Closed Type Families With Overlapping Equations (Extended Version)

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Abstract
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Keywords
Type families, Type-level computation, Haskell, System FC

Disciplines
Computer Engineering | Computer Sciences

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Abstract
Open, type-level functions are a recent innovation in Haskell that move Haskell towards the expressiveness of dependent types, while retaining the look and feel of a practical programming language. This paper shows how to increase expressiveness still further, by adding closed type functions whose equations may overlap, and may have non-linear patterns over an open type universe. Although practically useful and simple to implement, these features go beyond conventional dependent type theory in some respects, and have a subtle metatheory.

Categories and Subject Descriptors F.3.3 [Logics and Meanings of Programs]: Studies of Program Constructs—type structure; D.3.3 [Programming Languages]: Language Constructs and Features; F.4.2 [Mathematical Logic and Formal Languages]: Grammars and Other Rewriting Systems—parallel rewriting systems

General Terms Design, Languages, Theory

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1. Introduction
Type families are a relatively recent extension to Haskell that allows the programmer to express type-level computation (Chakravarty et al. 2005). For example, one can say

\[
\text{type family } \text{Elt} \, \langle a :: * \rangle :: * \\
\text{type instance } \text{Elt} \, \text{ByteString} = \text{Word8} \\
\text{type instance } \text{Elt} \, [b] = b
\]

The first line declares the type family \(\text{Elt}\) and gives its kind; the second and third are two independent declarations that give two equations for \(\text{Elt}\). Now the types \(\langle \text{Elt} \, \text{ByteString} \rangle\) and \(\text{Word8}\) are considered equivalent by the type inference engine, and likewise \(\langle \text{Elt} \, \text{[Int]} \rangle\) and \(\text{Int}\). Type families have proved to be a popular feature in Haskell, dovetailing particularly nicely with Haskell’s type classes. Type families are naturally partial and open. For example, there is no equation for \(\text{Elt} \, \text{Char}\) above, so \(\text{Elt} \, \text{Char}\) will never be equal to any other type. On the other hand, the author of a new library is free to add a new instance, such as this one:

\[
\text{type instance } \text{Elt} \, \langle \text{Set} \, b \rangle = b
\]

However, not all type-level functions can be defined by open type families. An important example is the equality function, which determines whether two types can be shown equal at compile-time:

\[
\text{type family } \text{Equal} \, a \, b :: \text{Bool} \\
\text{type instance } \text{Equal} \, a \, a = \text{True} \quad -- \text{Instance (A)} \\
\text{type instance } \text{Equal} \, a \, b = \text{False} \quad -- \text{Instance (B)}
\]

The programmer intends these equations to be read top-to-bottom, like a term-level function definition in Haskell. However, because GHC’s current type families are open, they must be defined by independent, un-ordered \textit{type instance} equations. The two equations overlap, so they are rightly rejected lest they be used to deduce unsound type equalities. For example, we could reduce the type \(\text{Equal} \, \text{Int} \, \text{Int}\) to both \text{True} and \text{False}, since both patterns match. Yet equality is a well-defined function, and a useful one too, as we discuss in Section 2. To fix this omission we introduce \textit{closed} type families with ordered equations, thus:

\[
\text{type family } \text{Equal} \, a \, b :: \text{Bool where} \\
\text{Equal} \, a \, a = \text{True} \\
\text{Equal} \, a \, b = \text{False}
\]

Now all the equations for the type family are given together, and can be read top-to-bottom. However, behind this simple idea lie a number of complexities. In this paper we describe these pitfalls and their sometimes non-obvious solutions. We make the following contributions:

\begin{itemize}
  \item We introduce closed type families with overlapping equations, and show how they can readily express programs that were previously inexpressible or required indirect encodings (Section 2).
  \item Our system supports non-linear left-hand sides, such as that for \(\text{Equal}\) above, where the variable \(a\) is repeated in the first equation. It also supports coincident overlap, which allows some lightweight theorem-proving capability to be incorporated in the definitional equality of types (Section 3.4).
  \item We give the subtle rules that govern type family simplification, including those that determine when a pattern cannot be matched by a type (Section 3).
  \item We describe a typed core language that includes both open and closed type families (Section 4), and prove that it is type-safe, assuming that type families terminate (Section 5). We do that by establishing a \textit{consistency} property of the type equations induced by type families.
  \item We identify the complications for consistency that arise from non-terminating type families and we expose a subtle oversight in GHC’s current rules for open type families in Section 6.
  \item We have implemented closed type families in GHC as well as a number of case studies, such as the units package, an extensible framework for dimensional analysis, presented in Appendix A. Closed type families are available now in GHC 7.8.
\end{itemize}
In short, the programmer sees a simple, intuitive language feature, but the design space (and its metatheory) is subtle. Although type families resemble the type-level computation and “large eliminations” found in full-spectrum dependently-typed languages like Coq and Agda, there are important semantic and practical differences. We discuss these in Section 8.

2. Closed type families

Haskell (in its implementation in GHC) has supported type families for several years. They were introduced to support associated types, a feature that Garcia et al.’s (2003) comparison between C++, Haskell, and ML, noted as a C++’s main superiority for generic programming.

Type families were designed to dovetail smoothly with type classes. For example, the type function\(^2\) \(\text{Elt}\) above could be used to specify the element type in a container class:

```haskell
class Container c where
  empty :: c
  member :: Elt c → c → Bool
instance Container [a] where ...
instance Container ByteString where ...
```

New instances for `Container` can be defined as new types are introduced, often in different modules, and correspondingly new equations for `Elt` must be added too. Hence `Elt` must be open (that is, can be extended in modules that import it), and distributed (can be scattered over many different modules). This contrasts with term-level functions where we are required to define the function all in one place.

The open, distributed nature of type families, typically associated with classes, requires strong restrictions on overlap to maintain soundness. Consider

```haskell
!type family \(F\ a\ b\ :: \star\)
!type instance \(F\ \text{Int}\ a\ =\ \text{Bool}\)
!type instance \(F\ a\ \text{Bool}\ =\ \text{Char}\)
```

Now consider the type \((F\ \text{Int}\ \text{Bool})\). Using the first equation, this type is equal to \(\text{Bool}\), but using the second it is equal to \(\text{Char}\). So if we are not careful, we could pass a \(\text{Bool}\) to a function expecting a \(\text{Char}\), which would be embarrassing.

GHC therefore brutally insists that the left-hand sides of two type instance equations must not overlap (unify). (At least, unless the right-hand sides would then coincide; see Section 3.4.)

2.1 Closed families: the basic idea

As we saw in the Introduction, disallowing overlap means that useful, well-defined type-level functions, such as type level equality, cannot be expressed. Since openness is the root of the overlap problem, it can be solved by defining the equations for the type family \(\text{all in one place}\.\) We call this a closed type family and define it using a `where` clause on the function’s original declaration. The equations may overlap, and are matched top-to-bottom. For example:

```haskell
!type family \(\text{And}\ (a\ ::\ \text{Bool})\ (b\ ::\ \text{Bool})\ ::\ \text{Bool}\ where
  \text{And}\ True\ True\ =\ True
  \text{And} a\ b\ =\ False
```

Since the domain of `And` is closed and finite, it is natural to write all its equations in one place. Doing so directly expresses the fact that no further equations are expected.

Although we have used overlap in this example, one can always write functions over finite domains without overlap:

```haskell
!type family \(\text{And’}\ (a\ ::\ \text{Bool})\ (b\ ::\ \text{Bool})\ ::\ \text{Bool}\ where
  \text{And’} True True = True
  \text{And’} False True = False
  \text{And’} True False = False
  \text{And’} False False = False
```

Nevertheless, overlap is convenient for the programmer, mirrors what happens at the term level, avoids a polynomial blowup in program size, and is more efficient (for the type checker) to execute. Furthermore, when defined over an open kind, such as \(\star\), closed type families allow a programmer to express relationships (such as inequality of types—see Section 2.4) that are otherwise out of reach.

2.2 Non-linear patterns

Let us return to our equality function, which can now be defined thus:

```haskell
!type family \(\text{Equal}\ (a\ ::\ \star)\ (b\ ::\ \star)\ ::\ \text{Bool}\ where
  \text{Equal} a a = True
  \text{Equal} a b = False
```

This declaration introduces the type function \(\text{Equal}\), gives its kind and, in the `where` clause, specifies all its equations. The first equation has a non-linear pattern, in which \(a\) is repeated, and it overlaps with the second equation. If the domain were finite we could avoid both features by writing out all the equations exhaustively, but new types can be introduced at any time, so we cannot do that here.

The issue becomes even clearer when we use kind polymorphism (Yorgey et al. 2012), thus:

```haskell
!type family \(\text{Equal}\ (a\ ::\ \kappa)\ (b\ ::\ \kappa)\ ::\ \text{Bool}\ where
  \text{Equal} a a = True
  \text{Equal} a b = False
```

For example, \((\text{Equal}\ \text{Maybe}\ \text{List})\) should evaluate to \(\text{False}\). It may seem unusual to define a function to compute equality even over types of function kind \((\star\rightarrow\bigstar)\). After all, there is no construct that can compare functions at the term level.

At the type level, however, the type checker decides equality at function kinds all the time! In the world of Haskell types there exist no anonymous type-level functions, nor can type families appear partially applied, so this equality test—which checks for definitional equality, in type theory jargon—is straightforward. All \(\text{Equal}\) does is reify the (non-extensional) equality test of the type checker.

In fact, Haskell programmers are used to this kind of equality matching on types; for example, even in Haskell 98 one can write

```haskell
instance \(\text{Num}\ a\Rightarrow\text{Num}\ (T\ a\ a)\) where ...
```

Because the type inference engine already supports decidable equality, it is very straightforward to implement non-linear patterns for type functions as well as type classes. Non-linear patterns are convenient for the programmer, expected by Haskell users, and add useful expressiveness. They do make the metatheory much harder, as we shall see, but that is a problem that has to be solved only once.

2.3 Type structure matching

In our experience, most cases where closed type families with overlapping equations are useful involve a variation on type equality. However, sometimes we would like to determine whether a type matches a specific top-level structure.

For example, we might want to look at a function type of the form \(\text{Int} \\rightarrow (\text{Bool} \\rightarrow \text{Char}) \\rightarrow \text{Int} \\rightarrow \text{Bool}\) and determine that this is a function of three arguments.
Because the equations are tried in order, any function type will trigger the first equation and any ground non-function type (that is, a type that is not a type variable or an arrow type) will trigger the second. Thus, the type family effectively counts the number of parameters a function requires.

When might this be useful? We have used this type family to write a variable-arity zipWith function that infers the correct arity, assuming that the result type is not a function type. Other approaches that we are aware of (Fridlender and Indrika 2000; McBride 2002; Weirich and Casinghino 2010) require some encoding of the desired arity to be passed explicitly. A full presentation of this extension is given in Appendix B. To achieve the same functionality in a typical dependently typed language like Agda or Coq, we must pattern-match over some inductive universe of codes that can be interpreted into types.

2.4 Observing inequality

Type families such as \( \text{Equal} \) allow programmers to observe when types do not match. In other words, \( \text{Equal} \ a \ b \) automatically reduces to \( \text{False} \), via the second equation. With open type families, we could only add a finite number of reductions of un-equal types to \( \text{False} \).

However, the ability to observe inequality is extremely useful for expressing failure in compile-time search algorithms. This search could be a simple linear search, such as finding an element in a list. Such search underlies the HList library and its encoding of heterogeneous lists and extensible records (Kiselyov et al. 2004). It also supports Swierstra’s solution to the expression problem via extensible datatypes (Swierstra 2008). Both of these proposals use the extension \(-\text{XOverlappingInstances}\) to implement a compile-time equality function.¹

Type families can directly encode more sophisticated search algorithms than linear list searching, including those requiring backtracking, simply by writing a functional program. For example, the following closed type family determines whether a given element is present in a tree.

\[
\begin{align*}
\text{data} & \quad \text{Tree} \ a = \ \text{Leaf} \ a \mid \ \text{Branch} \ (\text{Tree} \ a) \ (\text{Tree} \ a) \\
\text{type family} & \quad \text{TMember} \ (e :: \ \kappa) \ (\text{set} :: \ \text{Tree} \ \kappa) :: \ \text{Bool} \ \text{where} \\
& \quad \text{TMember} \ e \ (\text{Leaf} \ x) \ = \ \text{Equal} \ e \ x \\
& \quad \text{TMember} \ e \ (\text{Branch} \ \text{left} \ \text{right}) \ = \\
& \quad \text{Or} \ (\text{TMember} \ e \ \text{left}) \ (\text{TMember} \ e \ \text{right})
\end{align*}
\]

Implementing this search using overlapping type classes, which do not support backtracking, requires an intricate encoding with explicit stack manipulation.

2.5 Summary

Type-level computation is a powerful idea: it allows a programmer to express application-specific compile-time reasoning in the type system. Closed type families fill in a missing piece in the design space, making type families more expressive, convenient, and more uniform with term-level functional programming.

3. Simplifying closed family applications

We have shown in the previous sections how type family reduction can be used to equate types. For example, a function requiring an argument of type \( T \ True \) can take an argument of type \( T \ (\text{And} \ True \ True) \), because the latter reduces to the former.

Because the definition of type equality is determined by type family reduction, the static semantics must precisely define what reductions are allowed to occur. That definition turns out to be quite subtle, so this section develops an increasingly refined notion of type family reduction, motivated by a series of examples. The presentation gives a number of definitions, using the vocabulary of Figure 1, but we eschew full formality until Section 4. We use the term \"target\" to designate the type-function application that we are trying to simplify. We say that a type \( \tau_1 \ \text{\"simplifies\"} \) or \"reduces\" to another type \( \tau_2 \) if we can rewrite the \( \tau_1 \) to \( \tau_2 \) using a (potentially empty) sequence of left-to-right applications of type family equations. We also use the notation \( \tau_1 \leadsto \tau_2 \) to denote exactly one application of a type family equation and \( \tau_1 \leadsto^* \tau_2 \) to denote an arbitrary number of reductions. Type equality is defined to be roughly the reflexive, symmetric, transitive, congruent closure of type reduction; details are in Section 4.3.

We frequently refer to the example in the introduction, repeated below, with the variables renamed to aid in understanding:

\[
\begin{align*}
\text{type family} & \quad \text{Equal} \ (a :: \ \kappa) \ (b :: \ \kappa) :: \ \text{Bool} \ \text{where} \\
& \quad \text{Equal} \ a \ = \ \text{True} \quad -- \ \text{Eqn} \ (A) \\
& \quad \text{Equal} \ b \ c \ = \ \text{False} \quad -- \ \text{Eqn} \ (B)
\end{align*}
\]

3.1 No functions on the LHS

If we wish to simplify \( \text{Equal} \ a \ b \), equation (A) of the definition matches, so we can safely \"fire\" the equation (A) to simplify the application to \( \text{True} \).

Even here we must take a little care. What happens if try this?

\[
\begin{align*}
\text{type family} & \quad F \ (a :: \ \text{Bool}) \ \text{where} \\
& \quad F \ False \ = \ \text{False} \\
& \quad F \ True \ = \ \text{True} \\
& \quad F \ (\text{Equal} \ x \ y) \ = \ \text{True}
\end{align*}
\]

Then \( F \ (\text{Equal} \ Int \ Bool) \) superficially appears to match only the third equation. But of course, if we simplify the argument of \( F \) in the target, it would become \( F \ False \), which matches the first equation.

The solution here is quite standard: in type family definitions (both open and closed) we do not allow functions in the argument types on the LHS. In terms of Figure 1, the LHS of a function axiom must be a pattern \( \rho \). This is directly analogous to allowing only constructor patterns in term-level function definitions, and is already required for Haskell’s existing open type families.

We then propose the following first attempt at a reduction strategy:

\[\]
\[\]

³This extension allows class instances, but not type family instances, to overlap. If the type inference engine chooses the wrong class instance, a program may have incoherent behavior, but it is believed that type safety is not compromised. See Morris and Jones (2010) for relevant discussion.
Candidate Rule 1 (Closed type family simplification). An equation for a closed type family $F$ can be used to simplify a target $(F \, \tau)$ if (a) the target matches the LHS of the equation, and (b) no LHS of an earlier equation for $F$ matches the target.

The formal definition of matching follows:

Definition 1 (Matching). A pattern $\rho$ matches a type $\tau$, written $\text{match}(\rho, \tau)$, when there is a well-kindled substitution $\Omega$ such that $\Omega(\rho) = \tau$. The domain of $\Omega$ must be a subset of the set of free variables of the pattern $\rho$.

3.2 Avoiding premature matches with apartness

Suppose we want to simplify $\text{Equal Bool} \, d$. Equation (A) above fails to match, but (B) matches with a substitution $\Omega = \{ b \mapsto \text{Bool}, \_ \mapsto d \}$. But it would be a mistake to simplify $\text{Equal Bool} \, d$ to $\text{False}$. Consider the following code:

```plaintext
type family FunIf (b :: Bool) :: * where
  FunIf True = Int \rightarrow Int
  FunIf False = ()
bad :: d \rightarrow FunIf (Equal Bool d)
bad _ = ()
segFault :: Int
segFault = bad True 5
```

If we do simplify the type $\text{Equal Bool} \, d$ to $\text{False}$ then we can show that $\text{bad}$ is well typed, since $\text{FunIf False}$ is $(\)$. But then $\text{segFault}$ calls $\text{bad}$ with $d$ instantiated to $\text{Bool}$. So $\text{segFault}$ expects $\text{bad True}$ to return a result of type $\text{FunIf}$, which reduces to $\text{Int} \rightarrow \text{Int}$, so the call in $\text{segFault}$ type-checks too: Result: we apply $(\)$ as a function to $5$, and crash.

To rule out our counterexample to type soundness, apartness is a substitution $\rho$ such that $\tau \rightsquigarrow \tau'$: $\text{match}(\rho, \tau')$.

3.3 A definition of apartness

We have so far sketched necessary properties that the apartness check must satisfy—otherwise, our type system surely is not sound. We have also described why a simple unification-based test does not meet these conditions, but we have not yet given a concrete definition of this check.

Note that we cannot use Property 4 to define $\text{apart}(\rho, \tau)$ because it would not be well founded. We need $\text{apart}(\rho, \tau)$ to define how type families should reduce, but Property 4 itself refers to type family reduction. Furthermore, even if this were acceptable, it seems hard to implement. We have to ensure that, for any substitution, no reducts of a target can possibly match a pattern; there can be exponentially many reducts in the size of the type and the substitution.

Hence we seek a conservative but cheap test. Let us consider again why unification is not sufficient. In the example from the previous section, we showed that type $\text{Equal Int} (\text{G Bool})$ does not match equation (A), nor does it unify with (A). However, $\text{Equal Int} (\text{G Bool})$ can simplify to $\text{Equal Int} \, \text{Int}$ and now equation (A) does match the reduct.

To take the behavior of type families into account, we first flatten any type family applications in the arguments of the target (i.e., the types $\tau$ in a target $F \, \tau$) to fresh variables. Only then do we check that the new target is not unifiable with the pattern. This captures the notion that a type family can potentially reduce to any type—anything more refined would require advance knowledge of all type families, impossible in a modular system. In our example, we must check $\text{apart}((a, a), (\text{Int}, \text{G Bool}))$ when trying to use the second equation of $\text{Equal}$ to simplify $\text{Equal Int} (\text{G Bool})$. We first flatten $\text{(Int, G Bool)}$ into $\text{(Int, x)}$ (for some fresh variable $x$). Then we check whether $(a, a)$ cannot be unified with $(\text{Int}, x)$. We quickly discover that these types can be unified. Thus, $(a, a)$ and $(\text{Int, G Bool})$ are not apart and simplifying $\text{Equal Int} (\text{G Bool})$ to $\text{False}$ is prohibited.

What if two type family applications in the target type are syntactically identical? Consider the type family $F$ below:

```plaintext
type family F a b where
  F Int Bool = Char
  F a a = Bool
```

Should the type $F (G \, \text{Int}) (G \, \text{Int})$ be apart from the left-hand-side $F \, \text{Int Bool}$? If we flatten to two distinct type variables then it is not apart; if we flatten using a common type variable then it becomes apart. How can we choose if flattening should preserve sharing or not? Let us consider the type $F \, b \, b$, which matches

---

Footnote 4 For instance, the implementation of unify can be the standard first-order unification algorithm of Robinson.
the second equation. It is definitely apart from $F \text{ Int } \text{ Bool}$ and can indeed be simplified by the second equation. What happens, though, if we substitute $G \text{ Int}$ for $b$ in $F b b'$? If flattening did not take sharing into account, $(G \text{ Int}, G \text{ Int})$ would not be apart from $(\text{Int}, \text{Bool})$, and $F (G \text{ Int})$ $(G \text{ Int})$ wouldn’t reduce. Hence, the ability to simplify would not be stable under substitution. This, in turn, threatens the preservation theorem.

Thus, we must identify repeated type family applications and flatten these to the same variable. In this way, $F (G \text{ Int}) (G \text{ Int})$ is flattened to $F x x$ (never $F x y$), will be apart from the first equation, and will be able to simplify to $\text{Bool}$, as desired.

With these considerations in mind, we can now give our implementation of the apartness check:

**Definition 5** (Flattening). To flatten a type $\tau$ into $\tau'$, written $\tau' = \text{flatten}(\tau)$, process the type $\tau$ in a top-down fashion, replacing every type family application with a type variable. Two or more syntactically identical type family applications are flattened to the same variable; distinct type family applications are flattened to distinct fresh variables.

**Definition 6** (Apartness). To test for apart $(\rho, \tau)$, let $\tau' = \text{flatten}(\tau)$ and check $\text{unify}(\rho, \tau')$. If this unification fails, then $\rho$ and $\tau$ are apart. More succinctly: apart $(\rho, \tau) \equiv \neg \text{unify}(\rho, \text{flatten}(\tau))$.

We can show that this definition does indeed satisfy the identified necessary properties from Section 3.2. In Section 5.1 we will also identify the sufficient conditions for type soundness for any possible type-safe implementation of apartness, show that these conditions imply the properties identified in the previous section (a useful sanity check!) and prove that the definition of apartness that we just proposed meets these sufficient conditions.

3.4 Allowing more reductions with compatibility

Checking for apartness in previous equations might be unnecessarily restrictive. Consider this code, which uses the function $\text{And}$ from Section 2.1:

$$
f :: T a \rightarrow T b \rightarrow T (\text{And } a \ b)$$
$$tt :: T \text{ True}$$
$$g :: T a \rightarrow T a$$
$$g x \equiv f \ x \ tt$$

Will the definition of $g$ type-check? Alas no: the call $(f \ x \ tt)$ returns a result of type $T (\text{And } a \ True)$, and that matches neither of the equations for $\text{And}$. Perhaps we can fix this by adding an equation to the definition of $\text{And}$, thus:

$$\text{type family And } (a :: \text{Bool}) (b :: \text{Bool}) :: \text{Bool} \text{ where}$$
$$\text{And True True } = \text{True } -- (1)$$
$$\text{And a True } = a -- (2)$$
$$\text{And a b } = \text{False } -- (3)$$

But that does not work either: the target ($\text{And } a \ True$) matches $2$ but is not apart from $1$, so $2$ cannot fire. And yet we would like to be able to simplify ($\text{And } a \ True$) to $a$, as Eqn (2) suggests. Why should this be sound? Because anything that matches both $1$ and $2$ will reduce to $\text{True}$ using either equation. We say that the two equations coincide on these arguments. When such a coincidence happens, the apartness check is not needed.

We can easily formalize this intuition. Let us say that two equations are compatible when any type that matches both left-hand sides would be rewritten by both equations to the same result, eliminating non-convergent critical pairs in the induced rewriting system:

**Definition 7** (Compatibility). Two type-family equations $p$ and $q$ are compatible iff $\Omega_1(lhs_p) = \Omega_2(lhs_q)$ implies $\Omega_1(rhs_p) = \Omega_2(rhs_q)$.

For example, $(1) \ and \ (2)$ are compatible because a type, such as $\text{And True True}$, would be rewritten by both to the same type, namely $\text{True}$. It is easy to test for compatibility:

**Definition 8** (Compatibility implementation). The test for compatibility, written $\text{compat}(p, q)$, checks that $\text{unify}(\text{lhs}_p, \text{lhs}_q) = \Omega \implies \Omega(\text{rhs}_p) = \Omega(\text{rhs}_q)$. If $\text{unify}(\text{lhs}_p, \text{lhs}_q)$ fails, $\text{compat}(p, q)$ holds vacuously.

The proof that $\text{compat}(p, q)$ implies that $p$ and $q$ are compatible appears in Appendix G and is straightforward. We can now state our final simplification rule for closed type families:

**Rule 9** (Closed type family simplification). An equation $q$ of a closed type family can be used to simplify a target application $F \tau$ if the following conditions hold:

1. The target $\tau$ matches the type pattern $\text{lhs}_q$.
2. For each earlier equation $p$, either $\text{compat}(p, q)$ or $\text{apart}(\text{lhs}_p, \tau)$.

For example, we can fire equation $(2)$ on a target that is not apart from $(1)$, because $(1)$ and $(2)$ are compatible. We show that Rule 9 is sufficient for establishing type soundness in Section 5.

Through this use of compatibility, we allow for a limited form of theorem proving within a closed type family definition. The fact that equation $(2)$ is compatible with $(1)$ essentially means that the rewrite rule for $(2)$ is admissible given that for $(1)$. By being able to write such equations in the closed type family definition, we can expand Haskell’s definitional equality to relate more types.

3.5 Optimized matching

In our original Candidate Rule 2 above, when simplifying a target $F \tau$ with an equation $q$, we are obliged to check $\text{apart}(\text{lhs}_p, \tau)$, for every earlier equation $p$. But much of this checking is wasted duplication. For example, consider

$$\text{type family F a where}$$
$$F \text{ Int} = \text{Char } -- (1)$$
$$F \text{ Bool} = \text{Bool } -- (2)$$
$$F x \ = \text{Int } -- (3)$$

If a target matches $(2)$ there is really no point in checking its apartness from $(1)$, because anything that matches $(2)$ will be apart from $(1)$. We need only check that the target is apart from any preceding equations that could possibly match the same target.

Happily, this intuition is already embodied in our new simplification rule Rule 9. This rule checks $\text{compat}(p, q) \lor \text{apart}(\text{lhs}_p, \tau)$ for each preceding equation $p$. But we can precompute $\text{compat}(p, q)$ (since it is independent of the target), and in the simplification rule we need check apartness only for the pre-computed list of earlier incompatible equations. In our example, equations $(1)$ and $(2)$ are vacuously compatible, since their left-hand sides do not unify, and hence no type can match both. Thus, there is no need to check for apartness from $(1)$ of a target matching $(2)$.

3.6 Compatibility for open families

As discussed in the introduction, type instance declarations for open type families must not overlap. With our definition of compatibility, however, we can treat open and closed families more uniformly by insisting that any two instances of the same open type family are compatible:

**Definition 10** (Open type family overlap check). Every pair of equations $p$ and $q$ for an open type family $F$ must satisfy $\text{compat}(p, q)$.

Notice that this definition also allows for coincident right-hand sides (as in the case for closed type families, Section 3.4). For example, these declarations are legal:
Expressions:
\[ e ::= x \mid \lambda x: \tau. e \mid e_1 \ e_2 \mid \Lambda \alpha: \kappa. e \mid e \tau \mid e \gg \gamma \mid \ldots \]

Types:
\[ \tau, \sigma ::= \alpha \mid \tau_1 \rightarrow \tau_2 \mid \forall \alpha: \kappa. \tau \]
\[ \psi, \nu ::= \gamma \mid \tau_1 \sim \tau_2 \]

Propositions:
\[ \phi ::= \gamma \sim \tau_2 \]

Coercions:
\[ \psi, \nu ::= \gamma \sim \tau_2 \]

Contexts:
\[ \Sigma ::= \cdot \mid \Sigma_1 \rightarrow \tau_2 \mid \Sigma, \tau \mid \Sigma \]

Variables:
\[ \Delta ::= \cdot \mid \Delta, x: \tau \mid \Delta, \alpha: \kappa \]

Combined:
\[ \Gamma ::= \Sigma; \Delta \]

Substitutions:
\[ \Omega ::= \tau \rightarrow \tau \]


4. System µFC: formalizing the problem

This far we have argued informally. In this section we formalize our design and show that it satisfies the usual desirable properties of type preservation and progress, assuming termination of type family reduction. It is too hard to formulate these proofs for all of Haskell, so instead we formalize µFC, a small, explicitly-typed subsystem of Haskell into System FC (Sulzmann et al. 2005; Schrijvers et al. 2008), how do we deal with coercions: Thankfully, this turns out to be remarkably easy: we simply use Rule 9 to simplify closed families in exactly the same manner as open families. The implementation in GHC is accordingly quite straightforward.

Despite the ease of implementation, there are perhaps complex new possibilities opened by the use of closed families—these are explored in Section 7.6.

4.1 System µFC

System µFC is an extension of System F, including kinds and explicit equality coercions. Its syntax is presented in Figure 2. This syntax is very similar to recent treatments of System FC (Weirich et al. 2013). We omit from the presentation the choice of ground types and their constructors and destructors, as they are irrelevant for our purposes.

There are a few points to note about type families, all visible in Figure 2. A type family has a particular arity, and always appears saturated in types. That explains the first-order notation \( F(\sigma) : \kappa \) in ground contexts \( \Sigma \) and \( F(\psi) \) in types.

A closed type family appears in µFC as a kind signature \( F(\sigma) : \kappa \), and a single axiom \( C : \Psi \), both in the top-level ground context \( \Sigma \). The “type” \( \Psi \) of the axiom is a list of equations, each of form \( \frac{\kappa}{\alpha} \). \( F(\tau) \sim \sigma \) just as we have seen before except that the quantification is explicit. For example, the axiom for \( \text{Equal} \) (restricted for simplicity to kind with \( \alpha \) ) looks like this:

\[
\text{axiomEq} : \quad \frac{\alpha \times \times \beta \times \times \cdots \times \tau}{\alpha \times \times \beta \times \times \cdots \times \tau} \sim \text{True} ; \quad \frac{\alpha \times \times \beta \times \times \cdots \times \tau}{\alpha \times \times \beta \times \times \cdots \times \tau} \sim \text{False}
\]

Although our notation for lists does not make it apparent, we restrict the form of the equations to require that \( \tilde{F} \) refers to only one type family—that is, there are no independent \( F \). We use subscripts on metavariables to denote which equation they refer to, and refer to the types \( \tilde{\sigma} \) as the type patterns of the \( \tilde{f} \) equation. We assume that the variables \( \tilde{\tau} \) bound in each equation are distinct from the variables bound in other equations.

An open type family appears as a kind signature and zero or more separate axioms, each with one equation.

4.2 Static semantics

Typing in µFC is given by the judgments in Figure 3. Most of the rules are uninteresting and are thus presented in Appendix C. The typing rules for expressions are entirely straightforward. The only noteworthy rule is the one for casting, which gives the raison d'être for coercions:

\[
\Gamma \vdash \text{tm} \ e : \tau \quad \text{Expression typing} \\
\Gamma \vdash \text{ty} \ e : \tau \quad \text{Type kinding} \\
\Gamma \vdash \text{co} \ e : \phi \quad \text{Coercion typing} \\
\Gamma \vdash \text{gnd} \ e : \tau \quad \text{Ground context validity} \\
\Gamma \vdash \text{ctx} \ e : \tau \quad \text{Variables context validity} \\
\Gamma \vdash \text{ctx} \ e : \tau \quad \text{Context validity}
\]

Here, we see that a cast by a coercion changes the type of an expression. This is what we mean by saying that a coercion witnesses the equality of two types—if there is a coercion between \( \tau_1 \) and \( \tau_2 \), then any expression of type \( \tau_1 \) can be cast into one of type \( \tau_2 \).

The rules for deriving the kind of a type are straightforward and are omitted from this presentation.

4.3 Coercions and axiom application

Coercions are less familiar, so we present the coercion typing rules in full, in Figure 4. The first four rules say that equality is congruent—that is, types can be considered equal when they are formed of components that are considered equal. The following three rules assert that coercibility is a proper equivalence relation. The \text{CO_LEFT} and \text{CO_RIGHT} rules assert that we can decompose complex equal-

Figure 3. Typing judgments for System µFC
Coercion typing

\[ \Gamma \vdash \alpha \gamma : \phi \]

Co_Arrow

\[ \Gamma \vdash \alpha \gamma_1 : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_2 : \tau_2 \leadsto \tau'_2 \]
\[ \Gamma \vdash \alpha \gamma : \tau_1 \leadsto \tau_2 \]

Co_Forall

\[ \Gamma \vdash \forall \alpha, \kappa \gamma : (\forall \alpha, \kappa \tau_1) \leadsto (\forall \alpha, \kappa \tau_2) \]

Co_App

\[ \Gamma \vdash \alpha \gamma_1 : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_2 : \tau_2 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma : \tau_1 \leadsto \tau_2 \]

Co_Trans

\[ \Gamma \vdash \alpha \gamma_1 : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_2 : \tau_2 \leadsto \tau_3 \]

Co_Sym

\[ \Gamma \vdash \alpha \gamma_1 : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_2 : \tau_2 \leadsto \tau_1 \]

Co_Refl

\[ \Gamma \vdash \alpha \gamma : \tau_1 \leadsto \tau \]
\[ \Gamma \vdash \alpha \gamma : \tau \leadsto \tau \]

Co_TyFam

\[ \Gamma \vdash \alpha \gamma : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_1 : \tau_1 \leadsto \tau_2 \]
\[ \Gamma \vdash \alpha \gamma_2 : \tau_2 \leadsto \tau_2 \]

CO_AXIOM

\[ C : \Psi \in \Sigma \]
\[ \Psi = \frac{\pi \sigma_i \mid \rho_i \mid \psi / \alpha_i}{} \]
\[ \Sigma \vdash \Delta \vdash \alpha_i \]
\[ \forall i, j, \text{no_conflict}(\Psi_i, \Psi_j) \]
\[ \text{compat}(\Psi_i, \Psi_j) \]
\[ \text{NC_COMPATIBLE} \]
\[ \text{NC_ARPART} \]
\[ \text{Figure 4. Coercion formation rules} \]

5. Metatheory

A summary of the structure of the type safety proof, highlighting the parts that are considered in this paper, is in Figure 5. Our main goals are to prove (i) the substitution lemma of types into coercions (Section 5.2), and (ii) a consistency property that ensures we never equate two types such as Int and Bool (Section 5.3). The substitution and consistency lemmas lead to the preservation and progress theorems respectively, which together ensure type safety. We omit the operational semantics of µFC as well as the other lemmas in the main proofs of preservation and progress, because these are all direct adaptations from previous work (Weirich et al. 2011; Sulzmann et al. 2007a).

We stress that, as Figure 5 indicates, we have proved type safety only for terminating type families. What exactly does that mean? We formally define the rewrite relation, now written \( \Sigma \vdash \cdot \leadsto \cdot \) to explicitly mention the set of axioms, with the following rule:

\[ C : \Psi \in \Sigma \]
\[ \Psi = \frac{\pi \sigma_i \mid \rho_i \mid \psi / \alpha_i}{} \]
\[ \Sigma \vdash \Delta \vdash \alpha_i \]
\[ \forall i, j, \text{no_conflict}(\Psi_i, \Psi_j) \]
\[ \text{compat}(\Psi_i, \Psi_j) \]
\[ \text{NC_COMPATIBLE} \]
\[ \text{NC_ARPART} \]
\[ \text{Figure 4. Coercion formation rules} \]

Figure 4. Coercion formation rules

\[ \text{Figure 4. Coercion formation rules} \]

The last rule of coercion formation, CO_AXIOM, is the one that we are most interested in. The coercion \( C[i] \rightarrow \) witnesses the equality obtained by instantiating the \( i \)’th equation of axiom \( C \) with the types \( \pi \). For example,

\[ \text{axiomEq}[0] \text{ Int } : \text{ True } \]

This says that if we pick the first equation of axiomEq (we index from 0), and instantiate it at Int, we have a witness for Equal Int Int True.

Notice that the coercion \( C[i] \rightarrow \) specifies exactly which equation is picked (the \( i \)’th one); µFC is a fully-explicit language. However, the typing rules for µFC must reject unsound coercions like

\[ \text{axiomEq}[1] \text{ Int } : \text{ False } \]

and that is expressed by rule CO_AXIOM. The premises of the rule check to ensure that \( \Sigma; \Delta \) is a valid context and that all the types \( \pi \) are of appropriate kinds to be applied in the \( i \)’th equation. The last premise implements Rule 9 (Section 3.4), by checking no_conflict for each preceding equation \( j \). The no_conflict judgment simply checks that either (NC_COMPATIBLE) the \( i \)’th and \( j \)’th equation for \( C \) are compatible, or (NC_ARPART) that the target is apart from the LHS of the \( j \)’th equation, just as in Rule 9.

In NC_COMPATIBLE, note that the compat judgment does not take the types \( \pi \); compatibility is a property of equations, and is independent of the specific arguments at an application site. The two rules for compat are exactly equivalent to Definition 8.

These judgments refer to algorithms apart and unify. We assume a correct implementation of unify and propose sufficient properties of apart in Section 5.1. We then show that our chosen algorithm for apart (Definition 6) satisfies these properties.

As a final note, the rules do not check the closed type family axioms for exhaustiveness. A type-family application that matches no axiom simply does not reduce. Adding an exhaustiveness check based on the kind of the arguments of the type family might be a useful, but orthogonal, feature.
In the conclusion of this rule, \( C[\cdot] \) denotes a type context with exactly one hole. Its use in the rule means that a type family can simplify anywhere within a type. Note that the no\textsubscript{\text{conflict}} premise of this rule is identical to that of the Co\textsubscript{\text{AXIOM}} rule. By “terminating type families” we mean that the \( \Sigma \vdash \cdot \sim \cdot \) relation cannot have infinite chains. We discuss non-terminating type families in Section 6.

As a notational convention, we extend the relation to lists of types by using \( \Sigma \vdash \pi_1 \sim \pi_2 \) to mean that exactly one of the types in \( \pi_1 \) steps to the corresponding type in \( \pi_2 \); in all other positions \( \pi_1 \) and \( \pi_2 \) are identical.

### 5.1 Preliminaries: properties of unification and apartness

In order to prove properties about no\textsubscript{\text{conflict}}, we must assume the correctness of the unification algorithm:

**Property 11** (unify correct). If there exists a substitution \( \Omega \) such that \( \Omega(\overline{\tau}) = \Omega(\overline{\pi}) \), then unify(\( \overline{\pi}, \overline{\tau} \)) succeeds. If unify(\( \overline{\pi}, \overline{\tau} \)) = \( \Omega \) then \( \Omega \) is a most general unifier of \( \overline{\pi} \) and \( \overline{\tau} \).

In Section 3.2, we gave some necessary properties of apart, namely Properties 2 and 4. To prove type soundness we need sufficient properties, such as the following three. Any implementation of apart that has these three properties would lead to type safety. We prove (in Appendix F) that the given algorithm for apart (Definition 6) satisfies these properties. Due to flattening in the definition of apart, this proof is non-trivial. As a sanity check, we also prove that the sufficient properties imply the necessary ones of Section 3.2.

**Property 12** (Apartness is stable under type substitution). If apart(\( \overline{\pi}, \overline{\tau} \)), then for all substitutions \( \Omega \), apart(\( \overline{\pi}, \Omega(\overline{\tau}) \)).

**Property 13** (No unifiers for apart types). If apart(\( \overline{\pi}, \overline{\tau} \)), then there exists no substitution \( \Omega \) such that \( \Omega(\overline{\pi}) = \Omega(\overline{\tau}) \).

The final property of the apartness check is the most complex. It ensures that, if an equation can fire for a given target and that target steps, then it is possible to simplify the reduce even further so that the same equation can fire on the final reduce.

**Property 14** (Apartness can be regained after reduction). If \( \tau = \Omega(\overline{\rho}) \) and \( \Sigma \vdash \tau \sim \tau' \), then there exists a \( \tau'' \) such that

1. \( \Sigma \vdash \tau'' \sim \tau''' \),
2. \( \tau''' = \Omega'(\overline{\rho'}) \) for some \( \Omega' \), and
3. for every \( \overline{\rho} \) such that \( \text{apart}(\overline{\rho}, \tau) \): apart(\( \overline{\rho}', \tau'' \)).

Here is an example of Property 14 in action. Consider the following type families \( F \) and \( G \):

- **type family** \( F \) a where \( F(\text{Int}, \text{Bool}) = \text{Char} \) -- (A)
- **type family** \( G \times \text{ where } G \text{ Int} = \text{Double} \)

Suppose that our target is \( F(\text{Int}, \text{Int}) \), and that our particular implementation of apart allows equation (B) to fire; that is, apart(\( \text{Int}, \text{Bool} \)). (\( G \text{ Int}, \text{Int} \)). Now, suppose that instead of firing (B) we chose to reduce the first \( G \text{ Int} \) argument to \( \text{Double} \). The new target is now \( F(\text{Double}, \text{Int}) \). Now (B) cannot fire, because the new target simply does not match (B) any more. Property 14 ensures that there exist further reductions on the new target that make (B) firable again—in this case, stepping the second \( G \text{ Int} \) to \( \text{Double} \) does the job. Conditions (2) and (3) of Property 14 formalize the notion “make (B) firable again”.

### 5.2 Type substitution in coercions

System \( \muFC \) enjoys a standard term substitution lemma. This lemma is required to prove the preservation theorem. As shown in Figure 5, the term substitution lemma depends on the substitution lemma for coercions. We consider only the case of interest here, that of substitution in the rule Co\textsubscript{\text{AXIOM}}.

**Lemma 15** (Co\textsubscript{\text{AXIOM}} Substitution). If \( \Sigma; \Delta, \beta; \kappa; \Delta' \vdash_{\text{co}} C[i] \tau : F(\rho, [\overline{\tau}/\alpha_i]) \sim_{\text{co}} \nu_1([\overline{\tau}/\alpha_i]) \) and \( \Sigma; \Delta, \kappa \vdash_{\text{co}} \sigma : \kappa \), then \( \Sigma; \Delta, \beta; \kappa; \Delta; \kappa; \sigma/\beta \vdash_{\text{co}} C[i] \tau[\sigma/\beta] : F(\rho, [\overline{\tau}/\alpha_i])[\sigma/\beta] \sim_{\text{co}} \nu_1([\overline{\tau}/\alpha_i])[\sigma/\beta] \).

The proof of this lemma, presented in Appendix D, proceeds by case analysis on the no\textsubscript{\text{conflict}} judgment. It requires the use of the (standard) type substitution lemma and Property 12, but is otherwise unremarkable.

### 5.3 Consistency

As discussed at the beginning of this section, to establish progress we must show consistency. Consistency ensures that we can never deduce equalities between distinct value types, denoted with \( \xi \):

\[ \xi \quad ::= \quad H \tau \vdash \tau_1 \rightarrow \tau_2 \mid \forall \alpha : \kappa. \tau \]

For example, \( \text{Int}, \text{Bool} \), and \( \forall \alpha : \kappa. \alpha \rightarrow \alpha \) are all value types. A set of axioms is consistent if we cannot deduce bogus equalities like \( \text{Int} \rightarrow \text{Bool} \) or \( \text{Int} \rightarrow \forall \alpha : \kappa. \alpha \rightarrow \alpha \).

**Definition 16** (Consistent contexts). A ground context \( \Sigma \) is consistent if, for all coercions \( \gamma \) such that \( \Sigma; \gamma \vdash_{\text{co}} \gamma_1 \sim \gamma_2 \):

1. if \( \gamma_1 = H \tau \), then \( \gamma_2 = H \tau \).
2. if \( \gamma_1 = \tau_1 \rightarrow \tau_2 \), then \( \gamma_2 = \tau_2 \rightarrow \tau_1 \), and
3. if \( \gamma_1 = \forall \alpha : \kappa. \tau_1 \), then \( \gamma_2 = \forall \beta : \kappa. \tau_2 \).

How can we check whether an axiom set is consistent? It is extremely hard to do so in general, so instead, following previous work (Weirich et al. 2011), we place syntactic restrictions on the axioms that conservatively guarantee consistency. A set of axioms that pass this check are said to be Good. We then prove the consistency lemma:

**Lemma 17** (Consistency). If \( \text{Good} \Sigma \), then \( \Sigma \) is consistent.

Following previous proofs, we show that if \( \text{Good} \Sigma \) and \( \Sigma_i \vdash_{\text{co}} \gamma : \sigma_1 \sim \sigma_2 \), then \( \sigma_1 \) and \( \sigma_2 \) have a common reduct
in the \(\sim\) relation. Because the simplification relation preserves type constructors on the heads of types, we may conclude that \(\Sigma\) is consistent.

However, one of the cases in this argument is transitivity: the joinability relation must be transitive. That is, if \(\tau_1\) and \(\tau_2\) have a common reduct \(\sigma_1\), and if \(\tau_2\) and \(\tau_3\) have a common reduct \(\sigma_2\), then \(\tau_1\) and \(\tau_3\) must have a common reduct (they are \(\tau\)_joined). To show transitivity of joinability, we must show congruence of the rewrite relation, in order to find the common reduct of \(\sigma_1\) and \(\sigma_2\) (which share \(\tau_2\) as an ancestor).

Our approach to this problem is to show local confluence (see Figure 6) and then use Newman’s Lemma (1942) to get full confluence. Newman’s Lemma requires that the rewrite system is terminating—this is where the assumption of termination is used.

The full, detailed proof appears in Appendix E.

### 5.4 Good contexts

What sort of checks should be in our syntactic conditions, \(\text{Good}\)? We would like \(\text{Good}\) to be a small set of common-sense conditions for a type reduction system, such as the following:

**Definition 18 (Good contexts).** We have \(\text{Good} \Sigma\) whenever the following four conditions hold:

1. For all \(C:\Psi \in \Sigma: \Psi\) is of the form \(\exists x: \Phi\). \(F(y) \sim v\) where all of the \(F_i\) are the same type family \(F\) and all of the type patterns \(\pi_i\) do not mention any type families.
2. For all \(C:\Psi \in \Sigma\) and equations \(\exists x: \Phi\). \(F(y) \sim v\) in \(\Psi\): the variables \(\pi_i\) all appear free at least once in \(\pi_i\).
3. For all \(C:\Psi \in \Sigma\); if \(\Psi\) defines an axiom over a type family \(F\) and has multiple equations, then no other axiom \(C':\Phi' \in \Sigma\) defines an axiom over \(F\). That is, all type families with ordered equations are closed.
4. For all \(C_1: \Phi_1 \in \Sigma\) and \(C_2: \Phi_2 \in \Sigma\) (each with only one equation), \(\text{compat}(\Phi_1, \Phi_2)\). That is, among open type families, the patterns of distinct equations do not overlap.

The clauses of the definition of \(\text{Good}\) are straightforward syntactic checks. In fact, these conditions are exactly what GHC checks for when compiling type family instances. This definition of \(\text{Good}\) leads to the proof of Lemma 39, as described above.

### 6. Non-terminating type families

By default GHC checks every type family for termination, to guarantee that the type checker will never loop. Any such check is necessarily conservative; indeed, GHC rejects the 

**Figure 6.** Graphical representation of confluence properties. A solid line is a universally quantified input, and a dashed line is an existentially quantified output.

In the \(\sim\) relation. Because the simplification relation preserves type constructors on the heads of types, we may conclude that \(\Sigma\) is consistent.

However, one of the cases in this argument is transitivity: the joinability relation must be transitive. That is, if \(\tau_1\) and \(\tau_2\) have a common reduct \(\sigma_1\), and if \(\tau_2\) and \(\tau_3\) have a common reduct \(\sigma_2\), then \(\tau_1\) and \(\tau_3\) must have a common reduct (they are \(\tau\)_joined). To show transitivity of joinability, we must show congruence of the rewrite relation, in order to find the common reduct of \(\sigma_1\) and \(\sigma_2\) (which share \(\tau_2\) as an ancestor).

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By default GHC checks every type family for termination, to guarantee that the type checker will never loop. Any such check is necessarily conservative; indeed, GHC rejects the 

**Figure 7.** Counter-example to confluence

the check. This may make the type checker loop, but it should not threaten soundness.

However, the soundness result of Section 5 covers only terminating type families. Surprisingly (to us) non-termination really does lead to a soundness problem (Section 6.1). We propose a solution that (we believe) rules out this problem (Section 6.2), but explain why the main result of this paper is difficult to generalize to non-terminating type families, leaving an open problem for further work.

#### 6.1 The problem with infinity

Consider this type family, adapted from Huet (1980):

**type family** 

\[ D \times \text{where} \]

\[ D ([b], b) = \text{Bool} \]

\[ D (c, c) = \text{Int} \]

We wish to simplify the target \(D (a, a)\). The type \((a, a)\) matches the second pattern \((c, c)\), but is it apart from the first pattern \(([b], b)\)? Definition 6 asserts that they are apart since they do not unify: unification fails with an occurs check error. Accordingly, Rule 9 would simplify \(D (a, a)\) to \(\text{Int}\). But consider the following definitions, where type family \(\text{Loop}\) is a nullary (0-argument) type family:

**type family** \(\text{Loop}\)

**type instance** \(\text{Loop} = [\text{Loop}]\)

If we instantiate \(a\) with \(\text{Loop}\) we get \((\text{Loop}, \text{Loop})\) which can simplify to \((\text{Loop}, \text{Loop})\). The latter does match the pattern \(([b], b)\), violating Property 4, a necessary condition for soundness.

So, in a non-terminating system our apartness check is unsound. Concretely, using our apartness implementation from Definition 6, we can equate types \(\text{Int}\) and \(\text{Bool}\), thus:

\[ \text{Int} \sim D (\text{Loop}, \text{Loop}) \sim D ([\text{Loop}], \text{Loop}) \sim \text{Bool} \]

Conclusion: we must not treat \((a, a)\) as apart from the pattern \(([b], b)\), even though they do not unify. In some ways this is not so surprising. In our earlier examples, apartness was based on an explicit contradiction (“a \(\text{Bool}\) cannot be an \(\text{Int}\)”), but here unification fails only because of an occurs check. As the \(\text{Loop}\) example shows, allowing non-terminating type-family definitions amounts to introducing infinite types, and if we were to allow infinite types, then \((a, a)\) does unify with \(([b], b)\)!

#### 6.2 Fixing the problem

The problem with the current apartness check is that finite unification fails too often. We need to replace the unification test in the definition of apartness with unification over infinite types:

**Definition 19 (Infinite unification).** Two types \(\tau_1\) and \(\tau_2\) are infinitely unifiable, \(\text{unify}_\omega(\tau_1, \tau_2)\), if there exists a substitution \(\omega\) whose range may include infinite types, such that \(\omega(\tau_1) = \omega(\tau_2)\).

For example types \((a, a)\) and \(([b], b)\) are unifiable with a substitution \(\omega = [a \mapsto [[[...]]], b \mapsto [[[...]]]]\). Efficient algorithms
to decide unification over infinite types (and compute most general unifiers) have existed for some time and are based on well-established theory (Huet 1976; Courcelle 1983). See Jaffar (1984) for such an algorithm, and Knight (1989) for a general survey.

We conjecture that replacing all uses of unify with \( \text{unify}_\infty \) in our definitions guarantees soundness, even in the presence of non-terminating family equations. Alas, this conjecture turns out to be very hard to prove, and touches on open problems in the non-terminating family equations. Happily, if the conjecture of Section 6.2 holds true, we can apply the same fix for open families as we did for closed families: simply use \( \text{unify}_\infty \) instead of unify when checking for overlap. Indeed, this is exactly how we have corrected this oversight in GHC 7.8.

### 7. Discussion and Future Work

The study of closed type families opens up a wide array of related issues. This section discusses some of the more interesting points we came across in our work.

#### 7.1 Denotational techniques for consistency

We do not have a proof of consistency for a system with non-terminating, non-left-linear axioms (even when using \( \text{unify}_\infty \) instead of unify). We have seen that confluence is false, and hence cannot be used as a means to show consistency.

A possible alternative approach to proving consistency—side-stepping confluence—is via a denotational semantics for types. We would have to show that if we can build a coercion \( \gamma \) such that \( \Gamma \vdash \gamma : \tau \sim \sigma \), then \( \llbracket \tau \rrbracket = \llbracket \sigma \rrbracket \), for some interpretation of types into a semantic domain. The "obvious" domain for such a semantics, in the presence of non-terminating computations, is the domain that includes \( \bot \) as well as finite and infinite trees. Typically in denotational semantics, recursive type families would be interpreted as the limit of approximations of continuous functions. However, the "obvious" interpretation of type families in this simple domain is not monotone. Consider this type family:

\[
\text{type family } D' \times y \\
\text{type instance } D' \{ b \} = \text{Bool} \\
\text{type instance } D' \{ c \} = \text{Int}
\]

As described in Section 2, the type instance equations of an open type family are required to have non-overlapping left-hand sides, and GHC 7.6 believes that the two equations do not overlap because they do not unify. But, using certain flags, GHC also accepts the definition of \( \text{Loop} \), and the target \( (D' \text{ Loop Loop}) \) demonstrates that the combination is unsound precisely as described above.\(^5\)

Happily, if the conjecture of Section 6.2 holds true, we can apply the same fix for open families as we did for closed families: simply use \( \text{unify}_\infty \) instead of unify when checking for overlap. Indeed, this is exactly how we have corrected this oversight in GHC 7.8.

### 7.2 Conservativity of apartness

We note in Section 3.3 that our implementation of apartness is conservative. This conservativity is unavoidable—it is possible for open type families to have instances scattered across modules, and thus the apartness check cannot adequately simplify the types involved in every case. However, the current check considers none of the type family axioms available, even if one would inform the apartness check. For example, consider:

\[
\text{type family } G \ a \ where \\
G \text{ Int} = \text{Bool} \\
G \{a\} = \text{Char}
\]

and we wish to simplify target \( \text{Equal Double} \ (G \ b) \). It is clear that an application of \( G \) can never simplify to \( \text{Double} \), so we could imagine a more refined apartness check that could reduce this target to \( \text{False} \). We leave the details of such a check to future work.

#### 7.3 Conservativity of coincident overlap: partial knowledge

It is worth noting that the compatibility check (Definition 8) is somewhat conservative. For example, take the type family:

\[
\text{type family } F \ a \ b \ where \\
F \text{ x x} = \text{Int} \\
F \{x\} (\text{Maybe x}) = \text{Char}
\]

Consider a target \( F \ g \text{ Int} \). The target matches the second equation, but not the first. But, the simplification rule does not allow us to fire the second equation—the two equations are not compatible, and the target is not apart from the first equation. Yet it clearly would be safe to fire the second equation in this case, because even if \( g \) turns out to be \( \text{Bool} \), the first equation would give the same result.

It would, however, be easy to modify \( F \) to allow the desired simplification: just add a new second equation \( F \ a \text{ Int} = \text{Int} \). This new equation would be compatible with the first one and therefore would allow the simplification of \( F \ g \text{ Int} \).

#### 7.4 Conservativity of coincident overlap: requiring syntactic equality

The compatibility check is conservative in a different dimension: it requires syntactic equality of the RHSs after substitution. Consider this tantalizing example:

\[
\text{type family } \text{Plus} \ a \ b \ where \\
\text{Plus Zero} \ a = a \quad \text{(A)} \\
\text{Plus (Succ b) c} = \text{Succ (Plus b c)} \quad \text{(B)} \\
\text{Plus d Zero} = d \quad \text{(C)} \\
\text{Plus e (Succ f)} = \text{Succ (Plus e f)} \quad \text{(D)}
\]

If this type family worked as one naively expect, it would simplify an addition once either argument’s top-level constructor were known. (In other dependently typed languages, definitions

\(^5\) Akio Takano has posted an example of how this can cause a program to fail, at http://ghc.haskell.org/trac/ghc/ticket/8162.
like this are not possible and require auxiliary lemmas to reduce when the second argument’s structure only is known.) Alas, it does not work as well as we would hope. The problem is that not all the equations are compatible. Let’s look at (B) and (C). To check if these are compatible, we unify \((\text{Succ } b), c\) with \((d, \text{Zero})\) to get \([c \mapsto \text{Zero}, d \mapsto \text{Succ } b]\). The right-hand sides under this substitution are \(\text{Succ } (\text{Plus } b \text{ Zero})\) and \(\text{Succ } b\). However, these are not syntactically identical, so equations (B) and (C) are not compatible, and a target such as \(\text{Plus } g \text{ Zero}\) is stuck.

Why not just allow reduction in the RHSs before checking for compatibility? Because doing so is not obviously well-founded! Reducing the \(\text{Succ } (\text{Plus } b \text{ Zero})\) type that occurred during the compatibility check above requires knowing that equations (B) and (C) are compatible, which is exactly what we’re trying to establish. So, we require syntactic equality to support compatibility, and leave the more general check for future work.

7.5 Lack of inequality evidence

One drawback of closed type families is that they sometimes do not compose well with generalized algebraic datatypes (GADTs). Consider the following sensible-looking example:

```plaintext
data X a where
  XInt :: X Int
  XBool :: X Bool
  XChar :: X Char

type family Collapse a where
  Collapse Int = Int
  Collapse x = Char

collapse :: X a \to X (Collapse a)
collapse XInt = XInt
collapse _ = XChar
```

The type function \(\text{Collapse}\) takes \(\text{Int}\) to itself and every other type to \(\text{Char}\). Note the type of the term-level function \(\text{collapse}\). Its implementation is to match \(\text{XInt}\)—the only constructor of \(X\) parameterized by \(\text{Int}\)—and return \(\text{XInt}\); all other constructors become \(\text{XChar}\). The structure of \(\text{collapse}\) exactly mimics that of \(\text{Collapse}\).

Yet, this code does not compile.

The problem is that the type system has no evidence that, in the second equation for \(\text{collapse}\), the type variable \(a\) cannot be \(\text{Int}\). So, when type-checking the right-hand side \(\text{XChar}\), it is not type-safe to equate \(\text{Collapse } a\) with \(\text{Char}\). The source of this problem is that the type system has no notion of inequality. If the \textbf{case} construct were enhanced to track inequality evidence and axiom application could consider such evidence, it is conceivable that the example above could be made to type-check. Such a notion of inequality has not yet been considered in depth, and we leave it as future work.

7.6 Type inference

The inclusion of closed type families to Haskell opens up new possibilities in type inference. By definition, the full behavior of a closed type family is known all at once. This closed-world assumption allows the type inference engine to perform more improvement on types than would otherwise be possible. Consider the following type family:

```plaintext
type family Inj a where
  Inj Int = Bool
  Inj Bool = Char
  Inj Char = Double
```

Type inference can discover in this case that \(\text{Inj}\) is indeed an injective type function. When trying to solve a constraint of the form \(\text{Inj } Int \sim \text{Inj } q\) the type inference engine can deduce that \(q\) must be equal to \(\text{Int}\) for the constraint to have a solution. By contrast, if \(\text{Inj}\) were not identified as injective, we would be left with an unsolved constraint as in principle there could be multiple other types for \(q\) that could satisfy \(\text{Inj } \text{Int} \sim \text{Inj } q\).

Along similar lines, we can imagine improving the connection between \(\text{Equal}\) and \(\sim\). Currently, if a proof \(a \sim b\) is available, type inference will replace all occurrences of \(a\) with \(b\), after which \(\text{Equal } a \ b\) will reduce to \(\text{True}\). However, the other direction does not work: if the inference engine knows \(\text{Equal } a \ b \sim \text{True}\), it will not deduce \(a \sim b\). Given the closed definition of \(\text{Equal}\), though, it seems possible to enhance the inference engine to be able to go both ways.

These deductions are not currently implemented, but remain as compelling future work.

8. Related work

8.1 Previous work on System FC

The proof of type soundness presented in this paper depends heavily on previous work for System FC, first presented by Sulzmann et al. (2007a). That work proves consistency only for terminating type families, as we do here.

In a non-terminating system, local confluence does not imply confluence. Therefore, previous work (Weirich et al. 2011) showed confluence of the rewrite system induced by the (potentially non-terminating) axiom set by establishing a \textit{local diamond} property (see Figure 6). However, the proof took a shortcut: the requirements for good contexts effectively limited all axioms to be left-linear. The local diamond proof relies on the fact that, in a system with linear patterns, matching is preserved under reduction. For instance, consider these axioms:

```plaintext
type instance F a b = H a

type instance G Int = Bool
```

The type \(F \ (G \text{ Int}) \ (G \text{ Int})\) matches the equation for \(F\) and can potentially simplify to \(F \ (G \text{ Int}) \text{ Bool}\) or to \(F \text{ Bool} \ (G \text{ Int})\) or even to \(F \text{ Bool} \text{ Bool}\). But, in all cases the reduct also matches the very same pattern for \(F\), allowing local diamond property to be true.

What is necessary to support a local diamond property in a system with \textit{closed} type families, still restricted to linear patterns? We need this property: if \(F \equiv \tau\) can reduce by some equation \(q\), and \(\tau \sim \tau'\), then \(F \equiv \tau'\) can reduce by that same equation \(q\). With only open families, this property means that matching must be preserved by reduction. With closed families, however, both matching and \textit{apartness} must be preserved by reduction. Consider the definition for \(F'\) below (where \(H\) is some other type family):

```plaintext
type family F’ a b where
  F’ Int Bool = Char
  F’ a b = H a
```

We know that \(F’ \ (G \text{ Int}) \ (G \text{ Int})\) matches the second equation and is apart (Definition 6) from the first equation. The reduct \(F’ \ (G \text{ Int}) \text{ Bool}\) also matches the second equation but is \textit{not} apart from the first equation. Hence, \(F’ \ (G \text{ Int}) \text{ Bool}\) cannot simplify by either equation for \(F’\), and the local diamond property does not hold. Put simply, our apartness implementation is not preserved by reduction.

In a terminating system, we are able to get away with the \textit{weaker} Property 14 for apart (where apartness is not directly preserved under reduction), which our implementation does satisfy. We have designed an implementation of apart which is provably stable under reduction, but it is more conservative and less intuitive for programmers. Given that this alternative definition of apart brought

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6 Actually, under \textit{parallel} reduction; see (Weirich et al. 2011).
a proof of type safety only for potentially non-terminating but 
linear patterns (prohibiting our canonical example Equal), and that it often led to stuck targets where a reduction naively seemed possible, we have dismissed it as being impractical. We thus seek out a proof of type safety in the presence of non-terminating, non-linear axioms.

8.2 Type families vs. functional dependencies

Functional dependencies (Jones 2000) (further formalized by Sulzmann et al. (2007b)) allow a programmer to specify a dependency between two or more parameters of a type class. For example, Kiselevov et al. (2004) use this class for their type-level equality function:

```haskell
class HEq x y (b :: Bool) | x y -> b
instance HEq x x True
instance (b ~ False) => HEq x y b
```

The annotation \( x \rightarrow y \rightarrow b \) in the class header declares a functional dependency from \( x \) and \( y \) to \( b \). In other words, given \( x \) and \( y \), we can always find \( b \).

Functional dependencies have no analogue in GHC’s internal language, System FC; indeed they predate it. Rather, functional dependencies simply add extra unification constraints that guide type inference. This can lead to very compact and convenient code, especially when there are multiple class parameters and bi-directional functional dependencies. However, functional dependencies do not generate coercions witnessing the equality between two types. Hence they interact poorly with GADTs and, more generally, with type-level programming.

8.3 Controlling overlap

Morris and Jones (2010) introduce instance chains, which obviate the need for overlapping instances by introducing a syntax for ordered overlap among instances. Their ideas are quite similar to the ones we present here, with a careful check to make sure that one instance is impossible before moving onto the next. However, the proof burden for their work is lower than ours—a flaw in instance selection may lead to incoherent behavior (e.g., different instances selected for the same code in different modules), but it cannot violate type safety. This is because class instances are compiled solely into term-level constructs (dictionaries), not type-level constructs. In particular, no equalities between different types are created as part of instance compilation.

8.4 Full-spectrum dependently typed languages

Type families resemble the type-level computation supported by dependently typed languages. Languages such as Coq (Coq development team 2004) and Agda (Norell 2007) allow ordinary functions to return types. As in Haskell, type equality in these languages is defined to include \( \beta \)-reduction of function application and \( \iota \)-reduction of pattern matching.

However, there are several significant differences between these type-level functions and type families. First is that Coq and Agda do not allow the elimination of their equivalents of kind \( \star \). There is no way to write a Coq/Agda function analogous to the closed type family below, which returns True for function types and False otherwise.

```haskell
type family IsArrow (a :: \star) :: Prop where
    IsArrow (a -> b) = True
    IsArrow a = False
```

Instead, pattern matching is only available for inductive datatypes. The consistency of these languages prohibits the elimination of non-inductive types such as \( \star \) (or \( Set, Prop, \) and \( Type \)).

Furthermore, pattern matching in Coq and Agda does not support non-linear patterns. As we discussed above, non-linear patterns allow computation to observe whether two types are equal. However, the equational theory of full spectrum languages is much more expressive than that of Haskell. Because these languages allow un-saturated functions in types, it must define when two functions are equal. This comparison is intensional, and allowing computation to observe intensional equality is somewhat suspicious. However, in Haskell, where all type functions must always appear saturated, this issue does not arise.

Due to the lack of non-linear patterns, Coq and Agda programmers must define individual functions for every type that supports decidable equality. (Coq provides a tactic—\texttt{decide equality}—to automate this definition.) Furthermore, these definitions do not immediately imply that equality is reflexive; this result must be proved separately and manually applied. In contrast, the closed type family \texttt{Equal a a} immediately reduces to \texttt{True}.

Similarly, functions in Coq and Agda do not support coincident overlap at definition time. Again, these identities can be proven as lemmas, but must be manually applied.

8.5 Other functional programming languages

Is our work on closed type families translatable to other functional programming languages with rich type-level programming? We think so. Though the presentation in this paper is tied closely to Haskell, we believe that the notion of apartness would be quite similar (if not the same) in another programming language. Accordingly, the analysis of Section 3 would carry over without much change. The one caveat is that, as mentioned above, non-linear pattern matching depends on the saturation of all type-level functions. If this criterion is met, however, we believe that other languages
could adopt the syntax and behavior of closed type families as presented here without much change.

9. Conclusions
Closed type families improve the usability of type-level computation, and make programming at the type level more reminiscent of ordinary term-level programming. At the same time, closed type families allow for the definition of manifestly-reflexive, decidable equality on types of any kind. They allow automatic reductions of types with free variables and allow the user to specify multiple, potentially overlapping but coherent reduction strategies (such as the equations for the `And` example).

On the theoretical side, the question of consistency for non-terminating non-left-linear rewrite systems is an interesting research problem in its own right, quite independent of Haskell or type families, and we offer it as a challenge problem to the reader.

Acknowledgments
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References


A. Description of units package

Using closed type families, we have written a library units, for strongly-typed dimensional analysis. For example, we want to write functions like this:

\[
\text{curPos} :: \text{Pos} \rightarrow \text{Velocity} \rightarrow \text{Acceleration} \rightarrow \text{Time} \rightarrow \text{Pos}
\]

\[\text{curPos} x0 v a t = x0 + (v \cdot \text{t}) + ((0.5 \cdot a \cdot \text{t}^2) \cdot (t \cdot \text{pTwo}))\]

The above code works with our library and type-checks. However, if we were to make an expression that does not respect physical units (say, by forgetting the \( t \) in the \( v \cdot t \)), we get a type error at compile time. For that particular case, the error says `Couldn’t match type ‘Meter’ with ‘Second’`, rather helpfully.

Importantly, this library is fully extensible. There are no wired-in units, except for `Scalar`. This way, users can apply the library to situations beyond just physics. For example, it might be sensible to

\[\text{cabal install units; you will need GHC 7.8.}\]
have HPixel and VPixel units when writing a drawing program, to
make sure that you don’t ever add a height with a width.

In order to support extensibility, new units are represented by
new datatypes, in kind *. For example, here are the definitions for
two units:

```haskell
data Meter = Meters
instance Unit Meter where
type BaseUnit Meter = Canonical
data Foot = Feet
instance Unit Foot where
type BaseUnit Foot = Meter
canversionRatio Φ = 0.3048
type Pos = MkDim Meter
```

It is the library’s extensibility that requires closed type families. It
needs to reason about type-level structures without being able
to be well-versed in type families. There are two independent ways
that closed type families are required in the design of the library:
manipulating dimension specifications as type-level sets (which is
similar to the example in Section 2.4) and managing the hierarchies
of inter-convertible units.

**Building a hierarchy of units with a distinguished root** In the
definition of Meter and Foot above, we also defined their relation-
ship. The code says that Meter is a canonical unit—that is, it is
not defined in terms of something else. On the other hand, Foot is
defined in terms of Meter, so that we can write code like

```
height :: Double
height = (1.8 % Meters) # Feet
```

and height will have the value 5.9055. Of course, we could convert
feet to meters simply by reversing the statement.

Say a library has been written on top of units that defines
different length measurements, such as Meters, Feet, and
LightYears. Now, a user of that library realizes that she needs to
define Inches. She would like to define inches in terms of Feet,
because she knows that conversion ratio. But, she doesn’t know
which of the existing length units is the canonical one. Part of the
design principle behind the units library is that she does not need to
know—she can define Inches in terms of any of the available
length units.

With this design in hand, we still need a way to compute the
conversion from our internal representation of a length—which
will be in Meters, the canonical unit—to Inches. We can see that
the declared units form a tree, rooted at Meters, and each new
unit refers to its BaseUnit, or parent in the tree. To find the right
conversion ratio, we simply have to walk up the tree from the
desired unit, multiplying all of the conversion ratios together.

But, how to implement this in Haskell? Recall that this tree
is a tree of types, which are erased at runtime. We should use a
class Unit that defines the conversion ratios, and we can have an
associated type BaseUnit the defines a unit’s parent in the tree. We
introduce an empty type Canonical to serve as a canonical unit’s
(i.e., Meter’s) parent, or BaseUnit. Then, we can (seemingly)
implement the conversion ratio calculation straightforwardly:

```
class (Unit (BaseUnit u)) ⇒ Unit u where
type BaseUnit u :: *
canversionRatio :: u → Double
-- ratio from u to u’s parent
canonicalConvRatio :: u → Double
-- ratio from u to canonical unit,
-- with default implementation
```

(Example for the Canonical type breaks the recursion in
canonicalConvRatio by overriding the default definition.)

There is a major problem with Unit as defined here—it has
a superclass cycle. The header states that every Unit’s BaseUnit
must also be a Unit, which is clearly ill-founded. Yet, this idea is
sensible, because we need to be able to call canonicalConvRatio
on a BaseUnit. What to do?

The full answer would take up too much space to describe (and
is available if you download the units package), but it boils down to
this:

```
type family CheckCanonical (unit :: *) :: Bool where
CheckCanonical Canonical = True
CheckCanonical unit = False
```

Using CheckCanonical, we can define a conditional constraint,
especially saying that every non-canonical unit must have a unit as
its parent. This breaks the type-level recursion and brings us back
onto solid footing.

It is never wise to say that an alternate encoding is impossible
in Haskell, but we were unable to find another one that works
smoothly and presents a very easy interface to users.

**B. zipWith with inferred arity**

Using the CountArgs closed type family from Section 2.3, we can
define a variable-arity zipWith function that infers the correct arity
from its first argument.

We first need a definition of the natural numbers. This definition
will only be used as a promoted datakind.

```
data Nat = Zero | Succ Nat
```

In our description, we will abbreviate these unary numbers with
ordinary decimals.

What will the type of our final zipWith be? It will first take a
function and then several lists. The types of these lists is determined
by the type of the function passed in. For example, suppose our
function `f` has type `Int → Bool → Double`, then the type of
zipWith should be `(Int → Bool → Double) → [Int] → [Bool] → [Double]`. Thus, we wish to take the type of the function and
apply the list type constructor `[` to each component of it.

Before we write the code for this operation, we pause to note
an ambiguity in this definition. Both of the following are sensible
concrete types for a zipWith over the function `f`:

```
zipWith :: (Int → Bool → Double)
       → [Int] → [Bool] → [Double]
```

The first of these is essentially map; the second is the classic
function `zipWith` that expects two lists. Thus, we must pass in the
desired number of arguments to apply the list type constructor to.
(The inferred arity comes in later.) The function to apply these list
 constructors is named Listify:

```
type family Listify (n :: Nat) arrows where
Listify Zero a = [a]
Listify (Succ n) (a → b) = [a] → Listify n b
```

We now need to create some runtime evidence of our choice
for the number of arguments. This will be used to control the
runtime operation of zipWith—after all, our function must have
both the correct behavior and the correct type. We use a GADT
NumArgs that plays two roles: it controls the runtime behavior
as just described, and it also is used as evidence to the type checker

```
that the number argument to Listify is appropriate. After all, we do not want to call Listify 2 (Int → Bool), as that would be stuck. By pattern-matching on the NumArgs GADT, we get enough information to allow Listify to fully reduce.

```haskell
data NumArgs :: Nat → * → * where
  NAZero :: NumArgs Zero a
  NASucc :: NumArgs n b → NumArgs (Succ n) (a → b)
```

We now write the runtime workhorse listApply, with the following type:

```haskell
listApply :: NumArgs n a → [a] → Listify n a
```

The first argument is the encoding of the number of arguments to the function. The second argument is a list of functions to apply to corresponding elements of the lists passed in after the second argument. Why do we need a list of functions? Consider evaluating `zipWith (+) [1, 2] [3, 4]`, where we recur not only on the elements in the list, but on the number of arguments. After processing the first list, we have to be able to apply different functions to each of the elements of the second list. To wit, we need to apply the functions `[(1+):1: (2+)]` to corresponding elements in the list `[3, 4]`. (Here, we are using Haskell’s “section” notation for partially-applied operators.)

Here is the definition of listApply:

```haskell
listApply NAZero fs = fs
listApply (NASucc na) fs =
  λargs → listApply na (apply fs args)
  where apply :: [a → b] → [a] → [b]
        apply (f : fs) (x : xs) = (f x : apply fs xs)
        apply _ _ = []
```

It first pattern-matches on its first argument. In the NAZero case, the list of functions passed in has 0 arguments, so we just return them. In the NASucc case, we process one more argument (args), apply the list of functions fs respectively to the elements of `args`, and then recur. Note how the GADT pattern-matching is essential for this to type-check — the type checker gets just enough information for Listify to reduce enough so that the second case can expect one more argument than the first case.

**Inferring arity** As explained in Section 2.3, here is the closed type family that counts the number of arguments in a function type:

```haskell
type family CountArgs (f :: *) :: Nat where
  CountArgs (a → b) = Succ (CountArgs b)
  CountArgs result Zero = result
```

We still need to connect this type-level function with the term-level GADT NumArgs. We use Haskell’s method for reflecting type-level decisions on the term-level, type classes. The following definition essentially repeats the definition of NumArgs, but because this is a definition for a class, the instance is inferred rather than given explicitly:

```haskell
class CNumArgs (numArgs :: Nat) (arrows :: *) where
  getNA :: NumArgs numArgs arrows
instance CNumArgs Zero a where
  getNA = NAZero
instance CNumArgs n b ⇒ CNumArgs (Succ n) (a → b) where
  getNA = NASucc getNA
```

Note that the instances do not overlap; they are distinguished by their first parameter.

It is now straightforward to give the final definition of `zipWith`, using the extension `-XScopedTypeVariables` to give the body of `zipWith` access to the type variable `f`:

```haskell
zipWith :: ∀ f. CNumArgs (CountArgs f) f ⇒ f → Listify (CountArgs f) f
```

The standard Haskell function `repeat` creates an infinite list of its one argument.

The following examples show that `zipWith` indeed infers the arity:

```haskell
example1 = zipWith (λ) [(False, True), (False, True)]
example2 = zipWith (+) :: Int → Int → Int
  [1, 2, 3] [4, 5, 6]
```

In Haskell, built-in numerical operations are generalized over a type class `Num`. In this case, the operator `+` has the type `Num a ⇒ a → a → a`. Because it is theoretically possible (but deeply strange!) for `a` to be instantiable with a function type, using `+` without an explicit type will not work — there is no way to infer an unambiguous arity. Specifically, `CountArgs` gets stuck.

```
CountArgs (a → a → a) simplifies to Succ (Succ (CountArgs a))
```

but can go no further; `CountArgs a` will not simplify to Zero, because `a` is not apart from `b → c`.

# C. Typing judgments for System µFC

<table>
<thead>
<tr>
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<td>Δ</td>
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<tr>
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<td>Γ</td>
<td>Δ ⊢ τ : k</td>
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<tr>
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<td>Γ</td>
<td>Δ ⊢ τ : k</td>
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<tr>
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<td>Γ</td>
<td>Γ ⊢ e : τ</td>
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<td>Γ</td>
<td>Γ ⊢ e : τ</td>
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<tbody>
<tr>
<td>e : τ</td>
<td>Γ</td>
<td>Γ ⊢ e : τ</td>
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D. Proof of substitution lemma

The kinding judgment for types, the proposition validity judgment, and the context validity judgments are all mutually recursive. They all support a standard substitution lemma, which we do not prove here:

**Lemma 20 (Type substitution).** Assume \( \Gamma \vdash y : \kappa \). Then, the following are true:

1. If \( \Gamma, \alpha : \Delta \vdash y : \kappa \), then \( \Gamma, \alpha : \Delta \vdash \alpha \vdash y : \kappa \).
2. If \( \Gamma \vdash \alpha, \Delta \vdash y : \kappa \), then \( \Gamma, \alpha : \Delta \vdash \alpha \vdash y : \kappa \).
3. If \( \Gamma, \alpha : \Delta \vdash y : \kappa \), then \( \Gamma, \alpha : \Delta \vdash \alpha \vdash y : \kappa \).

**Lemma (CO_AXIOM Substitution [Lemma 15]).** If \( \Sigma; \Delta; \beta : \kappa, \Delta' \vdash \alpha \) \( C[i]; \tau \vdash F(\rho_i[\tau/\alpha_i]) \sim \nu_i[\tau/\alpha_i] \) and \( \Sigma; \Delta \vdash y : \kappa \), then \( \Sigma; \Delta; \beta : \kappa; \Delta' \vdash \alpha \vdash C[i]; \tau \vdash F(\rho_i[\tau/\alpha_i][\sigma/\beta]) \sim \nu_i[\tau/\alpha_i][\sigma/\beta] \).

**Proof.** We invert \( \Sigma; \Delta; \beta : \kappa; \Delta' \vdash \alpha \vdash C[i]; \tau \vdash F(\rho_i[\tau/\alpha_i]) \sim \nu_i[\tau/\alpha_i] \) to get the following:

\[
\begin{align*}
& C : \Psi \in \Sigma \quad \Psi = \frac{\alpha_i \in \Sigma}{F(\rho_i)} \sim v \\
& \Sigma; \Delta; \beta : \kappa; \Delta' \vdash y : \kappa_i \\
& \Gamma, \alpha : \Delta \underline{\vdash} \Delta' \vdash \beta : \kappa_i \\
& \forall j < i, \text{no_conflict}(\Psi, i, \tau, j)
\end{align*}
\]

**Lemma 20** gives \( \Sigma; \Delta; \beta : \kappa; \Delta' \vdash y : \kappa_i \), and \( \Gamma, \alpha : \Delta \vdash \beta : \kappa_i \). Let \( \phi = F(\rho_i) \sim \nu_i, \) and it remains only to show that \( \forall j < i, \text{no_conflict}(\Psi, i, \tau, \beta : \kappa_i) \) and \( \phi[\tau/\alpha_i][\sigma/\beta] = \phi[\tau/\alpha_i][\sigma/\beta] \), and then we can use **CO_AXIOM** to get the desired result.

E. Proof of consistency

As described in Section 5.3, we use a rewrite relation, defined in Figure 8, such that it is complete with respect to \( \Sigma; \Delta \vdash \gamma : \tau \sim \tau_2 \), and then conclude that \( \Sigma \) must be consistent, as rewriting preserves non-type-family head forms.

**Type contexts** Throughout this proof, we use a notion of type contexts, or types with holes. The notation \( \Sigma[\cdot] \) denotes a type with exactly one hole in it. Similarly, \( \Sigma[\cdot][\cdot] \) denotes a type with any number of holes (possibly 0) in it. We generalize these definitions to lists, saying that \( \Sigma[\cdot][\cdot][\cdot] \) denotes a list of types with exactly one hole (in one specific type, not one hole per type) and that \( \Sigma[\cdot][\cdot][\cdot] \) denotes a list of types with any number of holes.

**E.1 Rewrite relation**

The only form of reduction is type family simplification, using the same no_conflict judgment that appears in the **CO_AXIOM** rule. The use of \( \Sigma[\cdot] \) in the conclusion states that a type family application can reduce anywhere within the structure of a type. As \( \Sigma[\cdot] \) denotes a type context with exactly one hole, only one type family reduction happens in one step. Note that this rule is non-deterministic.

We use the notation \( \Sigma \vdash \alpha \sim \beta \) to mean the reflexive, transitive closure of the relation \( \Sigma \vdash \alpha \sim \beta \). We write single-step joinability of \( \alpha \) and \( \beta \) as \( \Sigma \vdash \alpha \sim \beta \), this fact holds whenever there exists \( \sigma \) such that \( \Sigma \vdash \beta \sim \alpha \), or \( \Sigma \vdash \beta \sim \alpha \), or \( \Sigma \vdash \alpha \sim \beta \), or \( \Sigma \vdash \beta \sim \alpha \), or \( \Sigma \vdash \alpha \sim \beta \). General joinability is written \( \Sigma \vdash \alpha \equiv \beta \), this fact holds whenever there exists \( \sigma \) such that \( \Sigma \vdash \beta \equiv \beta \) and \( \Sigma \vdash \alpha \equiv \alpha \).

We generalize the relation to hold over lists of types, written \( \Sigma \vdash \tau \sim \sigma \), to say that the list \( \tau \) is identical to the list \( \sigma \) except for one element which takes one step. We also say \( \Sigma \vdash \tau \equiv \sigma \), which is identical to \( \Sigma \vdash \tau \equiv \sigma \).

**Definition 21 (Confluence).** Our rewrite system is confluent if, for all \( \alpha, \beta, \gamma \) such that \( \Sigma \vdash \alpha \sim^* \beta \) and \( \Sigma \vdash \alpha \sim^* \gamma \), \( \Sigma \vdash \beta \sim \gamma \).

The second fact above is immediate from the fact that the variable \( \beta \) must not be free in \( \phi \), invoking the Barendregt variable convention and noting that \( \beta \) is introduced separately from any of the variables in scope in \( \phi \).

Thus, we must only show \( \forall j < i, \text{no_conflict}(\Psi, i, \tau[\sigma/\beta], j) \). Thus, given \( j < i \) (and knowing \( \text{no_conflict}(\Psi, i, \tau, j) \)), we must show \( \text{no_conflict}(\Psi, i, \tau[\sigma/\beta], j) \). We proceed by case analysis on \( \text{no_conflict}(\Psi, i, \tau, j) \):

**Case NC_APART:** We must show only that apart \( \varphi_i, \rho_i[\tau[\sigma/\beta]/\alpha_i] \), assuming \( \varphi_i, \rho_i[\tau/\alpha_i] \). The result is immediate after invoking Property 12, with \( \Omega = \beta \rightarrow \sigma \) and noting that \( \beta \) cannot be free in \( \varphi_i \).

**Case NC_COMPATIBLE:** We note that \( \tau \) appears nowhere else in the premises of this rule. Therefore, changing \( \tau \) has no effect, and we are done.

Figure 8. The type rewriting rule
In order to show the completeness of the rewrite relation for transitivity coercions, we need to show the transitivity of the joinability relation—that is, that $Σ \vdash σ₁ \odot σ₂$ and $Σ \vdash σ₂ \odot σ₃$ implies $Σ \vdash σ₁ \odot σ₃$. This fact requires confluence of the rewrite system.

E.2 Local confluence

Newman’s lemma (Newman 1942) states that a terminating rewrite system is confluent if it is locally confluent.

**Definition 22** (Local confluence). Our rewrite system is locally confluent if, for all $σ₀$, $σ₁$, and $σ₂$ such that $Σ \vdash σ₀ \sim σ₁$ and $Σ \vdash σ₁ \sim σ₂$, then $Σ \vdash σ₀ \sim σ₂$.

A diagrammatic presentation of different confluence properties is in Figure 6.

Because we have assumed termination, we need only show local confluence to show confluence. As usual, we will need a small menagerie of supporting lemmas before we can get to the main proof.

**Lemma 23** (Stability of choice of substitution of lists). If $τ[σ/α] = τ[σ′/α]$ and all the $α$ are free in $τ$, then $σ = σ′$.

**Proof.** By induction on the structure of $τ$:

Case $τ = α$: It must be that $α = α$, a one-element list. Thus, we know that $σ$ and $σ′$ have the same free variables. The given equality reduces to $σ = σ′$, so we are done.

Case $τ = σ₁ \rightarrow σ₂$: Divide the variables $α$ into three groups:

- $α_τ$ are the variables free in $σ₁$ but not free in $σ₂$.
- $α_2$ are the variables free in $σ₂$ but not free in $σ₁$.
- $α_3$ are the variables free in both $σ₁$ and $σ₂$.

Divide $σ$ and $σ′$ accordingly. Then, we can use the induction hypothesis to get that $σ_τ, σ_τ = σ′_τ, σ′_τ$ and that $σ_2, σ_3 = σ′_2, σ′_3$. Thus, we can conclude that $σ = σ′$ as desired.

Cases $τ = \forall α.κ.τ, τ = v₁ v₂$, and $τ = F(τ)$: Similar.

Case $τ = H$: The list of variables $α$ must be empty, as must be $σ$ and $σ′$, so we are done.

**Lemma 24** (Stability of choice of substitution of lists in lists). If $τ[σ/α] = τ[σ′/α]$ and all the $α$ are free in $τ$, then $σ = σ′$.

**Proof.** By induction on the length of $τ$, appealing to Lemma 23 and using logic as above to manage the free variables.

**Lemma 25** (One step/one hole context substitution). If $Σ \vdash τ \rightarrow τ′$, then $Σ \vdash C[τ] \rightarrow C[τ′]$.

**Proof.** Straightforward induction on the structure of $C[τ]$.

**Lemma 26** (One step/many holes context substitution). If $Σ \vdash τ \rightarrow τ′$, then $Σ \vdash C[τ] \rightarrow C[τ′]$.

**Proof.** Straightforward induction on the structure of $C[τ]$.

**Lemma 27** (Multistep/multihole context substitution). If $Σ \vdash τ \rightarrow* τ′$, then $Σ \vdash C[τ] \rightarrow* C[τ′]$.

**Proof.** Straightforward induction on the length of the reduction $Σ \vdash τ \rightarrow* τ′$, appealing to Lemma 26.

**Lemma 28** (One step/one variable substitution). If $Σ \vdash τ \rightarrow τ′$, then $Σ \vdash σ[τ/α] \rightarrow σ[τ′/α]$.

**Proof.** By Lemma 25.

**Lemma 29** (One step/many variables substitution). $Σ \vdash τ \rightarrow τ′$. Then $Σ \vdash σ[τ/α] \rightarrow σ[τ′/α]$.

**Proof.** Straightforward induction on the length of the reduction $Σ \vdash τ \rightarrow τ′$, appealing to Lemma 29.

**Lemma 30** (Multistep/many variables substitution). $Σ \vdash τ \rightarrow* τ′$. Then $Σ \vdash σ[τ/α] \rightarrow* σ[τ′/α]$.

**Proof.** By induction on the structure of $ρ$, where the linearity assumption is needed when dividing up the variables and combining the results when appealing to multiple induction hypotheses.

**Lemma 31** (One step linear type pattern anti-substitution). If $σ$ is the set of free variables in linear pattern $ρ$ and $Σ \vdash ρ[σ/α] \rightarrow ρ[σ′/α]$, then $Σ \vdash σ \rightarrow σ′$.

**Proof.** By induction on the length of the reduction $Σ \vdash ρ[σ/α] \rightarrow* ρ[σ′/α]$, appealing to Lemma 31 in the inductive case and Lemma 23 in the base case.

**Lemma 32** (Multistep linear type pattern anti-substitution). If $σ$ is the set of free variables in linear pattern $ρ$ and $Σ \vdash ρ[σ/α] \rightarrow* ρ[σ′/α]$, then $Σ \vdash σ \rightarrow* σ′$.

**Proof.** Let $ρ′$ be the result of replacing all variables in $ρ$ with fresh variables. Thus $ρ′$ is a linearized version of $ρ$. Let the set of free variables in $ρ′$ be $σ$. We can see that for some list of types $ψ$, $ρ[σ/α] = ρ[ψ/α]$. (The list of types $ψ$ is just like $σ$ but with some repetitions to account for the linearization.) Similarly, we have $ρ[σ/α] = ρ[ψ/α]$. Thus, we know $Σ \vdash ρ[ψ/α] \rightarrow* ρ[ψ/α]$. We then appeal to Lemma 32 to get $Σ \vdash σ \rightarrow* σ′$. Recall that this notation means that $Σ \vdash σ \rightarrow* σ′$. Thus, we conclude that $Σ \vdash σ \rightarrow* σ′$ (because each $ψ$ and $ψ′$ has an equal $σ$ or $σ′$) and then $Σ \vdash σ \rightarrow* σ′$.

**Lemma 33** (Good, the rewrite relation $Σ \vdash τ \rightarrow* ρ$ is locally confluent).

**Proof.** We assume $Σ \vdash σ₀ \rightarrow* σ₁$ and $Σ \vdash σ₀ \rightarrow* σ₂$ and we must find $σ₃$ such that $Σ \vdash σ₁ \rightarrow* σ₃$ and $Σ \vdash σ₂ \rightarrow* σ₃$. We proceed by induction on the structure of $σ₀$.

**Case $σ₀ = τ₁ \rightarrow τ₂$: Inverting $Σ \vdash τ₁ \rightarrow τ₂$ and $Σ \vdash σ₀ \rightarrow σ₂$ tells us that $(τ₁ \rightarrow τ₂) = C₁[F₁(τ₁)] (\text{and } τ₁ \rightarrow τ₂) = C₂[F₂(τ₂)]$, with $σ₁ = C₁[ψ₁]$ and $σ₂ = C₂[ψ₂]$. We now do case analysis on $C₁[ψ₁]$ and $C₂[ψ₂]$:

Case $C₁[ψ₁] → C₁[ψ₁] \rightarrow τ₂, C₂[ψ₂] \rightarrow τ₂$: Note that $C₁[F₁(τ₁)] = τ₁ = C₁[F₁(ψ₁)]$, therefore, using the other conditions known from inverting the original steps from $σ₀$, we know that $Σ \vdash τ₁ \rightarrow* τ₃$ and $Σ \vdash τ₂ \rightarrow* τ₄$, where $τ₁ = C₁[ψ₁]$ and $τ₂ = C₂[ψ₂]$. Use the induction hypothesis to get $τ₃$ such that $Σ \vdash τ₁ \rightarrow* τ₃$ and $Σ \vdash τ₂ \rightarrow* τ₄$. Then, by Lemma 27 to lift this result back to $τ₁ \rightarrow τ₂$, we are done, showing that $σ₃ = τ₃ \rightarrow τ₂$.

Case $C₁[ψ₁] = C₁[ψ₁] \rightarrow τ₂, C₂[ψ₂] = C₂[ψ₂] \rightarrow τ₂$: Let $τ₁ = C₁[ψ₁]$ and $τ₂ = C₂[ψ₂]$. Then, $σ₁ = τ₁ \rightarrow τ₂$ and $σ₂ = τ₁ \rightarrow τ₂$ with $Σ \vdash τ₁ \rightarrow* τ₃$ and $Σ \vdash τ₂ \rightarrow* τ₄$. We let $σ₃ = τ₃ \rightarrow τ₄$, and we are done.
Other cases: Similar to the cases above.

Case $σ_0 = \forall α: C$: Similar to the case for $τ_1 \rightarrow τ_2$.

Case $σ_0 = τ_1. τ_2$: Similar to the case for $τ_1 \rightarrow τ_2$.

Case $σ_0 = F(τ)$: Inverting $Σ ⊢ τ_0 \rightarrow σ_1$ and $Σ ⊢ σ_0 \rightarrow σ_2$ gives us $σ_0 = C′[F(τ)]$ and $σ_0 = C′'[F′(τ)]$. If $C′[\cdot] \neq \cdot$ and $C′'[\cdot] \neq \cdot$, then we are in a case similar to the case for $τ_1 \rightarrow τ_2$, and we simply use induction. Otherwise, we are left with three cases:

Case $C′[\cdot] = F(\mathcal{C}[\cdot])$, $C′'[\cdot] = \cdot$: In this case, $τ = C[\mathcal{G}[F(τ)]]$. Let $τ′$ be the top-level reduct of $F(τ)$. Thus, $σ_1 = F(\mathcal{C}[τ′])$. Let $τ′ = \mathcal{C}[τ′]$. We also know that $Σ ⊢ F(τ) \rightarrow σ_2$ by a top level reduction. Inverting gives us the following:

- $C′: Ψ ∈ Σ$
- $Ψ = ψ/α ∈ Σ$, $F(\mathcal{C}[\cdot]) \rightarrow σ′$
- $τ′ = ψ/α ≈ σ′$
- $σ_2 = σ_1[ψ/α]$
- $∀ j < i, no_\text{conflict}(Ψ, i, \overline{ψ}, j)$

We want to find a common reduct of $σ_2$ (which might not be headed by $F$) and $F(τ′)$. Thus, we must find a way to reduce $F(τ′)$ at the top level. We now use Property 14 to get $\overline{τ′}$ such that $Σ ⊢ \overline{τ′} \rightarrow ψ/α = Ω(τ′)$ for some $Ω$, and for every $ψ′$ such that apart$(ψ′, τ′)$, apart$(ψ′, \overline{τ′})$.

Instead of reducing $F(τ)$ directly, we step $F(τ)$ to $F(τ′)$ (getting $Σ ⊢ F(τ′) \rightarrow F(Ω(τ′))$ from repeated application of Lemma 27) and then show that $F(Ω(τ′))$ can reduce at the top level by the same equation at $F(τ)$ reduced to form $σ_2$.

Thus, we must prove that $τ′ = ψ/α ≈ σ′$ (for some $ψ′$) and that, for all $j < i$, $no_\text{conflict}(Ψ, i, ψ′, j)$. We know that $Ψ′ = Ω(τ′)$. We also know that, by assumption, the free variables in $τ′$ are distinct from the free variables in $Ψ′$. Thus, $Ω′$ must map every free variable in $Ψ′$ to some other type. Thus, we have $τ′ = ψ/α ≈ σ′$ for the $ψ′$ taken from the range of $Ω′$.

We then perform invariance on the known facts, that for all $j < i, no_\text{conflict}(Ψ, i, ψ′, j)$. We now fix $j$, and repeat this argument for all $j < i$.

Case NC\_APART: We see that apart$(\overline{ψ}, ρ_1[ψ′/α])$. From Property 14, we see that apart$(\overline{ψ}, ρ_1[ψ′/α])$ as desired.

Case NC\_COMPATIBLE: The check $\text{compat}(Ψ[i], Ψ[j])$ does not depend on the types $ψ$ or $ψ′$, and thus we are done.

Thus, $F(\overline{τ′})$ reduces at the top level to $σ_3 = σ_1[ψ/α]$. It remains to show that $σ_2$ (which equals $σ_1[ψ/α]$), the initial top-level reduct of $F(τ′)$ reduces to $σ_3$. We know that $Σ ⊢ ρ_1[ψ/α] \rightarrow ρ_1[ψ/α]$. Thus, by repeated application of Lemma 33 (and appealing to clause 2 of Good to show that every $ψ$ is considered), we get $Σ ⊢ ψ ≈ \overline{ψ}$.

By Lemma 30, we can conclude $Σ \vdash \sigma_1[ψ/α] \rightarrow σ_1[ψ/α]$, as desired.

Case $C′'[\cdot] = F(\mathcal{C}[\cdot])$: Similar to the case above.

Case $C′'[\cdot] = \cdot, C′'[\cdot] = \cdot$: We will show a stronger property than local confluence in this case; we will show that if $Σ ⊢ F(τ) \rightarrow v_1$ and $Σ ⊢ F(τ) \rightarrow v_2$, both at the top level, then $v_1 = v_2$.

We invert both reductions to get the following facts, along with $\text{Γ}_\text{gpd}$:

<table>
<thead>
<tr>
<th>Case</th>
<th>$Σ \vdash F(τ) \rightarrow v_1$</th>
<th>$Σ \vdash F(τ) \rightarrow v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1 \cap \text{Ψ}_1 \vdash Σ$</td>
<td>$\Psi_1 = [\overline{α_1</td>
<td>ψ_1}]$, $F(\overline{ψ_1}) \rightarrow σ_1'$</td>
</tr>
<tr>
<td>$τ = ρ_1[ψ_1/α_1]$</td>
<td>$v_1 = σ_1', [ψ_1/α_1]$</td>
<td>$v_2 = σ_2', [ψ_2/α_2]$</td>
</tr>
<tr>
<td>$∀ k &lt; i$, no_\text{conflict}(Ψ, i, \overline{ψ}, k)</td>
<td>$∀ k &lt; j$, no_\text{conflict}(Ψ, j, \overline{ψ}, k)</td>
<td></td>
</tr>
</tbody>
</table>

Thus, we must show that $σ_1'[ψ_1/α_1] = σ_2'[ψ_2/α_2]$. From clause 3 of Good, we see that either $i = j = 0$ (open family) or $C_1 = C_2$ (closed family). We will tackle these cases separately:

**Open family**: In this case, the axioms $C_1$ and $C_2$ have one equation each and thus we simply drop the $i$ and $j$ subscripts. Let $Ψ_1$ and $Ψ_2$ be the the equations of $C_1$ and $C_2$, respectively.

We know from the inferences that $ρ_1[ψ_1/α_1] = ρ_2[ψ_2/α_2]$. Let $Ω_2 = [α_1 → ψ_1, α_2 → ψ_2]$. We can see that $Ω_2$ is a unifier of $\overline{ψ_1}$ and $\overline{ψ_2}$. Then, clause 4 of Good tells us that $\text{compat}(Ψ_1, Ψ_2)$. Here, we have two cases:

**Case COMPAT\_COINCIDENT**: We know that $Ω_1$ is a most general unifier of $\overline{ψ_1}$ and $\overline{ψ_2}$ (appealing to Property 11) and $Ω_1(σ_1′) = Ω_2(σ_1′)$. Thus, there must be some $Ω$ such that $Ω(σ_1′) = Ω ∩ Ω_1$.

We must show that $σ_1'[ψ_1/α_1] = σ_2'[ψ_2/α_2]$. This equation is equivalent to $Ω(σ_1′) = Ω_1(σ_1′)$, which in turn is $Ω_1(σ_1′) = Ω(σ_1′)$. But, we know that $Ω_1(σ_1′) = Ω_2(σ_1′)$, so we are done.

**Case COMPAT\_DISTINCT**: We know that unify($\overline{ψ_1}, \overline{ψ_2}$) fails. Yet, we have $Ω_2$ as a unifier of these types. Appealing to Property 40, we have a contradiction, and thus this case cannot happen.

**Closed family**: We know $C_1 = C_2$ and, by $\text{Γ}_\text{gpd}$, $Σ$, there can be only one axiom of the same name in the context, so $Ψ_1 = Ψ_2$, and thus we can drop the 1 and 2 subscripts, except on the $ψ$, which do not appear in the axiom types.

Thus, we must show $σ_1'[ψ_1/α_1] = σ_2'[ψ_2/α_2]$. Now, we must examine the indices $i$ and $j$. If $i = j$, then we are done by an application of Lemma 24, using $ρ_1[ψ_1/α_1] = ρ_2[ψ_2/α_2]$ and clause 2 of Good. So, we assume, without loss of generality, that $i > j$. Inverting $no_\text{conflict}(Ψ, i, \overline{ψ}, j)$ leads us to three cases:

**Case NC\_APART**: We see here that apart$(\overline{ψ}, ρ_1[ψ′/α])$.

Yet, we know from the original inferences that $ρ_1[ψ_1/α_1] = ρ_1[ψ_2/α_2]$ and clause 2 of Good. So, we assume, without loss of generality, that $i > j$. Inverting $no_\text{conflict}(Ψ, i, \overline{ψ}, j)$ leads us to three cases:

**Case NC\_COMPATIBLE**: Here, we know that $Ω$ is a most general unifier (appealing to Property 11) for $\overline{ψ_1}$ and $\overline{ψ_2}$ and that $Ω(σ_1′) = Ω(σ_2′)$.

From the original inferences, we know $ρ_1[ψ_1/α_1] = ρ_1[ψ_2/α_2]$. Let $Ω_2 = [α_1 → ψ_1, α_2 → ψ_2]$. We can say $Ω_2 = Ω ∩ Ω_1$ for some $Ω′$. We can rewrite our goal as showing that $Ω_1(Ω(σ_1′)) = Ω′(Ω_1(σ_1′))$. This is immediate from the fact that $Ω(σ_1′) = Ω(σ_1′)$, and so we are done.

**Case NC\_COMPATIBLE\_COMPAT\_COINCIDENT**: We know that unify($\overline{ψ_1}, \overline{ψ_2}$) fails. Yet, we know from the original in-
versions that \( \rho[\psi_1 / \alpha_i] = \rho[\psi_2 / \alpha_i] \). The substitution 
\( \alpha_i \mapsto \psi_1, \alpha_j \mapsto \psi_2 \) is then a unifier of \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), leading 
to a contradiction, appealing to Property 11.

Lemma 35 (Confluence of terminating systems). If Good \( \Sigma \) and \( \Sigma \vdash \cdot \leadsto \cdot \) is a terminating rewrite relation, then it is confluent.

Proof. By appealing to Newman’s lemma (Newman 1942) and 
Lemma 34.

E.3 From confluence to consistency

Lemma 36 (Transitivity). If \( \Sigma \vdash \cdot \leadsto \cdot \) is a terminating rewrite relation, Good \( \Sigma, \Sigma \vdash \tau_1 \Leftrightarrow^* \tau_2 \) and \( \Sigma \vdash \tau_2 \Leftrightarrow^* \tau_3 \), then \( \Sigma \vdash \tau_1 \Leftrightarrow^* \tau_3 \).

Proof. By Lemma 35.

Lemma 37 (Congruence). If \( \Sigma \vdash \tau_1 \Leftrightarrow^* \tau_2 \), then \( \Sigma \vdash C[\tau_1] \Leftrightarrow^* C[\tau_2] \).

Proof. By appealing to Lemma 27.

Lemma 38 (Completeness). If \( \Sigma \vdash \cdot \leadsto \cdot \) is a terminating rewrite relation, Good \( \Sigma \) and \( \Sigma; \Delta \vdash \gamma : \sigma_1 \sim \sigma_2 \), then \( \Sigma \vdash \sigma_1 \Leftrightarrow^* \sigma_2 \).

Proof. We proceed by induction on \( \Sigma; \Delta \vdash \gamma : \sigma_1 \sim \sigma_2 \):

Cases CO_ARROW, CO_FORALL, CO_APP, and CO_TYFAM:
By the induction hypothesis, appealing to Lemma 37 and 
Lemma 36.

Cases CO_REFL, CO_SYM, and CO_TRANS: From the fact that 
\( \Sigma \vdash \cdot \Leftrightarrow^* \cdot \) is an equivalence relation, appealing to 
Lemma 36.

Cases CO_LEFT and CO_RIGHT: The induction hypothesis gives 
us that \( \Sigma \vdash \tau_1 \Leftrightarrow^* \tau_2 \Leftrightarrow^* \sigma_1 \sim \sigma_2 \), and we can see that 
you may produce a type application must also have a type application. Thus, 
the common case must be \( \nu_1 \nu_2 \) (for some \( \nu_1 \) and \( \nu_2 \)) where \( \nu_1 \) joins \( \tau_1 \) and \( \nu_1 \) and \( \nu_2 \) joins \( \tau_2 \) and \( \sigma_2 \). Thus, we do.

Case COAXIOM: From clause 1 of Good and the COAXIOM 
rule, we know that \( \sigma_1 = F(v_1, [\rho/\alpha_i]) \) and that \( \sigma_2 = v_2'[\rho/\alpha_i] \). We conclude that 
\( \Sigma \vdash \sigma_1 \Leftrightarrow^* \sigma_2 \), as the premises of the rule 
RED are all given by the premises of the rule COAXIOM.

Lemma 39 (Consistency). If \( \Sigma \vdash \cdot \leadsto \cdot \) is a terminating rewrite relation and Good \( \Sigma \), then \( \Sigma \) is consistent.

Proof. A consistent coercion equates two types with the same 
ground head forms. By Lemma 38, these two types must be 
joinable under the rewrite relation. Yet, the rewriting rule preserves 
all head forms except for type families. As type families are not 
ground head forms, we are done.

F. Proof of properties of apart

This appendix includes the proofs that our concrete definition of 
apart, as given in Definition 6, satisfies the properties stated in 
Section 5.1. Then, we show that these properties, along with the 
assumption of termination, imply the high-level (sanity-check) properties 
from Section 3.2. It is well-founded to use our confluence 
result for these later proofs as those properties are not used anywhere 
in other proofs—they simply serve as a higher-level check 
on our formal results.

F.1 Proofs of Properties 12–14

We restate our implementation of apart:

Definition (Apartness [Definition 6]).

\[
\text{apart}(\rho, \tau) = \neg \text{unify}_\infty(\rho, \text{flatten}(\tau))
\]

Recall that \( \text{flatten} \) (Definition 5) replaces all type family 
capplications in a (finite) type with fresh variables, maximally preserving 
sharing. That is, flattening the same type family application twice 
in the same type (or list of types) converts both applications to the 
same fresh variable. In order for \( \text{flatten} \) to be a well-defined 
function, it must refer to a mapping from every possible type headed by a 
type family to fresh variables. This mapping is countably infinite, 
but we can assume, as usual, a countably infinite set of fresh 
variables. Furthermore, we assume that the set of variables in the range 
of this mapping is distinct from variables used elsewhere (particularly, 
in patterns). If this assumption is violated for some use of 
\( \text{flatten} \), we simply rename the variables accordingly.

The above definition of flattening with respect to an infinite 
mapping of type families to variables, means that flattening 
commutes with type constructors. For example, \( \text{flatten}(\tau_1 \rightarrow \tau_2) = \text{flatten}(\tau_1) \rightarrow \text{flatten}(\tau_2) \).

For completeness, we also restate the correctness of unification, 
but now for \( \text{unify}_\infty \).

Property 40 (unify\_\infty correct). If and only if there exists a substitution 
\( \omega \) (whose range may include infinite types) such that \( \omega(\sigma) = \omega(\tau) \), then \( \text{unify}_\infty(\sigma, \tau) \) succeeds, returning \( \omega \). Furthermore, \( \omega \) is 
a most general unifier of \( \sigma \) and \( \tau \).

Before getting to the properties themselves, we must prove 
some properties about flatten. First, we extend flatten to apply to 
substitutions and define an inverse operation:

Definition 41 (Flattening a substitution). If \( \Omega = [\alpha_1 \mapsto \tau_1] \), we 
say \( \text{flatten}(\Omega) \) for \( [\alpha_1 \mapsto \text{flatten}(\tau_1)] \), where sharing is maximally 
preserved between the different types \( \tau \).

Definition 42 (Inverse flattening). We let \( \text{flatten}^{-1} \) denote the 
inverse operation to flattening, implemented by doing a reverse 
lookup in the map from type family applications to variables.

Note that \( \text{flatten}^{-1} \) is a substitution, infinite in extent, but 
ordinary in other respects. In particular, note that the elements in the 
range of \( \text{flatten}^{-1} \) are finite—that is, \( \text{flatten}^{-1} \) could be denoted 
by the metavariable \( \Omega \).

Lemma 43 (Flattened substitutions). For all type patterns \( \rho \) and 
substitutions \( \Omega \), \( \text{flatten}(\Omega(\rho)) = (\text{flatten}(\Omega))(\rho) \).

Proof. The pattern \( \rho \) contains no type families, so flatten 
does not affect the parts of \( \rho \) unchanged by the application of \( \Omega \). Because 
\( \text{flatten} \) preserves maximal sharing, it must be the case that applying 
a flattened substitution yields the same result as flattening an 
substituted pattern. This can be shown by straightforward induction 
on \( \rho \).

Lemma 44 (Flattening commutes with substitution). For all 
\( \Omega \), there exists an \( \Omega' \) such that, for all \( \tau \), \( \text{flatten}(\Omega(\tau)) = \Omega'(\text{flatten}(\tau)) \).

Proof. We can say that 
\( \text{flatten}(\Omega(\tau)) = \text{flatten}(\Omega(\text{flatten}^{-1}(\text{flatten}(\tau)))) \)

Because \( \text{flatten}^{-1} \) is a substitution, and appealing to Lemma 43 
(noting that \( \text{flatten}(\tau) \) is a pattern), we can rewrite this as \( \text{flatten}(\Omega(\text{flatten}^{-1}(\text{flatten}(\tau)))) \). Thus, we let \( \Omega' \) be the substitution \( \text{flatten}(\Omega(\text{flatten}^{-1}(\text{flatten}(\tau)))) \) and we are done.
Lemma 45 (Flattening a list commutes with substitution). For all $\Omega$, there exists an $\Omega'$ such that, for all $\tau$, $\text{flatten}(\Omega') = \Omega'(\text{flatten}(\Omega))$.

Proof. By induction on the length of the list, appealing to Lemma 44.

Property 45 (Apartness is stable under type substitution [Property 12]). If $\text{apart}(\tau_1, \tau_2)$, then for all substitutions $\Omega$, $\text{apart}(\Omega(\tau_1), \Omega(\tau_2))$.

Proof. Expanding definitions, we must show that
\[\neg \text{unify}_\infty(\tau_1, \text{flatten}(\tau_2))\]
implies
\[\neg \text{unify}_\infty(\Omega(\tau_1), \text{flatten}(\Omega(\tau_2))).\]
We prove the contrapositive, that is, that $\text{unify}_\infty(\tau_1, \text{flatten}(\Omega(\tau_2)))$ implies $\text{unify}_\infty(\Omega(\tau_1), \text{flatten}(\tau_2))$. Thus, we have a substitution $\omega$ such that $\omega(\tau_1) = \text{flatten}(\Omega(\tau_2))$ and must find an $\omega'$ such that $\omega'(\tau_1) = \omega'(\text{flatten}(\tau_2))$.

By Lemma 45, we can say $\text{flatten}(\Omega(\tau_2)) = \Omega'(\text{flatten}(\tau_2))$ for some $\Omega'$. Then, choose $\omega' = \omega \circ \Omega'$. We can see that $\omega'(\tau_1)$ and $\omega'(\text{flatten}(\tau_2))$ (noting that the variables in $\tau_1$ are fresh from those in $\text{flatten}(\tau_2)$) as desired.

Property 46 (No unifiers for apart types [Property 13]). If $\text{apart}(\tau_1, \tau_2)$, then there exists no substitution $\Omega$ such that $\Omega(\tau_1) = \Omega(\tau_2)$.

Proof. Expanding definitions, we must show that $\neg \text{unify}(\tau_1, \text{flatten}(\tau_2))$ implies $\neg \text{unify}(\tau_1, \tau_2)$. We will show the contrapositive. Thus, we assume $\Omega$ such that $\Omega(\tau_1) = \Omega(\tau_2)$ and must find $\Omega'$ such that $\Omega'(\tau_1) = \Omega'(\text{flatten}(\tau_2))$.

Choose $\Omega' = \Omega \circ \text{flatten}^{-1}$. Because the free variables in $\tau_1$ are distinct from the variables in $\text{flatten}^{-1}(\tau_2)$, we have $\Omega'(\text{flatten}(\tau_2)) = \Omega(\tau_2)$. We also have
\[\Omega'(\text{flatten}(\tau_2)) = \Omega(\text{flatten}^{-1}(\text{flatten}(\tau_2))) = \Omega(\tau_2)\]
and we are done.

Property 47 (Apartness can be regained after reduction). If $\tau = \Omega(\tau)$ and $\Sigma \vdash \tau \leadsto^* \tau'$, then there exists a $\tau''$ such that
1. $\Sigma \vdash \tau'' \leadsto^* \tau$,
2. $\tau'' = \Omega'(\tau)$ for some $\Omega'$, and
3. for every $\rho$ such that $\text{apart}(\rho', \tau)$: $\text{apart}(\rho', \tau'')$.

Proof. We know that $\tau$ matches some pattern $\tau'$ and that one element in $\tau'$ steps, forming $\tau''$. Suppose that one element is $\tau_k$. Thus, $\Sigma \vdash \tau_k \leadsto^* \tau_k'$. Inverting this step relation gives us that $\tau_k = C[F(\tau)]$ and $\tau_k' = C[\tau']$, where $F(\tau)$ reduces to $\tau'$ at the top level.

Define $\mathcal{C}[\tau]$ to be the list of types $\tau$ such that every occurrence of $F(\tau)$ is replaced by $\tau$. Thus, $\mathcal{C}[F(\tau)] = \tau$. We choose $\tau''$ (from the statement of the property) to be $\mathcal{C}[\tau']$. We must show the following:

\* $\Sigma \vdash \tau'' \leadsto^* \tau'$: Straightforward application of the rule RED.

\* $\tau'' = \Omega'(\tau)$ for some $\Omega'$: Because $\tau'$ cannot contain type families, it must be that $\Omega$ maps some variables to types containing $F(\tau)$. Choose $\Omega'$ to be $\Omega$ with all occurrences of $F(\tau)$ replaced by $\tau'$. Because all occurrences of $F(\tau)$ in $\tau'$ have been replaced by $\tau'$, we can see that $\Omega'(\tau) = \tau''$.

\* For every $\rho'$ such that $\text{apart}(\rho', \tau)$, we have $\text{apart}(\rho', \tau'')$: Assume we have $\rho'$ such that $\text{apart}(\rho', \tau)$. Unfolding definitions (and taking the contrapositive) gives us $\omega$ such that $\omega(\tau') = \omega(\text{flatten}(\tau'))$, and we must find $\omega'$ such that $\omega'(\tau') = \omega'(\text{flatten}(\tau'))$.

Let $\alpha$ be the variable mapped from $F(\tau')$. Thus, $\text{flatten}(F(\tau')) = \alpha$. Let $\Omega_0 = [\alpha \mapsto \tau']$ and choose $\omega = \omega \circ \Omega_0$. Noting that $\alpha$ does not appear in $\rho'$, we see that $\omega(\tau') = \omega'(\tau')$. Now, we must only show that $\omega'(\text{flatten}(\tau')) = \omega'(\text{flatten}(\tau'))$. By our choice of $\omega'$, we know $\omega'(\text{flatten}(\tau')) = \omega'(\Omega_0(\text{flatten}(\tau')))$, thus we must show $\Omega_0(\text{flatten}(\tau')) = \text{flatten}(\tau')$. By its definition, $\text{flatten}$ takes all occurrences of $F(\tau')$ in $\tau'$ to $\alpha$. Then, $\Omega_0$ takes all of these occurrences of $\alpha$ to $\tau'$. Since the only difference between $\tau$ and $\tau'$ is that all occurrences of $F(\tau')$ are replaced by $\tau'$, we can see that $\Omega_0(\text{flatten}(\tau'))$ is indeed $\text{flatten}(\tau')$, and we are done.

F.2 Proofs of Properties 2 and 4

Property (Apartness through substitution [Property 2]). If $\text{apart}(\rho, \tau)$ then there exists no $\Omega$ such that $\text{match}(\rho, \Omega(\tau))$.

Proof. We shall prove by contradiction: assume $\Omega$ and $\Omega'$ such that $\Omega(\rho) = \Omega(\tau)$. We can simplify a bit and combine these substitutions, because the free variables of $\rho$ are distinct from those in $\tau$: we can say $\Omega_0(\rho) = \Omega_0(\tau)$. Then, this is a contradiction, appealing to Property 13, and we are done.

The next property (Property 4) requires an important auxiliary lemma.

Lemma 48 (Matching can be regained after reduction). If $\Sigma$ and $\Sigma \vdash \rho \leadsto^* \tau$ then there exists an $\Omega'$ such that $\Sigma \vdash \tau \leadsto^* \Omega'(\rho)$.

Proof. Throughout this proof, we will consider types as abstract syntax trees. We will use “type” and “tree” interchangeably.

Define the operation linearize to take a pattern and freshen all the type variables therein, thus producing a linear pattern. Our first step is to show that $\tau$ matches linearize($\rho$). How does $\Omega(\rho)$ step to $\tau$? It must be through a series of type family reductions. Because $\rho$ does not mention type families, these type families must occur in $\Omega(\rho)$ at or beneath where variables appear in the tree $\rho$. Thus, as $\Omega(\rho)$ steps, the tree structure imposed by $\rho$ does not change. However, it is possible that a type family application, say $F(\tau)$ steps in two different ways throughout the tree $\Omega(\rho)$ as $\Omega(\rho)$ is reducing. Thus, we can claim only that $\tau$ matches linearize($\rho$), not $\rho$ itself.

When comparing the trees $\tau$ and $\rho$, define a mismatch to be two locations in the respective trees where $\rho$ has a repeated variable and $\tau$ has two different sub-trees. Count only those matches that involve the left-most occurrence of a variable in $\rho$. We proceed by induction on the number of mismatches between $\rho$ and $\tau$.

Base case: If there are no mismatches, then we know that $\rho$ must match $\tau$ with a substitution $\Omega$. We are done.

Inductive case: Choose the left-most mismatch. Say that the repeated variable in $\rho$ is $\alpha$ and the disagreeing types in $\tau$ are $\sigma_1$ and $\sigma_2$. We know that $\Sigma \vdash \Omega(\rho) \leadsto^* \tau$, and thus that $\Sigma \vdash \Omega(\alpha) \leadsto^* \sigma_1$ and $\Sigma \vdash \alpha \leadsto^* \sigma_2$. By confluence (Lemma 35), we know that there exists a $\sigma_3$ such that $\Sigma \vdash \sigma_1 \leadsto^* \sigma_3$ and $\Sigma \vdash \sigma_2 \leadsto^* \sigma_3$. Let $\tau'$ be $\tau$, except that both $\sigma_1$ and $\sigma_2$ in $\tau$ are replaced by $\sigma_3$ in $\tau'$. We know that $\Sigma \vdash \tau' \leadsto^* \tau'$ by congruence of the rewrite relation and thus that $\Sigma \vdash \Omega(\rho) \leadsto^* \tau'$. Thus, we can use the induction hypothesis to get $\Omega'$ such that $\Sigma \vdash \tau' \leadsto^* \Omega'(\rho)$. Then, by transitivity of $\Sigma \vdash \tau' \leadsto^* \tau'$, we are done.

\[\square\]
Lemma 47 (Matching normal forms). If \( \text{Good} \Sigma \) and \( \Sigma \vdash \Omega(\rho) \rightharpoonup^* v \) where \( v \) is a normal form, then there exists \( \Omega' \) such that \( v = \Omega'(\rho) \).

Proof. We apply Lemma 46 to see that there exists an \( \Omega' \) such that \( \Sigma \vdash v \rightharpoonup^* \Omega'(\rho) \). But, we know that \( v \) cannot step, and thus \( v = \Omega'(\rho) \).

Lemma 48 (Longest reduction). Suppose \( \text{Good} \Sigma \). For every type \( \tau \) and its normal form \( v \) (whose uniqueness is guaranteed by the combination of confluence and termination), there exists a number \( n \) such that all reductions from \( \tau \) to \( v \) are of length at most \( n \).

Proof. König’s lemma states that every tree with infinitely many vertices, each having finite degree, has at least one infinite simple path. Here, we are considering trees of reductions, rooted at \( \tau \). We will use the contrapositive of König’s lemma: that if every node in a tree has finite degree and all simple paths are finite, then there are finitely many vertices. For any type \( \sigma \), there are finitely many types \( \alpha' \) such that \( \Sigma \vdash \sigma \rightharpoonup^* \alpha' \), because there are finitely many locations within \( \sigma \) that can be headed by a type family and finitely many equations that type family application might match. By termination, we know all simple paths in the tree of reductions are finite. Thus, the contrapositive of König’s lemma tells us that the tree has a finite number of nodes. Thus, we can simply enumerate all paths from \( \tau \) to \( v \) to discover the one with the longest path. This path’s length is our result.

Lemma 49 (Apartness and normal forms). If \( \text{apart}(\rho, \tau) \) and \( \Sigma \vdash \tau \rightharpoonup^* v \) where \( v \) is a normal form, then \( \text{apart}(\rho, v) \).

Proof. Let the longest path from \( \tau \) to \( v \) be of length \( n \) (Lemma 48). We perform induction on \( n \).

Base case: Trivial.

Inductive case: We know that \( \tau \) can step to some \( \tau' \); that is, \( \Sigma \vdash \tau \rightharpoonup \tau' \). We then appeal to Property 14 (choosing \( \rho = \text{flatten}(\tau) \), but the choice is irrelevant) to get \( \tau'' \) such that \( \Sigma \vdash \tau' \rightharpoonup^* \tau'' \) and \( \text{apart}(\rho, \tau'') \). By our assumption that \( n \) is the length of the longest path from \( \tau \) to \( v \) and the fact that \( \Sigma \vdash \tau \rightharpoonup^* \tau' \) by at least one step, we know that the longest path from \( \tau'' \) to \( v \) has length less than \( n \). Thus, we can use the induction hypothesis, and we are done.

Lemma 50 (Apartness implies no match). If \( \text{apart}(\rho, \tau) \), then \( \neg \text{match}(\rho, \tau) \).

Proof. We prove by contradiction. Assume \( \Omega \) such that \( \Omega(\rho) = \tau \). By the assumption that pattern variables are fresh, we can say \( \Omega(\rho) = \Omega(\tau) \). Then, by Property 13, we have a contradiction.

Property (Apartness through reduction and substitution [Property 4]). If \( \text{apart}(\rho, \tau) \), then for any \( \tau' \) such that \( \tau \rightharpoonup^* \tau' \), \( \neg \text{match}(\rho, \tau') \).

Proof. Let \( v \) be the unique normal form of \( \tau \). By Lemma 49, we know \( \text{apart}(\rho, v) \). By Lemma 50, \( \neg \text{match}(\rho, v) \). Note that the uniqueness of normal forms, we know \( \Sigma \vdash \tau' \rightharpoonup^* v \). By the contrapositive of Lemma 47, we see that \( \neg \text{match}(\rho, \tau') \) as desired.

G. Proof of compatibility soundness

In this appendix, we show that the concrete implementation of compatibility (Definition 8) satisfies the definition of compatibility (Property 7). We use the implementation of compatibility included in our formal inference rules, as it separates Definition 8 into its two cases:

<table>
<thead>
<tr>
<th>Equation compatibility</th>
<th>( \text{compat}(\Phi_1, \Phi_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi_1 = [\alpha_1:\kappa_1], \ F(\mathfrak{p}_1) \sim \tau_1 )</td>
<td></td>
</tr>
<tr>
<td>( \Phi_2 = [\alpha_2:\kappa_2], \ F(\mathfrak{p}_2) \sim \tau_2 )</td>
<td></td>
</tr>
<tr>
<td>( \text{unify}(\mathfrak{p}_1, \mathfrak{p}_2) = \Omega )</td>
<td></td>
</tr>
<tr>
<td>( \Omega(\tau_1) = \Omega(\tau_2) )</td>
<td></td>
</tr>
</tbody>
</table>

We generalize Property 7 to work with \( \text{unify}_\Omega \).

Property 51 (Compatibility with infinite unification). Two type-family equations \( p \) and \( q \) are compatible iff \( \omega_1(\text{lhs}_p) = \omega_2(\text{lhs}_q) \) implies \( \omega_1(\text{rhs}_p) = \omega_2(\text{rhs}_q) \).

Proof. For all type family equations \( \Phi_1 \) and \( \Phi_2 \), where \( \Phi_1 = [\alpha_1:\kappa_1], \ F(\mathfrak{p}_1) \sim \tau_1 \) and \( \Phi_2 = [\alpha_2:\kappa_2], \ F(\mathfrak{p}_2) \sim \tau_2 \), we must show that \( \text{compat}(\Phi_1, \Phi_2) \) implies that, for all \( \omega_1 \) and \( \omega_2 \) such that \( \omega_1(\mathfrak{p}_1) = \omega_2(\mathfrak{p}_2) \), it is the case that \( \omega_1(\tau_1) = \omega_2(\tau_2) \).

We have two cases:

Case \( \text{COMPAT_COINCIDENT} \): Here, we know that \( \omega(\mathfrak{p}_1) = \omega(\mathfrak{p}_2) \) and, by Property 40, that \( \omega \) is a most general unifier. We further know that \( \omega(\tau_1) = \omega(\tau_2) \). By assumption, \( \omega(\mathfrak{p}_1) = \omega(\mathfrak{p}_2) \). By the assumption that all patterns in type families have distinct variables, we know that the domains of \( \omega_1 \) and \( \omega_2 \) are distinct. Thus, we can write \( \omega' = \omega_1 \cup \omega_2 \), and say \( \omega'(\mathfrak{p}_1) = \omega'(\mathfrak{p}_2) \). Similarly, we can say that we wish to show \( \omega'(\tau_1) = \omega'(\tau_2) \). Because \( \omega \) is a most general unifier, we can say that \( \omega' = \omega'' \circ \omega \) for some \( \omega'' \). Thus, we wish to show \( \omega''(\omega(\tau_1)) = \omega''(\omega(\tau_2)) \). But, we know that \( \omega(\tau_1) = \omega(\tau_2) \) so we are done.

Case \( \text{COMPAT_DISTINCT} \): Here, we know that there exists no \( \omega \) such that \( \omega(\mathfrak{p}_1) = \omega(\mathfrak{p}_2) \). Yet, we have assumed that \( \omega_1(\mathfrak{p}_1) = \omega_2(\mathfrak{p}_2) \) and by an argument similar to the last case, we can combine \( \omega_1 \) and \( \omega_2 \) to \( \omega' \). This substitution \( \omega \) is then a unifier, leading to a contradiction.