Poisson structures in the ordinary sense. Results for symplectic structures are generalized to Poisson structures.

Chapter 3 concerns framed mapping spaces. It contains the main results of this thesis, particularly those given in this introduction.

Chapter 4 is about the spaces of monopoles $\text{Map}(\mathbb{P}^1, p, G/P)$. The Poisson structure on this space is constructed.
Chapter 2

Shifted Symplectic Structures

In the following, $k$ will be the base field, of characteristic 0.

Shifted symplectic structures are first defined in [PTVV]. We will recall some definitions and results.

Let $X$ be a derived Artin stack. We can form the de Rham algebra $\Omega^*_X = \text{Sym}_X^*(L_X[1])$. This is a weighted sheaf whose weight $p$ piece is $\Omega^p_X = \text{Sym}_X^p(L_X[1]) = \wedge^p L_X[p]$.

**Definition 2.1.** The space of $p$-forms of degree $n$ on $X$ is

$$\mathcal{A}^p(X, n) = \tau_{\leq 0} \text{Hom}_{L_{QCoh}(X)}(\mathcal{O}_X, \wedge^p L_X[n]).$$

Here $L_{QCoh}(X)$ is the $\infty$-category of chain complexes of quasicoherent $\mathcal{O}_X$-modules.

Let $d_{L_X}$ denote the differential on $L_X$ or the induced differential on $\wedge^* L_X$. Let $d_{dR}$ be the de Rham differential on $\wedge^* L_X$. Then we construct the weighted negative cyclic chain complex $NC^w$, whose degree $n$, weight $p$ part is

$$NC^n(\Omega_X)(p) = (\bigoplus_{i \geq 0} \wedge^{p+i} L_X[n-i], d_{L_X} + d_{dR}).$$

**Definition 2.2.** The space of closed $p$-forms of degree $n$ is

$$\mathcal{A}^{p, cl}(X, n) = \tau_{\leq 0} \text{Hom}_{L_{QCoh}(X)}(\mathcal{O}_X, NC^n(\Omega_X)(p)).$$

---

1 Here I use “space”, “simplicial set”, and “connective chain complex” interchangeably.
There is a natural “underlying form” map \( \mathcal{A}^{p,cl}(X,n) \to \mathcal{A}^p(X,n) \) corresponding to the projection
\[
\bigoplus_{i \geq 0} \wedge^{p+i} \mathbb{L}_X[n-i] \to \wedge^p \mathbb{L}_X[n].
\]

Intuitively, a “closed \( p \)-form” \( \omega \) of degree \( n \) consists of forms \( \omega_{p+i} \in \wedge^{p+i}(\mathbb{L}_X)_{n-i} \) for \( i \geq 0 \) such that \((d_{\mathbb{L}_X} + d_{dR})(\omega_p + \omega_{p+1} + \cdots) = 0\). This has the following interpretation. The underlying form of \( \omega \) is \( \omega_p \); we require \( d_{\mathbb{L}_X} \omega_p = 0 \) so that \( \omega_p \) defines a class in cohomology. For de Rham closedness, we do not require that \( d_{dR} \omega_p = 0 \), but that it is homotopic to zero with specified homotopy: \( d_{dR} \omega_p = -d_{L_X} \omega_{p+1} \) for some \( \omega_{p+1} \). We then require that \( \omega_{p+1} \) be de Rham closed, again in the sense that \( d_{dR} \omega_{p+1} = -d_{L_X} \omega_{p+2} \), et cetera. Note that closedness of a \( p \)-form is not a condition on a \( p \)-form, but an extra structure consisting of the forms \( \omega_{p+i} \) for \( i \geq 1 \). In the case of a (0-shifted) \( p \)-form on an ordinary smooth variety, \( \mathbb{L}_X \) is concentrated in degree 0, so we must have \( \omega_{p+i} = 0 \) for \( i \geq 1 \). Thus the structure reduces to a condition in this case.

Now we can define symplectic structures:

**Definition 2.3.** A 2-form \( \omega : \mathcal{O}_X \to \wedge^2 \mathbb{L}_X[n] \) of degree \( n \) is *nondegenerate* if the adjoint map \( T_X \to \mathbb{L}_X[n] \) is a quasi-isomorphism. Let \( \mathcal{A}^2(X,n)^{nd} \) denote the non-degenerate 2-forms of degree \( n \) on \( X \).

An *\( n \)-shifted symplectic form* on \( X \) is a closed 2-form whose underlying form is nondegenerate. The space of \( n \)-shifted symplectic forms is the (homotopy\(^2\)) product
\[
\text{Symp}(X,n) = \mathcal{A}^{2,cl}(X,n) \times_{\mathcal{A}^2(X,n)} \mathcal{A}^2(X,n)^{nd}.
\]

Let us give a few examples of spaces with shifted symplectic structures.

- If \( X \) is an ordinary (underived) smooth scheme, then a 0-shifted symplectic structure on \( X \) is precisely the same as a symplectic structure in the ordinary sense.

- Let \( G \) be a reductive affine smooth group scheme over \( k \). Then the classifying stack \( BG \) has a 2-shifted symplectic structure. A 2-shifted symplectic form on \( G \) is the same as a nondegenerate \( G \)-invariant quadratic form on \( \mathfrak{g} \).

- Let \( X \) be a derived Deligne-Mumford stack locally of finite presentation over \( k \). Then we can define the \( n \)-shifted cotangent stack \( T^\vee X[n] = \mathcal{R}\text{Spec Sym}_{\mathcal{O}_X}(T_X[-n]) \). Then \( T^\vee X[n] \) has

\( ^2\mathcal{A}^2(X,n)^{nd} \) is a union of connected components of \( \mathcal{A}^2(X,n) \), so the ordinary product is the homotopy product.
a natural $n$-shifted symplectic form defined analogously to the canonical symplectic form on $T^\vee X$ for a smooth scheme $X$.

- The quotient stack $[g^\vee / G]$ has a canonical 1-shifted symplectic form ([Ca], 1.2.3).

### 2.1 Lagrangian Structures

Let $Y$ be a derived Artin stack with an $n$-shifted symplectic form $\omega$ and let $f : X \to Y$ be a morphism.

**Definition 2.4.** The space of isotropic structures on $f$ (with respect to $\omega$) is

$$\text{Isot}(f, \omega) = \text{Path}_{0, f^*(\omega)}(A^{2, cl}(X, n))$$

the space of paths from 0 to $f^*(\omega)$ in $A^{2, cl}(X, n)$.

Let $T_f$ be the relative tangent complex of $f$, so that we have a distinguished triangle

$$T_f \to T_X \to f^*(T_Y).$$

An isotropic structure $h \in \text{Isot}(f, \omega)$ provides a homotopy between the morphism

$$T_X \wedge T_X \to f^*(T_Y) \wedge f^*(T_Y) \to \mathcal{O}_X[n]$$

and 0. Then we also get a homotopy from

$$T_f \otimes T_X \to T_X \wedge T_X \to f^*(T_Y) \wedge f^*(T_Y) \to \mathcal{O}_X[n]$$

to 0. There is another homotopy between this morphism and 0, coming from the canonical homotopy from $T_f \to f^*T_Y$ to 0. Composing these yields a loop in $\text{Hom}_{LQCoh(X)}(T_f \otimes T_X, \mathcal{O}_X[n])$, which is a point of $\text{Hom}_{LQCoh(X)}(T_f \otimes T_X, \mathcal{O}_X[n - 1])$. By adjunction, we get a map $\Theta_h : T_f \to \mathbb{L}_X[n - 1]$.

**Definition 2.5.** We say $h$ is Lagrangian if $\Theta_h$ is a quasi-isomorphism of complexes.

Note that an isotropic or Lagrangian structure is indeed a structure on a map, rather than a property as in the ordinary case. As with the closedness structure, the structure reduces to a condition when we restrict to ordinary varieties.
Remark 2.5.1. Note that if $\bullet_n$ denotes the point with trivial $n$-shifted symplectic structure, then a Lagrangian structure on $X \to \bullet_n$ is just an $(n-1)$-shifted symplectic structure on $X$.

Lagrangian structures are useful in generating new symplectic spaces:

**Theorem 2.6** ([PTVV], Theorem 2.9). Let $X, L_1, L_2$ be derived Artin stacks, $\omega \in \text{Symp}(X,n)$ an $n$-shifted symplectic structure on $X$, and $f_i : L_i \to X$ a morphism with Lagrangian structure $h_i$ for $i = 1, 2$. Then the product $L_1 \times_X^L L_2$ has a natural $(n-1)$-shifted symplectic structure, which we denote by $R(\omega, h_1, h_2)$.

**Proof.** We briefly show the construction of $R(\omega, h_1, h_2)$; the complete proof is given in [PTVV]. Let $Z = L_1 \times_X^L L_2$, and $\pi_i : Z \to L_i$ the projection for $i = 1, 2$. Let $u : f_1 \circ \pi_1 \Rightarrow f_2 \circ \pi_2$ be the natural homotopy.

The Lagrangian structure $h_i$ yields a homotopy $0 \sim f_i^* \omega$ in $A^{2,cl}(L_i, n)$. Pulling back by $\pi_i$ gives homotopies

$$\pi_i^* h_i : 0 \sim h_i^* f_i^* \omega,$$

for $i = 1, 2$. The homotopy $u$ gives a homotopy

$$u^* \omega : h_1^* f_1^* \omega \sim h_2^* f_2^* \omega.$$

Concatenating these paths yields a loop in $A^{2,cl}(L_i, n)$; for concreteness we take $\pi_1^* h_1 + u^* \omega - \pi_2^* h_2$. This defines an element

$$R(\omega, h_1, h_2) \in \pi_1(A^{2,cl}(Z, n)) \simeq \pi_0(A^{2,cl}(Z, n-1)).$$

Remark 2.6.1. In particular, let $X_1$ and $X_2$ be derived Artin stacks with $n$-shifted symplectic structures $\omega_1$ and $\omega_2$, respectively. Considering $\omega_1$ and $\omega_2$ as Lagrangian structures on the maps $X_1, X_2 \to \bullet_{n+1}$, this theorem provides the symplectic structure $\pi_1^* \omega_1 - \pi_2^* \omega_2$ on $X_1 \times X_2$. 


2.2 Symplectic Structures on Mapping Stacks

Let $X$ and $Y$ be derived Artin stacks. The evaluation map $ev : X \times \text{Map}(X, Y) \to Y$ yields a pullback map

$$ev^* : \mathbb{R}\Gamma(Y, \Omega_Y^*) \to \mathbb{R}\Gamma(X \times \text{Map}(X, Y), \Omega_{X \times \text{Map}(X, Y)}^*).$$

If $X$ is $\mathcal{O}$-compact (see [PTVV], Definition 2.1), we have a map

$$\mathbb{R}\Gamma(X \times \text{Map}(X, Y), \Omega_{\text{Map}(X, Y)}^*) \to \mathbb{R}\Gamma(X, \mathcal{O}_X) \otimes \mathbb{R}\Gamma(\text{Map}(X, Y), \Omega_{\text{Map}(X, Y)}^*),$$

which is the K"unneth formula

$$\mathbb{R}\Gamma(X \times \text{Map}(X, Y), \Omega_{\text{Map}(X, Y)}^*) \simeq \mathbb{R}\Gamma(X, \mathcal{O}_X) \otimes \mathbb{R}\Gamma(\text{Map}(X, Y), \Omega_{\text{Map}(X, Y)}^*)$$

followed by projection $\mathbb{R}\Gamma(X, \Omega^*(X)) \to \mathbb{R}\Gamma(X, \mathcal{O}_X)$ onto the 0-forms.

Given a “fundamental class” $[X] : \mathbb{R}\Gamma(X, \mathcal{O}_X) \to k[-d]$, we can compose these morphisms to obtain

$$\mathbb{R}\Gamma(Y, \Omega_Y^*) \to \mathbb{R}\Gamma(\text{Map}(X, Y), \Omega_{\text{Map}(X, Y)}^*)[-d],$$

which will induce a map on (closed) $p$-forms:

$$\mathcal{A}^{p, (cl)}(Y, n) \to \mathcal{A}^{p, (cl)}(\text{Map}(X, Y), n - d).$$

If $[X]$ satisfies a certain nondegeneracy condition, the above map will preserve nondegeneracy of forms. We now describe the condition.

For any perfect complex $E$ on $X$, we let $E^\vee = \mathbb{R}\text{Hom}(E, \mathcal{O}_X)$, and we have a natural pairing

$$\mathbb{R}\Gamma(X, E) \otimes \mathbb{R}\Gamma(X, E^\vee) \xrightarrow{\cup} \mathbb{R}\Gamma(X, \mathcal{O}_X) \xrightarrow{[X]} k[-d],$$

which is adjoint to a map

$$\mathbb{R}\Gamma(X, E) \xrightarrow{- \circ [X]} \mathbb{R}\Gamma(X, E^\vee)[-d].$$

More generally, for any $A \in \text{cdga}_{\leq 0}$, we let $X_A = X \times \text{Spec} A$, and for any perfect complex $E$ on
we have a map

\[ \mathbb{R}\Gamma(X, E) \xrightarrow{\cap [X]} \mathbb{R}\Gamma(X, E^\vee)[-d] \].

**Definition 2.7.** We say \([X]\) is a \(d\)-orientation if for every \(A \in \text{cdga}_k^{<0}\) and perfect complex \(E\) on \(X_A\), the map \(- \cap [X]_A\) is a quasi-isomorphism.

Then we have

**Theorem 2.8** ([PTVV], Theorem 2.5). *Let \(Y\) be a derived Artin stack, and let \(X\) be an \(\mathcal{O}\)-compact derived stack with a \(d\)-orientation \([X]\). Assume the derived mapping stack \(\text{Map}(X, Y)\) is a derived Artin stack locally of finite presentation over \(k\). Then we have a map*

\[ \int_{[X]} ev^*(-) : \text{Symp}(Y, n) \to \text{Symp}(\text{Map}(X, Y), n - d). \]

Fix \(\omega \in \text{Symp}(Y, n)\). Let us describe the induced structure \(\int_{[X]} ev^*(\omega)\) now. For any \(f \in \text{Map}(X, Y)\), the tangent complex at \(f\) is \(T_f \text{Map}(X, Y) \simeq \mathbb{R}\Gamma(X, f^*T_Y)\). Then the pairing

\[ \bigwedge^2 T_f \text{Map}(X, Y) \to k[n - d] \]

is given by

\[ \mathbb{R}\Gamma(X, f^*T_Y) \wedge \mathbb{R}\Gamma(X, f^*T_Y) \xrightarrow{\cup} \mathbb{R}\Gamma(X, f^*T_Y \wedge f^*T_Y) \]

\[ \xrightarrow{f^*(\omega)} \mathbb{R}\Gamma(X, \mathcal{O}_X[n]) \xrightarrow{[X]} k[n - d] \]

Several examples of orientations are given in [PTVV], following Theorem 2.5. One particular example is the case that \(X\) is Calabi-Yau. If \(X\) has dimension \(d\) and we have an isomorphism \(\omega_X \simeq \mathcal{O}_X\), then projection of \(\mathbb{R}\Gamma(X, \mathcal{O}_X)\) onto the degree \(d\) cohomology \(H^d(X, \mathcal{O}_X)[-d]\), followed by the isomorphism

\[ H^d(X, \mathcal{O}_X) \simeq H^d(X, \omega_X) \simeq k \]

provides a map \([X] : \mathbb{R}\Gamma(X, \mathcal{O}_X) \to k[-d]\). This is an orientation by Serre duality.
2.2.1 Boundary Structures

The following is due to [Ca]. Let $X, Y,$ and $Z$ be derived Artin stacks. Given a map $f : Z \to X,$ we have a pullback map $(- \circ f) : \text{Map}(X, Y) \to \text{Map}(Z, Y).$ Assume that $X$ and $Z$ are $\mathcal{O}$-compact and $Z$ has a $d$-orientation $[Z],$ that $Y$ has an $n$-shifted symplectic form $\omega,$ and that both mapping spaces are Artin stacks. Then $\text{Map}(Z, Y)$ will have an $(n - d)$-shifted symplectic structure. It is natural to ask when the pullback map $(- \circ f)$ has an isotropic or Lagrangian structure.

**Definition 2.9.** The *space of boundary structures* on $f$ (with respect to $[Z]$) is

$$\text{Bnd}(f, [Z]) = \text{Path}_{0, f, [Z]}(\text{Hom}_k(\mathbb{R}\Gamma(X, \mathcal{O}_X), k[-d]))$$

the space of paths from 0 to $f_*[Z]$ in $\text{Hom}_k(\mathbb{R}\Gamma(X, \mathcal{O}_X), k[-d]).$

This definition is dual to the definition of isotropic structures, and it is clear that a boundary structure on $f$ will yield an isotropic structure on $- \circ f$ with respect to $\int_{[Z]} ev_2^* \omega,$ via the identity

$$(- \circ f)^* \int_{[Z]} ev_2^* \omega = \int_{f_*[Z]} ev_X^* \omega.$$ 

This can be extended to a dual notion of nondegeneracy (see [Ca], definition 2.8) which guarantees that the isotropic structure is Lagrangian:

**Theorem 2.10 ([Ca], Theorem 2.9).** Let $X, Y, Z$ be as above. Then we have a map

$$\text{Bnd}(f, [Z]) \to \text{Isot}(f^*, \int_{[Z]} ev_2^* \omega)$$

sending nondegenerate boundary structures to Lagrangian structures.

In particular we are interested in the following case. Let $X$ be a geometrically connected smooth proper algebraic variety of dimension $d + 1,$ and say it has a smooth anticanonical effective divisor $D.$ Then $D$ is a $d$-dimensional Calabi-Yau variety by the adjunction formula, and so has a $d$-orientation $[D] : \mathbb{R}\Gamma(D, \mathcal{O}_D) \to k[-d].$ Similarly, using

$$K_X \simeq \mathcal{O}_X(-D),$$

11
we get a map \([X] : \mathbb{R}\Gamma(X, \mathcal{O}_X(-D)) \to k[-d-1]\). Then the short exact sequence

\[
0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota^* \mathcal{O}_D \longrightarrow 0
\]

gives us a commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}\Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}\Gamma(D, \mathcal{O}_D) \\
\downarrow^{[D]} & & \downarrow^{[X]} \\
\mathbb{R}\Gamma(X, \mathcal{O}_X(-D))[1] & \longrightarrow & k[-d]
\end{array}
\]

Since the top row is naturally homotopic to 0, this provides a path between 0 and \(\iota_* [D]\), that is, a boundary structure on \(\iota\). In fact:

**Lemma 2.11 ([Ca], Claim 3.3).** This is a nondegenerate structure.

And so, using Theorem 2.10, we get

**Corollary 2.12.** Let \(X\) be a geometrically connected smooth proper algebraic variety of dimension \(d+1\), and let \(D\) be a smooth anticanonical effective divisor. Let \(Y\) have an \(n\)-shifted symplectic form \(\omega \in \text{Symp}(Y,n)\). Assume \(\text{Map}(X,Y)\) and \(\text{Map}(D,Y)\) are derived Artin stacks.

Then there exist a natural \((n-d)\)-shifted symplectic form on \(\text{Map}(D,Y)\) and Lagrangian structure on \(\text{Map}(X,Y) \to \text{Map}(D,Y)\).
Chapter 3

Shifted Poisson Structures

To motivate the definition of shifted Poisson structures, let us first look at ordinary Poisson structures in terms of shifted symplectic structures. Let $X$ be a smooth (underived) scheme with Poisson bivector field $\pi$. The Poisson structure is equivalently given by the sheaf map $\pi^\sharp : T_X^\vee \to T_X$.

Here is one way to get maps $T_X^\vee \to T_X$. Let $Y$ be a formal stack with a 1-shifted symplectic structure, and let $q : X \to Y$ have a Lagrangian structure $h$. Then the Lagrangian condition gives a quasi-isomorphism $T_q \simeq T^\vee X$, and composing with $T_q \to T_X$ yields a map $\pi^\sharp h : T^\vee X \to TX$.

This construction necessarily yields a Poisson structure, and in fact provides all Poisson structures:

Theorem 3.1. Let $X$ be a smooth scheme. Then:

1. Given a Poisson structure $\pi$ on $X$, there exist a formal derived stack $Y$ with 1-shifted symplectic form $\omega$, a map $q : X \to Y$, and Lagrangian structure $h$ on $q$ such that $\pi^\sharp = \pi^\sharp h$.

2. Let $Y$ be a formal derived stack with 1-shifted symplectic form $\omega$, and let $q : X \to Y$ be a map with Lagrangian structure $h$. Then $\pi^\sharp h$ is a Poisson structure on $X$.

Proof. For (1), consider the map $\pi^\sharp : TX \to T^\vee X$. This map extends to a map

$$\wedge^p \pi^\sharp : \wedge^p T_X^\vee \to \wedge^p T_X$$
such that the square

\[
\begin{array}{ccc}
\wedge^p T_X & \xrightarrow{\wedge^p \pi^\sharp} & \wedge^p T_X \\
\downarrow d & & \downarrow [\pi, -] \\
\wedge^{p+1} T_X & \xrightarrow{\wedge^{p+1} \pi^\sharp} & \wedge^{p+1} T_X
\end{array}
\]

commutes, where \([- , -]\) is the Schouten bracket. To see this, first note that \([\pi, -]\) has square 0: for \(a \in \Gamma(U, \wedge^p T X)\), we have \([\pi, [\pi, a]] = \frac{1}{2}[[\pi, \pi], a]\), but \([\pi, \pi] = 0\) is exactly equivalent to the Jacobi identity for \(\pi\). Additionally, \([\pi, -]\) is a derivation: \([\pi, ab] = [\pi, a]b + (-1)^a a [\pi, b]\). The claim holds almost by definition for \(p = 0\): for \(f \in \Gamma(U, \mathcal{O}_X)\), we have

\[[\pi, a] = \iota_d \pi = \pi^\sharp(df).

Assuming the claim for \(p - 1\), note that \(\Gamma(U, \wedge^p T\_X^{-1})\) is generated \(k\)-linearly by sections of the form \(f d\alpha\), for \(\alpha \in \Gamma(U, \wedge^{p-1} T\_X^{-1})\). Then

\[
\wedge^{p+1} \pi^\sharp(d(f d\alpha)) = \wedge^{p+1} \pi^\sharp(df \wedge d\alpha) = \pi^\sharp(df) \wedge (\wedge^p \pi^\sharp(d\alpha)),
\]

and

\[
[\pi, \wedge^p \pi^\sharp(f d\alpha)] = [\pi, f \wedge^p \pi^\sharp(d\alpha)]
\]

\[
= [\pi, f] \wedge [\pi, \wedge^{p-1} \pi^\sharp(\alpha)] + f[\pi, [\pi, \wedge^{p-1} \pi^\sharp(\alpha)]]
\]

\[
= \pi^\sharp(df) \wedge (\wedge^p \pi^\sharp(d\alpha)).
\]

Thus the map \(T^\vee X \to TX\) induces a morphism

\[
\text{Sym}^\bullet(\pi^\sharp[-1]) : (\text{Sym}^\bullet(T^\vee X[-1]), d) \to (\text{Sym}^\bullet(TX[-1]), [\pi, -])
\]

of graded mixed cdga.

We can then form the derived quotient \([X/\pi^\sharp]\). This is a formal stack equipped with a map \(q : X \to [X/\pi^\sharp]\). It satisfies the universal property that a map \(f : X \to F\) to a formal derived stack
\( F \) descends to \( \varphi : [X/\pi^2] \to F \) iff the map \( \text{Sym}^\bullet(\pi^2[1]) \) factors through

\[
\psi : (\text{Sym}^\bullet(L_{f,\text{big}}[-1]), d) \to (\text{Sym}^\bullet(TX[-1]), [\pi, -]),
\]

and a map \( \psi \) of mixed graded cdgas uniquely determines \( \varphi \).

The structure sheaf of this stack is \( (\text{Sym}^\bullet O_{X}(TX[-1]), [\pi, -]) \). Its tangent complex is

\[
\mathcal{T}_{[X/\pi^2]} \simeq \{ T^\vee X \xrightarrow{\pi^2} TX \},
\]

with \( TX \) sitting in degree 0. Looking at the 2-forms, we have

\[
\wedge^2 L_{[X/\pi^2]} \simeq \{ \wedge^2 T^\vee X \to T^\vee X \otimes TX \to \text{Sym}^2 TX \}.
\]

The degree 1 component, \( T^\vee X \otimes TX \), contains a canonical section \( \omega \) corresponding to the identity \( TX \to TX \). For this to define a 2-form we need \( d\omega = 0 \). To see this, note that the image of \( \omega \) via

\[
T^\vee X \otimes TX \to TX \otimes TX
\]

is precisely the bivector field \( \pi \); that this disappears in \( \text{Sym}^2 TX \) is precisely the fact that \( \pi \) is antisymmetric. Nondegeneracy is clear, as the map \( \mathcal{T}_{[X/\pi^2]} \to L_{[X/\pi^2]}[1] \) is literally the identity using the above representatives for \( \mathcal{T}_{[X/\pi^2]} \) and \( L_{[X/\pi^2]} \). For closedness, let

\[
\zeta \in (O_{[X/\pi^2]})_1 \otimes (L_{[X/\pi^2]})_0 \cong TX \otimes T^\vee X
\]

be the section corresponding to the identity on \( TX \). Then \( d_{\text{dR}}\zeta = \omega \), so we have \( d_{\text{dR}}\omega = 0 \). Thus we can take 0 as a closedness structure for \( \omega \). (Note that \( \zeta \) does not define a form on \([X/\pi^2]\), as generally \( d\zeta \neq 0 \); thus \( \omega \) is not necessarily exact.)

Thus \([X/\pi^2]\) has a canonical 1-shifted symplectic structure. Looking at \( q : X \to [X/\pi^2] \), we see that \( q^* \omega \) is a form of degree 1, so it is zero in \( \wedge^2 L_X \simeq \wedge^2 T^\vee X \). Thus \( q \) is isotropic with isotropic structure 0. Further, \( T_q \simeq T^\vee X \), and the induced map \( T_q \to L_X \) is the identity. So in fact, \( q \) has a Lagrangian structure. Finally, \( \pi^2_0 : T^\vee X \simeq T_q \to TX \) is exactly the map \( \pi^2 \).

Now consider the case of (2). The quasi-isomorphism \( T_q \to L_X \) gives us a map \( \pi^2 : T^\vee X \simeq T^\vee X \otimes \text{Sym}^\bullet L_X \otimes TX \to T^\vee X \).
\( T_q \to TX \). Using the fiber sequence

\[ T^\vee X \xrightarrow{\pi^\sharp} TX \longrightarrow q^* TY, \]

we have

\[ q^* TY \simeq \{ T^\vee X \xrightarrow{\pi^\sharp} TX \}, \]

with \( TX \) sitting in degree 0. Under this identification, we have

\[ q^*(\wedge^2 L_Y) \simeq \{ \wedge^2 T^\vee X \to T^\vee X \otimes TX \to \text{Sym}^2 TX \}, \]

with \( q^* \omega \) corresponding to the identity in \( T^\vee X \otimes TX \). As before, antisymmetry of \( \pi^\sharp \) is exactly the fact that \( d(q^* \omega) = 0 \). In particular, \( \pi^\sharp \) corresponds to a bivector field \( \pi \). For the Jacobi identity, look at the second infinitesimal neighborhood \( X_{q,2} \) of \( X \) along \( q \). Its structure sheaf is given by \( \mathcal{O}_{X_{q,2}} \simeq (\text{Sym}_{\mathcal{O}_X}^\leq 2 (TX[-1]), [\pi, -]) \). For this to be a dg-algebra, we need \([\pi, [\pi, -]] = \frac{1}{2}[[\pi, \pi], -] = 0\) on \( \mathcal{O}_X \), so \([\pi, \pi] = 0\), which is the Jacobi identity.

Note that the actual Poisson structure on \( X \) only depends on a formal neighborhood of \( X \) in \( Y \); in particular, all the relevant structures involve \( q^* L_Y \) and its various byproducts.

With this in mind, we define:

**Definition 3.2.** An \( n \)-shifted Poisson structure on \( X \) is \((Y, \omega, q, h)\), where \( Y \) is a formal derived stack with an \((n+1)\)-shifted symplectic structure \( \omega \in \text{Symp}(Y, n+1) \), and \( q : X \to Y \) is a map with Lagrangian structure \( h \). \( Y \) is called the Poisson base of \( X \).

An equivalence of Poisson structures \((Y, \omega, q, h) \to (Y', \omega', q', h')\) is a pair \((g, \gamma)\) consisting of a map \( g : (Y, q) \to (Y', q') \) (in the category of formal derived stacks under \( X \)), and a homotopy \( \gamma : q^* \omega \sim q'^* g^* \omega' \) in \([NC^2(\text{Sym}^\bullet q^* L_Y[1]])\) such that

\[ (q')^* L_{Y'} \simeq q'^* g^* L_{Y'} \to q'^* L_Y \]

is a quasi-isomorphism, and the image of \( \gamma \) in \( \mathcal{A}^{2\text{-cl}}(X, n) \) intertwines \( h \) and \( h' \).

**Remark 3.2.1.** As per the remark following Definition 2.5, if \( \bullet_{(n+1)} \) denotes the point with trivial \((n+1)\)-shifted symplectic structure, then a symplectic structure on a derived Artin stack \( X \) is
the same as a Lagrangian structure on the map $X \to \bullet_{(n+1)}$. Thus every symplectic structure is naturally Poisson.

**Remark 3.2.2.** Let $X$ be a smooth variety and $\pi = 0$ the zero Poisson structure. Then the space $[X/\pi] \ast$ constructed in the proof of Theorem 2 is the shifted cotangent space $T^\ast X[1]$.

For nonzero $\pi$, suppose the (classical) moduli space of symplectic leaves $Y$ is a derived Deligne-Mumford stack. Suppose there is a closed 2-form $\omega$ on $X$ whose pullback to any symplectic leaf is the form induced by $\pi$. Then the map $X \to T^\ast Y[1]$ with Lagrangian structure $\omega$ also defines the Poisson structure $\pi$; this map descends to $[X/\pi] \ast \to T^\ast Y[1]$, which gives an equivalence of Poisson structures.

Now consider a smooth variety $X$ with Poisson structure $\pi$, which we consider in terms of the 1-shifted symplectic structure $\omega$ on some $Y$ and the Lagrangian structure on $q : X \to Y$. Let us now characterize coisotropic subvarieties of $X$ in terms of the map $q$ with its Lagrangian structure.

**Theorem 3.3.** Let $X$ be a smooth variety. Let $Y$ be a formal derived stack with 1-shifted symplectic structure $\omega$, and let $q : X \to Y$ be a map with Lagrangian structure $h$. Let $\pi$ be the resulting Poisson structure.

1. Suppose that $W$ is a coisotropic subvariety of $X$, and let $s : W \to X$ be the inclusion. Then there exists a formal derived stack $X'$ and maps $s' : W \to X'$, $q' : X' \to Y$, such that $q'$ has a Lagrangian structure, $q \circ s = q' \circ s'$, and the induced map $a : W \to P := X' \times_Y X$ has a Lagrangian structure.

2. Conversely, say $s : W \to X$ is a subvariety, and suppose there exist a formal derived stack $X'$, maps $s' : W \to X'$, $q' : X' \to Y$, a Lagrangian structure on $q'$, a homotopy $q \circ s \sim q' \circ s'$, and a Lagrangian structure on $a : W \to P := X' \times_Y X$. Then $W$ is coisotropic in $X$.

**Proof.** For (1), let $s : W \to X$ be a coisotropic subvariety. That is, $W$ is also a smooth variety, and the Poisson structure restricted to the conormal bundle $N_{W|X}^\vee \to T^\vee X \to TX$ factors through the tangent space $TW$ of $W$. Let the adjoint of $N_{W|X}^\vee \to TW$ be $\pi_{W}^d : T^\vee W \to N_{W|X}$: one can show that the morphism of mixed graded cdgas induced by $\pi_{W}^d$ descends to $\pi_{W}^d$, so we have a formal quotient $X' := [W/\pi_{W}^d]$, with a projection $s' : W \to [W/\pi_{W}^d]$. From the universal property of $[W/\pi_{W}^d]$ there is a natural map $q' : [W/\pi_{W}^d] \to Y$ descending from $W \to X \to Y$.

We can write

$$T_{[W/\pi_{W}^d]} \simeq \{N_{W|X}^\vee \to TW\},$$
with $TW$ in degree 0; thus,

$$\Lambda^2 L_{[W/\pi^* W]} \simeq \{ \Lambda^2 T^v W \to T^v W \otimes N_{W/X} \to \text{Sym}^2 N_{W/X} \}.$$ 

I claim that $(q')^* \omega = 0$. To see this, recall that the pullback of $\omega$ to $(q')^* \Lambda^2 L_Y$ is the element of $T^v X \otimes TX$ corresponding to the identity on $TX$. But $\text{Hom}(TX, TX) \to \text{Hom}(TW, N_{W/X})$ sends the identity to the composition $TW \to TX \to N_{W/X}$, which is 0 by definition. Thus $q$ has isotropic structure 0. Further, we have

$$\mathbb{T}_q \simeq \{ T^v W \to N_{W/X} \},$$

with $T^v W$ sitting in degree 0. The map $\mathbb{T}_q \to \mathbb{L}_{[W/\pi^* W]}$ is clearly an isomorphism, so the isotropic structure on $q$ is Lagrangian. Then $P = [W/\pi^* W] \times_Y X$ is a Lagrangian intersection, so it has a 0-shifted symplectic structure. One can check that $T_P$ is an extension

$$0 \to TW \to T_P \to T^v W \to 0.$$

Let $a : W \to P$ be map induced by $s'$ and $s$; if $\omega_P$ is the symplectic form on $P$, then $a^* \omega_P = 0$, so $a$ is isotropic (with isotropic structure 0). However, we have $\mathbb{T}_a \simeq T^v W[-1]$, and $\mathbb{T}_a \to \mathbb{L}_W[-1]$ is the identity. So in fact $a : W \to P$ is Lagrangian.

For (2), let $pr_1 : P \to X'$ be the projection. Then we have an exact sequence

$$\mathbb{T}_a \to T_s \to a^* T_{pr_1}.$$ 

Using $\mathbb{T}_a \simeq \mathbb{L}_W[-1]$ from the Lagrangian structure, and $\mathbb{T}_{pr_1} \simeq pr_1^* \mathbb{T}_q \simeq pr_1^* T^v X$, we have

$$T^v W[-1] \to T_s \to s^* T^v X,$$

so that $T_s \simeq N^v_{W/X}$. Further, the diagram

$$\begin{array}{ccc}
\mathbb{T}_s & \longrightarrow & TW \\
\downarrow & & \downarrow \\
\, & & \\
s^* \mathbb{T}_q & \longrightarrow & s^* TX
\end{array}$$

commutes, that is, the Poisson map $N^v_{W/X} \to T^v X \to TX$ factors through $N^v_{W/X} \to TW$. So $W$ is
coisotropic in the usual sense.

This leads us to define coisotropic structures in general:

**Definition 3.4.** Let $X$ be a derived Artin stack with $n$-shifted Poisson structure given by $f : X \to Y$. Let $W$ be a derived Artin stack with a map $g : W \to X$. A *coisotropic structure* on $g$ consists of the following data:

- $X'$ a formal derived stack
- $f' : X' \to Y$
- $g' : W \to X'$
- An homotopy $\eta : f \circ g \simeq f' \circ g'$
- A Lagrangian structure $\alpha$ on $f'$

Note that the above data define a map $X' \to Y' \times_Y X$, and that $Y' \times_Y X$ has an $n$-shifted symplectic form by Theorem 2.6. We finally require:

- A Lagrangian structure $\beta$ on the map $a : X' \to Y' \times_Y X$. We refer to the map $f' : X' \to Y$ as the *coisotropic base* of $W \to X$.

Let $(X'_i, f'_i, g'_i, \eta_i, \alpha_i, \beta_i)$ be coisotropic structures for $i = 1, 2$. An equivalence of coisotropic structures is a pair $(h, \gamma)$, where $h : (X'_1, f'_1, g'_1, \eta'_1) \to (X'_2, f'_2, g'_2, \eta'_2)$ is a morphism in the appropriate slice category, and $\gamma : (g'_1)^*\alpha_1 \simeq (g'_2)^*h^*\alpha_2$ is a homotopy in $|NC^2(Sym^* (g'_1)^*\wedge_{X'_1} [1])|$, such that

$$(g'_2)^*\wedge_{X'_2} (g'_1)^*h^*\wedge_{X'_1} (g'_1)^*\wedge_{X'_1}$$

is a quasi-isomorphism, and if $\omega_i$ is the symplectic form on $X'_i \times_Y X$ for $i = 1, 2$, then the homotopy $a_1^*\omega_1 \simeq a_2^*\omega_2$ induced by $\gamma$ intertwines the homotopies $\beta_1, \beta_2$.

**Remark 3.4.1.** It is clear from the definition that if $X$ has an $n$-shifted symplectic structure, considered as an $n$-shifted Poisson structure via $X \to \bullet_{(n+1)}$, then any Lagrangian morphism $Y \to X$ is also coisotropic over $\bullet_{n} \to \bullet_{(n+1)}$.

Note, however, that the classical fact that a morphism which is both coisotropic and isotropic must be Lagrangian is not true. For example, the morphism $\mathbb{A}^1 \to \bullet_1$ is clearly isotropic (with
isotropic structure 0), and coisotropic over $T^\vee \mathbb{A}^1[1] \to \bullet_2$. However, a Lagrangian structure on $\mathbb{A}^1 \to \bullet_1$ would be a symplectic structure on $\mathbb{A}^1$ (in the classical sense), which clearly does not exist.

**Definition 3.5.** Let $X$ be a derived Artin stack with Poisson structure given by $\mathcal{P}_2 = (Y_2, \omega_2, f_2, h_2)$, and $g : W \to X$ with coisotropic structure $(X', f', g', \eta, \alpha, \beta)$. Let $\mathcal{P}_1 = (Y_1, \omega_1, f_1, h_1)$ be another Poisson structure and $(k, \gamma) : \mathcal{P}_1 \to \mathcal{P}_2$ an equivalence. The pullback of the coisotropic structure via $h$ is as follows. Let $\widetilde{Y}_1$ be a formal neighborhood of $X$ in $Y_1$.

- Let $X'_1 = X' \times_{Y_2} \widetilde{Y}_1$.
- The map $f'_1 : X'_1 \to Y_1$ is $X'_1 \to \widetilde{Y}_1 \to Y_1$.
- $g'_1 : W \to X'_1$ is induced by $W \to X'$ and $W \to X \to \widetilde{Y}_1$.
- $\eta_1 = id : f_1 \circ g \sim f'_1 \circ g'_1$.
- Let $pr : X'_1 \to X'$ be the projection. The homotopy $\gamma$ yields a homotopy

$$(f'_1)^* \omega_1 \sim (f'_1)^* k^* \omega_2 \sim pr^* (f')^* \omega_2.$$  

Then $\alpha_1$ is this homotopy followed by $pr^* \alpha$. The Lagrangian condition follows from the Lagrangian condition for $\alpha$, using the fact that $\widetilde{Y}_1 \to Y_2$ and $pr$ are etale.

- Let $\omega_2$ be the induced symplectic structure on $P_2 := X' \times_{Y_2} X$. Let $\omega_1$ be the induced symplectic structure on $P_1 := X'_1 \times_{Y_1} X$. Let $r : P_1 \to P_2$ be the natural map. The homotopy induced by $\gamma$ in the previous point also gives a homotopy $\omega_1 \sim r^* \omega_2$. Following this with $r^* \beta$ yields $\beta_1$. As in the previous point, the Lagrangian condition follows from etaleness of all relevant maps and Lagrangianness of $\beta$.

**Lemma 3.6.** Let $X_1, X_2, X_3, Y$ be derived Artin stacks, and let $\omega \in \text{Symp}(Y, n)$ be an $n$-shifted symplectic structure on $Y$. For $i = 1, 2, 3$, let $f_i : X_i \to Y$ be a morphism with Lagrangian structure $h_i$. Note that any product $X_i \times_Y X_j$ has a canonical $(n - 1)$-shifted symplectic structure. Let $g_{12} : L_{12} \to X_1 \times_Y X_2$ and $g_{23} : L_{23} \to X_2 \times_Y X_3$ be morphisms with Lagrangian structures $k_{12}, k_{23}$ respectively.

Then $g_{13} : L_{13} := L_{12} \times_{X_2} L_{23} \to X_1 \times_Y X_3$ has a canonical Lagrangian structure.

**Proof.** Let $\pi_1 : L_{12} \to X_1$ and $\pi_2 : L_{12} \to X_2$ be the projections, and let $\eta_{12} : f_1 \circ \pi_1 \to f_2 \circ \pi_2$ be the natural equivalence of morphisms. If $\omega_{12} \in \text{Symp}(X_1 \times_Y X_2, n - 1)$ is the symplectic form given
by Theorem 2.6, then \( g_{12}^{*} \omega_{12} = \pi_1^* h_1 + \eta_{12}^* \omega - \pi_2^* h_2 \). Then \( k_{12} \) gives a path from 0 to this form.

Similarly if \( \pi_2' : L_{23} \to X_2 \) and \( \pi_3 : L_{23} \to X_3 \) are the projections and \( \eta_{23} : f_2 \circ \pi_2' \to f_3 \circ \pi_3 \) the equivalence of morphisms, then \( k_{23} \) is a path from 0 to \( (\pi_2')^* h_2 + \eta_{23}^* \omega - \pi_3^* h_3 \) in \( A^{2,cl}(L_{23}, n - 1) \).

Now let \( \pi_a : L_{13} \to L_{12} \) and \( \pi_b : L_{13} \to L_{23} \) be the projections and \( \eta_{ab} : \pi_2 \circ \pi_a \to \pi_2' \circ \pi_b \) the natural equivalence. Then in \( A^{2,cl}(L_{13}, n - 1) \) we have paths

\[
\begin{align*}
\pi_a^* k_{12} : 0 & \sim \pi_a^* \pi_1^* h_1 + \pi_a^* \eta_{12}^* \omega - \pi_a^* \pi_2^* h_2 \\
\pi_b^* k_{23} : 0 & \sim \pi_b^* (\pi_2')^* h_2 + \pi_b^* \eta_{23}^* \omega - \pi_b^* \pi_3^* h_3 \\
\eta_{ab}^* h_2 : 0 & \sim \pi_a^* \pi_2^* h_2 + \eta_{ab}^* f_2^* \omega - \pi_b^* (\pi_2')^* h_2.
\end{align*}
\]

Composing these gives

\[
0 \sim \pi_a^* \pi_1^* h_1 + \eta_{13}^* \omega - \pi_b^* \pi_3^* h_3,
\]

(\*)

where

\[
\eta_{13} = (\pi_b \eta_{23}) \circ (\eta_{ab} f_2) \circ (\pi_a \eta_{12}) : \pi_b \circ \pi_3 \circ f_3 \to \pi_a \circ \pi_1 \circ f_1
\]

is the equivalence. If \( \omega_{13} \in \text{Symp}(X_1 \times_Y X_3, n - 1) \) is the symplectic form, then (\*) is exactly the isotropic structure \( 0 \sim g_{13}^{*} \omega_{13} \) we need.

For Lagrangianess, we use the diagram

\[
\begin{array}{ccc}
T_{L_{13}} & \longrightarrow & T_{L_{12}} \oplus T_{L_{23}} \\
\downarrow & & \downarrow \text{tr} \\
L_{g_{13}}[n-2] & \longrightarrow & (L_{g_{12}} \oplus L_{g_{23}})[n-2] \\
\downarrow & & \downarrow \text{tr} \\
& L_{f_2}[n-1] & / \rightarrow
\end{array}
\]

The rows are exact and two of the three vertical maps are quasi-isomorphisms, so the third is as
well.

Restated in Poisson language, the above is a generalization of Theorem 2.6:

**Corollary 3.7.** Let $X$ have an $n$-shifted Poisson structure given by $f : X \to Y$. For $i = 1, 2$, let $g_i : X'_i \to X$ be coisotropic with coisotropic base $h_i : Y'_i \to Y$. Then $X_1 \times_X X_2$ is $(n - 1)$-shifted Poisson with base $Y_1 \times_Y Y_2$.

Now let us generalize the situation of mapping spaces. It is relatively clear that Lagrangian structures descend to mapping spaces:

**Theorem 3.8** ([Ca], Theorem 2.10). Let $X, Y, Z$ be derived Artin stacks and $f : Y \to Z$ a map. Assume $X$ is $O$-compact with $d$-orientation $[X]$. Assume the stacks $\text{Map}(X, Y)$ and $\text{Map}(X, Z)$ are derived Artin stacks locally of finite presentation over $k$. Then we have a map

$$\int_{[X]} ev^*(-) : \text{Lagr}(f, \omega) \to \text{Lagr}(f \circ - , \int_{[X]} ev^*(\omega)),$$

that is, from Lagrangian structures on $f$ to Lagrangian structures on $(f \circ -)$.

Again, using the language of Poisson structures, we have

**Corollary 3.9.** Let $Y$ have an $n$-shifted Poisson structure with base $Z$. Let $X$ be $O$-compact with $d$-orientation $[X]$. Assume the stacks $\text{Map}(X, Y)$ and $\text{Map}(X, Z)$ are derived Artin stacks locally of finite presentation over $k$. Then $\text{Map}(X, Y)$ has an $(n - d)$-shifted Poisson structure with base $\text{Map}(X, Z)$.

We also have a variant of Theorem 3.8 to the coisotropic case:

**Theorem 3.10.** Let $Y$ be $n$-shifted Poisson with base $Z$, and let $g : Y' \to Y$ be coisotropic with base $h : Z' \to Z$. Let $X$ be $O$-compact with $d$-orientation $[X]$. Assume the stacks $\text{Map}(X, Y')$, $\text{Map}(X, Z)$, $\text{Map}(X, Y')$, and $\text{Map}(X, Z')$ are derived Artin stacks locally of finite presentation over $k$. Then $\text{Map}(X, Y') \to \text{Map}(X, Y)$ is coisotropic with base $\text{Map}(X, Z') \to \text{Map}(X, Z)$.

**Proof.** By Theorem 3.8, the maps $\text{Map}(X, Z') \to \text{Map}(X, Z)$, $\text{Map}(X, Y) \to \text{Map}(X, Z)$, and $\text{Map}(X, Y') \to \text{Map}(X, Z' \times_Z Y)$ have natural Lagrangian structures. But

$$\text{Map}(X, Z' \times_Z Y) \cong \text{Map}(X, Z') \times_{\text{Map}(X, Z)} \text{Map}(X, Y')$$
as symplectic spaces.

And similarly of Theorem 2.10:

**Theorem 3.11.** Let $Y$ be $n$-shifted Poisson given by $f : Y \to Z$ and Lagrangian structure $h : 0 \sim \omega$. Let $g : W \to X$ be a map of $\mathcal{O}$-compact derived Artin stacks, and let $[W]$ be a $d$-orientation on $W$ and $\gamma$ a boundary structure on $g$.

Then $\text{Map}(X,Y) \to \text{Map}(W,Y) \times_{\text{Map}(W,Z)} \text{Map}(X,Z)$ has a natural Lagrangian structure. Equivalently, $\text{Map}(X,Y) \to \text{Map}(W,Y)$ has a coisotropic structure over $\text{Map}(X,Z) \to \text{Map}(W,Z)$.

**Proof.** $\text{Map}(W,Z)$ has symplectic structure $\int_{[W]} \text{ev}_{W}^{*} \omega$. The Lagrangian structure on $\text{Map}(W,Y) \to \text{Map}(W,Z)$ is given by

$$
\int_{[W]} \text{ev}_{W}^{*} h : 0 \sim \int_{[W]} \text{ev}_{W}^{*} g^{*} \omega = (g \circ -)^{*} \int_{[W]} \text{ev}_{W}^{*} \omega.
$$

The Lagrangian structure on $\text{Map}(X,Z) \to \text{Map}(W,Z)$ is given by

$$
\int_{\gamma} \text{ev}_{X}^{*} \omega : 0 \sim \int_{f_{*}[W]} \text{ev}_{X}^{*} \omega = (- \circ f)^{*} \int_{[W]} \text{ev}_{W}^{*} \omega.
$$

Let $\tilde{\omega}$ be the induced symplectic structure on $\text{Map}(W,Y) \times_{\text{Map}(W,Z)} \text{Map}(X,Z)$, and let

$$
r : \text{Map}(X,Y) \to \text{Map}(W,Y) \times_{\text{Map}(W,Z)} \text{Map}(X,Z)
$$

be the natural map. Then

$$
r^{*} \tilde{\omega} = \int_{f_{*}[W]} \text{ev}_{X}^{*} h - \int_{\gamma} \text{ev}_{W}^{*} g^{*} \omega,
$$

and the isotropy is given by

$$
\int_{\gamma} \text{ev}_{X}^{*} h : 0 \sim r^{*} \tilde{\omega}.
$$

For the Lagrangian condition, fix a dga $A$ and $\sigma : \text{Spec } A \to \text{Map}(X,Y)$ corresponding to $\tilde{\sigma} : X \times \text{Spec } A \to Y$. Let $\pi_{2} : X \times \text{Spec } A \to \text{Spec } A$ be the projection. Then

$$
\sigma^{*} T_r \simeq (\pi_{2})_{*} \text{HoFib}(\tilde{\sigma}^{*} T_{g}) \to (f \times 1_{\text{Spec } A})_{*}((f \times 1_{\text{Spec } A})^{*} \tilde{\sigma}^{*} T_{g})
$$

and

$$
\sigma^{*} L_{\text{Map}(X,Y)} \simeq ((\pi_{2})_{*} \tilde{\sigma}^{*} T_{Y})^{\vee} = \text{Hom}((\pi_{2})_{*} \tilde{\sigma}^{*} T_{Y}, \mathcal{O}_{\text{Spec } A}).
$$
The map $\sigma^* T_r \to \sigma^* \mathbb{L}_{\text{Map}(X,Y)} [n - d - 1]$ is induced by the maps $T_g \to \mathbb{L}_Y [n]$, a quasi-isomorphism given by the Lagrangian structure $h$, and

$$(\pi_2)_* \text{HoFib}(\tilde{\sigma}^* \mathbb{L}_Y \to (f \times 1_{\text{Spec} A})_*(f \times 1_{\text{Spec} A})^* \tilde{\sigma}^* \mathbb{L}_Y) \to ((\pi_2)_* \tilde{\sigma}^* T_Y)^*[d - 1],$$

a quasi-isomorphism given by the nondegenerate boundary structure.

Specifically, we want to generalize the case of 2.12:

**Corollary 3.12.** Let $X$ be a geometrically connected smooth proper algebraic variety of dimension $d + 1$, and let $D$ be a smooth anticanonical effective divisor. Let $Y$ have an $n$-shifted Poisson structure given by $Y \to Z$. Assume $\text{Map}(X, Y)$, $\text{Map}(D, Y), \text{Map}(X, Z)$, and $\text{Map}(D, Z)$ are derived Artin stacks.

Then there exist a natural $(n - d)$-shifted Poisson structure on $\text{Map}(D, Y)$ (over $\text{Map}(D, Z)$) and coisotropic structure on $\text{Map}(X, Y) \to \text{Map}(D, Y)$ (over $\text{Map}(X, Z) \to \text{Map}(D, Z)$).

Finally, we need more technical “Poisson generalizations” of some results. The following is a generalization of the first statement of Theorem 3.11:

**Corollary 3.13.** Let $Y$ be $n$-shifted Poisson given by $f : Y \to Z$. Let $C \to Y$ be coisotropic over $Y' \to Z$. Let $g : W \to X$ be a map of $\mathcal{O}$-compact derived Artin stacks, and let $[W]$ be a $d$-orientation on $W$ and $\gamma$ a boundary structure on $g$.

Then $\text{Map}(X, C) \to \text{Map}(W, C) \times_{\text{Map}(W, Y)} \text{Map}(X, Y)$ has a natural coisotropic structure.

**Proof.** Recall that the Poisson structure on $\Psi := \text{Map}(W, C) \times_{\text{Map}(W, Y)} \text{Map}(X, Y)$ is given by

$$\Psi \to \Xi := \text{Map}(X, Z) \times_{\text{Map}(W, Z)} \text{Map}(W, Y'),$$

with a natural Lagrangian structure, as per Corollary 3.7. I claim that $\text{Map}(X, C) \to \Psi$ is coisotropic over $\text{Map}(X, Y') \to \Psi$. First, the Lagrangian structure on $\text{Map}(X, Y') \to \Psi$ is exactly the one given
by Theorem 3.11. Next, note that \( \text{Map}(X, Y') \times_{\Psi} \Xi \) is exactly the limit of the diagram

\[
\begin{array}{ccc}
\text{Map}(W, C) & \xrightarrow{} & \text{Map}(W, Y') \\
\downarrow & & \downarrow \\
\text{Map}(X, Y) & \xrightarrow{} & \text{Map}(W, Y) \\
\downarrow & & \downarrow \\
\text{Map}(X, Y') & \xrightarrow{} & \text{Map}(W, Y') \\
\downarrow & & \downarrow \\
\text{Map}(X, Z) & \xrightarrow{} & \text{Map}(W, Z)
\end{array}
\]

which we can rewrite as \( \text{Map}(X, Y \times_{Z} Y') \times_{\text{Map}(W, Y \times_{Z} Y')} \text{Map}(W, C) \). This space has a symplectic form arising from the form on \( Y \times_{Z} Y' \), which will agree with the structure on \( \text{Map}(X, Y') \times_{\Psi} \Xi \) up to sign. But the fact that

\[
\text{Map}(X, C) \rightarrow \text{Map}(X, Y \times_{Z} Y') \times_{\text{Map}(W, Y \times_{Z} Y')} \text{Map}(W, C)
\]

has a Lagrangian structure is precisely Theorem 3.11.

This is a generalization of Lemma 3.6:

**Corollary 3.14.** Let \( X_1, X_2, X_3, Y \) be derived Artin stacks, and let \( Y \) have an \( n \)-shifted Poisson structure given by \( Y \rightarrow Z \) for some \((n+1)\)-shifted symplectic \( Z \). For \( i = 1, 2, 3 \), let \( f_i : X_i \rightarrow Y \) be a morphism coisotropy over \( Y_i' \rightarrow Z \). Note that any product \( X_i \times_Y X_j \) has a canonical \((n-1)\)-shifted Poisson structure over \( Y_i' \times_Y Y_j' \). Let \( g_{12} : C_{12} \rightarrow X_1 \times_Y X_2 \) and \( g_{23} : C_{23} \rightarrow X_2 \times_Y X_3 \) be morphisms coisotropy over \( L_{12} \rightarrow Y_i' \times_Y Y_j' \) and \( L_{23} \rightarrow Y_2' \times_Y Y_3' \), respectively.

Then \( C_{13} := C_{12} \times_{X_2} C_{23} \rightarrow X_1 \times_Y X_3 \) has a canonical coisotropic structure over \( L_{12} \times_{Y_2'} L_{23} \rightarrow Y_1' \times_{Z} Y_3' \).

**Proof.** We need to show that

\[
C_{13} \rightarrow T := (X_1 \times_Y X_3) \times_{Y_1' \times_{Z} Y_3'} (L_{12} \times_{Y_2'} L_{23})
\]

has a Lagrangian structure. As in the previous proof, writing \( T \) as a limit gives us

\[
T \cong (X_1 \times_{Y_1'} L_{12}) \times_{Y \times_{Z} Y_2'} (X_3 \times_{Y_3'} L_{23}).
\]
The base is again symplectic, and the maps

\[ X_1 \times_{Y'_1} L_{12} \to Y \times_Z Y'_2 \leftarrow X_3 \times_{Y'_3} L_{23} \]

have Lagrangian structures provided by Lemma 3.6. Thus this expresses \( T \) as a Lagrangian intersection, which again has a symplectic structure that agrees with the original structure on \( T \) up to sign. Now, \( X_2 \to Y \times_Z Y'_2 \) has a Lagrangian structure by assumption, and further rearrangement gives a Lagrangian structure on

\[ C_{12} \to (X_1 \times_Y X_2) \times_{Y'_1 \times_Z Y'_2} L_{12} \cong (X_1 \times_{Y'_1} L_{12}) \times_{Y \times_Z Y'_2} X_2, \]

and similarly for \( C_{23} \).

But then we can apply Lemma 3.6 to get the Lagrangian structure on \( C_{13} \to T \).

\[ \square \]
Chapter 4

Framed Mapping Spaces

Definition 4.1. Let $D, X,$ and $Y$ be derived Artin stacks, and fix maps $i : D \to X$ and $f : D \to Y$. We define the framed mapping space $\text{Map}(X, D, Y, f) = \text{HoFib}_f(\text{Map}(X, Y) \to \text{Map}(D, Y))$, the homotopy fiber of $\text{Map}(X, Y)$ over $f \in \text{Map}(D, Y)$. Where $f$ is understood we will write $\text{Map}(X, D, Y)$.

In the following, $X$ will generally be a smooth scheme and $i : D \to X$ the inclusion of a divisor; or $X$ and $D$ will both be divisors in some smooth scheme.

Now, for any $g : X \to Y$ framed along $D$, let us consider $(\mathcal{T}_{\text{Map}(X, D, Y)})_g$. We have an exact sequence

\[
\begin{array}{c}
(\mathcal{T}_{\text{Map}(X, D, Y)})_g \to (\mathcal{T}_{\text{Map}(X, Y)})_g \\
\to (\mathcal{T}_{\text{Map}(D, Y)})(i \circ g) \\
\to \mathcal{R}\Gamma(X, g^*\mathcal{T}_Y) \\
\to \mathcal{R}\Gamma(D, i^*g^*\mathcal{T}_Y)
\end{array}
\]

so we can identify $(\mathcal{T}_{\text{Map}(X, D, Y)})_g \simeq \mathcal{R}\Gamma(X, (g^*\mathcal{T}_Y)_{-D})$, where $(g^*\mathcal{T}_Y)_{-D}$ is the subsheaf of $g^*\mathcal{T}_Y$ vanishing on $D$. In our cases we will be able to write $D = V(a)$ locally, so $(g^*\mathcal{T}_Y)_{-D} \simeq a(g^*\mathcal{T}_Y)$.

More globally, let $ev : X \times \text{Map}(X, D, Y) \to Y$ be the evaluation map and

\[
\pi : X \times \text{Map}(X, D, Y) \to \text{Map}(X, D, Y)
\]

the projection. Then $\mathcal{T}_{\text{Map}(X, D, Y)} \simeq \pi_*\left((ev^*\mathcal{T}_Y)_{-(D \times \text{Map}(X, D, Y))}\right)$.

For $p \geq 0$ we have a cup product map

\[
\wedge^p \mathcal{T}_{\text{Map}(X, Y)} \sim \wedge^p (\pi_*ev^*\mathcal{T}_Y) \to \pi_* \wedge^p (ev^*\mathcal{T}_Y).
\]
This induces a map
\[ \pi_*(\wedge^p ev^*L_Y) \to (\pi_*(\wedge^p ev^*T_Y))^\vee \to \wedge^p \mathbb{L}_{\text{Map}(X,Y)}. \]

This map is compatible with the mixed structure on both sides, so descends to the level of negative cyclic complexes:
\[ \pi_*ev^*(NC(Y)) \to NC(\text{Map}(X,Y)). \]

With this in mind, we define a special class of forms on \( \text{Map}(X,Y) \):

**Definition 4.2.** A \( p \)-form on \( \text{Map}(X,Y) \) (resp. closed \( p \)-form) is *multiplicative* if the corresponding map \( \mathcal{O}_{\text{Map}(X,Y)} \to \wedge^p \mathbb{L}_{\text{Map}(X,Y)} \) factors through \( \pi_*(\wedge^p ev^*\mathbb{L}_Y) \) (resp. factors through \( \pi_* ev^*(NC(Y)) \)).

Note that all forms obtained from the \( \int_X ev^*(-) \) map of Theorem 2.8 are multiplicative.

The importance of multiplicative forms is as follows. Suppose \( E_1, E_2 \to ev^*T_Y \) are two sheaves on \( \text{Map}(X,Y) \) which are orthogonal in the sense that the multiplication map \( E_1 \otimes E_2 \to \wedge^2 ev^*T_Y \) is 0. Then for any 2-form \( \omega \), we have a pullback via
\[ \wedge^2 \mathbb{L}_{\text{Map}(X,Y)} \to (\pi_* E_1)^\vee \otimes (\pi_* E_2)^\vee. \]

If \( \omega \) is multiplicative, then we can lift the pullback through \( \pi_*(\wedge^2 ev^*L_Y) \to \pi_*(E_1^\vee \otimes E_2^\vee) \), which is the 0 map. Thus the pullback will be 0.

We want to generalize the case of Theorem 2.8 and Corollary 3.9 to spaces \( \text{Map}(X,D,Y) \). The main theorem of this section is

**Theorem 4.3.** Let \( X \) be a \( d \)-dimensional proper smooth scheme and \( D \) an effective divisor. Suppose \( E \) is an effective divisor of \( X \) such that \( \bar{D} = 2D + E \) is anticanonical. Let \( Y \) be a derived Artin stack such that \( \text{Map}(X,Y) \), \( \text{Map}(\bar{D},Y) \), \( \text{Map}(D,Y) \), and \( \text{Map}(D + E,Y) \) are themselves derived Artin stacks of locally finite presentation over \( k \). Fix a base map \( f : D \to Y \).

1. Suppose \( Y \) is \( n \)-shifted symplectic and the projection \( \text{Map}(D + E,Y) \to \text{Map}(D,Y) \) is etale over \( f \). Then \( \text{Map}(X,D,Y) \) has an \((n - d)\)-shifted symplectic structure.

2. Suppose \( Y \) is \( n \)-shifted Poisson. Then \( \text{Map}(X,D,Y) \) has an \((n - d)\)-shifted Poisson structure.
Proof. Consider the fiber diagram

\[
\begin{array}{c}
\text{Map}(X, D, Y) \rightarrow \text{Map}(\tilde{D}, D, Y) \rightarrow \text{Map}(D, D, Y) \\
\downarrow \quad \downarrow \\
\text{Map}(X, Y) \rightarrow \text{Map}(\tilde{D}, Y) \rightarrow \text{Map}(D, Y)
\end{array}
\]

where both squares and the larger rectangle are Cartesian. Then Map(\(\tilde{D}, D, Y\)) is \((n - d + 1)\)-shifted symplectic (resp. Poisson) by Theorem 2.8 (Corollary 3.9), and Map(\(X, Y\) \(\rightarrow\) Map(\(\tilde{D}, D, Y\)) has a canonical Lagrangian structure (coisotropic structure) by Corollary 2.12 (Corollary 3.12). If we can show that Map(\(\tilde{D}, D, Y\) \(\rightarrow\) Map(\(\tilde{D}, Y\)) has a Lagrangian structure (coisotropic structure) as well, we will be done by Theorem 2.6 (Corollary 3.7). We state this as a separate lemma:

**Lemma 4.4.** Let \(X, Y, D, \tilde{D}\) be as in the theorem. Then

1. Suppose \(Y\) is \(n\)-shifted symplectic and the projection Map(\(D + E, Y\) \(\rightarrow\) Map(\(D, Y\)) is etale over \(f\). Then \(\varphi : \text{Map}(\tilde{D}, D, Y) \rightarrow \text{Map}(\tilde{D}, Y)\) has a canonical Lagrangian structure.

2. Suppose \(Y\) is \(n\)-shifted Poisson. Then \(\text{Map}(\tilde{D}, D, Y) \rightarrow \text{Map}(\tilde{D}, Y)\) has a canonical coisotropic structure.

**Proof.** Let \(i : D \rightarrow \tilde{D}\) be the inclusion, and let \(g : \tilde{D} \rightarrow Y\) be a map such that \(f = g \circ i\). Then \(T_g \text{Map}(\tilde{D}, D, Y) \simeq \mathbb{R}\Gamma(\tilde{D}, \text{HoFib}(g^*T_Y \rightarrow i_*i^*g^*T_Y)). \) Let us write this as \(T_g \text{Map}(\tilde{D}, D, Y) \simeq \mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-D}). \) Similarly, for any extension of \(f : D \rightarrow Y\) to \(\tilde{f} : D + E \rightarrow Y\), we have \(T_g \text{Map}(\tilde{D}, D + E, Y, \tilde{f}) \simeq \mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-(D + E)}). \)

Let us consider (1). The multiplication

\[
(g^*T_Y)_{-D} \otimes (g^*T_Y)_{-(D+E)} \rightarrow \wedge^2 g^*T_Y
\]

is zero; in an affine local patch of \(X\), if \(D = V(a)\) and \(E = V(b)\), then the map on sheaves is

\[
a(g^*T_Y) \otimes ab(g^*T_Y) \rightarrow \wedge^2 g^*T_Y
\]

and \(a^2b = 0\) on \(2D + E\).

Now, the symplectic structure on Map(\(\tilde{D}, Y\)) is a multiplicative form. Thus, \(\mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-D})\) and \(\mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-(D+E)})\) are orthogonal in \(T_g \text{Map}(\tilde{D}, Y) \simeq \mathbb{R}\Gamma(\tilde{D}, g^*T_Y)\) under this structure.
Thus the map
\[
\mathbb{R}\Gamma(D, (g^*T_Y)_{-D}) \to \mathbb{T}_g \text{Map}(\tilde{D}, Y) \to \mathbb{L}_g \text{Map}(\tilde{D}, Y)[n - d + 1]
\]
\[
\to \left(\tilde{D}, \mathbb{R}\Gamma((g^*T_Y)_{-(D+E)})\right)^\vee [n - d + 1]
\]
is 0, so
\[
\mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-D}) \to \mathbb{L}_g \text{Map}(\tilde{D}, Y)[n - d + 1] \simeq \left(\mathbb{R}\Gamma(g^*T_Y)\right)^\vee [n - d + 1]
\]
factors through a map
\[
\mathbb{R}\Gamma(\tilde{D}, (g^*T_Y)_{-D}) \to \left(\mathbb{R}\Gamma(\tilde{D}, g^*T_Y \otimes \mathcal{O}_{D+E})\right)^\vee [n - d + 1].
\] (*)

In fact, extending \( g \) to a map \( \tilde{g} : X \to Y \), consider the diagram

\[
\begin{array}{cccccc}
\tilde{g}^*T_Y(-2D - E) & \to & \tilde{g}^*T_Y(-2D - E) & \to & 0 \\
\downarrow & & \downarrow & & \\
\tilde{g}^*T_Y(-D) & \to & \tilde{g}^*T_Y & \to & \tilde{g}^*T_Y \otimes \mathcal{O}_D \\
\downarrow & & \downarrow & & \\
\tilde{g}^*T_Y(-D) \otimes \mathcal{O}_{D+E} & \to & \tilde{g}^*T_Y \otimes \mathcal{O}_{2D+E} & \to & \tilde{g}^*T_Y \otimes \mathcal{O}_D
\end{array}
\]

As all columns and the first two rows are distinguished triangles, so is the last row; restricting back to \( \tilde{D} \), we get a quasi-isomorphism
\[
(g^*T_Y)_{-D} \simeq g^*T_Y \otimes \mathcal{O}_{D+E}(-D) \simeq g^*T_Y \otimes j_*K_{D+E},
\]
where \( j : D + E \to \tilde{D} \) is the inclusion.

Then the map (*) can be rewritten as a map
\[
\mathbb{R}\Gamma(D + E, j^*g^*T_Y \otimes K_{D+E}) \to (\mathbb{R}\Gamma(D + E, j^*g^*T_Y))^\vee [n - d + 1].
\]
This is just the quasi-isomorphism

\[ \mathbb{R} \Gamma(D + E, j^* g^* T_Y \otimes K_{D+E}) \to \mathbb{R} \Gamma(D + E, j^* g^* L_{\mathcal{Y}} \otimes K_{D+E})[n] \]

given by the symplectic structure on \( Y \), followed by the Serre duality quasi-isomorphism

\[ \mathbb{R} \Gamma(D + E, (j^* g^* T_Y)^\vee \otimes K_{D+E})[n] \to \mathbb{R} \Gamma(D + E, j^* g^* T_Y)^\vee [n - d + 1]. \]

In particular, (*) is a quasi-isomorphism.

Now let us consider case (1). The etaleness assumption gives us that

\[ \mathbb{R} \Gamma(\tilde{D}, g^* T_Y \otimes O_{D+E}) \to \mathbb{R} \Gamma(\tilde{D}, g^* T_Y \otimes O_D) \]

is a quasi-isomorphism, and so

\[ \mathbb{R} \Gamma(\tilde{D}, (g^* T_Y)_{-(D+E)}) \to \mathbb{R} \Gamma(\tilde{D}, (g^* T_Y)_{-D}) \]

is as well. Thus the map

\[ \wedge^2 \mathbb{R} \Gamma(\tilde{D}, (g^* T_Y)_{-D}) \to \wedge^2 \mathbb{R} \Gamma(\tilde{D} g^* T_Y) \]

is 0, so 0 is an isotropic structure on \( \varphi \). In addition, the map \( T_{\varphi} \to L_{\text{Map}(\tilde{D}, D, Y)}[n - d] \) is precisely the map (*) shifted by 1. This is a quasi-isomorphism, so we have a Lagrangian structure on \( \varphi \).

Now consider case (2). Suppose \( Y \) has a Poisson structure given by \( p : Y \to Z \), where \( Z \) has an \((n + 1)\)-shifted symplectic structure \( \omega \) and \( p \) has a Lagrangian structure \( \gamma \). Recall that the \((n - d + 1)\)-shifted Poisson structure on \( \text{Map}(\tilde{D}, Y) \) is given by \( \text{Map}(\tilde{D}, Y) \to \text{Map}(\tilde{D}, Z) \), with the symplectic and Lagrangian structures induced from \( \omega \) and \( \gamma \). I claim that

\[ \text{Map}(\tilde{D}, D, Y) \to \text{Map}(\tilde{D}, Y) \]

is coisotropic. The base \( B \) will be the formal neighborhood of \( \text{Map}(\tilde{D}, D, Y) \) in \( \text{Map}(D + E, D, Y) \times_{\text{Map}(D+E, D, Z)} \text{Map}(\tilde{D}, D, Z) \), and \( q : B \to \text{Map}(\tilde{D}, D, Z) \) comes from the projection

\[ \text{Map}(D + E, D, Y) \times_{\text{Map}(D+E, D, Z)} \text{Map}(\tilde{D}, D, Z) \to \text{Map}(\tilde{D}, D, Z) \]
followed by \( \text{Map}(\tilde{D}, D, Z) \rightarrow \text{Map}(\tilde{D}, Z) \).

First let’s find a convenient representation of \( T_B \). Let \( g \in \text{Map}(\tilde{D}, Y) \). In the diagram

\[
\begin{array}{c}
\text{R} \Gamma(\tilde{D}, (g^*T_p)_{-(D+\epsilon)})  \\
\downarrow  \\
\text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})  \\
\downarrow  \\
(\mathcal{T}_B)_g  \\
\end{array}
\]

\[
\begin{array}{c}
\text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})  \\
\downarrow  \\
\text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D}) \oplus \text{R} \Gamma(D + E, (g^*T_Y)_{-D})  \\
\downarrow  \\
(\mathcal{T}_B)_g  \\
\end{array}
\]

the last two columns and all rows are triangles, so the first column is as well. Thus we have

\[
(\mathcal{T}_B)_g \simeq \text{HoCofib}(\text{R} \Gamma(\tilde{D}, (g^*T_p)_{-(D+\epsilon)}) \rightarrow \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})). \tag{**}
\]

For the Lagrangian structure on \( g \), let us identify \( g^*\Omega \), where \( \Omega \) is the symplectic structure on \( \text{Map}(\tilde{D}, Z) \). For \( \ell \geq 2 \), we have by (**):

\[
(\wedge^\ell L_B)_g \simeq \{ \wedge^\ell \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})^\vee \\
\rightarrow \text{R} \Gamma(\tilde{D}, (g^*T_p)_{-(D+\epsilon)})^\vee \otimes \wedge^{\ell-1} \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})^\vee \\
\rightarrow \cdots \}.
\]

(That is, the two are equivalent as dg-objects). Now, \( \Omega \) on \( \text{Map}(\tilde{D}, Z) \) is multiplicative, and pulling back a multiplicative form to

\[
\text{Sym}^s \text{R} \Gamma(\tilde{D}, (g^*T_p)_{-(D+\epsilon)})^\vee \otimes \wedge^{\ell-s} \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})^\vee
\]

with \( s > 0 \) yields 0. The weight \( \ell \) part of \( g^*\Omega \) corresponds to a map \( k \rightarrow (\wedge^\ell L_B)_g \), which in turn decomposes to a nonzero map \( k \rightarrow \wedge^\ell \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})^\vee \) and a 0 map to all later terms in the sequence. A homotopy from this to 0 is given by restricting the isotropic structure \( \int_{[\tilde{D}]} ev^* \gamma \) from \( \wedge^\ell \text{R} \Gamma(\tilde{D}, g^*T_Y)^\vee \) to \( \wedge^\ell \text{R} \Gamma(\tilde{D}, (g^*T_Y)_{-D})^\vee \), and taking the 0 homotopy on all later terms.
For the Lagrangian condition, using (**), consider the diagram

\[
\begin{array}{ccc}
\mathbb{R}\Gamma(\hat{D}, (g^*T_p) \otimes O_{D+E})[-1] & \to & \mathbb{R}\Gamma(\hat{D}, (g^*T_p) - (D+E)) \to \mathbb{R}\Gamma(\hat{D}, g^*T_p) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\Gamma(\hat{D}, (g^*T_Y) \otimes O_D)[-1] & \to & \mathbb{R}\Gamma(\hat{D}, (g^*T_Y) - D) \to \mathbb{R}\Gamma(\hat{D}, g^*T_Y) \\
(\mathbb{T}_q)_g & \to & (\mathbb{T}_B)_g \to \mathbb{R}\Gamma(\hat{D}, g^*p^*T_Z)
\end{array}
\]

The second two columns and all rows are triangles, so the first column is too. Thus we have

\[
(T_q)_g \simeq \text{HoCofib}(\mathbb{R}\Gamma(\hat{D}, (g^*T_p) \otimes O_{D+E})[-1] \to \mathbb{R}\Gamma(\hat{D}, (g^*T_Y) \otimes O_D)[-1]).
\]  (***)

Similarly to the map (*) above, we obtain a quasi-isomorphism

\[
\mathbb{R}\Gamma(\hat{D}, (g^*T_p) - D) \to \left(\mathbb{R}\Gamma(\hat{D}, g^*T_Y \otimes O_{D+E})\right)^\vee [n - d + 1],
\]

namely

\[
\mathbb{R}\Gamma(\hat{D}, (g^*T_p) - D) \simeq \mathbb{R}\Gamma(D + E, j^*g^*T_Y \otimes K_{D+E}) \to \mathbb{R}\Gamma(D + E, j^*g^*L_Y \otimes K_{D+E})[n],
\]

where \(T_p \to L_Y[n]\) is a quasi-isomorphism coming from the Lagrangian structure on \(p\), followed by the Serre duality quasi-isomorphism

\[
\mathbb{R}\Gamma(D + E, (j^*g^*T_Y)^\vee \otimes K_{D+E})[n] \to \mathbb{R}\Gamma(D + E, j^*g^*T_Y^\vee)[n - d + 1].
\]

Similarly, there is a natural quasi-isomorphism

\[
\mathbb{R}\Gamma(\hat{D}, (g^*T_p) - (D+E)) \to \left(\mathbb{R}\Gamma(\hat{D}, g^*T_Y \otimes O_D)\right)^\vee [n - d + 1].
\]
Then the map $\mathbb{T}_q \to \mathbb{L}_B[n - d]$ is given by the diagram

\[
\begin{array}{ccc}
(T_q)_g & \to & (\mathbb{L}_B)_g[n - d + 1] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p) \otimes \mathcal{O}_{D+E}) & \to & \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y)_{-D})^\vee[n - d + 1] \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y) \otimes \mathcal{O}_D) & \to & \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p)_{-(D+E)})^\vee[n - d + 1]
\end{array}
\]

the columns are triangles by (***) and (***). Then this map is a quasi-isomorphism, so the isotropic structure is Lagrangian.

Let $Q = \text{Map}(\tilde{D}, Y) \times_{\text{Map}(\tilde{D}, Z)} B$ be the product, and $r : \text{Map}(\tilde{D}, D, Y) \to Q$ the map. For any $g \in \text{Map}(\tilde{D}, D, Y)$, consider the diagram

\[
\begin{array}{ccc}
\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p) \otimes \mathcal{O}_{D+E})[-1] & \to & \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p)_{-(D+E)}) \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y)_{-D}) & \to & \mathbb{R}\Gamma(\tilde{D}, g^*\mathcal{T}_Y) \\
\downarrow & & \downarrow \\
(r^*\mathcal{T}_Q)_g & \to & \mathbb{R}\Gamma(\tilde{D}, g^*\mathcal{T}_Y) \oplus (r^*\pi_2^*\mathcal{T}_B) \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma(\tilde{D}, g^*\mathcal{T}_Z);
\end{array}
\]

again, everything but the first column is a triangle, so the first column is too.

Then

\[(r^*\mathcal{T}_Q)_g \simeq \text{HoCofib}(\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p) \otimes \mathcal{O}_{D+E})[-1] \to \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y)_{-D})).\]

The map

\[(T_{\text{Map}(\tilde{D}, D, Y)})_q \simeq \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y)_{-D}) \to (r^*\mathcal{T}_Q)_g\]

is precisely the structure morphism of the above cofiber. Letting $\omega_Q$ be the symplectic structure on $Q$, we get $r^*\omega_Q = 0$, so $r$ has 0 as isotropic structure. It is easy to check that $(T_r)_q \simeq \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p) \otimes \mathcal{O}_{D+E})[-2]$, and the map $(T_r)_q \to (L_{\text{Map}(\tilde{D}, D, Y)})_q[n - d - 1]$ is the quasi-isomorphism

\[\mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_p) \otimes \mathcal{O}_{D+E}) \to \mathbb{R}\Gamma(\tilde{D}, (g^*\mathcal{T}_Y)_{-D})^\vee[n - d + 1]\]

shifted by $-2$. \qed
**Remark 4.4.1.** Using similar methods, one can show that if $Y$ is (pre)symplectic, then $\text{Map}(X,D+E,Y)$ has a natural presymplectic structure.

Analogously to Theorems 3.8 and 3.10, we have:

**Theorem 4.5.** Let $X$ be a $d$-dimensional proper smooth scheme and $D$ an effective divisor. Suppose $E$ is an effective divisor of $X$ such that $\tilde{D} = 2D + E$ is anticanonical. Let $Y$ be a derived Artin stack such that $\text{Map}(X,Y)$, $\text{Map}(\tilde{D},Y)$, $\text{Map}(D,Y)$, and $\text{Map}(D+E,Y)$ are themselves derived Artin stacks of locally finite presentation over $k$. Fix a base map $f : D \to Y$. Let $W$ be a derived Artin stack and pick a map $s : W \to Y$.

1. Suppose $Y$ is $n$-shifted symplectic, that the projection $\text{Map}(D+E,Y) \to \text{Map}(D,Y)$ is étale over $f$, and that $\text{Map}(D+E,W) \to \text{Map}(D,W)$ is étale over any lift $\tilde{f}$ of $f$. Suppose $s : W \to Y$ has a Lagrangian structure. Then $\text{Map}(X,D,W) \to \text{Map}(X,D,Y)$ has a natural Lagrangian structure.

2. Suppose $Y$ is $n$-shifted Poisson, and $s : W \to Y$ has a coisotropic structure. Then $\text{Map}(X,D,W) \to \text{Map}(X,D,Y)$ has a natural coisotropic structure.

**Proof.** Similarly to the previous theorem, we use the fiber diagram

\[
\begin{array}{ccc}
\text{Map}(X,D,W) & \to & \text{Map}(\tilde{D},D,W) \\
\downarrow & & \downarrow \\
\text{Map}(X,D,Y) & \to & \text{Map}(\tilde{D},D,Y) \\
\downarrow & & \downarrow \\
\text{Map}(X,W) & \to & \text{Map}(\tilde{D},W) & \eta \\
\downarrow & & \downarrow \\
\text{Map}(X,Y) & \to & \text{Map}(\tilde{D},Y) & \zeta \\
\downarrow & & \downarrow \\
& \text{Map}(X,Y) & \to & \text{Map}(\tilde{D},W) & \theta
\end{array}
\]

The front and back faces are Cartesian squares. We have a Lagrangian (resp. coisotropic) structure on $\zeta$ by Corollary 2.12 (Corollary 3.12), on $\eta$ by Theorem 2.6 (Corollary 3.7), and on $\theta$ by Lemma 4.4. We also have a Lagrangian (coisotropic) structure on $\text{Map}(X,W) \to \text{Map}(X,Y) \times_{\text{Map}(\tilde{D},Y)} \text{Map}(\tilde{D},W)$.
by Theorem 3.11 (Corollary 3.13). If we show that

$$\text{Map}(\tilde{D}, D, W) \to \text{Map}(\tilde{D}, D, Y) \times_{\text{Map}(\tilde{D}, Y)} \text{Map}(\tilde{D}, W)$$

has a Lagrangian (coisotropic) structure, then we will be done by Lemma 3.6 (Corollary 3.14). As before, we put this in a separate lemma:

**Lemma 4.6.** Let $X, Y, W, D, \tilde{D}$ be as in the theorem. Then

1. Suppose $Y$ is $n$-shifted symplectic, that the projection $\text{Map}(D + E, Y) \to \text{Map}(D, Y)$ is étale over $f$, and that $\text{Map}(D + E, W) \to \text{Map}(D, W)$ is étale over any lift $\tilde{f}$ of $f$. Suppose $s : W \to Y$ has a Lagrangian structure. Then $r : \text{Map}(\tilde{D}, D, W) \to \text{Map}(\tilde{D}, D, Y) \times_{\text{Map}(\tilde{D}, Y)} \text{Map}(\tilde{D}, W)$ has a canonical Lagrangian structure.

2. Suppose $Y$ is $n$-shifted Poisson and that $s : W \to Y$ has a coisotropic structure. Then $r : \text{Map}(\tilde{D}, D, W) \to \text{Map}(\tilde{D}, D, Y) \times_{\text{Map}(\tilde{D}, Y)} \text{Map}(\tilde{D}, W)$ has a canonical coisotropic structure.

**Proof.** For (1), let $\gamma$ be the Lagrangian structure on $s$. If $\Omega$ is the induced symplectic structure on $\text{Map}(\tilde{D}, D, Y) \times_{\text{Map}(\tilde{D}, Y)} \text{Map}(\tilde{D}, W)$, one can check that $r^*\Omega = -\int_{[\tilde{D}]} e^*\gamma$. But this is a multiplicative form, so is already 0 on $$(\wedge^{\ell} T_{\text{Map}(\tilde{D}, D, W)})_g \simeq \wedge^{\ell} R\Gamma(\tilde{D}, (g^*T_W)_{-(D+E)} \odot O_D) \simeq \wedge^{\ell} R\Gamma(\tilde{D}, (g^*T_W)_{-(D+E)}), \quad (\ell \geq 2)$$

for any $g \in \text{Map}(\tilde{D}, D, W)$; here the second quasi-isomorphism comes from the étaleness condition. Thus 0 is an isotropic structure. For the Lagrangian condition, we have

$$(T_r)_g \simeq R\Gamma(\tilde{D}, g^*T_s \otimes O_D) \simeq R\Gamma(\tilde{D}, g^*T_s \otimes O_{D+E}),$$

and the map $T_r \to L_{\text{Map}(\tilde{D}, D, W)}[n-d]$ is the quasi-isomorphism

$$R\Gamma(D + E, g^*T_s) \to R\Gamma(D + E, g^*L_W)[n-1]$$

from the Lagrangian condition on $s$, followed by the Serre quasi-isomorphism

$$R\Gamma(D + E, g^*L_W) \to R\Gamma(D + E, g^*T_W \otimes K_{D+E})^\vee [1-d] \simeq R\Gamma(\tilde{D}, (g^*T_W)_{-(D+E)})^\vee [1-d].$$
as in the proof of the previous theorem.

For (2), let the Poisson structure on $Y$ be given by $p : Y \to Z$ with Lagrangian structure $\gamma$, and the coisotropic structure on $W \to Y$ given by $u : W \to Y \times_Z Y'$ with Lagrangian structure $\epsilon$, where $p' : Y' \to Z$ has Lagrangian structure $\gamma'$.

Letting $B$ be a formal neighborhood of $\text{Map}(\tilde{D}, D, Y)$ in $\text{Map}(D + E, D, Y) \times_{\text{Map}(D + E, D, Z)} \text{Map}(\tilde{D}, D, Z)$, recall that the coisotropic structure on $\text{Map}(\tilde{D}, D, Y) \to \text{Map}(\tilde{D}, Y)$ came from the map $B \to \text{Map}(\tilde{D}, Z)$. In our present case, the Poisson structure on $\Psi := \text{Map}(\tilde{D}, D, Y) \times_{\text{Map}(\tilde{D}, Y)} \text{Map}(\tilde{D}, W)$ comes from $
abla \to \Xi := B \times_{\text{Map}(\tilde{D}, Z)} \text{Map}(\tilde{D}, Y')$.

Let $B'$ be a formal neighborhood of $\text{Map}(\tilde{D}, D, W)$ in $\text{Map}(D + E, D, W) \times_{\text{Map}(D + E, D, Y') \text{Map}(\tilde{D}, D, Y') \to \text{Map}(D + E, D, Y) \times_{\text{Map}(D + E, D, Z)} \text{Map}(\tilde{D}, D, Z)$

give a map $q' : B' \to \Xi$. As in the previous theorem, one can give an isotropic structure on $q'$ basically arising from the isotropic structure on $\text{Map}(\tilde{D}, W) \to \text{Map}(\tilde{D}, Y \times_Z Y')$, and Lagrangianness comes from

\[
(T_{q'})_g \to (L_{B'})_g[n - d + 2] \to \Gamma(\tilde{D}, (g^*T_u) \otimes O_{D+E})^{-1} \to \Gamma(\tilde{D}, (g^*T_W) - D)^\vee[n - d + 2]
\]

Again similarly to the previous theorem, letting $\rho : \text{Map}(\tilde{D}, D, W) \to R := B' \times_{\Xi} \Psi$, if $\Omega_R$ is the induced structure on $R$, then $\rho^*\Omega_R$ is already 0, so 0 is an isotropic structure. For the Lagrangian condition, at any $g \in \text{Map}(\tilde{D}, D, W)$, the corresponding map $(T_{\rho})_g \to (L_{\text{Map}(\tilde{D}, D, W)})_g[n - d - 1]$ is just

\[
\Gamma(\tilde{D}, g^*T_u \otimes O_{D+E})[-1] \to \Gamma(\tilde{D}, (g^*T_W) - D)^\vee[n - d - 1],
\]
analogous to previous maps. This gives the coisotropic structure on $\text{Map}(\tilde{D}, D, W) \to \Psi$ we needed.

\[ \square \]

4.1 Framed Vector Bundles on Surfaces

As an application of these theorems, let $X$ be a smooth surface with effective anticanonical bundle, and take effective divisors $D$ and $E$ such that $2D + E$ is anticanonical. Let $G$ be a reductive group. Choose a map $D \to BG$, that is, a $G$-bundle $G \to D$. The space $\text{Map}(X, D, BG, G)$ has, by Theorem 4.3, a 0-shifted Poisson structure. This structure will be symplectic if $\text{Map}(D + E, BG) \to \text{Map}(D, BG)$ is etale over $G$. That is, if for every extension $\tilde{G} \to D + E$ of $G$, the map

$$H^*(D + E, ad(\tilde{G})) \to H^*(D.ad(G))$$

is an isomorphism in all degrees. Assuming the moduli space is a smooth variety (or looking at a semistable locus), this will be an ordinary Poisson or symplectic structure. Taking $\zeta \in H^0(X, E)$ to be a section vanishing on $E$, this is precisely Theorem 4.3 of [Bo].

In particular, let us consider the case where $X = \mathbb{P}^2$, $D = E$ is a line $L$, $G = SL_n$, and $G$ is the trivial bundle. The space $\text{Map}(\mathbb{P}^2, L, BSL_n, G)$ may be identified with the framed $SU(n)$-instantons on $S^4$ ([Do]). In this case, the only extension of $G$ to $2L$ is the trivial bundle again, and the failure of

$$H^*(2L, \mathfrak{sl}_n \otimes \mathcal{O}_{2L}) \to H^*(L, \mathfrak{sl}_n \otimes \mathcal{O}_L)$$

to be an isomorphism is given by

$$H^*(L, \mathfrak{sl}_n \otimes \mathcal{O}_L(-1)) = 0,$$

so we have a symplectic structure.
Chapter 5

Monopoles

Let $G$ be a semisimple complex Lie group and $B$ a Borel subgroup. Let $Y = G/B$ be the complete flag variety of $G$. Fix a point $p \in \mathbb{P}^1$. The space $\text{Map}(\mathbb{P}^1, p, Y)$ is the space of framed $G$-monopoles on $\mathbb{R}^3$ [Ja]. In [FKMM] the authors show that this space has a symplectic structure. More generally, let $P$ be a parabolic subgroup of $G$ and $Y = G/P$ the partial flag variety; then it is shown that $\text{Map}(\mathbb{P}^1, p, Y)$ has a Poisson structure. Here I will show that the Poisson and symplectic structures arise from the machinery of shifted structures on framed mapping spaces. In particular, they extend to framed maps which do not obey stability conditions.

5.1 Classical Construction of the Symplectic Structure

The following construction is described in [FKMM].

Let $g_Y = g \otimes \mathcal{O}_Y$ denote the trivial $g$-bundle on $Y$. Let $p_Y \subset g_Y$ be subbundle whose fiber over a parabolic $P$ is its Lie subalgebra $p \subset g$; similarly let $r_Y \subset p_Y$ be the subbundle whose fiber over $P$ is the nilpotent radical $r_P$. Let $r_Y = p_Y/r_Y$ be the bundle of abstract Levi algebras.

Recall that $TY$ is canonically isomorphic to $g_Y/p_Y$, and a $G$-invariant symmetric nondegenerate bilinear form on $g$ will give an isomorphism $T^\vee Y \cong r_Y$. In the Borel case, we note that $l_Y$ is trivial.

The Poisson structure on $\text{Map}(\mathbb{P}^1, p, Y)$ is defined as follows. First note that at any $f \in \text{Map}(\mathbb{P}^1, p, Y)$, we have

$$T_f \text{Map}(\mathbb{P}^1, p, Y) \cong H^0(f^*TY(-1), \mathbb{P}^1) \cong H^0(f^*(g/p)(-1), \mathbb{P}^1)$$
and

\[ T^\vee_I \text{Map}(\mathbb{P}^1, p, Y) \cong H^1(f^* T^\vee Y(-1), \mathbb{P}^1) \cong H^1(f^*(\tau Y)(-1), \mathbb{P}^1). \]

Now consider the complex

\[ \tau_Y \rightarrow g_Y \rightarrow g_Y/p_Y \]

on \( Y \). Pulling back by \( f \) and twisting by \(-1\) yields

\[ f^*(\tau_Y)(-1) \rightarrow f^*(g_Y)(-1) \rightarrow f^*(g_Y/p_Y)(-1). \]

Now we take the hypercohomology spectral sequence of this complex. At page 0, we get the sheaf cohomology of each of the sheaves. Since \( g_Y \) is trivial, \( f^*(g_Y)(-1) \) is a sum of \( O_{\mathbb{P}^1}(-1) \)s and its cohomology vanishes. Thus the \( d_1 \) differentials vanish, and at \( E_2 \) we get a differential

\[ d_2 : H^1(f^*(\tau_Y)(-1), \mathbb{P}^1) \rightarrow H^0(f^*(g_Y/p_Y)(-1), \mathbb{P}^1), \]

that is,

\[ T^\vee_I \text{Map}(\mathbb{P}^1, p, Y) \rightarrow T^\vee_I \text{Map}(\mathbb{P}^1, p, Y). \]

This is the Poisson structure. Verifying that this really is a Poisson structure is done by a complicated explicit calculation in [FKMM], or follows conceptually from Theorem 5.2.

**Remark 5.0.1.** Assuming this really is a Poisson structure, let us show nondegeneracy for the case \( P = B \). The complex

\[ f^*(\tau_Y)(-1) \rightarrow f^*(g_Y)(-1) \rightarrow f^*(g_Y/p_Y)(-1) \]

is quasi-isomorphic to

\[ 0 \rightarrow f^*(l_Y)(-1) \rightarrow 0, \]

which has zero hypercohomology, as \( l_Y \) is trivial. Thus in particular the differential

\[ d_2 : H^1(f^*(\tau_Y)(-1), \mathbb{P}^1) \rightarrow H^0(f^*(g_Y/p_Y)(-1), \mathbb{P}^1) \]

must be an isomorphism.
5.2 Construction via Shifted Poisson Structures

In this section we describe a construction of a Poisson structure on $\text{Map}(\mathbb{P}^1, p, G/P)$ using the machinery of shifted Poisson structures.

Our first hope might be to find a 1-shifted Poisson structure on $Y = G/P$ and use Theorem 4.3 to induce a structure on $\text{Map}(\mathbb{P}^1, p, Y)$. This should make us suspicious in the $P = B$ case, as we would expect a 1-shifted symplectic structure on $G/B$, which would yield a quasi-isomorphism between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{r}[1]$, which is clearly impossible.

In the general case, we can also show we can’t get our Poisson structure this way. Suppose $Z$ is 2-shifted symplectic, and $g : Y \to Z$ has a Lagrangian structure defining a Poisson structure on $Y$. Using the Lagrangian condition, we get $T_g \simeq L_Y[1] \cong \mathfrak{r}[1]$, and in particular the map $T_g \to T_Y$ is the zero map. For any $f \in \text{Map}(\mathbb{P}^1, p, Y)$, the map $T_f^\vee \text{Map}(\mathbb{P}^1, p, Y) \to T_f \text{Map}(\mathbb{P}^1, p, Y)$ will just be the map

$$H^1(\mathbb{P}^1, f^*(\mathfrak{r})(-1)) \simeq \mathbb{R}\Gamma(\mathbb{P}^1, f^*(T_g)(-1)) \to \mathbb{R}\Gamma(\mathbb{P}^1, f^*(T_Y)(-1)) \simeq H^0(\mathbb{P}^1, f^*(\mathfrak{g}/\mathfrak{p})(-1)),$$

so will also be zero, which we do not want.

Instead, we note that $G/P$ is already related to an existing shifted symplectic stack, $BG$, via the fiber diagram

$$\begin{array}{ccc}
G/P & \longrightarrow & BP \\
\downarrow & & \downarrow \\
\bullet & \longrightarrow & BG
\end{array} \quad (*)
$$

Recall that the symplectic structure on $BG$ is given by a $G$-invariant nondegenerate symmetric quadratic form on $\mathfrak{g}$. Fix such a form $\omega$.

Choose an opposite parabolic $P^-$ so that $P \cap P^- = L$ is a Levi subgroup of $G$. Letting $I = \text{Lie}(L)$, we can then write $\mathfrak{g} = \mathfrak{r}^- \oplus I \oplus \mathfrak{r}$. Since $I$ is orthogonal to $\mathfrak{r}$ and $\mathfrak{r}^-$, $\omega$ descends to $I$ and is also $L$-invariant and nondegenerate. Thus, $BL$ has a symplectic structure $\omega_L$ induced from $BG$. Recall that the identification $L \cong P/\text{rad}(P)$ gives us a map $P \to L$.

**Lemma 5.1.** The map $\iota : BP \to BG \times BL$ has a Lagrangian structure given by 0. Thus $BP \to BG$ has a coisotropic structure.

*Proof.* The claim that 0 is an isotropic structure reduces to the claim that $\iota : p \to \mathfrak{g} \oplus I$ is isotropic.
in the usual sense with respect to \( \omega - \omega_L \). Write \( p = l \oplus r \) and recall that \( r \) is orthogonal to itself and \( l \). Then
\[
(\omega - \omega_L)(\iota_*(\ell, r), \iota_*(\ell', r')) = \omega(\ell, \ell') - \omega(\ell, \ell') = 0,
\]
so we have isotropy.

For the Lagrangian condition, recall that \( \omega \) pairs up \( r \) nondegenerately with \( r \). Let \( \Delta, \overline{\Delta} : l \to g \oplus l \) be the diagonal and antidiagonal maps, and note that \( \omega - \omega_L \) pairs up \( \Delta(l) \) and \( \overline{\Delta}(l) \) nondegenerately. Then the map \( \mathbb{T}_l \to \mathbb{T}_{BP}[1] \) is just the adjoint \( \Delta(l) \oplus r \to (\Delta(l) \oplus r)' \). □

So \( BP \to BG \) has a coisotropic structure, and if \( \bullet \to BG \) had one too, we would get a 1-shifted Poisson structure on \( G/P \) by Corollary 3.7. As mentioned, there is no decent shifted Poisson structure on \( G/P \) and it is also easy to check that \( \bullet \to BG \) has no coisotropic structure. Instead, we apply the functor \( \text{Map}(\mathbb{P}^1, p, -) \) to \( * \) to get
\[
\begin{array}{ccc}
\text{Map}(\mathbb{P}^1, p, G/P) & \longrightarrow & \text{Map}(\mathbb{P}^1, p, BP) \\
\downarrow & & \downarrow \\
\text{Map}(\mathbb{P}^1, p, \bullet) & \longrightarrow & \text{Map}(\mathbb{P}^1, p, BG)
\end{array}
\]
and note that now \( \text{Map}(\mathbb{P}^1, p, BG) \) is 1-symplectic and \( \text{Map}(\mathbb{P}^1, p, BP) \to \text{Map}(\mathbb{P}^1, p, BG) \) is coisotropic by Theorem 4.5. Let \( G_{\mathbb{P}^1} \) denote the trivial \( G \)-bundle on \( \mathbb{P}^1 \), framed at \( p \). Then the map
\[
\bullet = \text{Map}(\mathbb{P}^1, p, \bullet) \to \text{Map}(\mathbb{P}^1, p, BG)
\]
is just the point \( G_{\mathbb{P}^1} \). And this is coisotropic, as
\[
T_{G_{\mathbb{P}^1}} \text{Map}(\mathbb{P}^1, p, BG) = \mathcal{E}xt^*(\mathcal{O}_{\mathbb{P}^1} \otimes g, \mathcal{O}_{\mathbb{P}^1} \otimes g)(-1)[1] \cong 0.
\]
Then the map \( \bullet \to \text{Map}(\mathbb{P}^1, p, BG) \) is trivially Lagrangian, hence coisotropic. Then Corollary 3.7 gives a 0-shifted Poisson structure on \( \text{Map}(\mathbb{P}^1, p, G/P) \).

To be a little more specific, the coisotropic bases are the maps \( \text{Map}(\mathbb{P}^1, p, BL) \to \bullet \) and \( \bullet \to \bullet \) respectively, so the Poisson structure comes from a map \( \text{Map}(\mathbb{P}^1, p, G/P) \to \text{Map}(\mathbb{P}^1, p, BL) \). In particular, in the case \( P \) is a Borel, \( L \) is a torus, and so \( \text{Map}(\mathbb{P}^1, p, BL) = \prod_{i=1}^r \text{Map}(\mathbb{P}^1, p, BG_m) \), where \( r \) is the rank of \( G \). But \( \text{Map}(\mathbb{P}^1, p, BG_m) \cong \mathbb{Z} \), so \( \text{Map}(\mathbb{P}^1, p, BL) \) is a disjoint union of points \( \bullet_1 \) with the trivial 1-symplectic structure. Thus, \( \text{Map}(\mathbb{P}^1, p, G/P) \) is in fact symplectic.
Theorem 5.2. The Poisson structures described above coincide.

Proof. Let us look at the construction using shifted Poisson structures first. Recall that, for $X$ an undervived smooth scheme with Poisson structure $g: X \to Z$, we recover the map $T^* X \to T_X$ by using the Lagrangian structure on $g$ to yield an isomorphism $T_g \cong L_X$; with respect to this isomorphism, the Poisson structure is given by $L_X \cong T g \to T X$. In the case $X = \text{Map}(\mathbb{P}^1, p, G/P)$, let’s look at the map $g: X \to Z = \text{Map}(\mathbb{P}^1, p, BL)$. For $f \in X$, the tangent map $g_*: T_f X \to g^*(TZ)_f$ is the composition

$$\mathbb{R} \Gamma(\mathbb{P}^1, f^*(g_Y/p_Y)(-1)) \xrightarrow{\partial} \mathbb{R} \Gamma(\mathbb{P}^1, f^*(p_Y)(-1))[1] \xrightarrow{\pi} \mathbb{R} \Gamma(\mathbb{P}^1, f^*(l_Y)(-1))[1]$$

Here $\partial$ is the connecting map coming from the short exact sequence

$$0 \to f^*(p_Y)(-1) \to f^*(g_Y)(-1) \to f^*(g_Y/p_Y)(-1) \to 0;$$

since $f^*(g_Y)(-1)$ is acyclic, $\partial$ is a quasi-isomorphism. Then from the sequence

$$0 \to f^*(r_Y)(-1) \to f^*(p_Y)(-1) \to f^*(l_Y)(-1) \to 0$$

we see that the fiber of $\pi$ (and thus of $g_*$) is $\mathbb{R} \Gamma(\mathbb{P}^1, f^*(r_Y)(-1))[1] \simeq H^1(\mathbb{P}^1, f^*(l_Y)(-1))$. This is identified with $T^*_f X$ in the canonical way, and the Poisson map is then

$$H^1(\mathbb{P}^1, f^*(r_Y)(-1)) \to H^1(\mathbb{P}^1, f^*(p_Y)(-1)) \cong H^0(\mathbb{P}^1, f^*(g_Y/p_Y(-1))).$$

For the spectral sequence, consider the map of complexes

$$\tau_Y \longrightarrow g_Y \longrightarrow g_Y/p_Y.$$
Pull back by $f$, twist by $-1$, and look at the induced map on the $E_2$ page:

$$
\begin{array}{ccc}
H^1(\mathbb{P}^1, f^*(r_Y)(-1)) & \rightarrow & H^0(\mathbb{P}^1, f^*(g_Y/p_Y)(-1)) \\
\downarrow & & \downarrow \\
H^1(\mathbb{P}^1, f^*(p_Y)(-1)) & \rightarrow & H^0(\mathbb{P}^1, f^*(g_Y/p_Y)(-1))
\end{array}
$$

The bottom map is the inverse to the connecting isomorphism

$$H^0(\mathbb{P}^1, f^*(g_Y/p_Y)(-1)) \rightarrow H^1(\mathbb{P}^1, f^*(p_Y)(-1))$$

from the long exact sequence. Thus the Poisson map is exactly the composition

$$H^1(\mathbb{P}^1, f^*(r_Y)(-1)) \rightarrow H^1(\mathbb{P}^1, f^*(p_Y)(-1)) \cong H^0(\mathbb{P}^1, f^*(g_Y/p_Y)(-1))$$

as above.

**Remark 5.2.1.** In either case, much of the “real work” of the Poisson structure lies in the identification

$$(\mathbb{P}^1, H^0(f^*(g_Y/p_Y)(-1)))^\vee \cong H^1(\mathbb{P}^1, f^*(r_Y)(-1))$$

arising from Serre duality and the isomorphism $(g_Y/p_Y)^\vee \cong r_Y$.

**Remark 5.2.2.** Given that $G/P$ doesn’t have a 1-shifted Poisson structure, what structure does it have? Recall that an $n$-shifted symplectic structure on $X$ is equivalent to a map $X \rightarrow \bullet_{n+1}$ with a Lagrangian structure. Generalizing this by replacing $\bullet_{n+1}$ with an arbitrary $(n+1)$-shifted symplectic derived stack $Z$ yields the notion of an $n$-shifted Poisson structure.

However, there is another generalization we can make: if $X \rightarrow \bullet_{n+1}$ only has an isotropic structure, we get an $n$-shifted presymplectic structure on $X$. Combining these two, we might say an $n$-shifted “pre-Poisson” structure on $X$ is an $(n+1)$-shifted symplectic derived stack $Z$ and a map $X \rightarrow Z$ with an isotropic structure. This is the structure $G/P$ has; specifically, $G/P \rightarrow BL$ has an isotropic structure.
5.3 Other Fibers

In the fiber square

\[
\begin{array}{c}
\text{Map}(\mathbb{P}^1, p, G/P) \quad \rightarrow \\
\downarrow \\
\bullet \quad \rightarrow \\
\downarrow \\
\text{Map}(\mathbb{P}^1, p, BG)
\end{array}
\]

the base point in Map(\mathbb{P}^1, p, BG) is the trivial G-bundle on \mathbb{P}^1 framed at p. This gives us Map(\mathbb{P}^1, p, G/P) as the fiber, but is also necessary to get a Poisson structure: for general \mathcal{G} \in \text{Map}(\mathbb{P}^1, p, BG), the corresponding \bullet \rightarrow \text{Map}(\mathbb{P}^1, p, BG) is not Lagrangian or even coisotropic.

Let's be a little more precise. Let aut(\mathcal{G}) denote the vector bundle whose fiber at \( x \) is \text{Lie}(\text{Aut}(\mathcal{G}_x)), and let aut_p(\mathcal{G}) be the sheaf of sections of aut(\mathcal{G}) vanishing at p. Then \( T_{\mathcal{G}} \simeq \mathbb{R}\Gamma(\mathbb{P}^1, aut_p(\mathcal{G}))[1] \). The automorphisms of \( \mathcal{G} \) within Map(\mathbb{P}^1, p, BG) are Aut_p(\mathcal{G}), the group of automorphisms of \( \mathcal{G} \) over \mathbb{P}^1 which are the identity above p. Then the corresponding map

\[
B(\text{Aut}_p(\mathcal{G})) \rightarrow \text{Map}(\mathbb{P}^1, p, BG)
\]

will be an isomorphism on \( T \) in degree \(-1\), and will be Lagrangian (with Lagrangian structure 0).

Then applying Corollary 3.7, we get

**Theorem 5.3.** The product \( B(\text{Aut}_p(\mathcal{G})) \times_{\text{Map}(\mathbb{P}^1, p, BG)} \text{Map}(\mathbb{P}^1, p, BP) \) will have a Poisson structure. This structure is symplectic if \( P = B \).

The space \( B(\text{Aut}_p(\mathcal{G})) \times_{\text{Map}(\mathbb{P}^1, p, BG)} \text{Map}(\mathbb{P}^1, p, BP) \) can be described as follows. \( G \) acts on \( \mathcal{G} \) from the right and \( G/P \) on the left, so we can form the balanced product \( \mathcal{G} \times_G G/P \), a \( G/P \)-bundle over \mathbb{P}^1. Note that \( \text{Aut}_p(\mathcal{G}) \) still acts on \( \mathcal{G} \times_G G/P \). Let \( \Gamma(\mathbb{P}^1, \mathcal{G} \times_G G/P(-1)) \) denote the sections of \( \mathcal{G} \times_G G/P \) sending p to a specified point of \( (\mathcal{G} \times_G G/P)_x \). Then

\[
B(\text{Aut}_p(\mathcal{G})) \times_{\text{Map}(\mathbb{P}^1, p, BG)} \text{Map}(\mathbb{P}^1, p, BP) \cong \Gamma(\mathbb{P}^1, \mathcal{G} \times_G G/P(-1))/\text{Aut}_p(\mathcal{G}).
\]
Bibliography


