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Dual Periodic Resource Model

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***Abstract*—The paper considers compositional scheduling for hierarchical real-time systems using periodic resource models, which has been extensively studied in the past. We identify an unrealistic assumption in the existing literature that can make the computed component interfaces unimplementable. Namely, resource bandwidth can be expressed using arbitrary rational numbers. We show that resource bandwidth, computed by an algorithm that removes this assumption becomes overly pessimistic, and offer a new notion of a *dual-periodic resource model (DPRM) interface* that improves resource bandwidth of the interface. We study composition using DPRM interfaces and show properties of the new approach in terms of required resource bandwidth and preemption overhead.**

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I. INTRODUCTION

Component-based design has become the widely used technology for the construction of complex computer-based systems. Component technologies allow us to apply the divide-and-conquer approach to reduce design complexity. Components provide well-defined interfaces that abstract away implementation details and enable reuse of a component in different applications. Furthermore, many modern systems are developed through collaboration of many independent providers; in this case, components allow us to encapsulate intellectual properties.

Increasingly, real-time systems are also built using independently developed components. However, unlike conventional systems, real-time components need to satisfy timing and resource constraints and thus have to be allocated sufficient computational resources for this purpose. Schedulability analysis is employed to check that all timing constraints of an application containing multiple real-time tasks will be satisfied in the implementation. However, classical schedulability analysis algorithms are global; that is, they need to know all the tasks that comprise the system. This global nature of schedulability analysis greatly reduces the benefits of component-based development.

Compositional schedulability analysis. techniques have been developed to allow component-based development to be used for systems where multiple independently developed components share a computational resource [1]. Interfaces of real-time components contain information about the resource needs of a component, and the system scheduler uses this information to allocate resources to components. Within a component, a separate component-level scheduler further allocates the resource to the component workload, which can contain real-time tasks or other subcomponents.

A common way to represent resource requirements in a component interface is to use a *resource model* [2]. Several resource models have been proposed in the literature, with the periodic resource model [1] being one of the most commonly used. A periodic resource model $\Gamma = (\Pi, \Theta)$ used as the interface of a component specifies that the component needs to be allocated at least Θ time units of resource access in every Π time units. A necessary part of a resource model-based compositional schedulability analysis framework is an algorithm to calculate parameters of the resource model sufficient to make the component schedulable. It is also desirable to make such an algorithm optimal so that the component is not allocated unnecessary resources.

An optimal algorithm for the calculation of resource interfaces has been introduced in [3]. The algorithm computes periodic resource model (Π, Θ) that minimizes the *resource bandwidth* Θ/Π. While theoretically optimal, the algorithm cannot always be used in practice, because it calculates Θ as a rational number. Practically, Θ should be an integer multiple of the time slice used by the operating system, which may not be under the control of the application developer. We thus restrict the set of acceptable periodic resource models to have integer values of both Π and Θ. While scaling both Π and Θ by the same factor may yield an acceptable resource model with the same bandwidth, we remind the reader that Π cannot be made arbitrarily large, otherwise the component will become unschedulable due to the *blocking interval* of the resource model [1]. It is clear that an approximation of the optimal resource model with integer values introduces additional overhead into the scheduling framework. One of the goals of this paper is to quantify this overhead.

Furthermore, it is not sufficient to round up the value calculated by the existing algorithm. Consider the following example. Let the optimal resource model for a component be (1,0.54). Rounding the result up, we obtain the resource model (1,1). However, this may not be the minimum bandwidth that can be obtained with integer values, as a periodic resource model (4,3) may be able to schedule the component. We thus set out to develop a new algorithm to calculate an acceptable periodic resource model with the minimum bandwidth.

We then show that it is possible to characterize resource demand of a component even more precisely. We introduce a *dual-periodic resource model* (DPRM) interface, which contains two periodic resource models instead of one. It can be shown that if rational numbers are used in periodic models, DPRM interfaces do not improve the total resource bandwidth [3]. However, when restricted to integer parameter values, we show that it is possible to reduce the overhead of the interface bandwidth by using DPRM. An extensive simulation study allows us to demonstrate the scale of the improvement.

Contributions. This paper makes three distinct contributions related to the use of periodic resource models in the interfaces of real-time components.

- We propose an efficient algorithm to calculate the minimum-bandwidth periodic resource model with integer parameter values.
- All algorithms for resource model calculation, including the one proposed here, rely on an upper bound on the value of the resource model period Π. In the literature, the upper bound is a parameter of the algorithm specified by the designer. In this paper, we derive a theoretical upper bound for the period of the minimum-bandwidth resource model.
- Finally, we propose a new resource-demand interface, DPRM, and show that it allows us to reduce resource utilization compared to the minimum-bandwidth periodic resource model with integer parameters. we further propose a composition technique for DPRM interfaces and evaluate context switch overhead of DPRM interfaces.

This paper is an extended version of [4]. Composition of DPRM interfaces and consideration for context switch overhead is added.

Related work. Since the first two-level hierarchical real-time scheduling framework introduced by Deng et al. [5] and its extension to multi-level hierarchical systems [2], several compositional analysis techniques have been proposed for such systems (see e.g., [1], [3], [6]). The majority of these techniques assume independent periodic task models – or their variations – for the components. However, these techniques have also recently been extended to analyze hierarchical systems with dependency, such as systems containing interacting tasks [7] and resource sharing [8]. Compositional analysis methods have also been investigated in the context of virtual machine (VM) environment [9]–[11].

Most of the existing compositional analysis frameworks represent component interfaces using one of the two resource models: periodic [1] and explicit deadline periodic [6]. The advantages of these two resource models are that they can be directly transformed into real-time tasks, which are required by the upper-level scheduler, and their supply bound functions have regular structures that allow for optimal interface generation. All the existing algorithms, however, assume that the resource model take rational parameter values, which cannot always be used in practice. Further, these algorithms rely on a pre-specified bound on the resource period that is manually chosen by the designer, which cannot guarantee the optimality of the output interfaces.

Organization. The next section revisits the hierarchical scheduling framework. Section III-A presents a bound on the resource period and a revised interface generation algorithm using this bound, followed by a more efficient algorithm in Section IV. Section V proposes the DPRM interface that is able to reduce this overhead suffered by the periodic resource interface. Finally, we present our evaluation of our proposed techniques in Section VI before concluding the paper.

II. Hierarchical scheduling background

In a hierarchical scheduling framework, a system is composed of a set of real-time *components* that are scheduled in a tree-like manner as shown in Figure 1. Each component C in the system is defined by a tuple (W, Γ, A), where W is the component's workload, Γ is the resource interface of the component, and A is the scheduling policy that is used to schedule W. The workload W consists of either (i) a finite set of real-time tasks $\{T_1, T_2, \ldots, T_n\}$, if C is a leaf-component; or (ii) a finite set of subcomponents $\{C_1, C_2, \ldots, C_n\}$, otherwise. The resource interface Γ captures the minimum amount of resource that must be given to C to feasibly schedule the tasks/components in W. The compositional analysis of the system involves (1) computing the resource interface for each leaf-component from the resource demands of its tasks, and (2) subsequently, computing the resource interface for each non-leaf component from the interfaces of its subcomponents. We will focus on the former; the latter can be done using similar techniques as in [1].

In this paper, we assume that all tasks are periodic tasks with relative deadlines equal to periods. Each task T is defined by a period (deadline) p, a worst-case execution time e, with $p \geq e > 0$. The scheduling policy A is assumed to be Earliest Deadline First (EDF) and all our discussions pertain to EDF (without mentioning it explicitly). Note, however, the methods developed here can easily be extended to the RM (Rate Monotonic) by substituting the schedulability condition of EDF with that of RM.

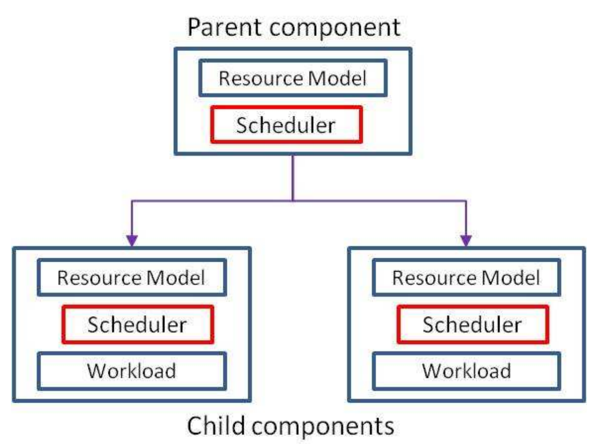

Fig. 1. A hierarchical scheduling system.

Schedulability condition. Given a workload W, the real-time resource requirement of W is characterized by a *demand bound function* (DBF) [12], denoted by $\mathsf{dbf}_W(t)$, which gives the maximum number of execution (resource) units required by the tasks/components of W in any time interval of length t for all $t \geq 0$. The DBF of a workload $W = \{T_1, T_2, \cdots, T_n\}$, with $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$, under EDF is [12]:

$$\forall\, t \geq 0, \;\; \mathsf{dbf}_W(t) = \sum_{i=1}^{n} \left(\lfloor \frac{t}{p_i} \rfloor e_i \right)$$

Similarly, the minimum resource guaranteed by a resource model Γ is captured by a *supply bound function* (SBF) [1], written as $\mathsf{sbf}_\Gamma(t)$, which gives the minimum number of execution units provided by Γ in any time interval of length t for all $t \geq 0$. Lemma 1 states the schedulability condition based on DBF and SBF [13]. In this lemma and the rest of the paper, LCM_W denotes the least common multiple (LCM) of all p_i where $1 \leq i \leq n$.

Lemma 1: Given a component $C = \big(W, \Gamma, EDF\big)$ with $W = \{T_1, T_2, \cdots, T_n\}$ and $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$. Then, C is schedulable (Γ can feasibly schedule W) iff

$$\forall\, t \;\, s.t. \;\; 0 \leq t \leq \mathsf{LCM}_W, \;\; \mathsf{sbf}_\Gamma(t) \geq \mathsf{dbf}_W(t). \tag{1}$$

Periodic Resource Model. A periodic resource model is defined by $\Gamma = (\Pi, \Theta)$ where Π is the resource period and Θ is the execution time guaranteed by Γ within every Π time units. The SBF of Γ is thus given by [1]:

$$\mathsf{sbf}_\Gamma(t) = \begin{cases} y\Theta + \max\big(0,\, t - x - y\Pi\big), & \text{if } t \geq \Pi - \Theta \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $x = 2(\Pi - \Theta)$ and $y = \lfloor \frac{t - (\Pi - \Theta)}{\Pi} \rfloor$.

An important concept associated with the periodic resource model is *bandwidth*. Specifically, the bandwidth of $\Gamma = (\Pi, \Theta)$ is given by $\mathsf{bw}_\Gamma = \frac{\Pi}{\Theta}$. A resource model is *bandwidth-optimal* for a workload W iff its bandwidth is the smallest among that of any resource model that can feasibly schedule W. In this paper, our notion of optimality refers to bandwidth-optimality

Definition 1 (Bandwidth-Optimal): A periodic resource model $\Gamma = (\Pi, \Theta)$ is bandwidth-optimal for a given workload W iff $\mathsf{bw}_\Gamma \leq \mathsf{bw}_{\Gamma'}$ for all Γ' that can feasibly schedule W.

Computation of the optimal periodic resource model. Algorithm 1 outlines the conventional procedure for computing the optimal resource model of a given workload (see e.g., [1], [14]). In this algorithm, $\Pi_{\max}$ is a predefined upper bound on the resource period. The function $\mathsf{MinExec}(\Pi,\, \mathsf{dbf}_W,\, \mathsf{LCM}_W)$ (Line 3) computes the minimum Θ for a given Π such that $\Gamma = (\Pi, \Theta)$ can feasibly schedule W (c.f. Lemma 1).

Algorithm 1 the optimal periodic resource model computation

Input: $\Pi_{\max}$, and dbf_W and LCM_W of a workload W

Output: The minimum bandwidth periodic resource model Γ

```
1: minBW = 1
2: for Π = 1 to Π_max do
3:     Θ = MinExec(Π, dbf_W, LCM_W)
4:     if Θ/Π < minBW then
5:         minBW = Θ/Π
6:         Γ = (Π, Θ)
7:     end if
8: end for
```

In existing work, the maximum bound $\Pi_{\max}$ of the resource period used in Algorithm 1 is either not discussed (and thus, ultimately infinite) or manually chosen by the designer. While the former approach is infeasible, the latter does not guarantee optimality, as illustrated in the example below.

Example 1: Consider a workload $W = \{T_1, T_2\}$ with $T_1 = (51, 23)$ and $T_2 = (130, 70)$. Suppose $\Pi_{\max}$ is chosen to be 80 in Algorithm 1. Then, the output given by Algorithm 1 is $\Gamma = (1, 1)$. However, this resource model is *not* optimal because there exists a periodic resource model $\Gamma' = (97, 96)$, which can feasibly schedule W (c.f. Lemma 1 and Equation 2) and has a lower bandwidth than that of Γ (because $\frac{96}{97} < \frac{1}{1}$).

Since the optimality of Algorithm 1 depends on how large $\Pi_{\max}$ is, the value chosen for $\Pi_{\max}$ must guarantee that the algorithm always outputs a minimum bandwidth model. Simultaneously, $\Pi_{\max}$ should be as small as possible to limit the computational complexity. In the next section, we present our method for computing the bound $\Pi_{\max}$ theoretically and a revised version of Algorithm 1 that uses this bound.

III. Bound on optimal resource period and a revised algorithm

A. An upper bound on the resource period

We first define the preliminary results that serve as foundation for our computation. Observe that any SBF of a periodic resource model can be upper bounded by a linear function. We define the *upper supply bound function* (USBF) [6] of a resource model Γ to be the linear function with the smallest slope among all linear functions that upper bound sbf_Γ. The USBF of a periodic resource model $\Gamma = (\Pi, \Theta)$ is [6]:

$$\forall\, t \geq 0: \; \mathsf{usbf}_\Gamma(t) = \max\left(\frac{\Theta}{\Pi}\big(t - (\Pi - \Theta)\big), 0\right). \tag{3}$$

Lemma 2: Given a component $C = (W, \Gamma, EDF)$ where $W = \{T_1, T_2, \cdots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$, and $\Gamma = (\Pi, \Theta)$. Then, C is schedulable only if

$$\forall\, t \;\, s.t. \;\; 0 \leq t \leq \mathsf{LCM}_W, \;\; \mathsf{usbf}_\Gamma(t) \geq \mathsf{dbf}_W(t). \tag{4}$$

Proof: Suppose Equation 4 does not hold. Then, there exists $t_0 \geq 0$ such that $\mathsf{usbf}_\Gamma(t_0) < \mathsf{dbf}_W(t_0)$. By definition, $\mathsf{sbf}_\Gamma(t) \leq \mathsf{usbf}_\Gamma(t)$ for all $t \geq 0$. Thus, $\mathsf{sbf}_\Gamma(t_0) \leq \mathsf{usbf}_\Gamma(t_0) < \mathsf{dbf}_W(t_0)$. In other words, C is not schedulable due to Lemma 1. ■

One can verify that if Γ satisfies the schedulability condition for W (see Lemma 1) then it satisfies Equation 4; however, the reverse does not hold. Thus Equation 4 gives a necessary condition for the schedulability of W under the resource model Γ. By abuse of notation, we refer to Equation 4 as the *USBF-schedulability condition* for W and we say that a model Γ can *potentially* schedule W iff it satisfies Equation 4.

Basic ideas. The upper bound on the resource period of the optimal periodic resource model for a given workload W can be derived based on dbf_W and its relationship with the USBFs of the resource models that can potentially schedule

W. Intuitively, let M be the set of resource models that can potentially schedule W. Suppose $\Gamma_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ is the optimal resource model for W. Then,

$$\mathsf{bw}_{\Gamma_{\mathsf{opt}}} \geq B_{\min} \stackrel{\mathsf{def}}{=} \min\big\{\mathsf{bw}_{\Gamma} \mid \Gamma \in M\big\}.$$

To derive the bound on Π_{opt}, we will find all the possible resource models in M that have the minimum bandwidth equal to $B_{\min}$. Towards this, we vary the resource period Π and compute for each Π a (unique) resource model $\Gamma_\Pi = (\Pi, \Theta_\Pi)$ belonging to M that has the minimum bandwidth among all the resource models in M with the same resource period Π. Then, $B_{\min}$ can be computed by $B_{\min} = \min\big\{\mathsf{bw}_{\Gamma_\Pi} \mid \Pi \in \mathbb{N}\big\}$. We will show that for all Π, the USBF of Γ_Π intersects dbf_W at exactly one special point – a *critical time point*. At the same time, Γ_Π has the largest bandwidth among all the resource models $\Gamma_{\Pi,t}$ with period Π that have their USBF intersecting dbf_W at critical time points t. In other words,

$$\mathsf{bw}_{\Gamma_\Pi} = \max_{\substack{t \in \mathsf{CrT}_W \\ \Gamma_{\Pi,t} = (\Pi, \Theta_{\Pi,t})}} \{\mathsf{bw}_{\Gamma_{\Pi,t}} \mid \mathsf{usbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t)\},$$

where CrT_W is the set of all critical time points of W, which is determined based solely on the structure of dbf_W. Further, for given any Π and any $t \in \mathsf{CrT}_W$, we compute the bandwidth of $\Gamma_{\Pi,t}$ directly from $\mathsf{dbf}_W(t)$, Π and t. From these values, we derive $B_{\min}$, which allows us to bound Π_{opt}.

Computation details. First, we define the set of critical time points of a workload W.

Definition 2: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$ where $n \in \mathbb{N}$ and $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$. The set of critical time points of W is defined by

$$\mathsf{CrT}_W = \left\{ \operatorname*{argmax}_{0<t<\mathsf{LCM}_W} \frac{\mathsf{dbf}_W(t)}{t-s} \mid s \in \{1, 2, \cdots, l\} \right\} \tag{5}$$

where $y = \min_{1\leq i\leq n} p_i$ and $l = y - \mathsf{dbf}_W(y)$.

Example 2: Consider $W = \{(5,1), (12,2), (17,1)\}$. The set of critical time points of W is $\mathsf{CrT}_W = \{5, 36\}$, which is shown in Figure 2.

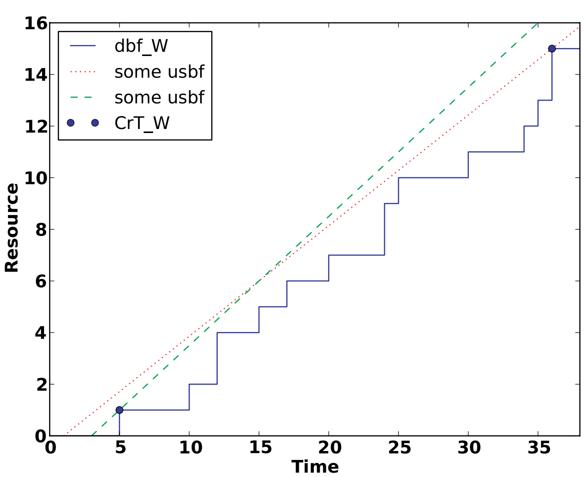

Fig. 2. Relationship between CrT_W and dbf_W.

Lemma 3: Given a component $C = (W, \Gamma, EDF)$ where $W = \{T_1, T_2, \cdots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$, and $\Gamma = (\Pi, \Theta)$. Suppose Γ satisfies the USBF-schedulability condition for W. Then, for all $t \geq 0$, if $\mathsf{usbf}_\Gamma(t) = \mathsf{dbf}_W(t)$ then $t \in \mathsf{CrT}_W$.

Proof: We will prove the lemma by contradiction. Suppose there exists $t_0 \notin \mathsf{CrT}_W$ such that $\mathsf{usbf}_\Gamma(t_0) = \mathsf{dbf}_W(t_0)$. Let $s = \Pi - \Theta$. Then, by Definition 2,

$$\exists\, t' \in \mathsf{CrT}_W : \frac{\mathsf{dbf}_W(t')}{t'-s} > \frac{\mathsf{dbf}_W(t_0)}{t_0-s} \tag{6}$$

On the other hand,

$$\begin{aligned} & \mathsf{usbf}_\Gamma(t') \geq \mathsf{dbf}_W(t') \\ \Rightarrow\ & \frac{\Theta}{\Pi}(t' - (\Pi - \Theta)) \geq \mathsf{dbf}_W(t') \\ \Rightarrow\ & \frac{\mathsf{dbf}_W(t_0)}{t_0 - s}(t' - s) \geq \mathsf{dbf}_W(t') \\ \Rightarrow\ & \frac{\mathsf{dbf}_W(t_0)}{t_0 - s} \geq \frac{\mathsf{dbf}_W(t')}{t' - s}. \end{aligned} \tag{7}$$

Since Equation 7 contradicts Equation 6, the lemma holds. ■

For any given Π and any given $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Pi,t}$ be the resource model with period Π such that its USBF intersects dbf_W at time point t. Then, $\Gamma_{\Pi,t}$ is unique and its bandwidth can be determined using Lemma 4.

Lemma 4: Given any $\Pi \in \mathbb{N}$ and any $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Pi,t} = (\Pi, \Theta)$ be the periodic resource model such that $\mathsf{usbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t)$. Then, $\Theta = \mathsf{Exec}(\Pi, t, d_t)$ and $\mathsf{bw}_{\Gamma_{\Pi,t}} = \mathsf{BW}(\Pi, t, d_t)$, where $d_t = \mathsf{dbf}_W(t)$ and

$$\mathsf{Exec}(\Pi, t, d_t) \stackrel{\mathsf{def}}{=} \frac{\Pi - t + \sqrt{(\Pi - t)^2 + 4\Pi d_t}}{2},$$

$$\mathsf{BW}(\Pi, t, d_t) \stackrel{\mathsf{def}}{=} \frac{\Pi - t + \sqrt{(\Pi - t)^2 + 4\Pi d_t}}{2\Pi}.$$

Proof: We have:

$$\begin{aligned} & \mathsf{usbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t) \\ \Leftrightarrow\ & \frac{\Theta}{\Pi}(t - (\Pi - \Theta)) = d_t \\ \Leftrightarrow\ & \Theta^2 + \Theta(t - \Pi) - \Pi d_t = 0 \end{aligned}$$

Since $\Theta \geq 0$, the above equation has a unique solution:

$$\Theta = \frac{-(t - \Pi) + \sqrt{(t - \Pi)^2 + 4\Pi d_t}}{2}.$$

In other words, $\Theta = \mathsf{Exec}(\Pi, t, d_t)$. As a result, the bandwidth of $\Gamma_{\Pi,t}$ is $\mathsf{bw}_{\Gamma_{\Pi,t}} = \frac{\Theta}{\Pi} = \mathsf{BW}(\Pi, t, d_t)$. Hence the lemma. ■

Lemma 5: For any $t \in \mathsf{CrT}_W$, the function $\mathsf{BW}(\Pi, t, d_t)$ defined in Lemma 4 is increasing on the domain of Π.

Proof: Since $\frac{d\mathsf{BW}(\Pi)}{d\Pi} \geq 0$ implies $\mathsf{BW}(\Pi)$ is increasing, we would like to show

$$\frac{d\mathsf{BW}(\Pi)}{d\Pi} \geq 0$$

$$\Leftrightarrow \Big(\frac{\Pi - t + \sqrt{(\Pi-t)^2 + 4d_t\Pi}}{\Pi}\Big)' \geq 0$$

$$\Leftrightarrow \Big(1 - \frac{t}{\Pi} + \sqrt{\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}}\Big)' \geq 0$$

$$\Leftrightarrow 0 + \frac{t}{\Pi^2} + \frac{1}{2}\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}^{(-\frac{1}{2})}\Big(\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}\Big)' \geq 0$$

$$\Leftrightarrow \frac{t}{\Pi^2} + \frac{1}{2}\sqrt{\frac{\Pi^2}{(\Pi-t)^2 + 4d_t\Pi}}\Big(\frac{(2t-4d_t)\Pi - 2\cdot t^2}{\Pi^3}\Big) \geq 0$$

$$\Leftrightarrow 2t\cdot\Pi + \sqrt{\frac{\Pi^2}{(\Pi-t)^2 + 4d_t\Pi}}\big((2t-4d_t)\Pi - 2\cdot t^2\big) \geq 0$$

$$\textit{by multiplying } 2\Pi^3 \geq 0$$

$$\Leftrightarrow \sqrt{\frac{\Pi^2}{(\Pi-t)^2 + 4d_t\Pi}}$$

$$\Big(\Big(\frac{\Pi^2}{(\Pi-t)^2 + 4d_t\Pi}\Big)^{(-\frac{1}{2})}2t\cdot\Pi + (2t-4d_t)\Pi - 2\cdot t^2\Big) \geq 0$$

$$\Leftrightarrow \sqrt{\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}}2t\cdot\Pi + (2t-4d_t)\Pi - 2\cdot t^2 \geq 0$$

$$\textit{by dividing } \sqrt{\frac{\Pi^2}{(\Pi-t)^2 + 4d_t\Pi}} \geq 0$$

$$\Leftrightarrow \sqrt{\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}}2t\cdot\Pi + (2t-4d_t)\Pi - 2\cdot t^2 \geq 0$$

$$\Leftrightarrow \sqrt{\frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2}} \geq \frac{t^2 - (t-2d_t)\Pi}{t\cdot\Pi}$$

$$\Leftrightarrow \frac{(\Pi-t)^2 + 4d_t\Pi}{\Pi^2} \geq \Big(\frac{t^2 - (t-2d_t)\Pi}{t\cdot\Pi}\Big)^2$$

by applying square in both sides(since the left side ≥ 0, *the equation trivailly holds if the right side* < 0*).*

$$\Leftrightarrow t^2(\Pi-t)^2 + t^2\cdot 4d_t\Pi \geq \big(t^2 - (t-2d_t)\Pi\big)^2$$

$$\Leftrightarrow 4d_t\cdot(t-d_t) \geq 0$$

which is obvious since $t \geq d_t = \mathsf{dbf}_W(t)$. ∎

The bound $\Pi_{\max}$ on the optimal period Θ_{opt} can now be computed based on CrT_W and a known resource model $\Gamma_c = (\Pi_c, \Theta_c)$ with $\Pi_c \leq \Pi$ that can feasibly schedule W. Theorem 1 formally specifies this bound.

Theorem 1: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$ with $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$. Suppose $\Gamma_c = (\Pi_c, \Theta_c)$ is the current periodic resource model obtained at some intermediate execution step of Algorithm 1. Then, the optimal periodic resource model $\Gamma_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ for W satisfies

$$\Pi_c \leq \Pi_{\mathsf{opt}} \leq \mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$$

where $\kappa = \frac{\Theta_c}{\Pi_c}$ and

$$\mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W) \stackrel{\text{def}}{=} \min_{t\in\mathsf{CrT}_W} \frac{\kappa t - \mathsf{dbf}_W(t)}{\kappa(1-\kappa)}.$$

Proof: Since Algorithm 1 finds the optimal resource period in an increasing manner, $\Pi_{\mathsf{opt}} \geq \Pi_c$. Further, that Γ_{opt} is optimal implies

$$\mathsf{bw}_{\Gamma_{\mathsf{opt}}} \leq \mathsf{bw}_{\Gamma_c} = \kappa. \tag{8}$$

Next, for any given $t \in \mathsf{CrT}_W$, let $\Gamma_{\Pi_{\mathsf{opt}},t} = (\Pi_{\mathsf{opt}}, \Theta_t)$ where $\Theta_t = \mathsf{Exec}(\Pi_{\mathsf{opt}}, t, d_t)$ and $d_t = \mathsf{dbf}_W(t)$. That is, the USBF of $\Gamma_{\Pi_{\mathsf{opt}},t}(t)$ intersects dbf_W at time point t. Let $\Gamma^*_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta^*_{\mathsf{opt}})$ be the resource model with the minimum bandwidth among all resource models with period Π_{opt} that can potentially schedule W. Then, its bandwidth must be at least equal to that of $\Gamma_{\Pi_{\mathsf{opt}},t}$ for all $t \in \mathsf{CrT}_W$ (otherwise, Γ^*_{opt} does not satisfy the USBF-schedulability condition). Thus,

$$\forall\, t \in \mathsf{CrT}_W :\ \mathsf{bw}_{\Gamma^*_{\mathsf{opt}}} \geq \mathsf{BW}(\Pi_{\mathsf{opt}}, t, d_t) \tag{9}$$

On the other hand, since Γ_{opt} can feasibly schedule W, its bandwidth must be at least equal to that of Γ^*_{opt}. That is,

$$\mathsf{bw}_{\Gamma_{\mathsf{opt}}} \geq \mathsf{bw}_{\Gamma^*_{\mathsf{opt}}} \tag{10}$$

Combine Equations 8, 9 and 10, we obtain: For all $t \in \mathsf{CrT}_W$, $\mathsf{BW}(\Pi_{\mathsf{opt}}, t, d_t) \leq \kappa$ is equivalent to

$$\Leftrightarrow \quad \sqrt{(\Pi_{\mathsf{opt}} - t)^2 + 4\Pi_{\mathsf{opt}}d_t} \leq 2\kappa\cdot\Pi_{\mathsf{opt}} + t - \Pi_{\mathsf{opt}}$$

$$\Leftrightarrow \quad (\Pi_{\mathsf{opt}} - t)^2 + 4\Pi_{\mathsf{opt}}d_t \leq \big((2\kappa-1)\Pi_{\mathsf{opt}} + t\big)^2$$

$$\Leftrightarrow \quad \Pi_{\mathsf{opt}} \leq \frac{\kappa t - d_t}{\kappa(1-\kappa)} = \frac{\kappa t - \mathsf{dbf}_W(t)}{\kappa(1-\kappa)}$$

The above can be rewritten as $\Pi_{\mathsf{opt}} \leq \min_{t\in\mathsf{CrT}_W} \frac{\kappa t - \mathsf{dbf}_W(t)}{\kappa(1-\kappa)}$ or $\Pi_{\mathsf{opt}} \leq \mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$. ∎

Example 3: Given a workload W with $\mathsf{CrT}_W = \{10\}$ and $\mathsf{dbf}_W(10) = 2$. Suppose that $\Gamma_c = (2,1)$ is the current minimum bandwidth periodic resource model that can feasibly schedule W among all models with period $\Pi \leq 2$. In this case, $\kappa = 0.5$. The upper bound on the resource period is computed to be $\Pi_{\mathsf{opt}} \leq \mathsf{MaxResPeriod}(0.5, \mathsf{dbf}_W, \mathsf{LCM}_W) = \frac{0.5\cdot 10 - 2}{0.5(1-0.5)} = 12$ by Theorem 1. As illustrated in Figure 3, the optimal periodic resource model for W is $\Gamma_{\mathsf{opt}} = (3,1)$, which indeed satisfies Theorem 1.

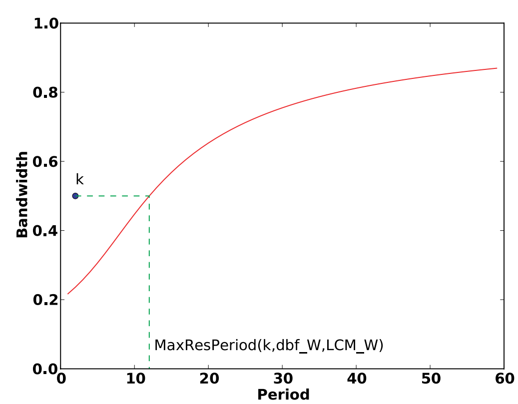

Fig. 3. The upper bound on the resource period in Example 3.

B. A revised algorithm using bound on resource period

Algorithm 2 gives an extension of Algorithm 1 by incorporating the upper bound on the resource periods $\mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W))$ defined in Section III-A.

Algorithm 2 A revised algorithm using resource period bound.

Input: dbf_W, LCM_W for a workload W

Output: The optimal periodic resource model Γ for W

```
1:  if dbf_W(LCM_W) ≥ LCM_W − 1 then
2:      Γ = (1, 1)
3:  else
4:      Θ′ = MinExec(LCM_W, dbf_W, LCM_W)
5:      κ = Θ′ / LCM_W
6:      Π_max = MaxResPeriod(κ, dbf_W, LCM_W)
7:      for Π = 1 to Π_max do
8:          Θ = MinExec(Π, dbf_W, LCM_W)
9:          if Θ/Π < κ then
10:             κ = Θ/Π
11:             Γ = (Π, Θ)
12:             Π_max=min(Π_max, MaxResPeriod(κ,dbf_W,LCM_W))
13:         end if
14:     end for
15: end if
```

In Algorithm 2, Line 1-2 handles the special case $\mathsf{dbf}_W(\mathsf{LCM}_W) \geq \mathsf{LCM}_W - 1$, which has $\Gamma = (1,1)$ as the minimum bandwidth resource model. This is because any resource model $\Gamma' = (\Pi, \Theta)$ that can feasibly schedule W must satisfy $2(\Pi - \Theta) \leq 1$ (due to $\mathsf{sbf}_{\Gamma'}(\mathsf{LCM}_W) \geq \mathsf{dbf}_W(\mathsf{LCM}_W) \geq \mathsf{LCM}_W - 1$) and hence $\Pi = \Theta$ (since $\Theta, \Pi \in \mathbb{N}$). In Line 4-5, Θ' denotes the minimum supply for $\Pi = \mathsf{LCM}_W$ and κ denotes the bandwidth of $(\mathsf{LCM}_W, \Theta')$. Since κ is not 1, we can find the initial $\Pi_{\max}$ in Line 6. The function $\mathsf{MinExec}(\Pi$, dbf_W, $\mathsf{LCM}_W)$ (Line 4,8) is the same as in Algorithm 1. The function $\mathsf{MaxResPeriod}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ in Line 6 and 12 computes the upper bound on the resource period as defined in Theorem 1. Finally, the minimum bandwidth acquired during algorithm execution is stored in κ and used to re-evaluate $\Pi_{\max}$ (Line 9-13).

Computation complexity. Observe that the time complexity of Algiorithm 2 is building CrT_W and $\mathsf{MaxResPeriod}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ times $\mathsf{MinExec}(\Pi$, dbf_W, $\mathsf{LCM}_W)$.

By definition of CrT_W, building CrT_W is O($\min_{P_i \in W} P_i \cdot LCM_W$). We know $\mathsf{MinExec}(\Pi$, dbf_W, $\mathsf{LCM}_W)$ is O(LCM_W) [1].

The rest is the complexity of $\mathsf{MaxResPeriod}(\kappa$, dbf_W, $\mathsf{LCM}_W)$. Since $\mathsf{dbf}(\mathsf{LCM}_W) < \mathsf{LCM}_W - 1$, at least the periodic resource model $(\mathsf{LCM}_W, \mathsf{LCM}_W - 1)$ can schedule W. Hence, our worst-case initial κ is $\frac{\mathsf{LCM}_W - 1}{\mathsf{LCM}_W}$. We know $y \in CrT_W$ where $y = \min_{p_i \in W} p_i$. Therefore,

$$
\begin{aligned}
& O(\mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)) \\
= \; & O(\frac{\frac{\mathsf{LCM}_W - 1}{\mathsf{LCM}_W} y - \mathsf{dbf}_W(y)}{\frac{\mathsf{LCM}_W - 1}{\mathsf{LCM}_W}(1 - \frac{\mathsf{LCM}_W - 1}{\mathsf{LCM}_W})}) \\
= \; & O(\frac{y - \mathsf{dbf}_W(y)}{(\frac{1}{\mathsf{LCM}_W})}) \\
= \; & O((y - \mathsf{dbf}_W(y)) \cdot \mathsf{LCM}_W) \\
= \; & O((y - 1) \cdot \mathsf{LCM}_W) \qquad \qquad since \; \mathsf{dbf}_W(y) \geq 1 \\
= \; & O(\min_{p_i \in W} p_i \cdot \mathsf{LCM}_W)
\end{aligned}
$$

Hence, the time complexity of Algorithm 2 is $\mathrm{O}\big(\min_{P_i \in W} P_i \cdot \mathsf{LCM}_W + \mathsf{LCM}_W \times (\min_{P_i \in W} P_i \cdot \mathsf{LCM}_W)\big)$, which is equal to $\mathrm{O}\big((\mathsf{LCM}_W)^2 \cdot \min_{P_i \in W} P_i\big)$.

IV. A NEW ALGORITHM FOR COMPUTING THE OPTIMAL PERIODIC RESOURCE MODEL

In this section, we present a new algorithm for computing the optimal resource model that is more efficient than the revised algorithm in the previous section. Observe that in searching for the optimal resource model for a workload W, Algorithm 2 iterates the resource period Π from 1 to the period bound $\Pi_{\max}$, which is computed using $\mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$ and updated with respect to the minimum bandwidth κ obtained thus far. Since computing the resource execution time Θ for any given period Π has a constant time complexity, the algorithm's time complexity is proportional to the number of iterations of Π, which is $\mathsf{MaxResPeriod}(\kappa_0$,$\mathsf{dbf}_W$, $\mathsf{LCM}_W)$ in the worst case where κ_0= $\mathsf{MinExec}(\mathsf{LCM}_W$,$\mathsf{dbf}_W$,$\mathsf{LCM}_W)$/$\mathsf{LCM}_W$. Since $\Theta \leq \Pi$, the upper bound on Θ will always be less than or equal to the upper bound on Π. Further, computing the resource period Π for any given Θ has the same time complexity as that of computing Θ from Π. As a result, we can reduce the search space by iterating Θ instead of Π.

Based on the above observation, Algorithm 3 gives a new procedure for computing the optimal resource model. We first explain the different steps involved in the algorithm and then present theoretical results supporting its correctness. Note that the result for the special case when $\mathsf{dbf}_W(\mathsf{LCM}_W) \geq \mathsf{LCM}_W - 1$ is $\Gamma = (1, 1)$ for the same reason as in Algorithm 2.

In Algorithm 3, the function $\mathsf{MinExec}(\Pi$, dbf_W, $\mathsf{LCM}_W)$ (Line 1) is the same as in Algorithm 1. The variable κ (Line 1) indicates the bandwidth of (LCM_W,$\mathsf{MinExec}(\mathsf{LCM}_W$, dbf_W, LCM_W)). The function $\mathsf{MaxResExec}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ (Line 2 and 8) computes the upper bound of Θ based on Theorem 3. The initial value of $\Theta_{\max}$ is in Line 2. The function $\mathsf{MaxPeriod}(\Theta$, dbf_W, $\mathsf{LCM}_W)$ (Line 4) computes – for any given Θ – an upper bound on the resource period Π of any resource model (Π, Θ) that can feasibly schedule W. The functions $\mathsf{MaxPeriod}(\Theta$, dbf_W, $\mathsf{LCM}_W)$ and $\mathsf{MaxResExec}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ are computed as below.

Algorithm 3 A new interface generation algorithm

Input: dbf_W, LCM_W for a workload W with $\mathsf{dbf}_W(\mathsf{LCM}_W) < \mathsf{LCM}_W - 1$

Output: The optimal periodic resource model Γ for W

$\kappa = \mathsf{MinExec}(\mathsf{LCM}_W, \mathsf{dbf}_W, \mathsf{LCM}_W)/\mathsf{LCM}_W$
$\Theta_{\max} = \mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$
for $\Theta = 1$ to $\Theta_{\max}$ **do**
  $\Pi = \mathsf{MaxPeriod}(\Theta, \mathsf{dbf}_W, \mathsf{LCM}_W)$
  if $\frac{\Theta}{\Pi} < \kappa$ **then**
    $\kappa = \frac{\Theta}{\Pi}$
    $\Gamma = (\Pi, \Theta)$
    $\Theta_{\max} = \min(\Theta_{\max}, \mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W))$
  end if
end for

Computation of $\mathsf{MaxPeriod}(\Theta, \mathsf{dbf}_W, \mathsf{LCM}_W)$**.** Theorem 2 gives the upper bound on the period Π of any resource model $\Gamma = (\Pi, \Theta)$ that can feasibly schedule W.

Theorem 2: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$ where $T_i = (p_i, e_i)$ for all $1 \le i \le n$. For any given Θ, the resource model $\Gamma = (\Pi, \Theta)$ can feasibly schedule W iff

$$\Pi \le \min_{0 \le t \le \mathsf{LCM}_W} \mathsf{IntPeriod}(\Theta, t) \stackrel{\mathsf{def}}{=} \mathsf{MaxPeriod}(\Theta, \mathsf{dbf}_W, \mathsf{LCM}_W)$$

$$\text{where: } \mathsf{IntPeriod}(\Theta, t) \stackrel{\mathsf{def}}{=} \begin{cases} \lfloor \frac{t+\Theta}{m-1} \rfloor, & \text{if } \mathsf{sbf}_\Gamma(t) \ge \mathsf{dbf}_W(t) \\ \lfloor \frac{t+\Theta}{m} \rfloor, & \text{otherwise} \end{cases}$$

with $m = \lceil \frac{\mathsf{dbf}_W(t)+\Theta}{\Theta} \rceil$.

Before presenting the proof of Theorem 2, we state some notations. For any given Θ and any given t such that $1 \le t \le \mathsf{LCM}_W$, $\mathsf{Period}(\Theta, t)$ denotes a period value such that the resource model $R_{\Theta,t} = (\mathsf{Period}(\Theta, t), \Theta)$ satisfies $\mathsf{sbf}_{R_{\Theta,t}}(t) = \mathsf{dbf}_W(t)$. Then, the following corollary holds.

Corollary 1: For all $\Theta \ge 0$, and all t s.t. $1 \le t \le \mathsf{LCM}_W$,

$$\lfloor \mathsf{Period}(\Theta, t) \rfloor = \mathsf{IntPeriod}(\Theta, t).$$

Proof: For any given Θ and any given t such that $1 \le t \le LCM_W$, We can compute $\mathsf{Period}(\Theta, t)$ by the definition. Let $d = \mathsf{dbf}_W(t)$.

$$\begin{aligned} & \mathsf{sbf}_{(\mathsf{Period}(\Theta,t),\Theta)}(t) = d & (11) \\ \Leftrightarrow \quad & \lfloor \frac{t - (\mathsf{Period}(\Theta,t) - \Theta)}{\mathsf{Period}(\Theta,t)} \rfloor \Theta + rem(t) = d \\ \Leftrightarrow \quad & \lfloor \frac{t+\Theta}{\mathsf{Period}(\Theta,t)} \rfloor = \frac{d - rem(t) + \Theta}{\Theta} \\ \Leftrightarrow \quad & \frac{t+\Theta}{\mathsf{Period}(\Theta,t)} = \lceil \frac{d - rem(t) + \Theta}{\Theta} \rceil \\ \Leftrightarrow \quad & \mathsf{Period}(\Theta,t) = \frac{t+\Theta}{w} \ where \ w = \lceil \frac{d - rem(t) + \Theta}{\Theta} \rceil \end{aligned}$$

Let $\mathsf{IntPeriod}(\Theta, t) = \lfloor \mathsf{Period}(\Theta, t) \rfloor$. Then, we have $\mathsf{sbf}_{(\mathsf{IntPeriod}(\Theta,t),\Theta)}(t) \ge sbf_{(Period(\Theta,t),\Theta)}(t) = \mathsf{dbf}_W(t)$ which implies

$$\mathsf{sbf}_{(\mathsf{IntPeriod}(\Theta,t),\Theta)}(t) \ge \mathsf{dbf}_W(t) \qquad (12)$$

$\mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{w} \rfloor$ where $w = \left\lceil \frac{d-rem(t)+\Theta}{\Theta} \right\rceil$ Let $m = \lceil \frac{d+\Theta}{\Theta} \rceil$. Then, $w = m$ or $w = m-1$ since $0 \le rem(t) < \Theta$. Hence, $\mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{m-1} \rfloor$ or $\lfloor \frac{t+\Theta}{m} \rfloor$. Since we would like to compute maximum period to schedule W in Theorem 2, large $\mathsf{IntPeriod}(\Theta, t)$ is better.

i) Suppose that $\mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{m-1} \rfloor$ first. We should check whether assumption is right since we don't know $rem(t)$ yet. Therefore, Eqation 12 should hold. Let $\Pi' = \lfloor \frac{t+\Theta}{m-1} \rfloor$. If $\mathsf{sbf}_{(\Pi',\Theta)}(t) \ge \mathsf{dbf}_W(t)$, assumption is right. Therefore, $\lfloor \mathsf{Period}(\Theta, t) \rfloor = \mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{m-1} \rfloor$ where $m = \lceil \frac{\mathsf{dbf}_W(t)+\Theta}{\Theta} \rceil$ if $\mathsf{sbf}_{(\Pi',\Theta)}(t) \ge \mathsf{dbf}_W(t)$.

ii) Otherwise, assume that $\mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{m} \rfloor$. Agian, we should check whether assumption is right. Therefore, Eqation 12 should hold. Since $\lfloor \frac{t+\Theta}{m} \rfloor \le \lfloor \frac{t+\Theta}{w} \rfloor$ where $w = \left\lceil \frac{d-rem(t)+\Theta}{\Theta} \right\rceil$ and $m = \lceil \frac{d+\Theta}{\Theta} \rceil$, we know

$$\begin{aligned} & \mathsf{sbf}_{(\lfloor \frac{t+\Theta}{m} \rfloor,\Theta)}(t) \ge \mathsf{sbf}_{(\lfloor \frac{t+\Theta}{w} \rfloor,\Theta)}(t) \\ \Leftrightarrow \ & \mathsf{sbf}_{(\mathsf{IntPeriod}(\Theta,t),\Theta)}(t) \ge \mathsf{sbf}_{(\mathsf{Period}(\Theta,t),\Theta)}(t) \\ \Leftrightarrow \ & \mathsf{sbf}_{(\mathsf{IntPeriod}(\Theta,t),\Theta)}(t) \ge \mathsf{dbf}_W(t) \end{aligned}$$

which implies Equation 12 holds. Therefore, $\lfloor \mathsf{Period}(\Theta, t) \rfloor = \mathsf{IntPeriod}(\Theta, t) = \lfloor \frac{t+\Theta}{m} \rfloor$ where $m = \lceil \frac{\mathsf{dbf}_W(t)+\Theta}{\Theta} \rceil$ if $\mathsf{sbf}_{(\Pi',\Theta)}(t) < \mathsf{dbf}_W(t)$. ∎

Proof of Theorem 2: ($\Rightarrow$) Recall the SBF of Γ defined in Equation 2. One can easily verify that for all Π_1, Π_2,

$$\Pi_1 \le \Pi_2 \ \Leftrightarrow \ \mathsf{sbf}_{(\Pi_1,\Theta)}(t) \ge \mathsf{sbf}_{(\Pi_2,\Theta)}(t) \ \ \forall t \ge 0. \qquad (13)$$

Suppose $\Gamma = (\Pi, \Theta)$ can feasibly schedule W, i.e.,

$$\forall\, 0 \le t \le \mathsf{LCM}_W : \ \mathsf{sbf}_\Gamma(t) \ge \mathsf{dbf}_W(t).$$

By definition, $\mathsf{dbf}_W(t) = \mathsf{sbf}_{R_{\Theta,t}}(t)$ where $R_{\Theta,t} = (\mathsf{Period}(\Theta, t), \Theta)$ for all $0 \le t \le \mathsf{LCM}_W$. Hence,

$$\forall\, 0 \le t \le \mathsf{LCM}_W : \ \mathsf{sbf}_\Gamma(t) \ge \mathsf{sbf}_{R_{\Theta,t}}(t). \qquad (14)$$

Since Γ and $R_{\Theta,t}$ have the same execution time Θ, and due to Equation 13, Equation 14 is equivalent to

$$\forall\, 0 \le t \le \mathsf{LCM}_W : \ \Pi \le \mathsf{Period}(\Theta, t).$$

Since $\Pi \in \mathbb{N}$, $\Pi \le \mathsf{Period}(\Theta, t)$ is equivalent to $\Pi \le \lfloor \mathsf{Period}(\Theta, t) \rfloor = \mathsf{IntPeriod}(\Theta, t)$ due to Corollary 1. Hence,

$$\Pi \le \min_{0 \le t \le \mathsf{LCM}_W} \mathsf{IntPeriod}(\Theta, W, t)$$

($\Leftarrow$) Suppose $\Pi \le \min_{0 \le t \le \mathsf{LCM}_W} \mathsf{IntPeriod}(\Theta, t)$. Then,

$$\Pi \le \mathsf{IntPeriod}(\Theta, t) \le \mathsf{Period}(\Theta, t), \ \forall\, 0 \le t \le \mathsf{LCM}_W.$$

Denote $\Pi_t = \mathsf{Period}(\Theta, t)$. Apply Equation 13, we have

$$\forall\, 0 \le t \le \mathsf{LCM}_W, \ \mathsf{sbf}_{(\Pi,\Theta)}(t) \ge \mathsf{sbf}_{(\Pi_t,\Theta)}(t).$$

Since $\mathsf{sbf}_{(\Pi_t,\Theta)} = \mathsf{dbf}_W(t)$ by the definition of Π_t, we imply $\mathsf{sbf}_{(\Pi,\Theta)}(t) \ge \mathsf{dbf}_W(t)$ for all $0 \le t \le \mathsf{LCM}_W$. In other words, $\Gamma = (\Pi, \Theta)$ can feasibly schedule W. ∎

Computation of $\mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$**.** For any given current minimum bandwidth κ at some intermediate execution step of Algorithm 3, we can compute the upper bound $\mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$ on the value Θ of the optimal resource model $\Gamma_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ in a similar fashion as done in Section III-A. Theorem 3 formally defines this bound.

Theorem 3: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$ that is scheduled under EDF, with $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$. Suppose $\Gamma_c = (\Pi_c, \Theta_c)$ is the current periodic resource model obtained at some intermediate execution step of Algorithm 3. Then, the optimal periodic resource model $\Gamma_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ for W satisfies $\Theta_c \leq \Theta_{\mathsf{opt}} \leq \mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$ with $\kappa = \frac{\Theta_c}{\Pi_c}$, and

$$\mathsf{MaxResExec}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W) \stackrel{\mathsf{def}}{=} \min_{t \in \mathsf{CrT}_W} \frac{\kappa t - \mathsf{dbf}_W(t)}{1-\kappa}.$$

Before presenting proof, we show basic idea behind finding the upper bound of resource execution time of the optimal periodic resource model.

The upper bound on the resource execution time of the optimal periodic resource model for a given workload W can be derived based on the DBF of the workload and its relationship with the USBFs of the resource models that can potentially schedule W. Intuitively, let M be the set of resource models that can potentially schedule W (i.e., they satisfy the USBF-schedulability condition). Suppose $\Gamma_{\mathsf{opt}} = (\Pi_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ is the bandwidth-optimal resource model for W. Then,

$$\mathsf{bw}_{\Gamma_{\mathsf{opt}}} \geq B_{\min} \stackrel{\mathsf{def}}{=} \min\{\mathsf{bw}_{\Gamma} \mid \Gamma \in M\}$$

To derive the bound on Θ_{opt}, we will find all the possible resource models in M that have the minimum bandwidth $B_{\min}$. Towards this, we vary the resource execution time Θ and compute for each Θ a (unique) resource model $\Gamma_\Theta = (\Pi_\Theta, \Theta)$ belonging to M that has the minimum bandwidth among all the resource models in M with the same resource execution time Θ. Then, $B_{\min}$ can be computed by

$$B_{\min} = \min\{\mathsf{bw}_{\Gamma_\Theta} \mid \Theta \in \mathbb{N}\}.$$

We know that for all Θ, the USBF of Γ_Θ intersects dbf_W at *critical time point*. Further, Γ_Θ has the largest bandwidth among all the resource models $\Gamma_{\Theta,t}$ with period Θ that have their USBF intersecting dbf_W at critical time points t. In other words,

$$\mathsf{bw}_{\Gamma_\Theta} = \max_{\substack{t \in \mathsf{CrT}_W \\ \Gamma_{\Theta,t} = (\Pi_{\Theta,t}, \Theta)}} \{\mathsf{bw}_{\Gamma_{\Theta,t}} \mid \mathsf{usbf}_{\Gamma_{\Theta,t}}(t) = \mathsf{dbf}_W(t)\},$$

where CrT_W is the set of all critical time points of W. Further, for given any Θ and any $t \in \mathsf{CrT}_W$, we can compute the bandwidth of $\Gamma_{\Theta,t}$ directly from $\mathsf{dbf}_W(t)$, Θ and t. From these values, we can derive $B_{\min}$, which in turn allow us to bound the resource period Θ_{opt} of the bandwidth-optimal resource model for W. Below we present the details.

For any given Θ and any given $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Theta,t}$ be the resource model with execution time Θ such that its USBF intersects dbf_W at time point t. Then, $\Gamma_{\Theta,t}$ is unique and its bandwidth can be determined using Lemma 6.

Lemma 6: Given any $\Theta \in \mathbb{N}$ and any $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Theta,t} = (\Pi, \Theta)$ be the periodic resource model such that $\mathsf{usbf}_{\Gamma_{\Theta,t}}(t) = \mathsf{dbf}_W(t)$. Then, $\Pi = \mathsf{Period}(\Theta, t, d_t)$ and $\mathsf{bw}_{\Gamma_{\Theta,t}} = \mathsf{BW}(\Theta, t, d_t)$ where

$$\mathsf{Period}(\Theta, t, d_t) \stackrel{\mathsf{def}}{=} \frac{\Theta(t+\Theta)}{d_t + \Theta}$$

and

$$\mathsf{BW}(\Theta, t, d_t) \stackrel{\mathsf{def}}{=} \frac{d_t + \Theta}{t + \Theta}$$

where $d_t = \mathsf{dbf}_W(t)$.

Proof: We have:

$$\begin{aligned} & \mathsf{usbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t) = d_t \\ \Leftrightarrow\ & \frac{\Theta}{\Pi}(t - (\Pi - \Theta)) = d_t \\ \Leftrightarrow\ & \Theta^2 + \Theta(t - \Pi) - \Pi d_t = 0 \end{aligned}$$

Since $\Theta \geq 0$, the above is equivalent to

$$\Pi = \frac{\Theta(t+\Theta)}{d_t + \Theta}.$$

In other words, $\Pi = \mathsf{Period}(\Theta, t, d_t)$. As a result, the bandwidth of $\Gamma_{\Theta,t}$ is $\mathsf{bw}_{\Gamma_{\Theta,t}} = \frac{\Theta}{\Pi} = \mathsf{BW}(\Theta, t, d_t)$. Hence the lemma. ■

Lemma 7: For any given $t \in \mathsf{CrT}_W$, the bandwidth function $\mathsf{BW}(\Theta, t, d_t)$ defined in Lemma 6 is increasing on the domain of Θ.

Proof: for $\forall \Theta_1, \Theta_2$ s.t. $\Theta_1 < \Theta_2$, $\mathsf{BW}(\Theta_1, t, d_t) < \mathsf{BW}(\Theta_1, t, d_t)$.

$$\begin{aligned} & \frac{d_t + \Theta_1}{t + \Theta_1} < \frac{d_t + \Theta_2}{t + \Theta_2} \\ \Leftrightarrow\ & (d_t + \Theta_1)(t + \Theta_2) < (d_t + \Theta_2)(t + \Theta_1) \\ \Leftrightarrow\ & \Theta_1 t + \Theta_2 d_t < \Theta_2 t + \Theta_1 d_t \\ \Leftrightarrow\ & \Theta_1 (t - d_t) < \Theta_2 (t - d_t) \\ \Leftrightarrow\ & \Theta_1 < \Theta_2 \ since\ (t - d_t) \geq 0 \end{aligned}$$

which is exactly assumption. ■

Proof of Theorem 3: Since Algorithm algo:new finds the optimal resource period in an increasing manner, $\Theta_{\mathsf{opt}} \geq \Theta_c$. Further, since Γ_{opt} is bandwidth-optimal,

$$\mathsf{bw}_{\Gamma_{\mathsf{opt}}} \leq \mathsf{bw}_{\Gamma_c} = \kappa. \tag{15}$$

Next, for any given $t \in \mathsf{CrT}_W$, let $\Gamma_{\Theta_{\mathsf{opt}},t} = (\Pi_t, \Theta_{\mathsf{opt}})$ where $\Pi_t = \mathsf{Period}(\Theta_{\mathsf{opt}}, t, d_t)$ where $d_t = \mathsf{dbf}_W(t)$. That is, the USBF of $\Gamma_{\Theta_{\mathsf{opt}},t}(t)$ intersects dbf_W at time point t. Let $\Gamma^*_{\mathsf{opt}} = (\Pi^*_{\mathsf{opt}}, \Theta_{\mathsf{opt}})$ be the resource model with the minimum bandwidth among all resource models with period Θ_{opt} that can potentially schedule W. Then, its bandwidth must be greater or equal to that of $\Gamma_{\Theta_{\mathsf{opt}},t}$ for all $t \in \mathsf{CrT}_W$ (otherwise, Γ^*_{opt} does not satisfy the USBF-schedulability condition). In other words, for all $t \in \mathsf{CrT}_W$,

$$\mathsf{bw}_{\Gamma^*_{\mathsf{opt}}} \geq \mathsf{BW}(\Theta_{\mathsf{opt}}, t, d_t) \tag{16}$$

On the other hand, since Γ_{opt} can feasibly schedule W, its bandwidth must be greater or equal to that of Γ^*_{opt}. That is,

$$\mathsf{bw}_{\Gamma_{\text{opt}}} \geq \mathsf{bw}_{\Gamma^*_{\text{opt}}} \tag{17}$$

Combine Equations 15, 16 and 17, we obtain: For all $t \in \mathsf{CrT}_W$,

$$\begin{aligned} &\mathsf{BW}(\Theta_{\text{opt}}, t, d_t)\frac{d_t + \Theta_{\text{opt}}}{t + \Theta_{\text{opt}}} \leq \kappa \\ \Leftrightarrow\quad & d_t + \Theta_{\text{opt}} \leq \kappa(t + \Theta_{\text{opt}}) \\ \Leftrightarrow\quad & \Theta_{\text{opt}} \leq \frac{\kappa t - d_t}{1-\kappa} \end{aligned}$$

Then, we have

$$\Theta_{\text{opt}} \leq \frac{\kappa t - \mathsf{dbf}_W(t)}{1-\kappa}, \ \forall\, t \in \mathsf{CrT}_W \tag{18}$$

which can be rewritten as $\Theta_{\text{opt}} \leq \min_{t\in\mathsf{CrT}_W} \frac{\kappa t - \mathsf{dbf}_W(t)}{1-\kappa}$ or $\Theta_{\text{opt}} \leq \mathsf{MaxResExec}(\kappa, W)$. ■

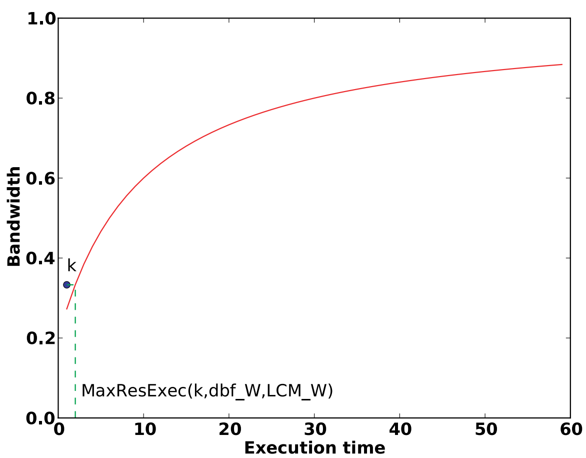

Fig. 4. The upper bound on the resource execution units in Example 4.

Example 4: Given a workload W with $\mathsf{CrT}_W = \{10\}$ and $\mathsf{dbf}_W(10) = 2$. Suppose that $\Gamma_c = (3,1)$ is the current minimum-bandwidth periodic resource model that can feasibly schedule W given by Algorithm 3. In this case, $\kappa = \frac{1}{3}$. The upper bound on the resource execution units is computed using Theorem 3 to be $\Theta_{\text{opt}} \leq \mathsf{MaxResExec}(\frac{1}{3}, \mathsf{dbf}_W, \mathsf{LCM}_W) = \frac{\frac{1}{3}10-2}{1-\frac{1}{3}} = 2$. As illustrated in Figure 4, the optimal resource model for W is $\Gamma_{\text{opt}} = (3,1)$, which satisfies Theorem 3.

Computation complexity compared to Algorithm 2. Observe that the time complexity of Algiorithm 3 is building CrT_W and $\mathsf{MaxResExec}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ times $\mathsf{MaxPeriod}(\Theta$, dbf_W, LCM_W Note that $\mathsf{MaxPeriod}(\Theta$, dbf_W, $\mathsf{LCM}_W)$ is $\mathsf{O}(LCM_W)$, which is identical to the complexity of $\mathsf{MinExec}(\Pi$, dbf_W, $\mathsf{LCM}_W)$.

Only difference compared to Algorithm 2 is $\mathsf{MaxResExec}(\kappa$, dbf_W, $\mathsf{LCM}_W)$ instead of $\mathsf{MaxResPeriod}(\kappa$, dbf_W, $\mathsf{LCM}_W)$. From Theorem 3 and 1, $\mathsf{MaxResExec}(\kappa$, dbf_W, $\mathsf{LCM}_W) = \kappa\mathsf{MaxResPeriod}(\kappa$, dbf_W, $\mathsf{LCM}_W)$. Since $\kappa \leq \frac{\mathsf{LCM}_W - 1}{\mathsf{LCM}_W}$, one can hence verify that Algorithm 3 is at least $\frac{\mathsf{LCM}_W}{\mathsf{LCM}_W - 1}$ times faster than Algorithm 2.

V. Dual Periodic Resource Model

A. Overhead of periodic resource interface with integer values

When assuming rational parameter values for resource interfaces, the periodic resource interface with period of 1 and execution time equal to the utilization of the workload always has the minimum bandwidth among that of all resource interfaces [3]. However, this optimality of periodic resource model is no longer achievable when it is restricted to have only integer parameters. As an example, consider a workload W composed of only one task $T = (5,1)$. The ideal minimum bandwidth resource interface (i.e., with rational parameter values) given by Algorithm 1) is $(1, 0.2)$. Hence, the minimum bandwidth of W is 0.2. On the other hand, the minimum bandwidth resource interface with integer parameter values for W (given by Algorithm 2) is $(3,1)$, which has a bandwidth of $\frac{1}{3}$. Thus, the minimum bandwidth periodic resource interface with integer parameter values incurs at least 66% overheads compared to the ideal one with rational parameter values. By the same reason, the new algorithm (Algorithm 3) also experiences similar bandwidth overhead.

The above overhead introduced by the integer constraints has prompted a need for new resource interfaces with integer parameters and their associated interface computation techniques that can achieve better resource utilization than the periodic resource interface do. In the coming sections, we present such an interface and its computation. Here, we discuss the computation for leaf-components only; the computation for non-leaf components can be established using a similar technique as in the case of periodic resource interface [1].

B. Dual periodic resource model (DPRM)

A dual periodic resource model (DPRM) interface is defined by $\Omega = (\Gamma_1, \Gamma_2)$ where Γ_1 and Γ_2 are periodic resource models. Semantically, each DPRM offers the same amount of resource as the total resource units given by the two resource models Γ_1 and Γ_2. Thus, its bandwidth is given by $\mathsf{bw}_\Omega = \mathsf{bw}_{\Gamma_1} + \mathsf{bw}_{\Gamma_2}$. Its SBF and schedulability condition are given by Lemma 8 and 9, respectively.

Lemma 8: The SBF of a DPRM $\Omega = (\Gamma_1, \Gamma_2)$ where $\Gamma_1 = (\Pi_1, \Theta_1)$ and $\Gamma_2 = (\Pi_2, \Theta_2)$ is given by:

$$\mathsf{sbf}_\Omega(t) = \mathsf{sbf}_{\Gamma_1}(t) + \mathsf{sbf}_{\Gamma_2}(t), \ \forall t \geq 0. \tag{19}$$

Proof: By SBF definition, $\mathsf{sbf}_\Omega(t)$ gives the minimum number of execution (resource) units that are provided by Ω in any time interval of length t for all $t \geq 0$ (see Section II). Since the minimum number of execution units provided by Γ_1 and by Γ_2 are $\mathsf{sbf}_{\Gamma_1}(t)$ and $\mathsf{sbf}_{\Gamma_2}(t)$, respectively, the minimum number of execution units provided by Ω in any interval of length $t \geq 0$ is $\mathsf{sbf}_{\Gamma_1}(t) + \mathsf{sbf}_{\Gamma_2}(t)$. In other words, for all $t \geq 0$, $\mathsf{sbf}_\Omega = \mathsf{sbf}_{\Gamma_1}(t) + \mathsf{sbf}_{\Gamma_2}(t)$. ■

Lemma 9: Given a component $C = (W, \Omega, EDF)$ where $W = \{T_1, T_2, \ldots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$, and

$\Omega = (\Gamma_1, \Gamma_2)$ is a DPRM interface. C is schedulable under Ω iff

$$\forall t \ s.t. \ 0 \le t \le \mathsf{LCM}_W, \ \mathsf{dbf}_W(t) \le \mathsf{sbf}_\Omega(t). \quad (20)$$

Proof: Immediate from Lemma 1 and Lemma 8. ■

DPRM interface computation. Given a component with workload W that is scheduled under EDF, we would like to compute the optimal DPRM interface $\Omega = (\Gamma_1, \Gamma_2)$ for W (i.e., Ω has the minimum bandwidth among that of all DPRM interface $\Omega' = (\Gamma'_1, \Gamma'_2)$ that can feasibly schedule W). The basic idea is to iterate the period Π_1 and execution time Θ_1 of the first element Γ_1 as well as the periodic Π_2 of the second element Γ_2 of the DPRM interface $\Omega = (\Gamma_1, \Gamma_2)$. For each running Π_1, Θ_1 and Π_2, we compute the corresponding Θ_2 such that $\Omega = \big((\Pi_1, \Theta_1), (\Pi_2, \Theta_2)\big)$ has the minimum bandwidth among that of all DPRM interface $\Omega' = \big((\Pi_1, \Theta_1), (\Pi_2, \Theta'_2)\big)$ that can feasibly schedule W. We then keep track of the interface Ω with the minimum bandwidth during our iteration.

Algorithm 4 shows the procedure for computing the minimum-bandwidth DPRM interface Ω. The functions $\mathsf{MinExec}(\Pi,\ \mathsf{dbf}_W,\ \mathsf{LCM}_W)$ and $\mathsf{MaxResPeriod}(\kappa,\ \mathsf{dbf}_W)$ (Line 1 and 2, respectively) are the same as in Algorithm 3. The function $\mathsf{MaxResExecDPRM}(\Pi,\ \mathsf{dbf}_W,\ \mathsf{LCM}_W)$ (Line 4) gives an upper bound on the value of Θ_1. The function $\mathsf{getResModel}(\mathsf{dbf}_W,\ \mathsf{LCM}_W,\ \Gamma_1)$ (Line 7) gives the optimal periodic resource model for the remaining resource demand of W after W has been served by the resource model Γ_1. These two new functions are computed as below.

Algorithm 4 DPRM interface computation

Input: dbf_W, LCM_W for a workload W with $\mathsf{dbf}_W(\mathsf{LCM}_W) < \mathsf{LCM}_W - 1$
Output: The optimal DPRM $\Omega = (\Gamma_1, \Gamma_2)$ for W

1: $\kappa = \mathsf{MinExec}(\mathsf{LCM}_W, \mathsf{dbf}_W, \mathsf{LCM}_W)/\mathsf{LCM}_W$
2: $\Pi_1^{\max} = \mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W)$
3: **for** $\Pi_1 = 1$ to $\Pi_1^{\max}$ **do**
4: $\quad\Theta_1^{\max} = \mathsf{MaxResExecDPRM}(\Pi, \mathsf{dbf}_W, \mathsf{LCM}_W)$
5: $\quad$**for** $\Theta_1 = 1$ to Θ_1^{max} **do**
6: $\quad\quad\Gamma_1 = (\Pi_1, \Theta_1)$
7: $\quad\quad\Gamma_2 = \mathsf{getResModel}(\mathsf{dbf}_W, \mathsf{LCM}_W, \Gamma_1)$
8: $\quad\quad$**if** $\mathsf{bw}_{\Gamma_1} + \mathsf{bw}\Gamma_2 < \kappa$ **then**
9: $\quad\quad\quad\kappa = \mathsf{bw}_{\Gamma_1} + \mathsf{bw}_{\Gamma_2}$
10: $\quad\quad\quad\Omega = (\Gamma_1, \Gamma_2)$
11: $\quad\quad\quad\Pi_1^{max} = \min(\Pi_1^{max}, \mathsf{MaxResPeriod}(\kappa, \mathsf{dbf}_W, \mathsf{LCM}_W))$
12: $\quad\quad$**end if**
13: $\quad$**end for**
14: **end for**

Computation of $\mathsf{MaxResExecDPRM}(\Pi, \mathsf{dbf}_W, \mathsf{LCM}_W)$**.** Given any Π_1, Θ_1 and Π_2, the execution time Θ_2 of Ω is determined such that the resource supplied by the periodic resource model (Π_2, Θ_2) must be at least equal to the remaining demand of the workload W after W has been serviced by (Π_1, Θ_1). Towards this, we define the *remaining demand bound function* (RDBF) as below.

Definition 3: Given a workload $W = \{T_1, T_2, ..., T_n\}$ with $1 \le i \le n$. The RDBF of W after being serviced by a resource model R, denoted by $\mathsf{rdbf}_{W-R}(t)$, specifies the maximum number of remaining execution units required by W in any time interval of length t after W has been serviced by W.

One can easily verify that

$$\forall\, t \ge 0: \ \mathsf{rdbf}_{W-R}(t) = \max\big(0, \mathsf{dbf}_W(t) - \mathsf{sbf}_R(t)\big). \quad (21)$$

Lemma 10 gives the schedulability condition for W under Ω.

Lemma 10: Given a component $C = (W, \Omega, EDF)$ where $W = \{T_1, T_2, \ldots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \le i \le n$, and $\Omega = (\Gamma_1, \Gamma_2)$ is a DPRM. Then, C is schedulable under Ω (Ω can feasibly schedule W) iff

$$\forall t \ s.t. 0 < t \le \mathsf{LCM}_W, \ \ \mathsf{rdbf}_{W-\Gamma_1}(t) \le \mathsf{sbf}_{\Gamma_2}(t). \quad (22)$$

Proof: It follows directly from Lemma 9 and Equation 21. ■

We define the LSBF (lower supply bound function) of a periodic resource model Γ to be the linear function with the smallest slope that lower bounds sbf_Γ, given by [1]:

$$\forall\, t \ge 0: \ \mathsf{lsbf}_\Gamma(t) = \max\Big(\frac{\Theta}{\Pi}\big(t - 2(\Pi - \Theta)\big), 0\Big). \quad (23)$$

The following lemma is derived from the schedulability condition of Γ (see Lemma 1) and the definition of lsbf_Γ.

Lemma 11: Given a component $C = (W, \Gamma, EDF)$ where $W = \{T_1, T_2, \cdots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \le i \le n$, and $\Gamma = (\Pi, \Theta)$. Then, C is schedulable if

$$\forall\, t \ s.t. \ \ 0 \le t \le \mathsf{LCM}_W, \ \ \mathsf{lsbf}_\Gamma(t) \ge \mathsf{dbf}_W(t). \quad (24)$$

One can verify that if Γ satisfies Equation 24 then it satisfies the schedulability condition for W (see Lemma 1) ; however, the reverse does not hold. Thus Equation 24 gives a sufficient condition for the schedulability of W under the resource model Γ. By abuse of notation, we refer to Equation 24 as the *LSBF-schedulability condition* for W and we say that a model Γ satisfies Equation 24 iff it can *sufficiently* schedule W.

Lemma 12: Given a component $C = (W, \Gamma, EDF)$ where $W = \{T_1, T_2, \cdots, T_n\}$, $T_i = (p_i, e_i)$ for all $1 \le i \le n$, and $\Gamma = (\Pi, \Theta)$. Suppose Γ satisfies LSBF-schedulability condition for W. Then, for all $t \ge 0$, if $\mathsf{lsbf}_\Gamma(t) = \mathsf{dbf}_W(t)$ then $t \in \mathsf{CrT}_W$.

Proof: We will prove the lemma by contradiction. Suppose there exists $t_0 \notin \mathsf{CrT}_W$ such that $\mathsf{lsbf}_\Gamma(t_0) = \mathsf{dbf}_W(t_0)$. Let $s = 2(\Pi - \Theta)$. Then, by Definition 2, there is a time point $t' \in \mathsf{CrT}_W$ such that

$$\frac{\mathsf{dbf}_W(t')}{t' - s} > \frac{\mathsf{dbf}_W(t_0)}{t_0 - s} \quad (25)$$

On the other hand, we have

$$\begin{aligned} \mathsf{lsbf}_\Gamma(t') \ge \mathsf{dbf}_W(t') \Rightarrow \ & \frac{\Theta}{\Pi}(t' - 2(\Pi - \Theta)) \ge \mathsf{dbf}_W(t') \\ \Rightarrow \ & \frac{\mathsf{dbf}_W(t_0)}{t_0 - s}(t' - s) \ge \mathsf{dbf}_W(t') \\ \Rightarrow \ & \frac{\mathsf{dbf}_W(t_0)}{t_0 - s} \ge \frac{\mathsf{dbf}_W(t')}{t' - s} \end{aligned} \quad (26)$$

which contradicts Equation 25. Hence, the lemma. ■

The maximum value of Θ_1 in the optimal DPRM interface $\Omega = (\Gamma_1, \Gamma_2)$ with $\Gamma_1 = (\Pi_1, \Theta_1)$ can now be computed using function $\mathsf{MaxResExecDPRM}(\Pi, \mathsf{dbf}_W, \mathsf{LCM}_W)$ defined in the following theorem.

Theorem 4: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$, with $T_i = (p_i, e_i)$ for all $1 \leq i \leq n$. For any given Π_1, the minimum bandwidth DPRM interface $\Omega = \big((\Pi_1, \Theta_1), (\Pi_2, \Theta_2)\big)$ for W satisfies $\Theta_1 \leq \Theta_1^{\max}$ where

$$\Theta_1^{\max} = \max_{t \in \mathsf{CrT}_W} \frac{(2\Pi_1 - t) + \sqrt{(2\Pi_1 - t)^2 + 8\Pi_1 d_t}}{4}.$$

Before presenting proof, we present basic idea of the theorem and make a helpful lemma first.

To compute optimal DPRM $\Omega = (\Gamma_1, \Gamma_2)$ for a given workload W, suppose that periodic resource model Γ_1 should be decided first. For any given Π_1 in DPRM $\Omega = (\Gamma_1, \Gamma_2) = \big((\Pi_1, \Theta_1), (\Pi_2, \Theta_2)\big)$, the upper bound on the resource execution time Θ_1 of the minimum bandwidth DPRM for a given workload W can be derived from the upper bound on resource execution time Θ_Π^* of the minimum bandwidth periodic resource model $\Gamma_\Pi^* = (\Pi, \Theta_\Pi^*)$ for W when $\Pi = \Pi_1$. If $\Theta_1 > \Theta_{\Pi_1}^*$, Then $\big((\Pi_1, \Theta_1), (\Pi_2, \Theta_2)\big)$ cannot have the minimum bandwidth since its bandwidth is greater than that of $\big((\Pi_1, \Theta_{\Pi_1}^*), (0, 0)\big)$. Therfore,

$$\Theta_1 \leq \Theta_{\Pi_1}^* \tag{27}$$

For any given Π, the upper bound on the resource execution time Θ of the minimum bandwidth periodic resource model $\Gamma = (\Pi, \Theta)$ for a given workload W can be derived from the DBF of the workload and its relationship with the LSBFs of the resource models that can sufficiently schedule W. Intuitively, let M be the set of resource models that can sufficiently schedule W (i.e., they satisfy the LSBF-schedulability condition). for any given Π, Suppose $\Gamma_\Pi^* = (\Pi, \Theta_\Pi^*)$ is the minimum bandwidth resource model for W. Then,

$$\Theta_\Pi^* \leq \Theta_\Pi^{\max} \stackrel{\mathsf{def}}{=} \min\{\Theta_\Pi \mid (\Pi, \Theta_\Pi) \in M\} \tag{28}$$

To derive the bound on Θ_Π^*, we will find all the possible resource models in M that has the minimum bandwidth periodic resource model $\Gamma_\Pi^{\max} = (\Pi, \Theta_\Pi^{\max})$. We will show that for a given Π the LSBF of $\Gamma_\Pi^{\max}$ intersects dbf_W at exactly *critical time point*. Further, $\Theta_\Pi^{\max}$ is the largest resource execution time among all the resource models $\Gamma_{\Pi,t} = (\Pi, \Theta_{\Pi,t})$ with period Π that have their LSBF intersecting dbf_W at critical time points t. In other words,

$$\Theta_\Pi^{\max} = \max_{t \in \mathsf{CrT}_W} \{\Theta_{\Pi,t} \mid \mathsf{lsbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t)\}, \tag{29}$$

where CrT_W is the set of all critical time points of W.

For any given Π and any given $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Pi,t}$ be the resource model with period Π such that its LSBF intersects dbf_W at time point t. Then, $\Gamma_{\Pi,t}$ is unique and its execution time can be determined using Lemma 13.

Lemma 13: Given any $\Pi \in \mathbb{N}$ and any $t \in \mathsf{CrT}_W$. Let $\Gamma_{\Pi,t} = (\Pi, \Theta_{\Pi,t})$ be the periodic resource model such that $\mathsf{lsbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t)$. Then, $\Theta_{\Pi,t} = \mathsf{LimitExec}(\Pi, t, d_t)$ where

$$\mathsf{LimitExec}(\Pi, t, d_t) \stackrel{\mathsf{def}}{=} \frac{(2\Pi - t) + \sqrt{(2\Pi - t)^2 + 8\Pi d_t}}{4}$$

where $d_t = \mathsf{dbf}_W(t)$.

Proof: We have:

$$\begin{aligned} &\mathsf{lsbf}_{\Gamma_{\Pi,t}}(t) = \mathsf{dbf}_W(t) = d_t \\ \Leftrightarrow\ &\frac{\Theta_{\Pi,t}}{\Pi}(t - 2(\Pi - \Theta_{\Pi,t})) = d_t \\ \Leftrightarrow\ &2\Theta_{\Pi,t}^2 + \Theta_{\Pi,t}(t - 2\Pi) - \Pi d_t = 0 \end{aligned}$$

Since $\Theta \geq 0$, the above is equivalent to

$$\Theta_{\Pi,t} = \frac{(2\Pi - t) + \sqrt{(2\Pi - t)^2 + 8\Pi d_t}}{4}.$$

In other words, $\Theta_{\Pi,t} = \mathsf{LimitExec}(\Pi, t, d_t)$. Hence the lemma. ■

Proof of Theorem 4: For any given Π_1, there is the minimum bandwidth periodic resource model $\Gamma_{\Pi_1}^* = (\Pi_1, \Theta_{\Pi_1}^*)$ for W. By Eqaution 27 and 28, $\Theta_1 \leq \Theta_{\Pi_1}^* \leq \Theta_{\Pi_1}^{\max}$.

Next, for any given $t \in \mathsf{CrT}_W$, let $\Gamma_{\Pi_1,t} = (\Pi_1, \Theta_{\Pi_1,t})$ where $\Theta_{\Pi_1,t} = \mathsf{LimitExec}(\Pi_1, t, d_t)$ where $d_t = dbf_W(t)$. That is, the LSBF of $\Gamma_{\Pi_1,t}(t)$ intersects dbf_W at time point t. Then, $\Theta_{\Pi_1}^{\min}$ must be greater or equal to that of $\Theta_{\Pi_1,t}$ for all $t \in \mathsf{CrT}_W$ by Eqaution 29(otherwise, $\Gamma_{\Pi_1}^{\min}$ does not satisfy the LSBF-schedulability condition). In other words, for all $t \in \mathsf{CrT}_W$,

$$\begin{aligned} &\Theta_{\Pi_1}^{\max} = \max_{t \in \mathsf{CrT}_W} \Theta_{\Pi_1,t} \\ \Leftrightarrow\ &\Theta_{\Pi_1}^{\max} = \max_{t \in \mathsf{CrT}_W} \frac{(2\Pi_1 - t) + \sqrt{(2\Pi_1 - t)^2 + 8\Pi_1 d_t}}{4}. \end{aligned}$$

In other words, $\Theta_1^{\max} = \Theta_{\Pi_1}^{\max}$. ■

Computation of $\mathsf{getResModel}(\mathsf{dbf}_W, \mathsf{LCM}_W, \Gamma_1)$. The function $\mathsf{getResModel}(\mathsf{dbf}_W, \mathsf{LCM}_W, \Gamma_1)$ computes a period resource model Γ_2 such that $\Omega = (\Gamma_1, \Gamma_2)$ is the minimum bandwidth DPRM interface that can schedule W. Let a workload W' be the remaining workload of W after being serviced by the resource model Γ_1. This Γ_2 can be obtained as the output of Algorithm 3 on the inputs $\mathsf{rdbf}_{W-\Gamma_1}(t)$ and LCM_W since $\forall t$, $\mathsf{dbf}_{W'}(t) = \mathsf{rdbf}_{W-\Gamma_1}(t)$ and LCM_W is sufficient for $LCM_{W'}$ by Theorem 5.

Theorem 5: For any given workload W, any given Π, and any given periodic resource model Γ_1, if $\Gamma = (\Pi, \Theta)$ s.t. $0 \leq t \leq \mathsf{LCM}_W$, $\mathsf{sbf}_\Gamma(t) \geq \mathsf{rdbf}_{W-\Gamma_1}(t)$, then $\forall t \geq 0, \mathsf{sbf}_\Gamma(t) \geq \mathsf{rdbf}_{W-\Gamma_1}(t)$, which implies schedulability condition for the remaining workload of W after being serviced by the resource model Γ_1.

Before presenting proof, we define helpful lemmas.

Lemma 14: For any periodic resource model Γ, any $i \in \mathbb{N}$, and any $t \in \mathbb{N}$ s.t. $t \geq i$,

$$\mathsf{sbf}_\Gamma(t) \geq \mathsf{sbf}_\Gamma(t - i) + \mathsf{sbf}_\Gamma(i) \tag{30}$$

Proof: Let $t = i + x$ for some $x \geq 0$.

We would like to prove

$$\begin{aligned} & \mathsf{sbf}_\Gamma(t) \geq \mathsf{sbf}_\Gamma(t-i) + \mathsf{sbf}_\Gamma(i) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(t) - \mathsf{sbf}_\Gamma(i) \geq \mathsf{sbf}_\Gamma(x) \end{aligned}$$

which is true since stravtion time for $\mathsf{sbf}_\Gamma(x)$ is $2(\Pi-\Theta)$ and stravation time for $\big(\mathsf{sbf}_\Gamma(t) - \mathsf{sbf}_\Gamma(i)\big)$ is at most $2(\Pi-\Theta)$ while both function shapes is same after strvation time. ■

Corollary 2: For any worload W, any periodic resource model Γ, any $i \in \mathbb{N}$, and any $t \in \mathbb{N}$ s.t. $i \cdot \mathsf{LCM}_W \leq t \leq (i+1)\cdot \mathsf{LCM}_W$,

$$\mathsf{sbf}_\Gamma(t) \geq \mathsf{sbf}_\Gamma(t - i\cdot\mathsf{LCM}_W) + \mathsf{sbf}_\Gamma(i\cdot\mathsf{LCM}_W) \quad (31)$$

Proof: Immediate from Lemma 14. ■

Corollary 3: For any worload W, any Γ, and any $i \in \mathbb{N}$,

$$\mathsf{sbf}_\Gamma(i\cdot\mathsf{LCM}_W) \geq i\cdot\mathsf{sbf}_\Gamma(\mathsf{LCM}_W) \quad (32)$$

Proof: By Corollary 2 ,

$$\begin{aligned} & \mathsf{sbf}_\Gamma(i\cdot\mathsf{LCM}_W) \\ & \quad \geq \mathsf{sbf}_\Gamma((i-1)\mathsf{LCM}_W) + \mathsf{sbf}_\Gamma(\mathsf{LCM}_W) \\ & \quad \geq \mathsf{sbf}_\Gamma((i-2)\mathsf{LCM}_W) + 2\cdot\mathsf{sbf}_\Gamma(\mathsf{LCM}_W) \\ & \quad \cdots \\ & \quad \geq \mathsf{sbf}_\Gamma(\mathsf{LCM}_W) + (i-1)\cdot\mathsf{sbf}_\Gamma(\mathsf{LCM}_W) \\ & \quad \geq i\cdot\mathsf{sbf}_\Gamma(\mathsf{LCM}_W) \end{aligned}$$

■

Proof of Theorem 5: We will prove the theorem by contradiction. let $L = \mathsf{LCM}_W$. Suppose $\exists t_0$ s.t. $\mathsf{sbf}_\Gamma(t_0) < \mathsf{rdbf}_{W-\Gamma_1}(t_0)$. Then, $t_0 > L$ since $0 \leq t \leq L$, $\mathsf{sbf}_\Gamma(t) \geq \mathsf{rdbf}_{W-\Gamma_1}(t)$. Let $t_0 = a\cdot L + x$ for some $a \in \mathbb{N}$ s.t. $a \geq 1$ and some $x \in \mathbb{N}$ s.t. $0 \leq x \leq L$.

Since $\forall t > LCM_W$, $dbf_W(t) = dbf_W(t-L) + dbf_W(L)$,

$$\mathsf{dbf}_W(t_0) = \mathsf{dbf}_W(x) + \mathsf{dbf}_W(a\cdot L) \quad (33)$$

Since assumption hold when $t = L$,

$$\begin{aligned} & \mathsf{sbf}_\Gamma(L) \geq \mathsf{rdbf}_{W-\Gamma_1}(L) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(L) \geq \mathsf{dbf}_W(L) - \mathsf{sbf}_{\Gamma_1}(L) \\ \Leftrightarrow \; & \mathsf{sbf}_{\Gamma_1}(L) + \mathsf{sbf}_\Gamma(L) \geq \mathsf{dbf}_W(L) \\ \Leftrightarrow \; & a\cdot\mathsf{sbf}_{\Gamma_1}(L) + a\cdot\mathsf{sbf}_\Gamma(L) \geq a\cdot\mathsf{dbf}_W(L) \\ \Leftrightarrow \; & a\cdot\mathsf{sbf}_{\Gamma_1}(L) + a\cdot\mathsf{sbf}_\Gamma(L) \geq \mathsf{dbf}_W(a\cdot L) \\ \Leftrightarrow \; & \mathsf{sbf}_{\Gamma_1}(a\cdot L) + \mathsf{sbf}_\Gamma(a\cdot L) \geq \mathsf{dbf}_W(a\cdot L) \end{aligned} \quad (34)$$

by Corollary 3.

By Equation 33 and 34,

$$\mathsf{dbf}_W(t_0) \leq \mathsf{dbf}_W(x) + \mathsf{sbf}_{\Gamma_1}(a\cdot L) + \mathsf{sbf}_\Gamma(a\cdot L) \quad (35)$$

With $\exists t_0$ s.t. $\mathsf{sbf}_\Gamma(t_0) < \mathsf{rdbf}_{W-\Gamma_1}(t_0)$, Corollary 2, and Equation 35,

$$\begin{aligned} & \mathsf{sbf}_\Gamma(t_0) < \mathsf{dbf}_W(t_0) - \mathsf{sbf}_{\Gamma_1}(t_0) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(t_0) < \mathsf{dbf}_W(t) - \mathsf{sbf}_{\Gamma_1}(t_0 - a\cdot L) - \mathsf{sbf}_{\Gamma_1}(a\cdot L) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(t_0 - a\cdot L) + \mathsf{sbf}_\Gamma(a\cdot L) < \\ & \qquad \mathsf{dbf}_W(t_0) - \mathsf{sbf}_{\Gamma_1}(t_0 - a\cdot L) - \mathsf{sbf}_{\Gamma_1}(a\cdot L) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(x) + \mathsf{sbf}_\Gamma(a\cdot L) < \\ & \qquad \mathsf{dbf}_W(x) + \mathsf{sbf}_{\Gamma_1}(a\cdot L) + \mathsf{sbf}_\Gamma(a\cdot L) \\ & \qquad - \mathsf{sbf}_{\Gamma_1}(x) - \mathsf{sbf}_{\Gamma_1}(a\cdot L) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(x) < \mathsf{dbf}_W(x) - \mathsf{sbf}_{\Gamma_1}(x) \\ \Leftrightarrow \; & \mathsf{sbf}_\Gamma(x) < \mathsf{rdbf}_{W-\Gamma_1}(x) \end{aligned}$$

which contradicts assumption when $t = x$ s.t. $0 \leq x \leq L$. ■

C. Composition of DPRM interfaces

Consider a composite component C_s consisting of multiple child components scheduled under EDF scheduling policy. The workload and DPRM interface of C_s can be computed from the DPRM interfaces of its child components using the method outlined in Definition 4. As proven by Theorem 6, our composition allows the construction of a hierarchical scheduling framework that supports *compositional real-time guarantees*, i.e., the real time guarantee of a composite component in the framework is satisfied if and only if the real-time guarantees of its child components are satisfied.

Definition 4 (DPRM Interface Composition): Given a composite component C_s consisting of multiple child components $C_1, C_2, \cdots, C_n$ that are scheduled under EDF. Suppose $\Omega_i = (\Gamma_{i,1}, \Gamma_{i,2})$ is the DPRM interface of C_i for all $1 \leq i \leq n$, where $\Gamma_{i,j} = (\Pi_{i,j}, \Theta_{i,j})$ and $j \in \{1,2\}$. Then, a feasible workload W_s and a DPRM interface Ω_s for C_s are defined by:

- $W_s = \{T_{1,1}, T_{1,2}, \cdots, T_{n,1}, T_{n,2}\}$, where $T_{i,j} = (p_{i,j}, e_{i,j})$, with $p_{i,j} = \Pi_{i,j}$ and $e_{i,j} = \Theta_{i,j}$, is the periodic task corresponding to the periodic model $\Gamma_{i,j}$ of Ω_i for all $1 \leq i \leq n$ and $j \in \{1,2\}$.
- Ω_s is the corresponding DPRM interface of W_s, computed by Algorithm 4.

Theorem 6: Consider a composite component $C_s = (W_s, \Omega_s, EDF)$ composed of n child components $C_1, C_2, \ldots, C_n$ that are scheduled under EDF, where W_s and Ω_s are defined by Definition 4. Then, C_s is schedulable if and only if all $C_1, C_2, \cdots, C_n$ are schedulable.

Proof: ($\Rightarrow$) Suppose C_s is schedulable. Then, for all $1 \leq i \leq n$ and $j \in \{1,2\}$, $T_{i,j}$ and its corresponding periodic model $\Gamma_{i,j}$ are guaranteed to receive $e_{i,j}$ time units every $p_{i,j}$ time units. In other words, C_i receives from C_s a resource allocation of $\Theta_{i,1}$ time units every $\Pi_{i,1}$ time units, in addition to a resource allocation of $\Theta_{i,2}$ time units every $\Pi_{i,2}$ time units. Thus, C_i is schedulable for all $1 \leq i \leq n$.

($\Leftarrow$) Suppose all $C_1, C_2, \cdots, C_n$ are schedulable, i.e., the combined timing requirement of $C_1, C_2, \cdots, C_n$ are satisfied. By Definition 4, for all $1 \leq i \leq n$ and $j \in \{1,2\}$, each $T_{i,j}$ in W_s has the same timing requirement as $\Gamma_{i,j}$ does, where

$\Omega_i = (\Gamma_{i,1}, \Gamma_{i,2})$ is the DPRM of C_i. Thus, the combined timing requirement of all $T_{i,j}$ in W_s are also satisfied. Since Ω_s is a feasible DPRM interface of W_s, it can schedule W_s. In other words, C_s is schedulable. ■

Example 5: Consider C_s consisting of two components, C_1 and C_2 where $C_i = (W_i, \Omega_i, EDF)$. Suppose that $W_1 = \{(11,1),(15,1),(60,7)\}$ and $W_2 = \{(16,1),(19,2),(50,6)\}$. Algorithm 4 calculates that $\Omega_1 = ((4,1),(20,1))$ and $\Omega_2 = ((4,1),(17,1))$. According to DPRM interface composition in Definition 4, $W_s = \{(4,1),(20,1),(4,1),(17,1)\}$ and Ω_s is calculated to $((3,2),(20,1))$ by Algorithm 4. Then, we have feasible component $C_s = (W_s, \Omega_s, EDF)$.

D. Context Switch Overhead

To evaluate context switch overhead, we compare DPRM interface to periodic resource model with rational number [3] in terms of the number of preemption. The following lemma computes the upper bound of the number of preemptions [15].

Lemma 15: Given a workload $W = \{T_1, T_2, \cdots, T_n\}$ where $T_i = (p_i, e_i)$ for all $1 \le i \le n$. The upper bound of the number of preemptions under EDF calculates as $NP_W(t) = \sum_{i=1}^{n} \lfloor t/p_i \rfloor$.

Lemma 16 shows that the number of preemptions in optimal DPRM is smaller than one in optimal periodic resource model with rational number.

Lemma 16: Consider a composite component C_s composed of n child components $C_1, C_2, \ldots, C_n$ that are scheduled under EDF. Algorithm in [3] calculates optimal periodic resource model with rational number for each component. In interface composition of periodic resource model [1], the workload for C_s for periodic resource model is $W_s^{PRM} = \{T_1, T_2, \cdots, T_n\}$, where $T_i = (1, e_i)$ for all $1 \le i \le n$. On the other hand, Algorithm 4 computes an optimal DPRM interface for each component. According to DPRM interface composition in Definition 4, the workload for C_s for DPRM is $W_s^{DPRM} = \{T_{1,1}, T_{1,2}, \cdots, T_{n,1}, T_{n,2}\}$ where $T_{i,j} = (p_{i,j}, e_{i,j})$ for all $1 \le i \le n$ and $j \in \{1,2\}$. If W_s^{PRM} and W_s^{DPRM} is scheduled in a dedicated system respectively, then the upper bound of the number of preemption in W_s^{DPRM} is smaller than the number of preemption in W_s^{PRM}.

Proof: Since the period of all tasks is 1, the number of preemption in W_s^{PRM} is $n \cdot t$ for any time inteval t.

The upper bound of number of preemption for W is $NP_{W_s^{DPRM}}(t)$ according to Lemma 15. Since scheduling tasks in W_D is repeated every $\mathsf{LCM}_{W_s^{DPRM}}$ (LCM of the workload W_s^{DPRM}) time units and the upper bound of the number of preemption per time units is largest at $\mathsf{LCM}_{W_s^{DPRM}}$,in short LCM_W, we would like to prove

$$NP_{W_s^{DPRM}}(\mathsf{LCM}_W) < n \cdot \mathsf{LCM}_W$$

$$\sum_{i=1}^{n} \sum_{j=1}^{2} \left\lfloor \frac{\mathsf{LCM}_W}{p_{i,j}} \right\rfloor < n \cdot \mathsf{LCM}_W$$

$$\frac{\mathsf{LCM}_W}{p_{1,1}} + \frac{\mathsf{LCM}_W}{p_{1,2}} + \cdots + \frac{\mathsf{LCM}_W}{p_{n,1}} + \frac{\mathsf{LCM}_W}{p_{n,2}} < n \cdot \mathsf{LCM}_W$$

which is obvious because all $p_{i,j} \ge 2$ and at least one $p_{i,j} > 2$ and otherwise, utilization of $W > 1$ which means that this hierarchical system is not feasible or W={(2,1),(2,1)} which is transformed from one component which contradicts assumption of multiple components. ■

VI. Experimental Results

A. Simulation Setup

To evaluate our improved algorithms and DPRM interface, we ran simulations on random component workloads, each consisting of at least three tasks. The task's periods were randomly chosen in the range of 10-100 following the uniform distribution. Each task's execution time was uniformly distributed random number from 1 to the task's period. We constrained the workload utilization to be no more than 0.8.

B. Generating Component Interface

This experiment evaluates the performance of the different resource model interfaces for 200 component workloads generated as above. For each workload, we computed its optimal periodic resource model (iPRM) and its optimal DPRM (iDPRM) with integer parameters using our algorithms. We additionally computed the optimal periodic resource model with rational number (rPRM) [3].

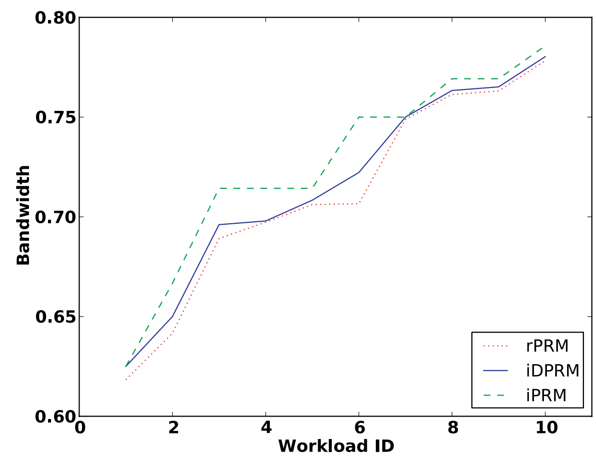

Fig. 5. Comparison between Periodic Resource Model and DPRM.

Figure 5 shows the bandwidth of the interfaces of the first ten component workloads. The X-axis is the workload identifier sorted by utilization whereas the Y-axis is the optimal bandwidth of the computed resource models. As shown in the figure, the iDPRM was always better than or as good as the iPRM: the iDPRM had smaller bandwidth than the iPRM did in 77% of the simulated workloads, with a bandwidth reduction($\frac{BW(iPRM)-BW(iDPRM)}{BW(iPRM)}$) of up to 12.5%. Further, with respect to the ideal bandwidth given by the rPRM, the iDPRM incurred only 1.25% bandwidth overhead in average whereas the iPRM suffered more than 2.56 times as much (3.22% overhead).

To evaluate the scalability of DPRM interface, we repeated the above experiment for larger workloads. Our simulation results showed that as the number of tasks increases, the above improvement of DPRM interface (over the periodic resource interface) also increases as Table II.

# of tasks in a workload	% of iDPRM with smaller bandwidth	Maximum bandwidth reduction	iPRM overhead (A)	iDPRM overhead (B)	(A)/(B)
3	77%	12.5%	3.22%	1.25%	2.58
4	81%	15.86%	3.27%	1.15%	2.84
5	86%	9.09%	1.81%	0.46%	3.93

TABLE I
SIMULATION RESULT DEPENDING THE NUMBER OF TASKS IN A WORKLOAD

This is expected due to the corresponding increase in complexity of the DBF function of the workload, which can be more effectively captured by the DPRM interface.

C. *Performance of Interface Composition*

This experiment evaluates the effectiveness of DPRM interface composition with 4 different scenarios. Experiment was performed on 200 two-level hierarchical real-time systems for each scenario. In the first three scenarios, each system consists of two components, with each containing three tasks and workload utilization no more than 0.6, 0.7, and 0.8, respectively. In the last scenario, each system consists of three components, with each containing three tasks and workload utilization no more than 0.8. For each system, we first generated the iPRM, rPRM and iDPRM interfaces for each component as in the above experiment (Section VI-B), and then computed the interfaces for the root component using the interface composition technique outlined in Section V-C.

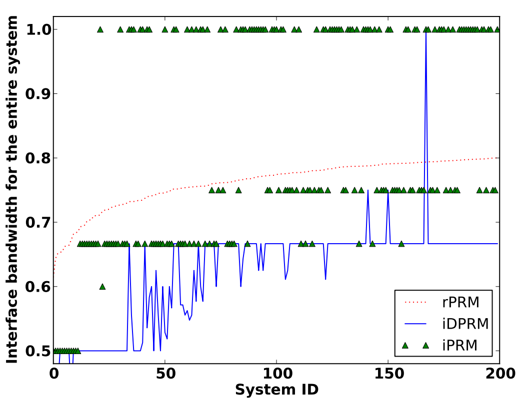

Fig. 6. Interface composition performance in Scenario 1

Figure 9 shows the bandwidth of the rPRM, iPRM and iDPRM interfaces of the simulated systems with three components and workload utilization less than or equal to 0.8. X-axis represents the system identification sorted by workload utilization. As shown in the figure, the iDPRM interface for the root component has smaller or equal bandwidth compared to the corresponding iPRM interface, and smaller bandwidth than the corresponding rPRM interfaces in most cases. It can

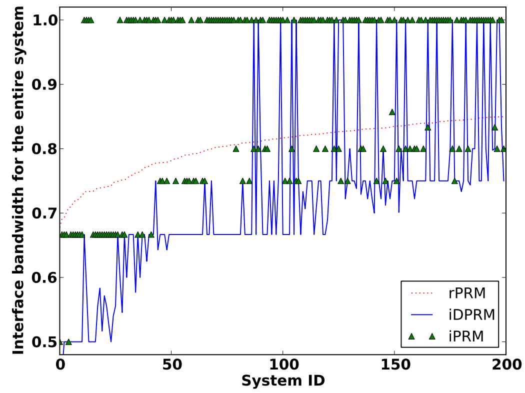

Fig. 7. Interface composition performance in Scenario 2

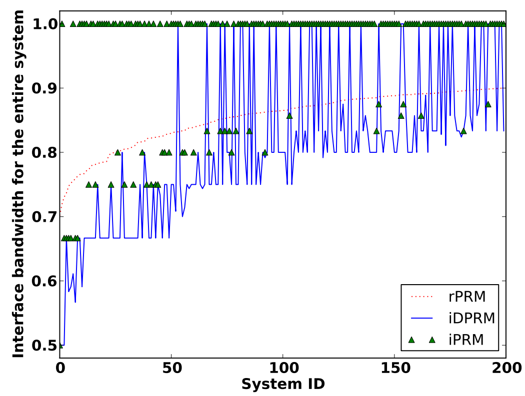

Fig. 8. Interface composition performance in Scenario 3

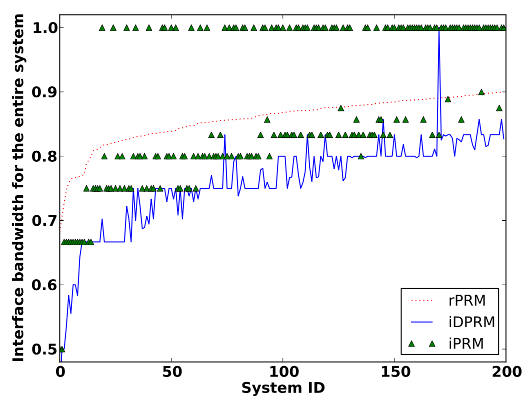

Fig. 9. Interface composition performance in Scenario 4

also be observed that the bandwidth reduction when using the iDPRM interfaces is more significant for system with smaller utilization or larger number of components or both, as further illustrated in Table II.

We also note that although it is possible to generate bandwidth-optimal rPRM interfaces for child components,

Scenario	1	2	3	4
Utilization of system workloads	0.6	0.7	0.8	0.8
Number of components	2	2	2	3
BW(iDPRM) $<$ BW(rPRM)	99.5%	89.0%	80%	99.5%
BW(iDPRM) $<$ BW(iPRM)	87.0%	83.0%	77.5%	94.0%
BW(iDPRM) = BW(iPRM)	13.0%	15.5%	16.0%	5.0%
BW(iDPRM) $>$ BW(iPRM)	0.0%	6.5%	4.5%	1.0%
Avg. BW reduction to rPRM	18.8%	11.4%	5.6%	11.6%
Avg. BW reduction to iPRM	22.0%	18.9%	14.9%	13.0%

TABLE II
BANDWIDTH REDUCTION OF IDPRM COMPARED TO RPRM AND IPRM.

rPRM still incurs bandwidth overhead in interface composition. If the period of parent component interface is close to the period of child component interface, parent component interface acquired by interface composition suffers high overheads. In rPRM, parent and child component interfaces have periods of 1. As workload utilization decreases, the overhead in rPRM interface composition increases.

On the other hand, there exist cases when iPRM and iDPRM cannot find the lower-bandwidth resource model except (1,1). For example, one interface is generated to (2,1). The interface is transformed into the task (2,1) in interface composition. With integer parameter, only interface (1,1) is feasible for the task, due to the worst-case starvation of resource model. The example is possible if workload utilization of one component in the system is larger or equal to $\frac{5}{11}$ in this simulation setup. Since the minimum period of a task in workload is 10, if a task (5,11) is included in the workload W_C in some component C. Then, it is possble that $\mathsf{dbf}_{W_C}(t) = 5$ for $t = 11$. There is also possibility that resource model (2,1) is computed to optimal periodic resource model with Algorithm 1 by following equation:

$$\mathsf{sbf}_{(2,1)}(t) \leq \mathsf{dbf}_{W_C}(t) \qquad if\ t = 11.$$

If resource model (2,1) is optimal periodic resource model for W_C, above cases can happen. The utilization of workload W_C is larger or eqaul to $\frac{5}{11}$. Based on this observation, if the workload utilization of some component is larger or eqaul to $\frac{5}{11}$, there is possibility that optimal periodic resource model is (2,1) or optimal DPRM includes (2,1). Hence, iPRM and iDPRM resource model for the system cannot find the lower-bandwidth resource model except (1,1).

For the system with larger number of components, DPRM interface composition can generate smaller bandwidth interface than other resource models because each component is transformed into two tasks in interface composition and the iDPRM interface can be generated effectively for larger number of tasks.

D. Interface Context Switch Overheads

This experiment evaluates the context switch overheads incurred by the components in a hierarchical system when using the iPRM and the iDPRM interfaces. We simulated the same set of two-level hierarchical systems in Scenario 3 (system with two components and workload utilization no less than 0.8) in Section VI-C. With the interface for each component in the system, we constructed the system workload W_s for each system according to interface composition. Then, we simulated scheduling W_s, which is equivalent to scheduling components in the system according to interface composition.

Figure 10 shows the number of context switches between the components within each system observed over a duration of 10,000 time units for the simulated systems. In the figure, the X-axis denotes the system identifier sorted by utilization, whereas the Y-axis denotes the number of context switches incured by iPRM interface less than iDPRM interface. It can be observed from the figure that the number of context switches incurred by the iDPRM interfaces is generally smaller than that of the iPRM interfaces. Specifically, the iDPRM interfaces incurred less or equal number of context switches compared to the iPRM interfaces for 69% of the 200 simulated systems.

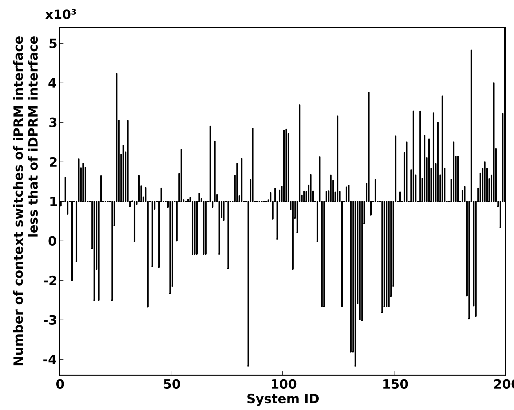

Fig. 10. Difference in the number of context switches of iPRM interfaces compared to iDPRM interfaces

Since the worst-case context switch overhead between components in existing hierarchical scheduling systems is observed to be less than 0.005 times of the scheduling resolution [11], we assumed three overhead values in our experiments: 0.01, 0.05 and 0.10 time unit. Figure 14 shows the total bandwidth of the iPRM and iDPRM interfaces (considering context switch overheads) of the components within the system for the first ten systems, given a context switch overhead of 0.1 time unit. As shown in the figure, and also observed in the remaining simulated systems, the iDPRM interface outperforms the iPRM in general. In particular, we observed that the iDPRM interface has a smaller or equal bandwidth compared to the iPRM interface in 94% of the total workloads (and similarly, 99% and 99.5% assuming an overhead of 0.05 time unit and 0.01 time unit, respectively).

VII. CONCLUSION

Traditional algorithms for computing the minimum-bandwidth resource model face two drawbacks: (i) they assume rational parameters for the resource model, which cannot always be used in practice, and (ii) the resource period is searched within a range specified by the designer, which

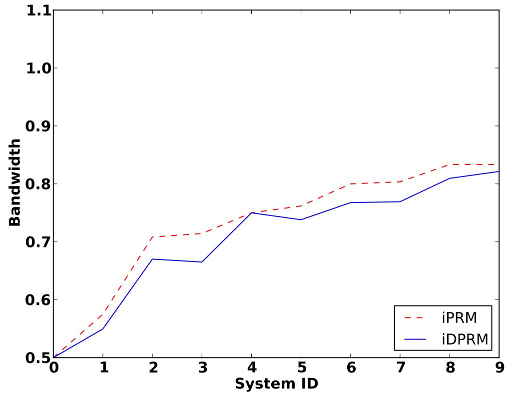

Fig. 11. Total component interface bandwidth without context switch overheads.

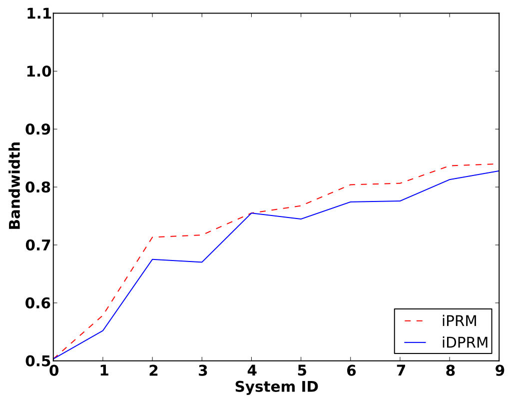

Fig. 12. Total component interface bandwidth with context switch overheads of 0.01 time unit.

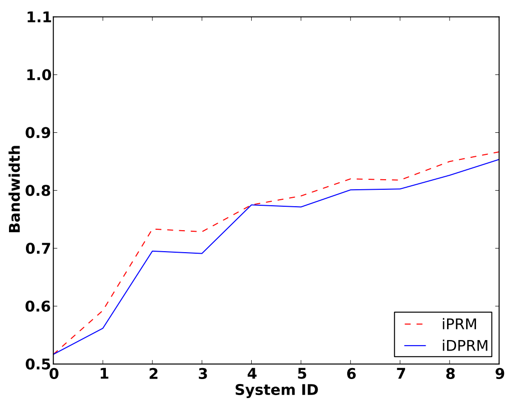

Fig. 13. Total component interface bandwidth with context switch overheads of 0.05 time unit.

cannot guarantee optimality. We have presented more efficient algorithms that tackle these drawbacks by considering integer parameters and a safe bound on the period. We further proposed the DPRM interface and an algorithm for computing

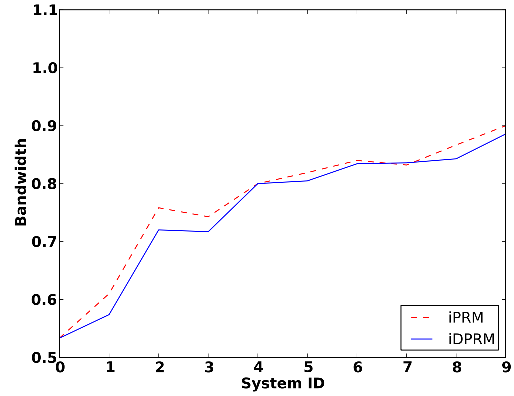

Fig. 14. Total component interface bandwidth with context switch overheads of 0.1 time unit.

the minimum bandwidth DPRM interface that is more accurate than the periodic resource interface when restricting the interface to have only integer parameters. Then, we have proposed a composition technique for DPRM interfaces.

Our simulation results showed that the DPRM achieved a lower bandwidth than the periodic resource model did in 77% of the workloads, reducing the overhead compared to the ideal case by more than half. Including context switch overhead, the DPRM had a smaller or equal bandwidth to the periodic model in 99.5% of the hierarchical systems. In interface composition, the DPRM interfaces were generated with a smaller bandwidth compared to the periodic model in 94% of the hierarchical systems.

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