A Modal Specification Theory for Timing Variability

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Abstract
Modal specifications are classical formalisms that can be used to express the functional variability of systems; it is particularly useful for capturing the stepwise refinement of component-based design. However, the extension of such formalisms to real-time systems has not received adequate attention. In this paper, we propose a novel notion of time-parametric modal specifications to describe the timing as well as functional variability of real-time systems. We present a specification theory on modal refinement, property preservation and compositional reasoning. We also develop zone-graph based symbolic methods for the reachability analysis and modal refinement checking. We demonstrate the practical application of our proposed theory and algorithms via a case study of medical device cyber-physical systems.

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Comments
A Modal Specification Theory for Timing Variability *

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Abstract. Modal specifications are classical formalisms that can be used to express the functional variability of systems; it is particularly useful for capturing the stepwise refinement of component-based design. However, the extension of such formalisms to real-time systems has not received adequate attention. In this paper, we propose a novel notion of time-parametric modal specifications to describe the timing as well as functional variability of real-time systems. We present a specification theory on modal refinement, property preservation and compositional reasoning. We also develop zone-graph based symbolic methods for the reachability analysis and modal refinement checking. We demonstrate the practical application of our proposed theory and algorithms via a case study of medical device cyber-physical systems.

1 Introduction

Modal specifications introduced by Larsen and Thomsen in 1980s [2] allow the stepwise refinement design of component-based systems. The classical formalism is modal transition systems (MTSs), which are essentially labelled transition systems equipped with two types of transitions: must transitions which are required in all implementations, and may transitions which are allowed (but not required). Refining a MTS is an iterative process that chooses to implement or omit each of the may transitions, until a fully specified implementation is achieved. Thus, a MTS permits a set of different implementations, making it possible to express the functional variability of systems. Indeed, modal specifications have proven useful in many applications, for example, software product lines [16] where a specification defines a family of various products. More recently, we applied modal specifications to on-demand medical device systems [14], allowing the design of complex medical device interfaces and the verification of safety properties at the system level.

One limitation of our previous work is that the focus was on the time-abstract behavior of systems. However, safety critical systems such as medical devices typically have real-time elements, and often the safety and/or effectiveness of such systems depends on doing the right thing at the right time. For example, we might require that an infusion pump starts delivering drugs within $t$ time units after it receives an on signal. This would be an example of a timeliness constraint; while we require that something

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happens by time $t$ it would be ok if it happened earlier. Another example could be a watchdog timer that puts an actuator into a fail-safe mode if it hasn’t received a recent control signal. In this situation we probably don’t want the watchdog to activate prior to time $t$, because it might disrupt the normal function of the system.

There are a few extensions on timed modal specifications [9, 8, 7], but none of them considers the timing variability of systems. For example, suppose we want to design a medical system that satisfies the following requirement: “the infusion pump should start infusing no later than 3 time units, once it receives an on signal”; we may express this requirement with a clock constraint $x \leq 3$ in the specification, meaning that the clock $x$ can progress within 3 time units. If we follow the theory of existing literature, then implementations must preserve the exact timing behavior of specification, i.e., $x \leq 3$ for all implementations. Nevertheless, from the system designer’s perspective, we think that an infusion pump requiring $x \leq 2$ is also a good implementation, since it does not violate the deadline. To capture this kind of timing variability, we propose the notion of time-parametric modal specifications which handles the above example by using a clock constraint $x \leq \alpha$, where $\alpha$ is a parameter satisfying $\alpha \leq 3$ and $\alpha \in \mathbb{N}$.

In our specification theory, an implementation is modeled as a timed I/O automaton (TIOA). Time-parametric modal specifications augment the notion of TIOAs, by distinguishing must and may transitions (for functional variability) and allowing parametric clock constraints (for timing variability). The relation between a specification and an implementation is captured by modal refinement. We prove that the satisfaction of safety and liveness properties on specifications is preserved on its implementations. To reason about the behavior of systems with multiple components, we also define the notion of composition and prove the compositional property of modal refinement. We develop a symbolic reachability analysis algorithm for time-parametric modal specifications, based on parametric zone-graphs. We also present a decision algorithm for the specification-implementation relation. Finally, we illustrate the applicability of our theory through a case study of medical device plug and play systems.

To the best of our knowledge, this is the first modal specification theory that can handle both timing and functional variability of real-time systems. In summary, there are three main contributions of this paper:

1. We propose a novel notion of time-parametric modal specifications, which can be used to describe the timing and functional variability of real-time systems; based on this notion, we develop a specification theory on modal refinement, property preservation and compositional reasoning.
2. We present a new symbolic semantical model as parametric zone-graphs, and develop two symbolic algorithms for the reachability analysis and modal refinement checking.
3. We demonstrate the practical application of our theory and algorithms with a case study of medical device systems.

The rest of the paper is organized as follows: we describe contributions (1), (2) and (3) in Sections 2, 3 and 4, respectively; we then make a conclusion and point out directions for future work in Section 5.
Related work. The seminal paper on modal transition systems (MTSs) was by Larsen and Thomsen [17]. Many extensions (e.g., modal specifications with data, MTSs with structured labels and disjunctive MTSs) have been developed over the years, for which an overview could be found in [2]. Note that the so-called parametric MTS [4], which uses boolean-valued parameters for expressing the functional variability of systems, should be distinguished from our notion where integer-valued parameters are used to capture timing variability.

There are a few extensions on timed modal specifications: a formalism in the CCS style was proposed in [9], but no logical characterization or model satisfaction relation is defined; Bertrand et al. [7] developed a complete specification theory for event-clock automata, a subclass of timed automata. However, none of these work considers the timing variability. There is also a formalism called MTS with durations [5], where controllable or uncontrollable intervals are used to express the duration of time; but this notion is not compositional, thus not suitable for component-based design.

Our work is also closely related to the timed automata theory [1, 12, 6]. The classical notion of parametric timed automata [13] is similar to our notion of time-parametric modal specifications in the sense that it also allows (real-valued) parameters in clock constraints, but its lack of the modal ingredient prevents the modeling of functional variability. Timed I/O automata are used as specifications in the theory of [10], but this work cannot handle timing variability.

2 Modal Specifications for Timing Variability

In this section, we propose a new notion of time-parametric modal specifications which can be used to describe the variability of timing and functional behavior in corresponding implementations. We first define parametric clock constraints in Section 2.1, and present the syntax and operational semantics of time-parametric modal specifications in Section 2.2. We develop the specification theory on modal refinement, property preservation and compositional reasoning in Sections 2.3, 2.4 and 2.5, respectively.

2.1 Clocks

As in the classical theory of timed automata [1, 12], we use a finite set of real-valued clock variables, called clocks for short, to describe the progress of time. Given a finite set of clocks $Clk$, we refer to a function $v : Clk \rightarrow \mathbb{R}_{\geq 0}$ as a clock valuation. Given $d \in \mathbb{R}_{\geq 0}$, let $v + d$ denote the clock valuation that assigns all clocks $x \in Clk$ to $v(x) + d$. The set of all clock valuations is denoted by $\mathbb{R}^{\geq 0}_{\text{Clk}}$. We use $0$ to denote the clock valuation that assigns $0$ to all clocks in $Clk$. Clocks can be reset to zero: for $r \subseteq Clk$, let $v[r \mapsto 0]$ denotes the clock valuation that resets all clocks in $r$ to $0$ and keeps the value $v(x)$ for clocks $x \in Clk \setminus r$. A clock constraint is a conjunctive formula of atomic predicates $x \sim c$, where $x \in Clk$ is a clock, $\sim \in \{\leq, <, =, >, \geq\}$ is an equality/inequality relation operator and $c \in \mathbb{N}$ is a constant. A clock valuation $v$ satisfies a clock constraint $g$, denoted by $v \models g$, iff the proposition formula of $g$ resolves to true when substituting all occurrence of clock $x$ with value $v(x)$. 

To reason about timing variability, we extend the notion of clock constraints to
parametric clock constraints, which are conjunctive formulae of predicates \( x \sim (c \pm \alpha) \)
where \( \alpha \) is a non-negative integer parameter bounded by a set of linear constraints \( C \).
Let \( \Theta \) be a set of parameters, we call a function \( f : \Theta \to \mathbb{N} \) a parameter assignment.
A parameter value \( f(\alpha) \) is valid if it satisfies the linear constraints \( C \) and \( c \pm f(\alpha) \in \mathbb{N} \). Let \( g \)
be a parametric clock constraint; by assigning a set of valid values \( f \) to its parameters
\( \Theta \), we obtain an instance of \( g \) as a clock constraint, denoted by \( g[f(\Theta)] \). For example,
a parametric clock constraint \( x \leq 1 + \alpha \) bounded by \( 1 \leq \alpha \leq 3, \alpha \in \mathbb{N} \) has three
instances: \( x \leq 2, x \leq 3 \) and \( x \leq 4 \). A clock valuation \( v \) satisfies a parametric clock
constraint \( g \), denoted by \( v \models g \), iff \( v \) satisfies all instances of \( g \); and we say that \( v \)
partially satisfies \( g \), denoted by \( v \models^p g \), iff \( v \) satisfies some instances of \( g \). Let \( g_i \) for
\( i = 1, 2 \) be two parametric clock constraints, each of which is bounded by a set of linear
constraints \( C_i \) over parameters \( \Theta_i \); their conjunction \( g = g_1 \land g_2 \) is then bounded by
the linear constraints \( C = C_1 \land C_2 \) over parameters \( \Theta = \Theta_1 \cup \Theta_2 \).

For the rest of the paper, we use \( B(\text{Clk}) \) (resp. \( P(\text{Clk}) \)) to denote the set of clock
constraints (resp. parametric clock constraints) over clocks \( \text{Clk} \). We have \( B(\text{Clk}) \subseteq P(\text{Clk}) \), since a clock constraint can be considered as a special case of parametric clock
constraints whose instance is itself.

2.2 Syntax and Operational Semantics

Now we propose the notion of time-parametric modal specifications, sometimes called
specifications for short in this paper, to capture the timing and functional variability of
different implementations modeled as timed I/O automata. A timed I/O automaton is a
timed automaton \([6]\) that distinguishes input, output and internal actions. Formally,

**Definition 1 (Timed I/O Automaton).** A timed I/O automaton (TIOA) \( A \) is a tuple
\((\text{Loc}, \tilde{l}, \text{Clk}, \text{Act}, \text{Inv}, \leadsto)\) where

- \( \text{Loc} \) is a finite set of locations, and \( \tilde{l} \in \text{Loc} \) is an initial location
- \( \text{Clk} \) is a finite set of clocks
- \( \text{Act} = \text{Act}^\text{i} \cup \text{Act}^\text{o} \cup \text{Act}^\tau \) is a finite set of actions partitioned into input \( \text{Act}^\text{i} \),
  output \( \text{Act}^\text{o} \) and internal \( \text{Act}^\tau \) actions
- \( \text{Inv} : \text{Loc} \to B(\text{Clk}) \) assigns invariants to locations\(^1\)
- \( \leadsto \subseteq \text{Loc} \times B(\text{Clk}) \times \text{Act} \times 2^{\text{Clk}} \times \text{Loc} \) is the transition relation

A transition \( \tilde{l} \xrightarrow{g, \alpha, r} \tilde{l}' \) is enabled at location \( \tilde{l} \) when guard \( g \in B(\text{Clk}) \) holds and action
\( \alpha \in \text{Act} \) occurs; any clock in \( r \subseteq \text{Clk} \) will be reset to 0 once the transition moves to
location \( \tilde{l}' \).

Syntactically, a time-parametric modal specification looks similar to a TIOA; it distin-
guishes must and may transition relations for functional variability, and allows para-
metric clock constraints for timing variability.

**Definition 2 (Time-Parametric Modal Specification).** A time-parametric modal spec-
ification \( \mathcal{M} \) is a tuple \((\text{Loc}, \tilde{l}, \text{Clk}, \text{Act}, \text{Inv}, \xrightarrow{\square}, \xrightarrow{\lozenge}, C(\Theta))\) where

\(^1\) We restrict invariants to be downwards closed as in the UPPAAL verification tool [3].
Fig. 1: An example of time-parametric modal specification.

- \(\text{Loc}, \bar{l}, \text{Clk}\) and \(\text{Act}\) are the same as in Definition 1
- \(\text{Inv}: \text{Loc} \rightarrow \mathcal{P}(\text{Clk})\) assigns downwards closed parametric clock constraints in the form of \(x \leq c \pm \alpha\) or \(x < c \pm \alpha\) to locations
- \(\leftarrow\rightarrow\subseteq \text{Loc} \times \mathcal{P}(\text{Clk}) \times \text{Act} \times 2^{\text{Clk}} \times \text{Loc}\) is the must transition relation describing required behavior
- \(\leftarrow\rightarrow\subseteq \text{Loc} \times \mathcal{P}(\text{Clk}) \times \text{Act} \times 2^{\text{Clk}} \times \text{Loc}\) is the may transition relation describing allowed behavior
- \(C(\Theta)\) is a set of linear constraints on a finite set of non-negative integer parameters \(\Theta\) that are used in \(\mathcal{P}(\text{Clk})\)

We consider only syntactically consistent specifications where \(\leftarrow\rightarrow\subseteq\leftarrow\rightarrow\), i.e., a required transition should also be allowed. Definition 2 coincides with Definition 1 if \(\leftarrow\rightarrow\subseteq\leftarrow\rightarrow\subseteq\), \(\mathcal{P}(\text{Clk}) = B(\text{Clk})\) and \(\Theta = \emptyset\).

A time-parametric modal specification abstracts the behavior of various TIOA implementations, which are required to contain all transitions from the set \(\leftarrow\rightarrow\) and allowed to optionally include transitions from the set \(\leftarrow\rightarrow\) (i.e., functional variability); transitions may occur in different timing intervals, depending on the values of specification parameters (i.e., timing variability).

Example 1. Fig. 1 shows a time-parametric modal specification for an infusion pump, which has 3 locations: detect, start and infusion. The initial location detect is indicated with an incoming arrow. There is only one clock variable \(x\). The actions include input (“bolus?”), output (“alarm!”, “flow!”, “off!”) and internal (“on”). There are two parameters \(\Theta = \{\alpha, \beta\}\), bounded by the linear constraints \(\alpha \leq \beta \leq 3\) and \(\alpha, \beta \in \mathbb{N}\). The invariants on detect and Infusion are both true (omitted in the figure), while the invariant on start is a parametric clock constraint \(x \leq \beta\). Must (resp. may) transitions are indicated by solid (resp. dashed) lines in the figure. For example, once a pump detects a “bolus?” request, it must move to the start location; then, when the guard \(x < \alpha\) is true, the pump may issue an “alarm!”, reset clock \(x\) and move back to location detect.

We follow the classical interpretation that defines the operational semantics of timed automata as timed transition systems [6].

Definition 3 (TIOA’s Operational Semantics). The operational semantics of a timed I/O automaton \(\mathcal{A} = \langle \text{Loc}, \bar{l}, \text{Clk}, \text{Act}, \text{Inv}, \rightarrow\rangle\), denoted by \([\mathcal{A}]\), is a timed transition system (TTS) represented as a tuple \((S, \pi, \Sigma, \rightarrow)\) where

- \(S = \{(l, v) \in \text{Loc} \times \mathbb{R}_{\geq 0}^{\text{Clk}} | v \models \text{Inv}(l)\}\) is an infinite set of states
- \(\pi = \langle \bar{l}, 0 \rangle\) is an initial state
Let $\Sigma = \text{Act} \cup \mathbb{R}_{\geq 0}$ be the alphabet.

We check whether a TIOA is an implementation of a time-parametric modal specifica-
tion by checking the transition relation defined by the following two rules:

- **Action transition:** $\langle l, v \rangle \xrightarrow{a} \langle l', v' \rangle$ for $a \in \text{Act}$ if there is transition $l \xrightarrow{g, a, r} l'$ in $\mathcal{M}$ such that $v \models g$, $v' \models \text{Inv}(l')$, and $v' = v[r \mapsto 0]$

- **Delay transition:** $\langle l, v \rangle \xrightarrow{d} \langle l, v + d \rangle$ for $d \in \mathbb{R}_{\geq 0}$ if $v + d \models \text{Inv}(l)$

To define the operational semantics of time-parametric modal specifications, we first extend the notion of TTSs to *modal timed transition systems*, by partitioning the transition relation into must action, may action and delay transitions.

**Definition 4 (Modal Timed Transition System).** A modal timed transition system (MTTS) $M$ is a tuple $(S, \pi, \text{Act}, \rightarrow_{\Box}, \rightarrow_{\Diamond}, \rightarrow_d)$ where

- $S$ is an (infinite) set of states, $\pi \in S$ is an initial state, and $\text{Act}$ is a set of actions
- $\rightarrow_{\Box} \subseteq S \times \text{Act} \times S$ is the must action transition relation, $\rightarrow_{\Diamond} \subseteq S \times \text{Act} \times S$ is the may action transition relation, and $\rightarrow_d \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is the delay transition relation

**Definition 5 (Specification’s Operational Semantics).** For a time-parametric modal specification $\mathcal{M} = (\text{Loc}, \overrightarrow{l}, \text{Clk}, \text{Act}, \text{Inv}, \rightarrow_{\Box}, \rightarrow_{\Diamond}, C(\Theta))$, its operational semantics, denoted by $[\mathcal{M}]$, yields a finite set of MTTSs such that there is a one-to-one mapping between a valid parameter assignment $f(\Theta)$ and a MTTS $M = (S, \pi, \text{Act}, \rightarrow_{\Box}, \rightarrow_{\Diamond}, \rightarrow_d)$ where

- $S = \{ \langle l, v \rangle \in \text{Loc} \times \mathbb{R}_{\geq 0}^{\text{Clk}} \mid v \models \text{Inv}(l)[f(\Theta)] \}$, and $\pi = \langle l, 0 \rangle \in S$
- $\langle l, v \rangle \xrightarrow{a} \langle l', v' \rangle$ (resp. $\langle l, v \rangle \xrightarrow{a} \langle l', v' \rangle$) if there is a must (resp. may) action transitions $l \xrightarrow{g, a, r} l'$ (resp. $l \xrightarrow{g, a, r} l'$) in $\mathcal{M}$ such that $v \models g[f(\Theta)]$, $v' \models v[r \mapsto 0]$ and $v' = v[f(\Theta)]$
- $\langle l, v \rangle \xrightarrow{d} \langle l, v + d \rangle$ for $d \in \mathbb{R}_{\geq 0}$ if $v + d \models \text{Inv}(l)[f(\Theta)]$

Note that $\text{Inv}(l)[f(\Theta)]$ and $g[f(\Theta)]$ are clock constraints obtained by substituting the occurrences of parameters $\Theta$ with values $f(\Theta)$. Computing the set of valid parameter assignments $f(\Theta)$ reduces to solving the linear constraints $C(\Theta)$ of $\mathcal{M}$, which can be efficiently solved by using a SMT solver such as Z3 [11].

### 2.3 Modal Refinement

We check whether a TIOA is an implementation of a time-parametric modal specification by checking the modal refinement relation.

**Definition 6 (Modal Refinement of MTTSs).** Let $M_i = (S_i, \pi_i, \text{Act}, \rightarrow_{\Box,i}, \rightarrow_{\Diamond,i}, \rightarrow_d,i)$ for $i = 1, 2$ be two MTTSs. We say that $M_1$ modally refines $M_2$, denoted by $M_1 \preceq M_2$, if there exists a binary relation $R \subseteq S_1 \times S_2$ containing $(\pi_1, \pi_2)$ such that for each $(s, t) \in R$ we have

- for all $(t, a, t') \in \rightarrow_{\Diamond,2}$ there is some $(s, a, s') \in \rightarrow_{\Diamond,1}$ with $(s', t') \in R$
- for all $(s, a, s') \in \rightarrow_{\Box,1}$ there is some $(t, a, t') \in \rightarrow_{\Diamond,2}$ with $(s', t') \in R$
Since a TTS (Definition 3) can be considered as a special case of MTTSs where \( \rightarrow □ = \rightarrow ♦ \), the above definition is also applicable for \( A ⪯ M \) where \( A \) is a TTS and \( M \) is a MTTS.

**Definition 7 (Implementation).** Let \( A \) be a TIOA and \( M \) be a time-parametric modal specification over the same actions. We say that \( A \) is an implementation of \( M \), denoted by \( A ⊑ M \), if there exist a MTTS \( M \in \mathcal{J}_M \) such that \( J_A K \| = M \).

A time-parametric modal specification \( M \) may admit a set of implementations with timing and functional variability. By fixing a parameter assignment for the specification, a MTTS \( M \in \mathcal{J}_M \) representing certain timing behavior is chosen; and by checking whether the operational semantics of \( A \) modally refines \( M \), we compare their functional behavior. \( A \) is an implementation of \( M \) when both the timing and functional requirements are met. Later in Section 3.3, we present a symbolic method to check the specification-implementation relation.

**Example 2.** Consider the time-parametric modal specification shown in Fig. 1. The TIOA in Fig. 2a is an implementation of the specification, where the may transition labelled by "alarm!" is implemented and the parameters are fixed for \( α = 1, β = 3 \).

The TIOA in Fig. 2b is not a valid implementation, because the invariant in location \texttt{start} is \( x ≤ 5 \), violating the specification’s constraints of \( x ≤ β \) and \( β ≤ 3 \).

### 2.4 Model Checking

**Property preservation** (i.e., the satisfaction of certain property on a specification implies that all implementations also satisfy the property) is a crucial for component-based design. In this section, we prove the property preservation for time-parametric modal specifications. We consider two most commonly used types of properties: **safety** and **liveness**.

Let \( \phi \) be state formulae about locations and time. A safety property claims “something bad will never happen”; formally, it can be written in temporal logic \( \mathcal{A} [\square \phi] \) (i.e., invariably \( \phi \)), or expressed as the negation of reachability property \( \neg \mathcal{E} [\phi] \) (i.e., never possibly \( \phi \)). A liveness property says “something will eventually happen”, expressed with path formulae \( \mathcal{A} [\phi] \) (i.e., always eventually \( \phi \)). We refer the readers to [6] for the details of temporal logic.

Let \( A \) be a TIOA implementation and \( \psi \) be a safety or liveness property; we say that \( A \) satisfies \( \psi \), denoted by \( A \models \psi \), iff its operational semantics \( [A] \models \psi \). Let \( M \)
be a MTTS, we define \( M \) must (resp. may) satisfy \( \psi \), denoted by \( M \models \Box \psi \) (resp. \( M \models \Diamond \psi \)), iff the formula \( \psi \) is true on paths that only contain must (resp. may) and delay transitions. \( M \models \Box \theta \) iff \( M \models \Diamond \phi \) is false. We say that a time-parametric modal specification \( M \) must satisfy a property \( \psi \), denoted by \( M \models \Box \psi \), iff \( M \models \Diamond \psi \) for all MTTSs \( M \in [\mathcal{M}] \); and \( M \) may satisfy \( \psi \), denoted by \( M \models \Diamond \psi \), iff \( M \models \Box \psi \) for some \( M \in [\mathcal{M}] \).

**Lemma 1.** Let \( A \) be a TTS and \( M \) be a MTTS such that \( A \preceq M \). Let \( \psi \) be a safety or liveness property. Suppose \( M \models \Box \psi \), then \( A \models \psi \).

**Proof.** See the appendix.

**Theorem 1.** Let \( A \) be a TIOA implementation of a time-parametric modal specification \( M \), i.e., \( A \subseteq M \). Let \( \psi \) be a safety or liveness property. Suppose \( M \models \Box \psi \), then \( A \models \psi \).

**Proof.** Since \( A \subseteq M \), based on Definition 7, there must exist a MTTS \( M \in [\mathcal{M}] \) such that \([A] \preceq M \). Given that \( M \models \Box \psi \), we know that \( M \models \Box \psi \) for every \( M \in [\mathcal{M}] \). Thus, based on Lemma 1, we have \([A] \models \psi \), meaning that \( A \models \psi \).

### 2.5 Compositional Reasoning

We now introduce the notion of composition, which is important for component-based design, and prove the property preservation compositionally. Let \( M_1 \) and \( M_2 \) be two time-parametric modal specifications. They are composeable iff they have disjoint sets of clocks and parameters, i.e., \( \text{Clk}_1 \cap \text{Clk}_2 = \emptyset \) and \( \Theta_1 \cap \Theta_2 = \emptyset \), and their actions only overlap on complementary types: \((\text{Act}_1^O \cup \text{Act}_1^I) \cap (\text{Act}_2^O \cup \text{Act}_2^I) = \emptyset \) and \((\text{Act}_1^O \cap \text{Act}_2^I) \cap (\text{Act}_1^I \cup \text{Act}_2^O) = \emptyset \).

**Definition 8** (Composition). Given two composeable time-parametric modal specifications \( M_i = (\text{Loc}_i, \overline{\tau}_i, \text{Clk}_i, \text{Act}_i, \text{Inv}_i, \leftrightarrow_{\Box,i}, \leftrightarrow_{\Diamond,i}, C_i(\Theta_i)) \) for \( i = 1, 2 \). Their composition product, denoted by \( M_1 || M_2 \), yields a specification \((\text{Loc}_1 \times \text{Loc}_2, \overline{\tau}_1 \cap \overline{\tau}_2, \text{Clk}_1 \cup \text{Clk}_2, \text{Act}, \text{Inv}, \leftrightarrow_{\Box, \Box}, \leftrightarrow_{\Box, \Diamond}, \leftrightarrow_{\Diamond, \Box}, \leftrightarrow_{\Diamond, \Diamond}, C(\Theta)) \) such that

- \( \text{Act} = \text{Act}_1^I \cup \text{Act}_2^O \cup \text{Act}_2^I \) where \( \text{Act}_i^I = (\text{Act}_1^I \setminus \text{Act}_2^O) \cup (\text{Act}_2^O \setminus \text{Act}_1^O) \), \( \text{Act}_i^O = (\text{Act}_1^O \setminus \text{Act}_2^I) \cup (\text{Act}_2^I \setminus \text{Act}_1^I) \) and \( \text{Act}_i^I = \text{Act}_1^I \cup \text{Act}_2^I \cup (\text{Act}_1^I \cap \text{Act}_2^I) \)
- \( \text{Inv}(l_1, l_2) = \text{Inv}_1(l_1) \cap \text{Inv}_2(l_2) \)
- \( \leftrightarrow_{\Box} \) and \( \leftrightarrow_{\Diamond} \) are defined by the following rules (interchangeable for \( M_1 \) and \( M_2 \)):

\[
\begin{align*}
(l_1, g_1, \alpha, r_1, l'_1) & \in \leftrightarrow_{\gamma, 1} (l_2, g_2, \alpha, r_2, l'_2) \in \leftrightarrow_{\gamma, 2} & \text{(synchronizing)} \\
(l_1, l_2, g_1 \land g_2, \alpha, r_1 \cup r_2, (l'_1, l'_2)) & \in \leftrightarrow_{\gamma} & \\text{(interleaving)}
\end{align*}
\]

where \( \gamma \in \{\Box, \Diamond\} \): if \( \leftrightarrow_{\gamma, 1} = \leftrightarrow_{\Box, 1} \) and \( \leftrightarrow_{\gamma, 2} = \leftrightarrow_{\Diamond, 2} \) in the synchronizing rule, or \( \leftrightarrow_{\gamma, 1} = \leftrightarrow_{\Box, 1} \) in the interleaving rule, then \( \leftrightarrow_{\gamma} = \leftrightarrow_{\Box} \); otherwise, \( \leftrightarrow_{\gamma} = \leftrightarrow_{\Diamond} \).
- $\Theta = \Theta_1 \cup \Theta_2$ and $C(\Theta) = C_1(\Theta_1) \land C_2(\Theta_2)$.

Since a TIOA can be considered as a special case of time-parametric modal specifications, the above definition is also applicable for the composition of two TIOAs (resp. a TIOA and a specification), which yields a product TIOA (resp. specification).

**Lemma 2.** Let $A$ be a TIOA implementation of a specification $M$, i.e., $A \trianglelefteq M$. Let $A'$ be a composable TIOA with $A$. Then $A \parallel A' \trianglelefteq M \parallel A'$.

**Proof.** See the appendix.

**Theorem 2.** Let $A$ be a TIOA implementation of a specification $M$, and $A'$ be a composable TIOA with $A$. Let $\psi$ be a safety or liveness property. Suppose $M \parallel A' \models \Box \psi$, then $A \parallel A' \models \psi$.

**Proof.** It’s straightforward from Theorem 1 and Lemma 2.

### 3 Symbolic Semantics and Verification

Recall that the operational semantics of a time-parametric modal specification consists of a (finite) set of MTTSs, each of which contains an infinite set of states. It’s not feasible to verify such infinite state-space models directly. To tackle this challenge, we define a new symbolic semantics as **parametric zone-graphs** in Section 3.1, and propose practical methods for symbolic reachability analysis and modal refinement checking in Sections 3.2 and 3.3, respectively.

#### 3.1 Symbolic Semantics: Parametric Zone-Graphs

Firstly, we extend the classical notion of zones and zone-graphs for timed automata [6] to parametric zones and zone-graphs. A **parametric zone** is in the form of a parametric clock constraint (defined in Section 2.1), representing the maximal set of clock valuations satisfying any instance of the parametric clock constraint. Let $v$ be a clock valuation and $D$ be a parametric zone, we define $v \in D$ iff $v$ partially satisfies the parametric clock constraint of $D$, i.e., $v \models^D D$. Given two parametric zones $D_1$ and $D_2$, if $v \in D_1$ implies $v \in D_2$, then zone $D_1$ is included in $D_2$, denoted by $D_1 \subseteq D_2$. We define $D^+ = \{v + d \mid v \in D, d \in \mathbb{R}_{\geq 0}\}$ for the zone progression, and $r(D) = \{v[r \mapsto 0] \mid v \in D\}$ for the clock reset of zones. A **parametric zone-graph** is a graph where each node consists of a location and a parametric zone. We define the symbolic semantics of time-parametric modal specifications based on parametric zone-graph as follows.

**Definition 9 (Specification’s Symbolic Semantics).** The symbolic semantics of a time-parametric modal specification $\mathcal{M} = (\text{Loc}, \bar{\mathcal{T}}, \mathcal{Clk}, \text{Act}, \mathcal{Inv}, \hookrightarrow, \hookrightarrow, C(\Theta))$, denoted by $\llbracket\mathcal{M}\rrbracket_z$, is a parametric zone-graph $(S, \pi, \text{Act}, \hookrightarrow, \hookrightarrow, C(\Theta))$ where

- $S = \{(l, D) \in \text{Loc} \times \mathcal{P} (\mathcal{Clk}) \mid D \subseteq \mathcal{Inv}(l)\}$ is a finite set of symbolic states, and
  - $\pi = (\bar{l}, D_0)$ is the initial state
- symbolic must action transition: $(l, D) \xrightarrow{\mathcal{g}, a, r} (l', r(D \land g) \land \mathcal{Inv}(l'))$ if there is a must transition $l \xrightarrow{a} l'$ in $\mathcal{M}$
Fig. 3: Parametric zone-graph of the specification in Fig. 1.

- **symbolic may action transition**: \( \langle l, D \rangle \xrightarrow{\gamma} \langle l', m(D \land g) \land Inv(l') \rangle \) if there is a may transition \( l \xrightarrow{\alpha, \beta} l' \) in \( M \)
- **symbolic delay transition**: \( \langle l, D \rangle \xrightarrow{d} \langle l, D' \rangle \) with \( \gamma \in \{\square, \Diamond, d\} \) implies that, for every \( v' \in D' \), there must exist at least one MTTS \( M' \in \left[ M \right] \) which has a transition \( \langle l, v \rangle \rightarrow \gamma \langle l', v' \rangle \) for some \( v \in D \).

A symbolic state \( \langle l, D \rangle \) in \( \left[ M \right]_z \) corresponds to a set of states in the operational semantics of \( M \). A symbolic transition \( \langle l, D \rangle \xrightarrow{\gamma} \langle l', D' \rangle \) with \( \gamma \in \{\square, \Diamond, \delta\} \) implies that, for every \( v' \in D' \), there must exist at least one MTTS \( M' \in \left[ M \right] \) which has a transition \( \langle l, v \rangle \rightarrow \gamma \langle l', v' \rangle \) for some \( v \in D \). Indeed, we show that the symbolic semantics given in Definition 9 is a correct and full characterization of the operation semantics given in Definition 5 as follows.

**Theorem 3.** Let \( M \) be a time-parametric modal specification, \( \left[ M \right]_z \) be its symbolic semantics and \( \left[ M \right] \) be its operational semantics.

- (Soundness) if the initial symbolic state \( \langle l, D_0 \rangle \) in \( \left[ M \right]_z \) must (resp. may) lead to a target state \( \langle l_f, D_f \rangle \), then for all \( v_f \in D_f \), state \( \langle l_f, v_f \rangle \) must (resp. may) be reachable from the initial state \( \langle l, 0 \rangle \) in some \( M \in \left[ M \right] \)
- (Completeness) if, in any \( M \in \left[ M \right] \), a target state \( \langle l_f, v_f \rangle \) must (resp. may) be reachable from the initial state \( \langle l, 0 \rangle \), then state \( \langle l, D_0 \rangle \) in \( \left[ M \right]_z \) must (resp. may) lead to \( \langle l_f, D_f \rangle \) for some \( D_f \) such that \( v_f \in D_f \).

**Proof.** See the appendix.

The clock valuations in a (parametric) zone-graph may drift unboundedly, inducing infinite symbolic transition relations and thus an infinite zone-graph. We can apply techniques such as the \( k \)-normalization [6] to normalize zones and guarantee the finiteness of transition relations. Note that the clock ceiling \( k \) of a parametric clock constraint \( x \sim c \pm \alpha \) is given by the maximum clock constant \( c \) plus the maximum value of parameter \( \alpha \).

**Example 3.** Fig. 3 illustrates the parametric zone-graph induced from the time-parametric modal specification shown in Fig. 1. There are 6 symbolic states in the zone-graph; for example, \( \langle \text{infusion}, x \leq \beta \rangle \) is a state associated with a location \text{infusion} and a
Algorithm 1 Must (resp. may) reachability analysis on parametric zone-graphs

Input: a parametric zone-graph $G$ whose initial state is $(l, D_0)$, a target $(l_f, \phi_f)$
Output: “YES”, if $(l_f, \phi_f)$ is reachable from $(l, D_0)$; “NO”, otherwise

1: PASSED=∅, WAIT={$(l, D_0)$}
2: while $\text{WAIT} \neq \emptyset$ do
3: take $(l, D)$ from $\text{WAIT}$
4: if $D = \emptyset$ for any valid parameter assignments $f(\Theta)$ then
5: prune state $(l, D)$ and its incoming/outgoing transitions from $G$
6: else
7: if $l = l_f$ and $D \cap \phi_f \neq \emptyset$ for all (resp. some) valid $f(\Theta)$ then
8: return “YES”
9: end if
10: if $D \not\subseteq D'$ for all $(l, D') \in \text{PASSED}$ then
11: add $(l, D)$ to $\text{PASSED}$
12: for all $(l', D')$ such that $((l, D), (l', D')) \in \leadsto_d$ and $\leadsto_\square$ (resp. $\leadsto_\diamond$) do
13: add $(l', D')$ to $\text{WAIT}$
14: end for
15: end if
16: end if
17: end while
18: return “NO”

parametric zone $\alpha \leq x \leq \beta$, which is bounded by the linear constraints: $\alpha \leq \beta \leq 3$ and $\alpha, \beta \in \mathbb{N}$.

In Fig. 3, solid lines represent symbolic must action transitions, e.g., there is a must transition labelled with action “bolus?” from state $(\text{detect}, x = 0)$ to $(\text{start}, x = 0)$; dashed lines are for the symbolic may action transitions, e.g., state $(\text{start}, x \leq \beta)$ may loop back to $(\text{detect}, x = 0)$ with action “alarm!”; and dash dotted lines are for symbolic delay transitions, e.g., state $(\text{detect}, x = 0)$ evolves to $(\text{detect}, x \geq 0)$ via zone progression.

3.2 Symbolic Reachability Analysis

Reachability analysis lies at the core of many verification problems, e.g., we can verify safety properties by checking whether some bad states are reachable. Inspired by the zone-graph based reachability algorithm in [6], we propose a symbolic algorithm for the reachability analysis of parametric zone-graphs, which is useful for the verification of time-parametric modal specifications.

As illustrated in Algorithm 1, we check whether a target state $(l_f, \phi_f)$ is reachable from the initial state $(l, D_0)$ by exploring the state-space of parametric zone-graph on-the-fly. Note that we twist Algorithm 1 for both the must and may reachability analysis (the differences are in Lines 7 and 12). The algorithm maintains two sets of states: PASSED for those having been traversed and WAIT for those to be considered. Starting with the initial state $(l, D_0)$, the algorithm processes each element $(l, D)$ of WAIT till

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2 We assume a normalized zone-graph with finite transition relations.
Algorithm 2 Checking the specification-implementation relation

**Input:** a TIOA $A$ and a time-parametric modal specification $M$

**Output:** “YES”, if $A$ is an implementation of $M$; “NO”, otherwise

1: obtain a finite set $F$ of valid parameter assignments by solving $C(\Theta)$ of $M$

2: while $F \neq \emptyset$ do

3: take a parameter assignment $f(\Theta)$ from $F$

4: obtain a pair of TIOAs $M^\square_f, M^\diamond_f$ by substituting all occurrences of parameters $\Theta$ with values $f(\Theta)$, and keeping only must and may transitions, respectively

5: if $M^\square_f$ simulates $A$, and $A$ simulates $M^\diamond_f$ then

6: return “YES”

7: end if

8: end while

9: return “NO”

the set becomes empty. If there is no clock valuation $v \in D$ for any parameter assignment $f(\Theta)$, i.e., $D = \emptyset$, then state $(l, D)$ and all transitions from/to it are pruned (see Line 5). If $l = l_f$ and $D \cap \phi_f \neq \emptyset$ for all valid parameter assignment $f(\Theta)$, then the target must be reachable; on the other hand, we say that $(l_f, \phi_f)$ may be reachable if $D \cap \phi_f \neq \emptyset$ is only true for some $f(\Theta)$, in the sense that the target may (not) be reachable for some implementation of the time-parametric modal specification. If $(l, D)$ does not hit the target and has not been traversed (see Line 10), then the algorithm adds $(l, D)$ to PASSED and all its successor states to WAIT. The algorithm terminates when WAIT is empty, and outputs that the target state $(l_f, \phi_f)$ is not reachable.

The termination of Algorithm 1 is guaranteed, because the parametric zone-graph $G$ has a finite set of symbolic states and finite transition relations, i.e., the size of WAIT is finite. The correctness of Algorithm 1 is given by Theorem 3.

**Example 4.** Consider the parametric zone-graph shown in Fig. 3. A target $(\text{infusion}, x \leq 5)$ must be reachable, because there is a path

$$(\text{detect}, x = 0) \rightarrow (\text{start}, x = 0) \rightarrow (\text{start}, x \leq \beta) \rightarrow (\text{infusion}, \alpha \leq x \leq \beta)$$

and, for any valid parameter values satisfying the constraint: $\alpha \leq \beta \leq 3$, we have $x \leq \beta \leq 5$. Suppose the target is $(\text{infusion}, x \leq 1)$, then it may be reachable, because $(\alpha \leq x \leq \beta) \land (x \leq 1) \neq \emptyset$ is true for some parameter assignments, e.g., $\alpha = \beta = 1$, but false for others, e.g., $\alpha = \beta = 3$.

3.3 Symbolic Modal Refinement Checking

Recall from Section 2.3 that we verify whether a TIOA is an implementation of a time-parametric modal specification via modal refinement check. We now propose a symbolic method to check the specification-implementation relation, by reducing the problem to the timed simulation check of timed automata, where standard zone-graph based algorithms [18] are available.

Algorithm 2 illustrates our method. Given a time-parametric modal specification $M$, we first solve the linear constraints $C(\Theta)$ and obtain a finite set of parameter as-
signments. For each parameter assignment \( f \), we substitute all the occurrences of parameters \( \Theta \) with values \( f(\Theta) \), resolving the timing variability of the specification. Then, by keeping only the must (resp. may) transition relations of \( \mathcal{M} \), we obtain a TIOA \( \mathcal{M}^f \) (resp. \( \mathcal{M}^f \)). We check the timed simulation relation between TIOAs \( \mathcal{M}^f, \mathcal{A} \) and \( \mathcal{M}^f \) (see Line 5), by applying the standard zone-graph based algorithm [18]. If both simulation checks are successful, we claim that \( \mathcal{A} \) is an implementation of \( \mathcal{M} \); otherwise, we continue with another parameter assignment. The termination of Algorithm 2 is guaranteed, because there are only finite parameter assignments; if we cannot find any assignment \( f \) that makes the condition in Line 5 true, then \( \mathcal{A} \) is not an implementation of \( \mathcal{M} \).

We argue the correctness of Algorithm 2 as follows. According to the definition of timed simulation [18], \( \mathcal{M}^f \) simulates \( \mathcal{A} \) iff every delay/action transition in \( \mathcal{M}^f \) is matched in \( \mathcal{A} \); similarly, \( \mathcal{A} \) simulates \( \mathcal{M}^f \) yields that every transition in \( \mathcal{A} \) has an exact match in \( \mathcal{M}^f \). Based on Definition 6, we can claim that \( \mathcal{A} \) modally refines MTTS \( \mathcal{M} \), whose must (resp. may) action transitions are given by the action transitions of \( \mathcal{M}^f \) (resp. \( \mathcal{M}^f \)) and delay transitions coincide with that of \( \mathcal{M}^f \) or \( \mathcal{M}^f \). Moreover, \( \mathcal{M} \) belongs to the set \( \mathcal{M} \). Therefore, based on Definition 7, \( \mathcal{A} \) is an implementation of \( \mathcal{M} \).

4 Case Study

In this section we describe how time parametric modal specifications can be used to specify and analyze on-demand medical systems. In an on-demand medical system clinicians assemble a composite medical device by coupling interoperable medical devices with applications that implement clinical algorithms. The devices carry a capabilities specification, while the applications carry requirements specifications. An underlying platform checks that only devices with capabilities compatible with an application are used; if a clinician attempts to couple an incompatible device they are notified and the coupling process is aborted [14]. As the specific device (i.e., make, brand, or model) to be coupled to the application is not known apriori by the application developer the safety assessment of an application must be done with respect to the application’s device requirements. Because of this, these specifications play an important role. Not only must the formalism be able to capture enough details about device behavior to enable a realistic safety assessment, they must also allow the developer to provide specifications that are as general as possible in order to ensure that a given application is compatible with as many devices as is safe.
For the rest of this section we will describe how such a system providing closed loop control of PCA therapy can be specified using time-parametric modal specifications, how refinement is used to check compatibility, and how the behavior of the overall application can be analyzed against its requirements. In this section we adopt a syntactic sugar by allowing each specification to contain state variables. Locations at the semantic level are then related to the cross-product of the state variable values in the standard way.

In PCA therapy a patient is attached to an infusion pump that provides a bolus of painkiller at the request of the patient. If the patient requests too much painkiller they run the risk of overdose. In the closed loop system (Fig. 4) a pulse oximeter is used to measure the blood oxygenation of the patient (SpO$_2$). The SpO$_2$ reading is forwarded over the network to the application which then determines if infusion should be canceled. If so, an ‘off’ signal is sent to the pump. First we adopt a simple patient model (Figure 5a) that relates the activity of the infusion pump to the SpO$_2$ of the patient. When infusion is off, the patient’s SpO$_2$ rises by 1 every two time units. When infusion is on, the SpO$_2$ decreases by 1 every time unit (the SpO$_2$ can never go above 100 or below 0). In each location, the patient model can emit the current SpO$_2$ value. Next we model the pulse oximeter which is a device that senses the SpO$_2$ of the patient then forwards its reading onto the application (Fig. 5b). Valid pulse oximeter implementations are allowed to sense and forward SpO$_2$ values as fast as they want, but they are required to read and forward a value at least once every 2 time units.

For the sake of space we omit a visualization of the safety interlock application. However, its functionality is simple. On every update from the pulse oximeter it determines if it should cancel any ongoing infusion. If the value is $< 95$ infusion is canceled. The application will also cancel infusion if it does not receive a SpO$_2$ update for more than 3 time units. Finally, the pump is as modeled previously in Fig. 1. If we wanted to verify a simple safety property, e.g. that the patient’s SpO$_2$ never goes below 85, we would compose the application, patient model, and the requirements specification on the pump and pulse oximeter and then apply the state reachability algorithm from Section 3.1. Compatibility is simply a refinement check between the device’s capabilities specification and the applications requirements specification.
5 Conclusion

In this paper we introduce time-parametric modal specifications which are timed I/O automata extended with must/may transitions and parametric clock constraints. This formalism enables system designers to specify a family of implementations who differ not only functionally but also in terms of timing behavior. We build a specification theory based on his formalism, including modal refinement, safety / liveness property preservation, and compositional reasoning. We also develop two symbolic algorithms for the reachability analysis and specification-implementation relation checking. The usefulness of our theory is demonstrated via a case study of medical device systems.

For the future work, there are several potential directions. Firstly, we aim to improve the modal refinement decision algorithm in 3.3; instead of checking the refinement for every possible parameter assignment, we would like to find a way to narrow down the choices of parameters when performing the refinement check on-the-fly. Secondly, we will consider weak modal refinement which concerns only the observable behavior of systems. Moreover, we plan on building a tool that implements all proposed algorithms. This tool would be used to both verify properties and check refinement of time-parametric specifications. Additionally, we plan to incorporate the refinement check into a prototype on-demand medical system platform.

References

Appendix

Lemma 1. Let $A$ be a TTS and $M$ be a MTTS such that $A \preceq M$. Let $\psi$ be a safety or liveness property. Suppose $M \models \psi$, then $A \models \psi$.

Proof. Case 1: $\psi = \neg E(\phi)$ is a safety property where $\phi$ is a state formula for good behavior. For the sake of contradiction, we assume $A \not\models \psi$, i.e., $A \models E(\neg \phi)$, meaning that there exist a path $\pi$ in $A$ reaching some bad state $s_f \models \neg \phi$ from the initial state $\pi$. Since $A \preceq M$, every transition in $A$ is also allowed in $M$; that is, every action transition $s \xrightarrow{a} s'$ along $\pi$ corresponds to a may transition $t \xrightarrow{a} t'$ in $M$, and every delay transition $s \xrightarrow{d} s'$ for $d \in \mathbb{R}_{\geq 0}$ along $\pi$ has an exact match $t \xrightarrow{d} t'$ in $M$. Thus, from the initial state $\pi$ of $M$, there is path $\pi'$ corresponding to $\pi$ and reaching some state $t_f \models \neg \phi$; since path $\pi'$ consists of only may action and delay transitions, we have $M \models \phi$. This is a contradiction to $M \models \neg E(\phi)$, which suggests that $M \models \phi$. Therefore, we shall have $A \models \psi$.

Case 2: $\psi = \mathcal{K}(\phi)$ is a liveness property where $\phi$ is a state formula. Given that $M \models \mathcal{K}(\phi)$, every path $\pi$, leaving the initial state $\overline{\pi}$ of $M$ would eventually reach some state $t_f \models \phi$ via a sequence of must action and delay transitions. Since $A \preceq M$, for every must action transition $t \xrightarrow{a} t'$, there is a corresponding transition $s \xrightarrow{a} s'$ in $A$; and for every delay transition $t \xrightarrow{d} t'$ with $d \in \mathbb{R}_{\geq 0}$, there is a counterpart $s \xrightarrow{d} s'$ in $A$. Thus, every path initiated from $\overline{\pi}$ in $A$ corresponds to a path $\pi$, in $M$, and would eventually reach some state $t_f \models \phi$; that is, $A \models \mathcal{K}(\phi)$, i.e., $A \models \psi$.

Lemma 2. Let $A$ be a TIOA implementation of a specification $\mathcal{M}$, i.e., $A \subseteq M$. Let $A'$ be a composable TIOA with $A$. Then $A\|A' \subseteq M\|A'$. Proof. Since $A \subseteq M$, based on Definition 7, there must exist a MTTS $M \in [\mathcal{M}]$ such that $[A] \preceq M$. Let $f$ be the parameter assignment corresponding to $M$. By substituting all occurrences of parameters in $M$ with values $f$, we obtain $M_f$ such that $[M_f] = M$ and $[M_f\|A']$ yields a single MTTS belonging to the set $[M\|A']$. In order to ensure $A\|A' \subseteq M\|A'$, we only need to prove that $[A\|A'] \preceq [M\|A']$ is true, by considering the following three cases.

1. For any must action transition $\langle l, v \rangle \xrightarrow{a} \mathcal{K}(l', v')$ in $[M_f\|A']$, we shall prove that there is a corresponding transition in $[A\|A']$.
   - Suppose $a \in \text{Act}_M$ and the transition is a product of the synchronizing rule in Definition 8. Let $l = (l_s, l_q)$ and $l' = (l'_s, l'_q)$, where $l_s, l'_s$ (resp. $l_q, l'_q$) are locations of $M_f$ and are locations of $A'$. There is a transition $\langle l_s, v_l \rangle \xrightarrow{a} \mathcal{K}(l'_s, v_{l'})$ in $M$. Since $[A] \preceq M$, there must exist some $\langle l_s, v_l \rangle \xrightarrow{a} \langle l'_s, v_{l'} \rangle$ in $[A]$ and hence a corresponding transition $\langle (l_s, l_q), \mu \rangle \xrightarrow{a} \langle (l'_s, l'_q), \mu' \rangle$ in $[A\|A']$.
   - If $a \in \text{Act}_M$, $l = (l_s, l_q)$ and $l' = (l'_s, l'_q)$, then the transition is obtained from an interleaving step on $M_f$, so that there is a transition $\langle l_s, v_l \rangle \xrightarrow{a} \langle l'_s, v_{l'} \rangle$ in $M$. Since $[A] \preceq M$, there must exist a corresponding transition $\langle l_s, v_l \rangle \xrightarrow{a} \langle l'_s, v_{l'} \rangle$ in $[A]$ and hence $\langle (l_s, l_q), \mu \rangle \xrightarrow{a} \langle (l'_s, l'_q), \mu' \rangle$ in $[A\|A']$.
   - If $a \notin \text{Act}_M$, then the transition is a result of the interleaving on $A'$, i.e., $l = (l_s, l_q)$ and $l' = (l'_s, l'_q)$. Thus, a corresponding interleaving transition $\langle (l_s, l_q), \mu \rangle \xrightarrow{a} \langle (l'_s, l'_q), \mu' \rangle$ shall take place in $[A\|A']$. 
2. For any transition \( (l, v) \xrightarrow{a} (l', v') \) in \([A][A']\), we can prove that there is may action transition in \([M_f][A']\), by reasoning about the correspondence between their projected transitions on \([A]\) and \([M_f]\) in a similar way as for the must action case.

3. For any delay transition \( (l, v) \xrightarrow{d} (l, v + d) \) in \([M_f][A']\), let its projection on \([M_f]\) (or \(M\)) be \( (l, v_s) \xrightarrow{d} (l, v_s + d) \). Since \([A] \subseteq M\), based on Definition 6, there must exist a transition \( (l, v) \xrightarrow{d} (l, v + d) \) in \([A]\) and hence a corresponding delay transition in \([A][A']\). Similarly, we can prove that there is a matching transition in \([M_f][A']\) for any delay transition in \([A][A']\).

According to Definition 6, the above three cases yield \([A][A'] \subseteq [M_f][A']\). Thus, we have proved that \([A][A'] \subseteq M\).

**Theorem 3.** Let \(M\) be a time-parametric modal specification, \([M]_z\) be its symbolic semantics and \([M]_z\) be its operational semantics.

- (Soundness) if the initial symbolic state \(\langle \emptyset, D_0 \rangle\) in \([M]_z\) must (resp. may) lead to a target state \(\langle \emptyset, D_f \rangle\), then for all \(v_f \in D_f\), state \(\langle \emptyset, v_f \rangle\) must (resp. may) be reachable from the initial state \(\langle \emptyset, 0 \rangle\) in some \(M \in [M]\)

- (Completeness) if, in any \(M \in [M]\), a target state \(\langle \emptyset, v_f \rangle\) must (resp. may) be reachable from the initial state \(\langle \emptyset, 0 \rangle\), then state \(\langle \emptyset, D_0 \rangle\) in \([M]_z\) must (resp. may) lead to \(\langle \emptyset, D_f \rangle\) for some \(D_f\) such that \(v_f \in D_f\)

**Proof.** We will prove by induction on the length of paths. Without loss of generality, we assume that all paths are expressed in the form of alternating (must or may) action transitions and delay transitions, i.e., \(\ldots \langle l_{n-1}, v_{n-1} \rangle \xrightarrow{a} \langle l_n, v_n \rangle \xrightarrow{d} \langle l_{n+1}, v_{n+1} \rangle \ldots\) for \(a \in Act\) and \(d \in R_{\geq 0}\).

(Soundness) Assume \(\langle \emptyset, D_0 \rangle \leadsto^* \langle l_n, D_n \rangle \leadsto \langle l_{n+1}, D_{n+1} \rangle\), where \(\leadsto^*\) represents a succession of transitions. By induction, we have \(\langle \emptyset, 0 \rangle \leadsto \langle l_n, v_n \rangle\) for all \(v_n \in D_n\). We need to prove for all \(v_{n+1} \in D_{n+1}\), there is a transition \(\langle l_n, v_n \rangle \xrightarrow{\sigma} \langle l_{n+1}, v_{n+1} \rangle\). There are two cases, since \(\sigma\) can be an action or a delay.

- Suppose \(\langle l_n, D_n \rangle \xrightarrow{a} \langle l_{n+1}, D_{n+1} \rangle\) for \(a \in Act\) and \(\gamma \in \{\square, \diamond\}\). Based on Definition 9, we have \(l_n \xrightarrow{g} a \xrightarrow{\gamma} l_{n+1}\) and \(D_{n+1} = r(D_n \land g) \land Inv(l_{n+1})\). By Definition 5, there is a transition \(\langle l_n, v_n \rangle \xrightarrow{\gamma} \langle l_{n+1}, v_{n+1} \rangle\) in some \(M \in [M]\) such that \(v_n \in g\). Thus, for all \(v_{n+1} \in D_{n+1}\), there is a \(v_n \in D_n\) such that \(v_n \in g\), \(v_{n+1} \in Inv(l_{n+1})\) and \(v_{n+1} = v_n[r \leadsto 0]\).

- Suppose \(\langle l_n, D_n \rangle \xrightarrow{d} \langle l_{n+1}, D_{n+1} \rangle\) for \(d \in R_{\geq 0}\). From Definition 9, we have \(l_n = l_{n+1}\) and \(D_{n+1} = D_n \land Inv(l_n)\). Due to the definition of zone progression, we have \(D_{n+1} = \{v_n + d \mid v_n \in D_n, d \in R_{\geq 0}\}\). Based on Definition 5, we have \(\langle l_n, v_n \rangle \xrightarrow{d} \langle l_n, v_n + d \rangle\) if \(v_n + d \in Inv(l_n)\). Thus, for all \(v_{n+1} \in D_{n+1}\), there is a \(v_n \in D_n\) such that \(v_{n+1} = v_n + d\) and \(v_{n+1} \in Inv(l_n)\).

(Completeness) Assume \(\langle \emptyset, 0 \rangle \leadsto \langle l_n, v_n \rangle \xrightarrow{\sigma} \langle l_{n+1}, v_{n+1} \rangle\). The induction step gives \(\langle \emptyset, D_0 \rangle \leadsto \langle l_n, v_n \rangle \xrightarrow{\sigma} \langle l_{n+1}, v_{n+1} \rangle\). We need to prove that \(\langle l_n, D_n \rangle \xrightarrow{\sigma} \langle l_{n+1}, D_{n+1} \rangle\) for some \(D_{n+1}\) and \(v_{n+1} \in D_{n+1}\). There are two cases:
– Suppose \( \langle l_n, v_n \rangle \overset{a}{\rightarrow} \gamma \langle l_{n+1}, v_{n+1} \rangle \) for \( a \in Act \) and \( \gamma \in \{\Box, \Diamond\} \) in any \( M \in \mathcal{M} \). Based on Definition 5, there is a transition \( l_n \overset{g,a,r}{\rightarrow} l_{n+1} \) in \( M \) and \( v_n \in g, v_{n+1} = v_n[r \mapsto 0] \) and \( v_{n+1} \in Inv(l_{n+1}) \). By Definition 9, we have \( \langle l_n, D_n \rangle \overset{\gamma}{\rightarrow} \langle l_{n+1}, D_{n+1} \rangle \) and \( D_{n+1} = r(D_n \land g) \land Inv(l_{n+1}) \). Thus, \( v_{n+1} \in D_{n+1} \).

– Suppose \( \langle l_n, v_n \rangle \overset{d}{\rightarrow} \langle l_{n+1}, v_{n+1} \rangle \) for \( d \in \mathbb{R}_{\geq 0} \). Then we have \( l_n = l_{n+1}, v_{n+1} = v_n + d \) and \( v_n, v_{n+1} \in Inv(l_n) \). From Definition 9, we get \( \langle l_n, D_n \rangle \overset{d}{\rightarrow} \langle l_{n+1}, D_{n+1} \rangle \) and \( D_{n+1} = D_n \uparrow Inv(l_n) = \{v_n + d \mid v_n \in D_n, d \in \mathbb{R}_{\geq 0} \land v_n + d \in Inv(l_n)\} \). Thus, \( v_{n+1} \in D_{n+1} \).