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Legged Self-Manipulation

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ABSTRACT This paper introduces self-manipulation as a new formal design methodology for legged robots with varying ground interactions. The term denotes a set of modeling choices that permit a uniform and body-centric representation of the equations of motion—essentially a guide to the selection and configuration of coordinate frames. We present the hybrid system kinematics, dynamics, and transitions in the form of a consistently structured representation that simplifies and unites the account of these, otherwise bewilderingly diverse differential algebraic equations. Cleaving as closely as possible to the modeling strategies developed within the mature manipulation literature, self-manipulation models can leverage those insights and results where applicable, while clarifying the fundamental differences. Our primary motivation is not to facilitate numerical simulation but rather to promote design insight. We instantiate the abstract formalism for a simplified model of RHex, and illustrate its utility by applying a variety of analytical and computational techniques to derive new results bearing on behaviors, controllers, and platform design. For each example, we present empirical results documenting the specific benefits of the new insight into the robot’s transitions from standing to moving in place and to leaping.

INDEX TERMS Legged locomotion, Robot control, Robot kinematics, Manipulator dynamics.

I. INTRODUCTION

Legged robots, such as the notional mechanism in Fig. 1, will necessarily experience a variety of changing contact conditions as they perform ever more complex tasks requiring greater autonomy on novel, and possibly shifting, terrain. As new conditions arise, it is crucial to formally (and therefore automatically) generate and analyze the system dynamics, in a manner requiring as little knowledge as possible about the geometry and mechanics of the substrate. In contrast, multi-fingered object manipulation has been well formalized for many decades [1]–[6], motivating our paper that leverages this established body of work and extends it to legged robots as they manipulate themselves through the world. We exploit these “self-manipulation” modeling choices to generate the diverse constrained quasi-static and Lagrangian dynamics (as well as the hybrid transition conditions that relate them) arising from the exponentially many possible contact conditions. The framework is body centric (does not require full knowledge of the world reference frame), allows for massless limbs (does not require an invertible mass matrix), and permits underactuation (does not require an invertible kinematic Jacobian). More than numerical simulation (e.g. [7]), our goal is the distillation of physically parametrized models into formal design results, utilizing a variety of analytical and numerical methods.

The central contribution of the paper is summarized in a single “master equation” (33), in addition to the equations and modeling decisions that lead up to it, expressing a rigid
robot’s Lagrangian dynamics together with the constraint forces required to sustain them for any combination of legs and other body parts contacting rigid ground. From the perspective of electromechanical platform design, this model elucidates the role of morphology and actuator characteristics in promoting or precluding certain desired motions of the body. From the perspective of behavior design, the model comprises a compact, uniform representation of the hybrid dynamical control system—the family of state spaces, controlled dynamics over them, and guard conditions and reset maps determining their adjacency—that must be exercised to achieve those motions.

The use of this formalism is illustrated with reference to a succession of tasks executed on the RHex robot [8], [9] wherein the interplay of controlled joint torques, leg contact conditions, and body reactions is particularly heightened by that machine’s very limited actuation. But the model is applicable to any legged machine, and we are convinced that specific power (W/kg) limitations exhibited by every available actuation scheme [10]–[12] will incur very similar, complex tradeoffs between various limb strategies for imparting work upon the body. We believe that the body-centric self-manipulation framework will reveal deeper, more formal results concerning the adjacency relations between a legged machine’s hybrid dynamics cells with strong behavioral implications, but these ideas go beyond the scope of the present paper and are merely hinted at in the conclusion through speculative remarks concerning further work presently in progress.

We intend the paper to be readily accessible to readers acquainted with modern texts on robot manipulation and control [1]–[6]. The modeling principles underlying self-manipulation are quite general, and as such are presented first in tutorial form. We believe their value and coherence is best conveyed in the context of a specific robot presented with a variety of specific tasks, spanning the energetic range from static to quasi-static to dynamic. The RHex robot is required to first stand in place (with the least possible energy on unknown terrain [13]), then manipulate itself in that place (while acting as a “tilt-scanning” sensor platform [14], [15]) and finally leap dynamically from that place (to “prepare” [16] various behaviors such as gap crossing [17] or pronking [18]). The value of this formal method becomes quite apparent when the great multiplicity, diversity and dynamically varying nature of the contact modes is considered across these seemingly disparate tasks. The framework generates automatically the equations of motion for all of them, whose consistent structure differs only in one term (the matrix A in Eqn. (32)) facilitating not only computational analysis but also formal proofs of the validity or optimality of behavioral choices, controller design, and robot morphology.

This paper is structured as follows: Section I-A compares and contrasts self-manipulation with traditional manipulation and locomotion models. Section I-B then introduces the motivating behaviors which illustrate the benefits of this analysis. Section II formalizes the modeling decisions and then review the quasi-static and dynamic equations of motion. The abstract formalism is instantiated in Section III for RHex, providing a concrete example. This model is put to task in Section IV, where the various behaviors are instantiated and analyzed, with specific new design results called out. Section V concludes the paper with a brief summary and look ahead.

A. BACKGROUND: SELF-MANIPULATION, MANIPULATION, AND LOCOMOTION

Most centrally, the self-manipulation methodology appeals to the key formulation of the manipulation literature, the “grasp map” or “grip transformation”, G [2], which relates wrenches at the contact points to wrenches on the object.1 Specifically, in Section II-B, we adopt the same modeling choices that lead to this map in the traditional setting. However in self-manipulation, the robot must itself move relative to the inertial world, and we focus attention on the consequences of this departure from the manipulation framework.

Of course such modeling decisions are not required to arrive at accurate kinematics or dynamics. Consider the robot in Fig. 2, a single kinematic chain similar to a 4 bar mechanism. The robot could be “cut” in many ways, in order to determine the identical e-DOF (degrees of freedom, here \( e = 1 \)) from various q-dimensional open-loop dynamics and \( c \) constraint forces. For example, one might make: one cut, at a toe \( (q = 3, c = 2, \text{Fig. 2(d)}) \), common in walking analysis [19], [6, Ch. 16]) or at the robot center \( (q = 4, c = 3, \text{Fig. 2(b)}) \), producing strong symmetries; two cuts at the hips \( (q = 5, c = 4, \text{Fig. 2(e)}) \), akin to parts feeding [20]) or at the toes \( (q = 5, c = 4, \text{Fig. 2(a)}) \), direct instantiation of manipulation [21], [22], as well as Fig. 2(f), sometimes called a “floating-base” [23]); four cuts, at the joints \( (q = 9, c = 8, \text{Fig. 2(c)}) \), a free body diagram. Each variation of this example results the same one-DOF mechanics model, however cutting at the toes (Fig. 2(f)) admits simple expressions for friction at each toe (as opposed to at the hip or body, Fig. 2(b), (e)),

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1We adopt the specific notation introduced in [1], though these ideas coincide with the formulation in [2]–[6]—and, indeed, most works on the subject make similar modeling decisions.

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generalizes across all contact conditions (as opposed to the walking model which must be separately instantiated in each mode, Fig. 2(d)), and with far fewer states and constraints than the more general problem (Fig. 2(c)).

Contrary to the oft encountered adage, locomotion is not the same as manipulation. To any reasonable level of precision, a legged robot is not moving the world with its feet (i.e. the “object” is Earth,\(^2\) Fig. 2(a), [21], [22]), nor is the world moving the robot (i.e. grounded legs reaching up and manipulating the robot Fig. 2(e), [20]). In particular there are three main differences between the usual manipulation formulation and the self-manipulation setting: the robot is the object (and so the “Palm” and “Object” frames are coincident, as in Fig. 1); we are concerned with motion of the robot and not what the robot is touching (and so the grasp map must be composed with an appropriate reflection, as appears in (15)); and the dynamics of the legs and body are not decoupled (so in particular the mass matrix is no longer block diagonal, as apparent in (25), complicating the dynamics). Why bother following a manipulation formulation if the problems are actually different? While not every result carries over exactly, the problems are similar enough that matching as closely as possible the modeling decisions that have emerged from this very successful and mature body of work facilitates the reuse, or slightly modified extension, of several valuable ideas and results (e.g. rolling contact [24]–[28], Section II-F).

Finally, it is worth a brief pause to consider the relationship between self-manipulation and locomotion, though they are often considered equivalent [29]. Typically, the locomotion problem domain focuses on the mobility of the body in machines or animals [30], and, most commonly, with respect to a greatly reduced or “template” [31] projection (although certainly not always, e.g. [19]). In contrast, the self-manipulation problem is to determine the simplest “anchor” [31] that fully expresses the actuators’ abilities perform to work upon that body, typically via ground reaction forces directed through the limbs (16), but, not unimportantly, often through momentum transfer in flight (67). Our “master equation” (33) formulates the self-manipulation problem under conditions of great enough generality to provide a complete anchoring model for virtually any locomotion problem where the substrate mechanics is well characterized by traditional lumped models (e.g. excluding flowing media [32]).

**B. RESULTS: CONTROLLERS, BEHAVIORS, DESIGNS**

We illustrate the utility of the self-manipulation formalism through a succession of increasingly energetic tasks implemented on the decade-old hexapod, RHex [8], [9]. The selected behaviors hopefully strike the reader as a plausible and coherent short “episode” of a kind likely to arise within an autonomous missions: the robot comes to a halt at some location of interest; once there, it actively engages its sensory payload; this new information provokes the sudden determine to leap up and escape that location. Notwithstanding the intuitively straightforward, even mundane nature of this simple vignette, in the absence of a systematic formalism along the lines this paper develops, such a succession of tasks would present the behavior designer with a diverse (and combinatorially numerous) array of seemingly unrelated mechatronic and sensorimotor control problems whose common implementation offers no unifying insight into what properties of the platform might help or hinder the mission. In contrast, we point out how this general methodology informs and simplifies the analysis of each of these constituent tasks regarded separately, and affords a unifying framework for analyzing the performance as a function of platform design parameters (here focused on leg shape [33], [34], but the formalism makes explicit the role of the various other morphological parameters (e.g., Table 2) in the behavioral consequences).

1) REACTIVE STANDING

The first behavioral example (Section IV-A) seeks to find a reduction in the power needed for stand-in-place tasks on unmodeled rough terrain [13]. The controller developed is quite simple and implements the intuitive notion that all actuators must resist external load while relaxing any relative imbalances between their individual efforts. However it is not immediately apparent under what circumstances this scheme is correct, nor even that it converges in all application situations. The analytical expression for internal and external torques facilitates the identification of the appropriate operating conditions and a proof of convergence assuming they prevail. As this proof covers both round and stick legs, the design choice makes no real difference for this behavior. The controller provides up to a 90% reduction in power use compared to an open-loop stand.

Why worry about the power used when the robot is idle? In one urban search and rescue study researchers discovered that for 49% of the robot’s deployment it remained stationary, as the operators needed that time to gain situational awareness [35]. This is corroborated by RHex tests in the Mojave desert, where in at least one specific instance during a trial in March of 2010, the operator paused the robot in a standing posture while deciding how to proceed, causing a motor to burn out after less than a minute. Robots operating on challenging terrain, especially in the heat of a desert, need a low-energy standing posture for health and mission runtime.

2) PITCHING SENSOR SWEEP

For almost any exteroceptive task, perceptual capability can be increased by extending a sensor’s field of view by moving it. Rather than (or in addition to) adding dedicated “neck” actuators, Section IV-B documents how RHex’s legs can be used to provide a change in pitch, increasing the vertical field of view of any payload. Here the formal setup of the constraints in different contact modes determine analytically both the pitching range as well as the pitching velocity in any mode. Furthermore the dynamical liftoff conditions provide a

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\(^{2}\)“The planet Earth’s radius and mass are \(R_0\) and \(M_0\),” [22].
speed bound for safe execution.

Our empirical example of this general idea features a horizontal (Dorsal) planar laser scanner, a sensor that has no vertical extent and so the only way to build a two dimensional depth map is to move it out of the plane. Prior work on this behavior used an ad-hoc geometric model of the half circle legs to numerically compute an operable scanning range of about 10° [14]. Without an analytical form, the geometry would have to be re-generated in each contact state. It turns out that this behavior benefits from a different contact mode (with the body sliding on the ground). Furthermore while both round and stick legs can reach this same peak pitch, the rounded legs require less torque to do so. Since the kinematics and dynamics are analytically derived with this formalism across all contact states, the pitching velocity can be controlled, and the maximal velocity the behavior can execute without breaking ground contact is found to be higher in the sliding contact mode. The final behavior has a range of ±17° that can be used for example to more easily detect stairwells and cliffs [15].

3) PITCH CONTROL IN LEAPING

Recent work has raised the question of how best to use a leap to “prepare” [16] a subsequent behavior by reaching some goal state, whether that goal is across a gap, onto an obstacle, or continuing into another leap [17]. In this prior work, individual actuators were assigned a trivial control policy (maximal torque), and the varied sequences of contact modes achievable through their relative timing dictated the resulting state. Here we consider a finer use of the actuators over the course of a prescribed route through the contact modes (notionally motivated by gap crossing or pronking [18], [36]) wherein the robot must at some point engage all its legs on the ground and reach an aerial state with large forward but low pitching kinetic energy.

We analyze three aspects of pitch control for such leaps in Section IV-C—the behavior design (focusing on the splay or the asymmetry in leg angle), the controller design (focusing on stubbing the toes and using the legs as tails), and the robot design (focusing on the shape and the mass of the legs). The underlying dynamics of a splayed posture is beneficial in several ways. While pushing with both legs equally maintains a level pitch at first, the front leg provably breaks its ground contact first, and so the robot pitches upwards. A splayed leap (as used before and adjusted via hand tuning [18], touchdown plane control [36], or exhaustive search [17]) minimizes this liftoff imbalance. Furthermore a splayed pronk has the added benefit of a higher possible velocity.

The best results for pronking on RHex have all resorted to decelerating or stubbing the rear legs near the end of stance [36], [37]. We show analytically that this negative work does cut down the pitch of the robot, though at the same time bleeding off some of the forward kinetic energy, as demonstrated experimentally in an extreme case.

Next, to illustrate the role of leg design in leaping, the dynamics are combined with the takeoff conditions in single support to derive a bound on maximum forward velocity. This bound suggests that the rounded leg design of RHex enables a higher maximum speed than an equivalent stick leg. Furthermore we consider the inertial effects of the nearly massless legs [38], which in the air act as “tails,” [39], [40], that are able to generate a non-trivial body rotation.

II. SELF-MANIPULATION

A. NOTATION

Table 1 summarizes the notation in this section, chosen where possible to match [1]. Denote a rigid frame B, expressed in the coordinates of rigid frame A (or, equivalently, a rigid transformation that takes frame A into frame B) \( g_{ab} \in G_a \) := SE(d), where \( d = 2 \) for a planar model and \( d = 3 \) for a spatial model. In local coordinates a rigid transformation will be written as a vector, for example \( \mathbf{x} = [x, y, z, \phi_x, \phi_y, \phi_z]^T \), where the Euler angles may be chosen as convenient but here will be ZYX—Yaw, Pitch, Roll. The group product will be denoted as \( g_{ab} \cdot g_{bc} \). The velocity of frame B relative to frame A as seen by A is \( V_{ab} = g_{ab}^{-1} \cdot \dot{g}_{ab} \), or in twist coordinates \([\omega_{ab}, s_{ab}]^T\) (the “spatial” velocity), while the same velocity written in the coordinates of frame B is \( V_{ab} = g_{ab}^{-1} \cdot \dot{g}_{ab} \) (the “body” velocity) [1, Section 2.4.2]. A body wrench (generalized force), \( \mathbf{F}_b := [\mathbf{f}_b, \mathbf{t}_b] \), is defined such that work is \( \mathbf{V}_b \cdot \mathbf{F}_b \). An adjoint transformation matrix, \( \mathbf{A}_{\mathbf{G}_{ab}} : T_{G_b} \rightarrow T_{G_a} \), relates the two expressions of velocity, \( \mathbf{V}_{ab} = \mathbf{A}_{\mathbf{G}_{ab}} V_{ab} \) [1, Chapter 2].

Denoting by \( \pi_x \) a projection down to the x component, \( \pi_{\mathbf{q}} \) a projection to some collection of components to be defined by the friction conditions as specified below, and the non-subscripted \( \pi \) the projection from an element of \( SE(d) \) (position and orientation) to \( \mathbb{R}^d \) (position only), we express the origin of a frame, \( \mathbf{g} \), as (e.g. when \( d = 2 \)) \( (x, z) = \pi(g) \in \pi(G) \approx \mathbb{R}^2 \). Similarly let \( R_{ab} := \pi_R g_{ab} \) be the rotational component of the rigid transformation, in matrix form. To convert between local coordinates and twist coordinates, define (e.g. when \( d = 2 \)) \( \mathbf{R} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), so that \( \mathbf{V}_{ab} = \mathbf{R}^T \dot{x} \).

B. MODELING DECISIONS

This self-manipulation model follows the usual conventions from manipulation [1]–[6]: the hand and object are separated at the fingertips; the wrench bases at the fingers (i.e. the motions that the contact resists) are considered in unison; and (through the separated velocity constraints of the grasp map and hand Jacobian) dictate the forces and torques on the object and robot. However since the robot is the object, we set the “Object” frame, \( O \), to be coincident with the robot’s “Palm” frame, \( P \).

Define the following coordinate frames, as shown in Fig. 1 (and corresponding to [1, Fig. 5.14]). Let \( P \) be attached to the COM of the robot body segment, and the object frame, \( O \) be co-located at \( P \), but attached to the world. The usual manipulation problem takes the palm frame as fixed, and so from the robot’s perspective it appears that the gravitational force is applied to the movable Earth (the object), and not the...
other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around. Co-locating the object and palm coordinate frames allows for wrenches and twists that are referenced to that point in the world, and the robot’s actual motion is simply the opposite sign (from a ground based observation), as that point in the world, and the robot’s actual motion is simply the other way around.
the robot actually know anything about their location.

There are a number of different types of contact possible once a contact point is known, for example RHx has both sliding and non-sliding point contacts (see [2, Table 2–3], [1, Table 5.2] for a full list of examples). In each contact frame define a subspace consisting of only degrees of freedom that friction keeps fixed, \(C_k \subseteq \mathbb{C}_k\), \(k \in \mathcal{K}\), with a projection (whose expression in coordinates is a sub-block of the identity matrix), \(x_{c_k} = \pi_{c_k}(g_{c_kw}(q))\). However the standard grasping analysis instead focuses on the wrenches that the contact can resist (as is required for the non-holonomic constraints of rolling contact), and so in that spirit define the wrench basis as the image of the pullback of \(\pi_{c_k}\) from the allowable contact wrenches to all possible contact wrenches at a contact point \([1, \text{Section 5.2.1}], B_{c_k} : T^*C_k \rightarrow T^*G_{c_k}\), (written in coordinates, \(B_{c_k}^T := D\pi_{c_k}\)). The contact conditions enforce zero motion in these directions as well—giving rise to the familiar dual pairing,

\[
\dot{x}_{c_k} = B_{c_k}^T V_b^{c_k}, \quad \dot{x}_{c_k} \in TC_k
\]

\[
F_{c_k} = B_{c_k} f_{c_k}, \quad f_{c_k} \in T^*C_k
\]

This contact constraint holds only when the contact forces are in some Coulomb friction cone relating the normal and tangential forces,\(^3\)

\[
U_k(f_{c_k}) \geq 0, \quad U_k : T^*C_k \rightarrow \mathbb{R}^{c_k}
\]

[1, Section 5.2.1], [6, Section 27.3], satisfying,

\[
U_k(f_a) \geq 0, \quad U_k(f_b) \geq 0
\]

\[
\Rightarrow U_k(af_a + \beta f_b) \geq 0 \quad \forall \alpha, \beta \in \mathbb{R}^+
\]

which in the planar case \((d = 2)\) is simply a matrix multiplication, \(U_k f_{c_k} \geq 0\).

For multi-finger robots, and now multi-legged robots, it is often convenient to deal with the collection of contact positions (and similarly for twists and wrenches) among all contact points,

\[
x_c := (x_{c_1}, \ldots, x_{c_n}) \in \mathcal{C} := \Pi_{k \in \mathcal{K}} C_k
\]

\[
f_c := (f_{c_1}, \ldots, f_{c_n}) \in T^*C
\]

\[
U(f_c) := (U_1(f_{c_1}), \ldots, U_n(f_{c_n})) \geq 0
\]

where \(\mathcal{C} := \text{dim}(\mathcal{C}) = \sum_{k \in \mathcal{K}} \text{dim}(C_k)\) is the total number of constraints on the system.

**D. KINEMATIC LOOP CLOSURES**

The friction holding the contact points in place along some dimensions sets up the following constraint functions, expressing the \(k^{th}\) contact condition by the equality,

\[
x_{c_k} = \pi_{c_k}(g_{c_kw}(q)) = \pi_{c_k}(g_{c_kw}(q) \cdot g_{c_kw}(x))
\]

(see further discussion in the Appendix Section A), motivating the definition of the constraint function

\[
a_k(q) := \pi_{c_k}(g_{c_kw}(q)) - \dot{x}_{c_k}
\]

whose zeros, \(a_k^{-1}[0]\), comprise the constraint set for some initial contact position \(\dot{x}_{c_k}\). Collectively, the c kinematic constraints are,

\[
0 \equiv a(q).
\]

This constraint sets up the initial pitching sensor behavior presented in Section IV-B.1, though this is quickly extended to include rolling contact as described in Section II-F.

**E. INFINITESIMAL KINEMATICS: THE GRASP MAP AND HAND JACOBIAN**

The infinitesimal kinematics over the base constraint (10), relates wrenches and twists between the body and joints and contacts, through induced tangent constraints. Given a constrained motion, \(q_d(t) : \mathbb{R} \rightarrow \mathcal{Q}\), satisfying (10), \(a \circ q_d \equiv 0\), these induced constraints can be given coordinate expression by differentiating the constraint equation, however by using the fact that the constraint equation must be true no matter where the world frame is, some of the interdependence is removed. Therefore we claim that,

\[
\frac{d}{dt} a_k \circ q_d \equiv 0
\]

\[
\Rightarrow A_k q \dot{q} := [-B_{c_k}^T A_{\mathcal{K}c_k} V_b^{c_k} - B_{c_k}^T A_{\mathcal{K}c_k} R_{wp}^T] q = 0
\]

(11)

where the leg Jacobian \(J_{c_k} \) is defined such that, \(V_{b_{c_k}} \) is \(J_{c_k}^T(\theta) \dot{\theta}\). The proof of this claim may be written out using either a homogeneous representation or a twist representation, as shown in the Appendix Section A. Here instead we show that this equality (11) is equivalent to the standard manipulation constraint,

\[
J_{\mathcal{K}} \dot{\theta} = G^T V_{p_{po}}
\]

[1, Eqn. 5.15], typically derived indirectly in terms of twists by defining the grasp map and hand Jacobian, and not by differentiating a base constraint (10).

In manipulation literature, the grasp map, \(G : T^*C \rightarrow T^*G_0\), takes wrenches at the contact points (i.e., forces at the contact points), \(f_c \in T^*C\) to wrenches on the object, \(F_o \in T^*G_0\), and its dual, \(G^T\), acts covariantly, taking body twists of the object, \(V_o \in T^*G_0\), to twists at the contact point, \(\dot{x}_c \in TC\), all expressed in coordinates as,

\[
G f_c = F_o, \quad G^T V_b^{p_{po}} = \dot{x}_c
\]

(13)

\[
G := [A_{c_1}^T \ldots B_{c_1} \ldots A_{c_n}^T B_{c_n}]
\]

[1, Section 5.2.2, Fig. 5.15], where \(G \in \mathbb{R}^{3 \times 3c}\) if \(d = 2\), and \(G \in \mathbb{R}^{6 \times 3c}\) if \(d = 3\). As this is a self-manipulation, \(V_{p_{po}}\) is the opposite of the body velocity one would normally consider, as it is the velocity of \(O\), attached to the world. This opposite direction comes from the more general one velocity identity, \(V_{b_{p_{po}}} = -A_{\mathcal{K}wp} V_{wp}\) [1, Lemma 2.16] (see Appendix Section F for proof), but in this case the adjoint matrix is simply identity. Therefore a self-manipulation “gasp” map is defined as,

\[
G_s := -G, \quad G_s^T V_{b_{p_{po}}} = \dot{x}_c, \quad G_s f_c = F_p
\]

(15)
where recall that \(\mathbf{V}^b_{\text{op}} = \mathbf{V}^b_{\text{sw}}\) is the body velocity of \(P\) relative to any world coordinate frame, and \(\mathbf{F}_p\) is the body wrench.

Next, the hand Jacobian, \(\mathbf{J}_h\) relates infinitesimal motion at the joints, \(\dot{\mathbf{q}} \in T\Theta\) to twists at the contact points, \(\mathbf{x}_c \in T\mathbf{C}\), and has a dual, the pullback from contact wrenches, \(\mathbf{f}_c \in T^*\mathbf{C}\) to hip torques, \(\tau \in T^*\mathbf{C}\), all expressed as,

\[
\mathbf{J}_h \dot{\mathbf{q}} = \mathbf{J}_h^T \mathbf{f}_c = \tau
\]

[1, Section 5.5.1, Fig. 5.15], where \(\mathbf{J}\) Jacobian definition carries over directly from manipulation, \(\mathbf{F}\) to any world coordinate frame, and \(\mathbf{V}\) to \(T^*\mathbf{C}\), all expressed as,

\[
\mathbf{J}_h \dot{\mathbf{q}} = \mathbf{J}_h^T \mathbf{f}_c = \tau
\]

Note that in the case of body contact, the hand Jacobian has a column of all zeros, that is contact wrenches at that point have no direct projection onto the joint torques (see Appendix Section B for further discussion).

Combining (13) and (16), and recalling that \(\mathbf{V}^b_{\text{sw}} = \mathbf{R}^T_{\text{sw}} \dot{\mathbf{x}}\), we see that as claimed in (11),

\[
\mathbf{A}(\mathbf{q}) \dot{\mathbf{q}} = \left[ -\mathbf{J}_h \quad \mathbf{G}_t \mathbf{R}^T_{\text{sw}} \right] \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{x}} \end{bmatrix} = \left[ -\mathbf{J}_h \quad \mathbf{G}_s^T \right] \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{V}}^b_{\text{sw}} \end{bmatrix} = 0
\]

asserting that the motion of the contact frames as seen from the robot and the world agree (so long as the friction constraints hold (7)).

**F. ROLLING CONTACT**

Rolling contact, when the contact frame \(\mathbf{C}_s\) is not fixed relative to either the body or object, can be treated as a simple extension to the above analysis. At each instant the velocity of the body and joints is as if the leg was a simple stick leg, with a toe at the point of contact,\(^4\), however the evolution of the contact location is dictated by the relative geometry. The underlying contact velocity constraint is still correct, as the contact instantaneously cannot move in the constrained direction, however in general this constraint is non-holonomic [24] (i.e. there is no corresponding base constraint (8)).

Therefore in general the velocity constraint components \(\mathbf{G}_s\) and \(\mathbf{J}_h\) also depend on parameterized contact coordinates, \(\eta\), which update as some function of the local geometry [26]. For RH\(\text{Ex}\), the geometry is simple enough that no extra \(\eta\) parameters are needed (the rolling contact frames can be described fully by elements of \(\mathbf{q}\)), and so in the interest of space we direct the reader to [1, Chapter 5.6] for a full derivation (with similar notation) as well as [24]–[26]. In addition see [27], [28] for higher order considerations—for now it is sufficient to note that while the constraint, \(\mathbf{A}\) (11), is the same for rolling contact as an equivalent stick leg, \(\dot{\mathbf{A}}\) may not be, and so while quasi-statically both follow the same trajectory, the dynamics are different in order to account for the changing constraints. This fact is used for example in Result C.4.

\(^4\)See the Appendix Section C for simple proof for RH\(\text{Ex}\) like circular legs.

**G. PARAMETERIZATION OF THE CLOSED-LOOP KINEMATICS**

In most of the application settings considered below, given the \(e\) constraints (10), it is convenient to work with a local parameterization of the \(e - q - c\) dimensional manifold of remaining mobility. This amounts to the choice of an implicit function, \(h : \mathcal{Y} \rightarrow \mathcal{Q}\), where \(\mathcal{Y}\) is some convenient open subset of \(\mathbb{R}^e\). To simply answer certain questions about the robot motion in terms of particular components of \(\mathbf{q}\), it is convenient to identify \(\mathcal{Y}\) with a problem-specific Euclidean submanifold of our generalized coordinates, i.e. \(\mathcal{Y} := \psi(\mathcal{Q})\), where, in coordinates, \(\psi\) is some fixed linear combination of the components of \(\mathbf{q}\) corresponding to the directions of interest in the problem (and often a projection whose matrix representation is a subcollection of columns from the identity matrix). The tangent map, \(\mathbf{Y} := D\psi, \mathbf{Y} \in \mathbb{R}^{q \times e}\), results in a combined constraint equation,

\[
\begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{Y} - \mathbf{I}_{dc} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{y}} \end{bmatrix} = 0
\]

such that the associated implicit function (split into hand and object components),

\[
\mathbf{h}_h(y) = \theta, \quad \mathbf{h}_s(y) = \mathbf{x}, \quad \mathbf{h} = (\mathbf{h}_h, \mathbf{h}_s) : \mathcal{Y} \rightarrow \mathcal{Q}
\]

is a local immersion—i.e., its Jacobian maps,

\[
\dot{\mathbf{h}} = \mathbf{H}_h \dot{\mathbf{y}}, \quad \dot{\mathbf{x}} = \mathbf{H}_s \dot{\mathbf{y}}, \quad \mathbf{H} := D_{\mathbf{y}} \mathbf{h}
\]

is full rank (never passing through the origin) in both tangent spaces (for some local region in \(\mathcal{Y}\)). In this paper we assume that such a parameterization exists. The matrix inverted in (20) is dimension \(q \times q\), and invertible except at singularities in the kinematics or parameterization. Note that \(\mathbf{H}_h\) can be thought of as the instantaneous gear ratios for \(n\) independent motor shafts coupled rigidly to an external output load with \(e\) DOF.

This implicit function is used to show that the change of basis in Section IV-A is a good approximation, and again in Section IV-B.3 that \(\mathbf{H}_h\) can be used to control the pitching rate of the robot.

**H. QUASI-STATIC FORCES AND TORQUES**

The wrench due to the gravitational potential field, \(\mathbf{F}_g\), is derived from the height \(\zeta : \mathcal{G}_p \rightarrow \mathbb{R}\) in that potential field and, at static equilibrium, it is exactly balanced by the contact forces through the grasp map (13),

\[
\mathbf{G}_t \mathbf{f}_c = -\mathbf{F}_g.
\]

If the number of DOF in the closed-chain analysis, \(e\), is less than the number of motors, \(n\), there is some “internal” force components, which lies in the subspace [1, Definition 5.3],

\[
\ker(\mathbf{G}_s) = \Im(\mathbf{f}_N) := \left\{ \sum_{i=1}^a \alpha_i \mathbf{f}_N | \alpha_i \in \mathbb{R} \right\}
\]
that forms the homogeneous solution, i.e. contact wrenches that are internal in that they can perform no work on the object, where $a := n - e$.

To invert (21) and determine the contact forces required to balance the external wrench, torque constraints of the form

$$\mathbf{t}^T \mathbf{J}_c^T \mathbf{t}_c = 0, \quad i \leq M,$$

must be imposed upon the hip joint torque vector, $\tau \in T^* \Theta$, (16). Pulling back through the infinitesimal kinematics, this now constrains the contact wrench magnitude vector $\mathbf{f}_c \in T^* C$ leading to a unique solution of the full rank augmented version of (21) taking the form,

$$\mathbf{t}^T \mathbf{J}_h \mathbf{t}_c = 0, \quad \mathbf{f}_p := - \left[ \mathbf{G}_s \mathbf{t}^T \mathbf{J}_h^T \right]^{-1} \left[ \mathbf{F}_c \right].$$

Thus the overall vector of contact wrench magnitudes is,

$$\mathbf{f}_c = \mathbf{f}_p + \sum_i a_i \mathbf{f}_{Ni},$$

and the internal and external components may be projected back into motor torques through (16).

These quasi-static internal and external forces are the key to the reactive standing behavior in Section IV-A, and calculating the torque requirement in the pitching sensor sweep (Section IV-B.2).

### I. DYNAMICS

This section derives a Lagrangian formulation for the robot dynamics in terms of some local coordinates, arriving at a relationship between $\mathbf{q}, \dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$ in (33). We allow for the option of having massless legs, and so expressions of the dynamics cannot involve directly inverting the mass matrix.

If the body of the robot has mass $m_b$ and inertia $I_b$, and each leg segment may have mass $m_l$ and inertia $I_l$, then observe that the total kinetic energy is,

$$T = \frac{1}{2} \mathbf{q}^T \mathbf{V}_{Vb} \mathbf{M}_b \mathbf{q} = \frac{1}{2} \mathbf{q}^T \mathbf{M} \mathbf{q}$$

and the inertia terms in (25) arise from the coupling inherent in self-manipulation.

Using these statements of the kinetic and potential energy, and applying Lagrange’s equations [1, Eqn. 6.4], [41, Sec. 5.1],

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \mathbf{q}^T \mathbf{M} \dot{\mathbf{q}} - V(\mathbf{q}) = \frac{1}{2} \sum_{i,j=1}^q \mathbf{M}_{ij} \dot{q}_i \dot{q}_j - V(\mathbf{q})$$

and the internal and external components may be projected back into motor torques through (16).

This formulation of the dynamics is sufficient to determine, for example, that the front leg always lifts off the ground before the rear leg in a symmetric leap (Section IV-C.1). However it is convenient to use the parameterization of Section II-G to separate the free dynamics from the constraint forces, as shown in the next section.

### I. REDUCED DYNAMICS

Instead of working with the complete dynamics we can instead consider only the $e$ free Lagrangian variables in $\mathbf{q}$, as introduced above. In this case the dynamics are,

$$\begin{align*}
\mathbf{M} \dot{\mathbf{q}} + \mathbf{C} \ddot{\mathbf{q}} + \mathbf{N} &= \mathbf{A}^T \lambda, \\
\mathbf{N} &= \mathbf{A} \dot{\mathbf{q}},
\end{align*}$$

where $\mathbf{N}$ depends only on $\theta$ and $\phi$ as the force due to gravity is position independent. Note that damping may also be modeled with $\mathbf{N}$, in which case it depends on $\dot{\mathbf{q}}$.

The constraint forces, $A^T \lambda$, arise from the closed-loop constraint (11), and the contact force magnitudes must satisfy the friction constraint (7), $U(\mathbf{q}) \geq 0$.

Rearranging (29) into the familiar form, where note that $A$ is the only term that varies with contact mode,

$$\begin{align*}
\mathbf{M}(\theta, \phi) \dot{\mathbf{q}} + \mathbf{C}(\theta, \phi, \dot{\mathbf{q}}) \ddot{\mathbf{q}} + \mathbf{N}(\theta, \phi) &= \mathbf{A}^T(\mathbf{q}) \lambda, \\
\mathbf{N}(\theta, \phi) &= \mathbf{A} \dot{\mathbf{q}}
\end{align*}$$

and the nonlinear (gravitational) and applied forces are,

$$\begin{align*}
\mathbf{M}(\theta, \phi) &= \frac{1}{2} \sum_{k=1}^q \left( \frac{\partial \mathbf{M}_{ik}}{\partial \dot{q}_k} + \frac{\partial \mathbf{M}_{ik}}{\partial \dot{q}_j} \right) \dot{q}_k \\
\mathbf{N}(\theta, \phi) &= \frac{1}{2} \sum_{j,k=1}^q \partial \mathbf{M}_{jk} \dot{q}_j \dot{q}_k - \partial \mathbf{V} \dot{q}_i.
\end{align*}$$

where we assume that the inverted matrix in (33) is nonsingular even if $\mathbf{M}$ is not (see Appendix Section D).

This formulation of the dynamics is sufficient to determine, for example, that the front leg always lifts off the ground before the rear leg in a symmetric leap (Section IV-C.1). However it is convenient to use the parameterization of Section II-G to separate the free dynamics from the constraint forces, as shown in the next section.
\[
\ddot{\mathbf{q}} + \mathbf{C}(\dot{\mathbf{q}}, \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{N} = \ddot{\mathbf{Y}}
\]
\[
\dot{\mathbf{y}} = \mathbf{M}^{-1}(\mathbf{q}) \left( \ddot{\mathbf{Y}} - \mathbf{N}(\mathbf{q}) \right) - \mathbf{M}^{-1}(\mathbf{q}) \mathbf{C}(\mathbf{q}, \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}}) \dot{\mathbf{y}}
\]

where,
\[
\mathbf{M} := \mathbf{H}^T \mathbf{M}, \quad \mathbf{C} := \mathbf{H}^T \mathbf{C} + \mathbf{H}^T \mathbf{M} \dot{\mathbf{q}}
\]
\[
\dot{\mathbf{N}} := \mathbf{H}^T \dot{\mathbf{N}}, \quad \ddot{\mathbf{y}} := \mathbf{H}^T \mathbf{Y}
\]

(see Appendix Section E for derivation). Note that while the base constraint is a function of the initial conditions (\(\mathbf{X}_0\)), the Jacobian constraint is not, and so any analysis that holds across any initial conditions needs the full \(\mathbf{q}\). Furthermore in general rolling contact requires non-holonomic constraints, and so \(\mathbf{h}\) may not exist. Therefore we have left \(\mathbf{q}\) explicitly in (35) and write out the reduced dynamics in terms of both the reduced variables, \(\mathbf{y}\), and original variables, \(\mathbf{q}\), (although \(\mathbf{h}(\mathbf{y})\) may be substituted for \(\mathbf{q}\) when possible).

To recover the original configuration space accelerations and Lagrange multipliers,
\[
\ddot{\mathbf{q}} = \mathbf{H} \ddot{\mathbf{y}} + \dot{\mathbf{H}} \dot{\mathbf{y}}
\]
\[
\lambda = \left[ \mathbf{A}^* (\mathbf{Y} - \dot{\mathbf{N}}) - \mathbf{A}^* (\mathbf{M} \ddot{\mathbf{q}} + \dot{\mathbf{C}}) \right] \dot{\mathbf{y}}
\]

where \(\mathbf{A}^*,\) a left inverse of \(\mathbf{A}\), is chosen to also satisfy \(\mathbf{A}^* \mathbf{M} = \mathbf{0}\), that is,
\[
\left[ \begin{array}{c}
\mathbf{A}^* \\
\mathbf{C}^*
\end{array} \right] = \left[ \begin{array}{c}
\mathbf{A}^T \\
\mathbf{C}^T
\end{array} \right]^{-1}
\Rightarrow \mathbf{A}^* \mathbf{A}^T = \mathbf{1}_{d_c}, \quad \mathbf{A}^* \mathbf{M} = \mathbf{0}
\]

(see Appendix Section E for derivation).

These reduced dynamics lead to, for example, a simple proof that stubbing the toe at the end of stance in prono is beneficial to the robot’s pitch, as described in Section IV-C.3.

**K. SUMMARY OF ASSUMPTIONS**

For convenience, the assumptions made by the self-manipulation framework just presented are summarized as follows:

A.1) There is a distinguished (pre-selected) rigid body on the robot with a frame, \(\mathbf{P}\), attached at the COM.

A.2) There is an inertial world frame, \(\mathbf{W}\), at some (possibly unknown) fixed location, and at each instant is rigidly connected to a frame, \(\mathbf{O}\), co-located at \(\mathbf{P}\).

A.3) The robot is in contact with the world at some finite number of points, \(\mathbf{C}_i\), and the set of all possible contact points is known (albeit, in general, not their location).

A.4) The combined mass matrix \(\mathbf{M}\) does not need to be full rank (i.e. there may be massless limbs), however the reduced mass matrix, \(\hat{\mathbf{M}}\), is (i.e. any massless links are constrained such that the overall system motion is well defined).

A.5) Some choice of generalized coordinates, i.e., a non-singular minimal parameterization of the free motion, \(\mathbf{y}\), is available in any mode of interest.

Furthermore the following assumptions are made in this paper, but not fundamental to the structure of the framework, B.1) There are no elastic or compliant components.

B.2) Any damping or air resistance is negligible.

Some additional assumptions are made when this general framework is instantiated in the next section.

**III. SELF-MANIPULATION FOR RHHex**

**A. MODEL PARAMETERS**

When RHHex uses pairs of contralateral legs in phase on level ground, it is very effectively anchored to the sagittal plane, and so here we develop a planar model for RHHex \((d := 2)\), with only two legs modeled, each with one joint \((n := 2, q = 5)\), as shown in Fig. 3. In addition the body is allowed to contact the ground at up to two locations (front and rear), so that \(k \leq 4\). The rubber treads on the toes have a relatively high coefficient of friction (especially on rough outdoor terrain) that we assume always resists tangential (sliding) motion, while the hard shell of RHHex’s body has a very low coefficient of friction and so we assume that the body always is able to slide. There are thus 2 contact wrenches at each toe and 1 at the front and rear of the robot, implying that \(c \leq 6\). We do not assume that the robot is endowed with any exteroceptive sensors, and as such must assume instead that the robot is on flat level ground, and so the contact normals are aligned with gravity.

The location and orientation of the various frames are shown in Fig. 3. In the palm frame, let the \(+\mathbf{x}\) axis be aligned with the robot, \(+\mathbf{z}\) in the “downward” direction from the robot, and thus \(+\mathbf{y}\) exiting the page (this is a standard “North, East, Down” orientation). Hip \(i(S_i)\) is located \(\ell_i\) away from the \(\mathbf{P}\) frame along the positive \(\mathbf{x}\) direction, and the leg length is \(\rho_i\) putting the \(F_i\) frame at \(\rho_i\) along the positive \(\mathbf{z}\) direction from \(M_i\), thus,

\[
\mathbf{g}_{pc_1}(\theta, \phi) := \begin{bmatrix} \ell_1 - \rho_1 \sin \theta_1, \rho_1 \cos \theta_1, -\phi \end{bmatrix}^T
\]
\[
\mathbf{g}_{pc_2}(\theta, \phi) := \begin{bmatrix} -\ell_2 - \rho_2 \sin \theta_2, \rho_2 \cos \theta_2, -\phi \end{bmatrix}^T
\]

when the leg is supported on its toe \((0 \leq \theta_i - \phi < \pi)\, as with leg 2 in Fig. 3). While the leg is in rolling contact (as with leg 1 in Fig. 3),

**FIGURE 3.** Coordinate frames and key dimensions for RHHex under the self-manipulation formulation.
TABLE 2. Physical quantities used in the RHex model. Note that \( m_I, l_I, \) and \( \tau_I \) are doubled in practice as contra-lateral legs are used in parallel in this paper, though all but the last behavior assume massless legs.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_b )</td>
<td>7.15 kg</td>
<td>Body Mass</td>
</tr>
<tr>
<td>( I_0 )</td>
<td>0.15 kg m(^2)</td>
<td>Body Inertia</td>
</tr>
<tr>
<td>( m_l )</td>
<td>0.063 kg</td>
<td>Leg Mass</td>
</tr>
<tr>
<td>( l_l )</td>
<td>0.00046 kg m(^2)</td>
<td>Leg Inertia</td>
</tr>
<tr>
<td>( l_1, l_2 )</td>
<td>20.5 cm</td>
<td>Body-Hip Length</td>
</tr>
<tr>
<td>( \rho_1, \rho_2 )</td>
<td>17 cm</td>
<td>Leg Length</td>
</tr>
<tr>
<td>( \rho_b )</td>
<td>5 cm</td>
<td>Body Radius</td>
</tr>
<tr>
<td>( \tau_s )</td>
<td>15 N-m</td>
<td>Saturated Maximum Torque</td>
</tr>
</tbody>
</table>

The body has semi-circular ends with radius \( \rho_b \) about the hips and so the two potential body contact points are,

\[
\begin{align*}
g_{pc_1}(\theta, \phi) &= \left[ \ell_1 - \frac{\rho_1}{2}(\sin \theta_1 + \sin \phi), \\
& \quad \frac{\rho_1}{2}(\cos \theta_1 + \cos \phi), -\phi \right]^T \\
g_{pc_2}(\theta, \phi) &= \left[ -\ell_2 - \frac{\rho_2}{2}(\sin \theta_2 + \sin \phi), \\
& \quad \frac{\rho_2}{2}(\cos \theta_2 + \cos \phi), -\phi \right]^T.
\end{align*}
\]

The body pitch is \( \phi = 0 \) when the robot is horizontal and a positive pitch when hip 1 is higher than hip 2. The leg angles are measured as \( \theta_1 \) and \( \theta_2 \) in the clockwise direction from the body +z direction. In Fig. 3, \( \theta_1 < 0, \theta_2 > 0, \phi > 0 \). Physical values used, including lengths and masses, are summarized in Table 2.

### B. RHex KINEMATICS

The definitions of the previous subsection completely describe the kinematics and dynamics of the robot in any contact configuration. A full list of the resulting matrix formulation of the kinematics and dynamics is included in the Appendix Section G, while here we only look at the base constraint and explore the quasi-static state space.

From Section II-D, the base kinematic constraint is defined by composing the inverse of the specified maps for \( g_{pc_k} \) with the body coordinates, \( x \), which for stick legs (0 \( \leq \theta - \phi < \pi \)),

\[
a_k(q) = \pi_k(g_{pc_k}(q)) - \hat{x}_{ck}, \quad k \in 1, 2
\]

while for rolling contact (\(-\pi \leq \theta - \phi < 0\)),

\[
a_k(q) = \pi_k(g_{pc_k}(q)) - \hat{x}_{ck}, \quad k \in 1, 2
\]

Similarly for the body contact, the base kinematic map is,

\[
a_k(q) = \pi_k(g_{pc_k}(q)) - \hat{x}_{ck}, \quad k \in 3, 4
\]

\[
\begin{bmatrix}
-z + \ell_k \sin(\phi) - \rho_b \\
-z + \ell_k \sin(\phi) - \frac{\rho_0}{2}(1 - \cos(\theta_k - \phi))
\end{bmatrix} - \hat{x}_{ck}
\]

which is the equivalent of setting \( \theta_k = \phi \).

As introduced in Section II-C, the active contact constraints can be found by checking the height of each potential contact frame, \( \zeta_{ck}(x) \). If however the robot does not know its exact place in the world (\( \pi(x) \)), the contact condition can still be found by relying on the assumptions of flat level ground and quasi-static operation. First since the robot is quasi-static, choose the world reference frame to be \( W \equiv O \), and so \( \pi(x) = 0 \). In this case the active contact points are simply those that are farthest from the body in their \( z \) direction, which recall always non-positive by convention,

\[
\pi_z(g_{ck}) < \pi_z(g_{ci}), \quad \forall k \in K, i \notin K.
\]

To consider the difference between rolling contact (on the rounded part of the half circle leg) and point contact (on the toe of the leg) a further condition is needed that specifies which part of the leg are rounded. In particular, the leg presents the rounded half of its shape when \(-\pi < \theta - \phi < 0 \) (as opposed to e.g. [42] which is rounded on the other half).

Once contact mode is determined, pitch is an implicit function of the equality of the constraint (10), 0 \( \equiv a_{ci}(q) \). As before, take \( \pi(x) \equiv 0 \), and so,

\[
\pi_z(a_{ci}) - \pi_z(a_{ck}) = 0, \quad \forall i, j \in K
\]

\[
\pi_z(g_{ci}) \equiv \pi_z(g_{ck})
\]

from which we can locally derive the body pitch as an implicit function of the joint angles. These conditions are combined

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Quasi-static contact modes over the entire state space of joint angles \((0 \sim \pi^2)\). Each region is distinguished by front and rear contact conditions that can be Body, Rolling, or Toe contact, as indicated by the respective letter.}
\end{figure}
to create Fig. 4, which shows the various quasi-static contact conditions across the entire joint space, \( \Theta \).

**C. ASSUMPTIONS FOR RHEx**

As a summary, the following assumptions, in addition to those listed in Section II-K, are made about the RHEx model,

C.1) RHEx is anchored to the sagittal plane by using contralateral legs together, and so \( d = 2 \).

C.2) The middle legs are not used, and so \( n = 2 \).

C.3) The body can contact in up to two locations but has a low coefficient of friction that never resists tangential friction forces.

C.4) The rubber toes have a high coefficient of friction and therefore always resist tangential forces (as the surface normal and frictional coefficients for novel terrain is unknown—if they are assumed to be known then the friction cone could be checked with (7)).

C.5) The robot is on flat level ground (\( C_k \) and \( W \), assumed everywhere except Section IV-A).

C.6) The legs are massless (\( m_l \equiv 0 \), \( l_1 \equiv 0 \), assumed everywhere except Section IV-C.5).

Furthermore the following simplifying assumptions are made in some of the following behaviors as marked,

D.1) The robot is near a nominally standing posture, i.e. the legs are “under” the robot, \( \theta_k \in U := [-30^\circ, 30^\circ] \), and the robot is close to level, \( \phi \in [10^\circ, 10^\circ] \).

D.2) The robot is symmetric (\( \ell_1 \equiv \ell_2 \equiv \ell \), \( \rho_1 \equiv \rho_2 \equiv \rho \), implicitly assumed for numerical calculations based on Table 2).

D.3) The robot is a point mass (\( I_b \equiv 0 \)).

**IV. BEHAVIORS**

**A. reactive standing**

In this section we document a quasi-static RHEx standing controller that delivers up to a 90% reduction in power use relative to an open-loop stand on unmodeled rough terrain [13]. The scheme is extraordinarily simple: the controller seeks simultaneously to reduce the variance of joint torques around their mean (right side of Fig. 5), while fighting to “lean up” against the mean load (left side of Fig. 5). The self-manipulation model is needed to formalize these insights and establish the correctness of the controller.

This notion of fighting an external force and relaxing the internal force has been used before on legged robots, usually without stating it in this way. For example, prior work on RHEx pushes the body uphill to be centered over the legs while climbing steep terrain [43], and separately regulate individual leg torques such that no one leg pushes harder than the rest [44]. These ideas were further developed on RiSE [45], [46] whose reactive gait phase adjustments were designed to balance forces within and between the sides. Similar internal force management strategies have been suggested on quadrupeds [47] and highly articulated bipeds [48], though without a proof of convergence.

To analyze this behavior, Section IV-A.1 decomposes the motor cost into average and difference terms, \( \epsilon \) and \( \delta \), and derive the quasi-static torques necessary to fight external (gravity) and any internal (legs fighting) wrenches. The key to this decomposition is to apply the closed-loop constraint to find the internal and external forces at the toes (Section II-H), and projecting that back into motor torques (Section II-E). Section IV-A.2 sets up the controller summarized in Fig. 5, in part by using an approximation to the closed-loop velocity constraint (Section II-G). Section IV-A.3 brings these parts all together and show convergence of the controller. This entire behavior is analyzed using stick legs with point contact, but as Section II-F has shown, the free motion of rolling contact is the same as that of a stick leg with appropriate radius, and so by showing Lyapunov convergence for any angle and any leg lengths (within appropriate bounds), the rolling contact can be ignored. Note that this proof does not rely on Assumption C.5 but instead simply uses Assumptions C.4 and D.1. Finally Section IV-A.4 tests this controller on a variety of indoor and outdoor terrain, with results documented in Table 3.

1) PROBLEM SETUP

For this problem we are trying to minimize the thermal cost needed to stand, which is proportional to the square of current (which in turn is proportional to torque). A natural goal to set is \( \Pi := \frac{1}{2} (\tau_1^2 + \tau_2^2) \). A key insight is to break apart the functional form of \( \Pi = \epsilon(\tau_1, \tau_2) + \delta(\tau_1, \tau_2) \), where \( \epsilon(\tau_1, \tau_2) := \tau_m^2 \) is the squared mean torque, and \( \delta(\tau_1, \tau_2) := \tau_d^2 \), is the squared difference in torques,

\[
\begin{bmatrix}
\tau_m \\
\tau_d
\end{bmatrix} := T \tau \quad T := \begin{bmatrix}
\tau_m^T \\
\tau_d^T
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]

(45)

and analogously \( [\theta_m \ \theta_d]^T := T \theta \). Section IV-A.3 shows that \( \epsilon \) captures the cost due to gravity while \( \delta \) captures the cost due to internal forces, and more importantly the proposed controller drives both costs to zero.

Similarly we may choose to parameterize the implicit function associated with the closed-loop constraint by the average velocity, with \( y \equiv \theta_m = \dot{\theta}_m \), i.e. \( \dot{y} = Y \ddot{q} := \frac{1}{2} [1 \ 0 \ 0 \ 0] \ddot{q} \).

This choice is motivated by the observation that motion has equal cost in both motors, though selection of \( y \equiv \theta_1 \) or
\( y \equiv \theta_2 \) also works well. The Jacobian of the associated implicit function (19) on some open neighborhood of the origin, \( U \subset \mathbb{R} \), is,

\[
H_h := \frac{2}{\gamma_1 + \gamma_2} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (46)
\]

\( \gamma_i := (\ell_1 + \ell_2) \rho_1 \cos \theta_j + \rho_1 \rho_2 \sin(\theta_2 - \theta_1), \quad j = 3 - i \) (47)

and is full rank (never passing through the origin) in both tangent spaces as the “gear ratios,” \( \gamma_i \), are always positive and nearly equal (bounded numerically by \( 0.83 \leq \gamma_1 / \gamma_2 \leq 1.19 \) under Assumption D.1 in [13]).

The “internal” component of the forces at the toes lies in the subspace defined by (22), which form the homogeneous solution, i.e., toe forces that are internal in that they can perform no work on the object. To specify a particular solution, apply (23) with \( \mathbf{t}^T \equiv \mathbf{t}_d^T \). This choice of particular solution, depicted in the lower sketches of Fig. 5, corresponds to toe forces that cancel gravity with the “right amount” of internal force, here defined by the difference condition.

The torque produced by these toe forces is given by the hand Jacobian, \( \mathbf{J}_f^h \), as in (16). It is convenient to work in a new basis for the joint-space torques, \( \tau \in T^*\Theta \), given by the scaled rotational transformation \( \mathbf{T} \) into the mean and difference of the torques as defined in (45). The torque implication of the particular solution (\( \alpha = 0 \)) is,

\[
\begin{bmatrix} \tau_m \\ \tau_d \end{bmatrix}_p := \alpha \mathbf{TJ}_f^h \mathbf{f}_p = -m_b g \rho_1 \rho_2 \begin{bmatrix} \ell_1 \cos \theta_1 \sin(\theta_2 - \phi) + \ell_2 \cos \theta_2 \sin(\theta_1 - \phi) \\ \gamma_1 + \gamma_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \] (48)

where \( \tau_{d,p} = 0 \) because the particular solution has no component in the \( \tau_d \) direction (23). Therefore all of the virtual work against gravity must show up in \( \tau_{m,p} \), so that \( \tau_{m,p} = \frac{1}{2} \mathbf{F}^T \mathbf{H}_h = \frac{1}{2} m_b g D\zeta o D_h \), (see [13] for proof).

The torque projection of the homogeneous solution is,

\[
\begin{bmatrix} \tau_m \\ \tau_d \end{bmatrix}_h := \alpha \mathbf{TJ}_f^h \mathbf{f}_N = \alpha \begin{bmatrix} \gamma_2 - \gamma_1 \\ \gamma_2 + \gamma_1 \end{bmatrix}. \] (49)

Here there is not an exact decomposition—we would like \( \tau_{m,h} \) to be zero so that \( \tau_m \) is exactly \( \tau_{m,p} \). However we have observed (in (47) and related discussion) that \( \gamma_1 \approx \gamma_2 \), and in any case if our controller is successful we can achieve this by simply canceling the internal force magnitude, \( \alpha \). Having derived an expression for the various torque components in (48)–(49), we can now apply a controller to this system and observe the effect on the component cost functions, \( \epsilon \) and \( \delta \).

2) CONTROLLER DESIGN

In this section, we show how direct current readings at the hips yield intrinsic sensors that approximate the gradient of two costs, \( \epsilon \) and \( \delta \), eliminating all need to know or compute the exact kinematics online. The change of basis in torque space, \( \mathbf{T} \), allows the robot to use these sum and difference torque measurements to closely approximate the gradient of its power-use cost function.

Because the motor controller is highly overdamped and rate limited we adopt “generalized damper” mechanics and model the motor as velocity controlled in general, \( \dot{\theta} = \kappa \sigma \dot{\theta} \), for some command \( \dot{\theta} \in T\Theta \). To guarantee that the system remains quasi-static, this command is rate limited to ensure \( \dot{\theta} \leq \kappa_\sigma \). However the motion is constrained by the closed-loop condition (12), and so the constrained motion is approximately,

\[
\dot{\theta}_m = \kappa_\sigma \dot{\theta}_m \quad (50)
\]

\[
\dot{\theta}_d = 0 \quad (51)
\]

since the system can move freely in approximately the \( \theta_m \) direction (exactly, in the \( \mathbf{H}_h \) (46) direction). In contrast, in the approximate \( \theta_d \) direction (exactly, along infinitesimal motions orthogonal to \( \mathbf{H}_h \)), motion is locked, hence generated torque must increase as,

\[
\dot{\epsilon}_d = \kappa_\sigma \dot{\theta}_d, \quad (52)
\]

i.e. any differentially applied command increases the torque as the system cannot move in that direction.

Again, we emphasize that this locked leg assumption is merely an approximation (as \( \partial \theta_d / \partial \theta_m \) is small), but no matter how large the shift, so long as \( \gamma > 0 \) for all angles, i.e. the sign of the direction of motion is correct, we can simply allow the internal force controller to compensate for this “disturbance” in \( \theta_d \) as the robot moves.\(^8\) Moreover this misalignment between the approximate and true parameterization of the free motion does not affect the zero point—in either case the zero has \( \tau_1 = \tau_2 = 0 \) and so the controller converges to the correct place, even if it does not take the “most direct” route.

3) INTERNAL AND EXTERNAL COST

Note that the internal cost, \( \delta = \dot{\tau}_d^2 = \sigma^2 (\gamma_1 + \gamma_2)^2 / 4 \) (49), vanishes if \( \alpha = 0 \). Therefore since the particular solution makes no contribution to \( \tau_d \) (48), based on our actuator model (51)–(52), it is straightforward to reduce \( \delta \) by asserting a control policy, \( \dot{\theta}_d := -\kappa \sigma \tau_d \), resulting in,

\[
\delta = 2 \tau_d \dot{\epsilon}_d = \left\{ \begin{array}{ll}
-2 \kappa_\sigma \kappa_1 \tau_d^2 & |\dot{\theta}_d| \leq \kappa_\sigma \\
-2 \kappa_\sigma \kappa_1 \kappa_\sigma |\tau_d| & |\dot{\theta}_d| > \kappa_\sigma \end{array} \right. \] (53)

Thus the positive definite function, \( \delta \), has a negative definite derivative along the motions of (52) under the specified control and, thus, as a Lyapunov function.

Result A.1: Relaxing the difference in torque (53) assures that \( \tau_d \), and therefore the internal cost \( \delta \), decays to zero from any nominally standing posture (Assumption D.1).

The exponential decay of the homogeneous (internal) torque solution leaves the second term of the cost function, \( \epsilon = \tau_{m,h}^2 + \tau_{m,p}^2 \), determined by the gravitational torque field through \( \tau_{m,p} \) (48), which can be minimized by bringing \( \theta_m \) to a critical point of \( \zeta_\sigma \). For \( \phi \in \mathcal{U} \), the closest

\(^8\)Without this dependence, the two controllers could be run sequentially.
critical point is a local maximum. Therefore, we consider minimizing the function \(-\zeta_0\), and implement the quasi-static dynamics \(\dot{\theta}_m := \kappa_m \tau_m\), since this implies,\(^9\)

\[-\dot{\zeta}_0 = -D\zeta_0, \quad \dot{\theta}_m = \begin{cases} -mbg\kappa_m\kappa_p|D\zeta_0|^2 |\dot{\theta}_m| & \leq \kappa_\sigma \\ -\kappa_p\kappa_\sigma |D\zeta_0| |\dot{\theta}_m| > \kappa_\sigma \end{cases} (54)\]

i.e., \(-\zeta_0\) (a smooth positive definite function in the neighborhood of a maximum), has a negative definite derivative under the control input as it enters the dynamics (50), and therefore,

**Result A.2:** Fighting the mean torque (54) assures that \(\theta_m\) converges to the local minimum of \(-\zeta_0\), the local maximum of \(\zeta_0\), and therefore the external cost \(\epsilon\), decays to zero from any nominally standing posture (Assumption D.1).

Furthermore, as discussed in Section II-F, since quasi-statically rolling is the same as an equivalent stick leg, and the above proof holds for any leg angle and length so long as \(\gamma_i > 0\), we find that,

**Result A.3:** The convergence of (53)–(54) holds even under rolling contact, and so either leg shape converges to near-zero cost.

Thus for this first behavior, either leg shape is acceptable.

4) EXPERIMENTAL RESULTS

We implement this controller on the robot by straightforward generalization of the difference torque controller (53), for each leg individually, and the mean torque controller (54), now applied to the mean of all six legs [13, Section III.D]. The controller was tested on a variety of terrains by having the robot execute a reactive stand from either a sitting or walking posture. Fig. 6 shows a test on a pile of rocks, and Table 3 summarizes the results.

Note that regardless of the initial conditions, the reactive power was reduced to around 4 W. In fact every trial except for one on the rocks reduced the power to below 5 W. In that outlier, with a final power usage of 21.2 W, the robot slipped partway through execution of the smart stand, and, as the current behavior executes for a fixed time, the robot did not have time to completely recover. Anecdotally, the entire robot can typically be turned off after this behavior runs and the robot remains standing (implying that the remaining 4 W may came from the control electronics or noise). Furthermore since the experiments that started from a walk contained a mixture of leg contact conditions (some legs on the rounded half and some on the toe), these results supports the claim that both round and stick legs converge to near-zero power under this controller.

**B. PITCHING SENSOR SWEEP**

The behavior developed in this section allows the robot to actuate around the pitch component of its body frame. While prior work has posed the problem and initiated an analysis [14] and empirical application [15] of such behaviors, both efforts introduced point solutions that focused on a single contact mode with ad-hoc geometry. Here the formal derivation of an expression for the pitch as an implicit function of the leg state, \(h_\phi(\theta)\), allows analytical solutions in any contact condition, with consequent formal insight into the implications of leg shape.

Section IV-B.1 uses the domains of the various quasi-static contact conditions for RHx (Section II-C, Fig. 4) and the implicit function for pitch based on the base kinematic constraint (Section II-D) as in (44) in order to evaluate across all contact conditions to find the maximum range (as shown in Fig. 7). This range is extended from about 10° of pitch when both legs are in rolling contact (as first discovered in [14]), to about 35° when all modes are considered, greatly increasing the view of the world afforded to any payload sensors. Furthermore Section IV-B.2 compares the torque (Section II-H) needed to hold a pitched pose with different leg shapes, finding that the rounded leg uses less power than the stick leg. Section IV-B.3 shows that the pitching rate is exactly controlled by the analytical expression (59), based on the closed-loop constraint (Section II-G), eliminating the need for a numerical solution to guarantee constant pitching velocity [14]. Finally, Section IV-B.4 checks the dynamic

**TABLE 3. Reactive stand power from seated position. Each row is an average of five trials, and idle ("hotel") power has been removed. First group started from a sitting posture, second group started from walking posture.**

<table>
<thead>
<tr>
<th>Terrain</th>
<th>Slope</th>
<th>Normal Power</th>
<th>Reactive Power</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asphalt</td>
<td>None</td>
<td>6.02 W</td>
<td>3.64 W</td>
<td>39.6%</td>
</tr>
<tr>
<td>Rocks</td>
<td>Various</td>
<td>6.32 W</td>
<td>3.73 W</td>
<td>40.1%</td>
</tr>
<tr>
<td>Grass</td>
<td>−14.0° pitch, 11.1° roll</td>
<td>5.89 W</td>
<td>4.12 W</td>
<td>30.0%</td>
</tr>
<tr>
<td>Grass</td>
<td>1.2° pitch, 5.5° roll</td>
<td>11.43 W</td>
<td>4.34 W</td>
<td>62.0%</td>
</tr>
<tr>
<td>Dirt</td>
<td>18.8–19.9° pitch</td>
<td>22.50 W</td>
<td>4.01 W</td>
<td>82.2%</td>
</tr>
<tr>
<td>Carpet</td>
<td>None</td>
<td>36.63 W</td>
<td>3.97 W</td>
<td>89.2%</td>
</tr>
<tr>
<td>Smooth</td>
<td>10.6° pitch</td>
<td>15.55 W</td>
<td>3.98 W</td>
<td>74.4%</td>
</tr>
<tr>
<td>Rocks</td>
<td>Various</td>
<td>31.25 W</td>
<td>7.30 W</td>
<td>76.6%</td>
</tr>
</tbody>
</table>
effects (Section II-J) to maintain the integrity of the sensory behavior by ensuring that the platform maintains frictional contact with the ground.

1) SCANNING RANGE

Within a given quasi-static contact mode (Fig. 4), pitch is an implicit function of the equality constraint (44), which for example in rolling contact on both legs (for simplicity written here for a symmetric robot, Assumption D.2). \(^{10}\)

\[
 h_\phi(\theta) = \arctan \left( \frac{4 \ell + \rho (\sin(\theta_2) - \sin(\theta_1))}{\rho (\cos(\theta_2) - \cos(\theta_1))} \right). \tag{55}
\]

The value for \( \phi \) for all quasi-static contact modes is shown in the contour plot of Fig. 7, as well as flow lines showing the constrained motion as a pushforward of a vector field over \( \mathcal{Y} \) through (19).

The behavioral design problem is then to find the extrema of this pitch, formally set up as either a constrained optimization in each contact mode, or an unconstrained optimization over the combined implicit value for \( \phi \). This combined function is continuous as the active constraints in two neighboring contact conditions are subsets of the constraints on the boundary. The easiest way to solve this analytically is to choose \( y = \theta_1 - \phi \), the global leg angle, and then parameterize the implicit function (55) in terms of \( y \), (here shown for the case of a stick leg in front and body contact in the rear), \(^{11}\)

\[
 h_\phi(y) = \arcsin \left( \frac{\rho_1 \cos(y) - \rho_2}{\ell_1 + \ell_2} \right) \tag{56}
\]

which is maximized at \( \phi = \arcsin(\rho_1 - \rho_2/\ell_1 + \ell_2) \approx 17^\circ \).

It is clear from Fig. 7 that this maximal upward pitch of \( \phi \approx 17^\circ \) is achieved at \( \theta_1 = 17^\circ \) and \( \theta_2 < -100^\circ \), and from sitting (\( \theta_1 = \theta_2 = -180^\circ \)) this pitch is only reachable when the rear body is sliding along the ground. Similarly the maximal downward pitch of \( \phi \approx -17^\circ \) is achieved at \( \theta_2 = -17^\circ \) and \( \theta_1 < -130^\circ \), i.e. when the front body is on the ground. This proves that,

Result B.1: The maximal pitching sensor sweep reaches \( \phi = \pm \arcsin(\rho_1 - \rho_2/\ell_1 + \ell_2) \approx \pm 17^\circ \) by using the sliding contact modes (56).

Such behaviors are shown in Fig. 8, where the robot is using a planar laser scanner to detect a stairwell and check for cliffs, as in [15]. However depending on the exact task a path through the double stance region may be useful in order to smoothly access both positive and negative pitches. If restricted to only rolling contact, then the maximal pitch is about \( 10^\circ \), as found in [14], and shown in Fig. 7.

2) TORQUE REQUIREMENT

While both round and stick legs reach these same extrema in pitch, the torque required is not the same. Under quasi-static operation the leg torque required to resist gravity is dictated by (16) and (23), however in these sliding contact modes \( n = e = 1 \), and so there are no internal forces, and \( \mathbf{G}_s \) is directly invertible. Thus,

\[
 \tau_1 = -\mathbf{J}_h^T \mathbf{G}_s^{-1} \mathbf{F}_g \tag{57}
\]

which can be compared for different leg shapes. Evaluating for round legs, \(^{12}\)

\[
 \tau_1 = -\frac{\ell_2 \rho_1 m_h g \cos(\phi) \sin(\theta_1 - \phi)}{2 (\ell_1 + \ell_2) \cos(\phi) - \rho_1 \sin(\theta_1 - \phi)} \tag{57}
\]

while for stick legs, \(^{13}\)

\[
 \tau_1 = -\frac{\ell_2 \rho_1 m_h g \cos(\phi) \sin(\theta_1 - \phi)}{(\ell_1 + \ell_2) \cos(\phi) - \rho_1 \sin(\theta_1 - \phi)}. \tag{58}
\]

While both leg shapes have the same zero torque point (when the leg is vertical, as found for the previous behavior), in general, \(^{14}\)

\(^{10}\)Calculated by solving (44) for \( \phi \) with the appropriate \( a \) (i.e. a concatenation of (42) for each leg).

\(^{11}\)Calculated by solving (44) for \( \phi \) with the appropriate \( a \) (i.e. a concatenation of (41) for the front and (43) in the rear) and replacing \( \theta_1 \) with \( y + \phi \).

\(^{12}\)Calculated with the appropriate \( \mathbf{G}_s \) (i.e. a concatenation (84) of (82) for the front and (83) for the rear) and \( \mathbf{J}_h \) (i.e. simply (86) for the front).

\(^{13}\)Calculated with the appropriate \( \mathbf{G}_s \) (i.e. a concatenation (84) of (81) for the front and (83) for the rear) and \( \mathbf{J}_h \) (i.e. simply (85) for the front).

\(^{14}\)Recall that these different contact conditions have different implicit relationships between \( \phi \) and \( \theta_1 \), however the factor of 2 in the denominator of (57) is the dominant factor.
Result B.2: A round leg (57) requires less torque to hold a pitched position than a stick leg (58) at the same angle. This is easy to see intuitively, as the rounded leg has a shorter effective lever arm.

3) VELOCITY CONTROL
As the robot is pitching its sensor, the data can be correlated by recording the pitch as measured or calculated at each instant. However it may be necessary to sweep the sensor with some constant pitching speed. In [14] this was achieved by numerically inverting the forward kinematics to choose a leg angle velocity that produced the desired pitch velocity.

Here we take advantage of the analytical expression of \( H_o \) (19), the velocity Jacobian that arises from the closed-loop constraint. Take for example rolling contact for the front leg and body contact in the rear, and let \( y := \theta_1 \), then a desired pitch velocity of \( \dot{\theta}_d \) can be achieved by setting the leg velocity, 15

\[
\dot{\phi} = H_o \dot{\theta}_1 := \pi \phi H \ddot{\theta}_1 \iff \dot{\theta}_1 = (H_o)^{-1} \dot{\phi},
\]

where

\[
(H_o)^{-1} = \frac{\rho_1 \sin(\theta_1 - \phi) - 2(\ell_1 + \ell_2) \cos(\phi)}{\rho_1 \sin(\theta_1 - \phi)} \tag{59}
\]

which can be implemented as an online feedback controller based on local measurements of \( \dot{\theta}_1 \) and \( \phi \), or can be calculated in advance numerically by using (18), (19), and thus,

Result B.3: A desired sensor pitch rate, \( \dot{\theta}_d \) can be exactly achieved by joint velocity tracking control around the reference signal (59) as derived from the velocity Jacobian (19).

Note that the denominator of (19) goes to zero when \( \theta_1 - \phi = 0 \), i.e. when the leg is vertical, as in fact the maximal pitch has been reached and the velocity control can no longer be applied.

4) DYNAMICS
The maximum speed at which the scanning behavior can be executed is limited by the takeoff condition, \( U \dot{\lambda} \geq 0 \), (7), based on the dynamics, (39) i.e. above a certain speed (\( \dot{y} := \dot{\theta}_1 \)) one of the contact points may lift off the ground. The maximum joint speed that avoids liftoff of either leg (with no torque, i.e. when the behavior reverses) is,

\[
\begin{align*}
\max \dot{y} & \\
\text{s.t.} \ U \dot{\lambda} & \geq 0 \\
\dot{\lambda} & = -\mathbf{D} \dot{y} + \mathbf{E} \lambda \\
\tau & = 0.
\end{align*}
\]

This speed is shown in Fig. 9 for a subset of the configuration space, along with a trace of the solution from [14]. Now we can bound the speed of the system to be below the minimum over this range, which is about 300 \(^{\circ}\)/s for the entire rolling contact range, or a little higher for most specific trajectories. In the rear body sliding contact case there is no solution to \( \lambda_{2,n} = 0 \), i.e. the front leg acting alone cannot lift the body off of the ground. However at high speeds the front leg loses contact, but those speeds are about twice that of the rolling contact case, and so, the maximum speed still consistent with full frictional contact is found numerically to be higher in the sliding modes (Fig. 9).

C. PITCH CONTROL IN LEAPING
Leaping with a low pitching velocity but high forward velocity (e.g. for gap crossing [17] or a pronking gait [18]) is very challenging for RHex because it entails in the high energy regime the delicate interaction of dynamics (Section II-I) with contact conditions (Section II-C) that we have already found to play a critical role in the quasi-static setting. For example, the pronk gait is known to have “severe pitch instability” on RHex [36], and as such several methods of pronking pitch control have been suggested—modifying torques during stance, adjusting the leg angles before touchdown, and rear leg stubbing (rapid leg deceleration at the end of stance). The first strategy applies a differential torque between the motors [18], but under rigid assumptions the robot motion is constrained to a single DOF (as noted in the standing behavior). Even assuming leg compliance, this control authority is near-singular for typical leg angles for pronking [36].

The second strategy considers the plane generated by the toes when landing, as the robot quickly pitches until all toes are on the ground [18]. This effect can be canceled by setting the toes to be parallel to the ground [36], or exploited by adjusting it to a desired pitch [18]. This had limited success, as it, “appeared to disturb the robot’s touchdown angles enough

15 The first two, [37] notes, “Unfortunately ... neither method could be used successfully to control pitch during pronking.”
to cause skidding” [37]. However the partial success of such strategies suggests that it may be beneficial in other ways, as it results in a splayed posture. The intuitive motivation to enforce symmetry between the legs does, at first, appear to control the pitch of the robot in stance as the parameterized free motion (19) has zero pitch. However when the friction constraints are considered (7), Section IV-C.1 proves that the front leg lifts off the ground first due to an unavoidable imbalance in toe normal forces (60), resulting in a positive pitch velocity. The same is not true when a fixed splay between the legs is moving, the more the robot pitches upwards (63). This bound is lower for rounded legs, and lower for level jumps than for a double stance period.

Thus the difference between the rear toe normal force (4th row, denoted “2n”) and front toe normal force (2nd row, denoted “Ln”) is (note that in the normal force direction, $U = -1$, as in (76)),

$$ (U\lambda)_{2n} - (U\lambda)_{1n} = \frac{2\tau}{t} $$

i.e. the rear normal force is always larger than the front normal force, and so,

**Result C.1**: In a symmetric jump ($\theta_1 = \theta_2$), the front leg lifts off the ground first (60) as an unavoidable consequence of the dynamics (19) and contact constraint (7) (under Assumption D.2).

If this were a bilateral constraint (e.g. pin joint), the front leg would continue to pull down on the robot to maintain the neutral pitch that the closed-loop constraint suggests.

2) SPLAYED LIFTOFF CONDITIONS

For the non-symmetric jumping case ($\theta_1 \neq \theta_2$), the analytical solution to $\lambda$ is complicated enough to obviate any benefit of direct visual inspection. However we can look at how the imbalance of (60) changes with the splay angle, $\theta_d := t_d\theta$ (45) (under Assumptions D.2 and D.3),

$$ \frac{\partial ((U\lambda)_{2n} - (U\lambda)_{1n})}{\partial \theta_d} \bigg|_{\theta_d = \theta_m} = -\frac{2\tau \sin^2(\theta_m)}{t^2 \cos^2(\theta_m)} < 0 $$

and (numerically true even for $I_b$ in Table 2), thus with otherwise equivalent conditions,

**Result C.2**: A positive splay angle ($\theta_d$) reduces the imbalance between the front and rear normal forces (61) for RHx performing a forward leap (under Assumptions D.2 and D.3).

This splayed posture result is supported by the leaping dataset [49]. In those experiments, the robot performed a family of forward leaps while applying maximally available motor shaft torque with variations only in the relative timing of torque onset, $t_2$, which can be seen in that dataset to be well correlated with $\theta_d$. Looking at the pitching moment at apex, as

The infinitesimal kinematics, $^{17}$

$$ H_\phi := \frac{\partial \phi}{\partial \theta_m} = \frac{2\rho^2 \sin(\theta_2 - \theta_1)}{\gamma_1 + \gamma_2} = 0 $$

implying the same condition holds for the pitch acceleration, $\ddot{\phi}$. Why then does this gait fail? The answer lies in the liftoff conditions, $U\lambda \geq 0$, $^{18}$

$$ \lambda = -D_\lambda \dot{q} + E_\lambda $$

$$ = \begin{bmatrix}
-\tau \tan(\theta_2) + \tau \cos(\theta_2) - \frac{1}{4}m_bg \sin(2\theta_2) - \frac{1}{4}m_b \rho \sin(\theta_2) \theta_2^2 \\
\tau + \frac{mg}{\rho} + \frac{\tau \sin(\theta_2)}{\rho} - \frac{1}{4}m_b \sin(2\theta_2) - \frac{1}{4}m_b \rho \cos(\theta_2) \theta_2^2 \\
\tau + \frac{mg}{\rho} + \frac{\tau \sin(\theta_2)}{\rho} + \frac{1}{4}m_b \sin(2\theta_2) + \frac{1}{4}m_b \rho \cos(\theta_2) \theta_2^2 \\
\end{bmatrix}. $$

17 Calculated from (20) with $y = \theta_m$ (as in (47)) and the appropriate $A$ (i.e. (17) where $G$ is a concatenation (84) of (82) for each leg and $J_b$ is block diagonal (87) with (86) for each leg), under Assumption D.2.

18 Calculated from (33), using (88)–(92) and the same $A$, or equivalently (39), under Assumption D.2.
listed in [49, Table I] and plotted in Fig. 10, it is clear that the pitching moment generated from a single leap decreases with increasing splay. Therefore a leap or prong with a higher pitch ($\dot{\theta}_d > 0$, i.e. below the main diagonal in the double rolling contact region of Fig. 7) on average incurs less deviation from that initial pitch during this first phase of the leap.

3) SINGLE LEG DYNAMICS

Once the front leg has lifted off the ground, the hybrid dynamics admit two degrees of freedom, and so we drop $\theta_1$ from the state as under Assumption C.6, the massless front leg can have arbitrary position in the air, which would otherwise violate Assumption A.4. Note that this contact mode is not accessible quasi-statically, and as such is not shown in Fig. 4. Choosing $y = [\theta_2, \dot{\theta}_2]^T$, the constrained motion of (19) evaluated with $A$ for a stick leg in the rear and no contact in the front,\(^{19}\)

$$\dot{q} = H\dot{y} = \begin{bmatrix} \dot{\theta}_2 \\ \rho_2 \cos(\theta_2 - \dot{\phi})(\dot{\theta}_2 - \dot{\phi}) - \ell_2 \sin(\phi) \dot{\phi} \\ \rho_2 \sin(\theta_2 - \phi)(\dot{\theta}_2 - \dot{\phi}) - \ell_2 \cos(\phi) \dot{\phi} \end{bmatrix}$$ \hspace{1cm} (62)

leading to the reduced dynamics of (35) for pitch,\(^{20}\)

$$\ddot{\phi} = -\tilde{D}_\phi \dot{q} + \tilde{E}_\phi = \ell_2 m_b \cos(\theta_2) \left( \rho_2 (\dot{\theta}_2 - \dot{\phi})^2 + \ell_2 \sin(\theta_2) \dot{\phi}^2 \right) / I_b + \ell_2^2 m_b \cos^2(\theta_2)$$ \hspace{1cm} (63)

where it is clear that,

**Result C.3:** Once the front leg has lifted off the ground, the harder the rear motor is pushing ($r_2$), and the faster the rear leg is moving ($\dot{\theta}_2$, when $|\dot{\phi}| < |\dot{\theta}_2|$), the more the robot accelerates counterclockwise in pitch (i.e. “upwards”, in the positive $\phi$ direction) (63).

Thus the controller design problem for a prong-like leap becomes a balancing act between the conflicting criteria to both move quickly and maintain pitch control. Some prong implementations on RHex reconcile this conflict by significantly reducing the torque at the end of stance, rapidly decelerating the leg [36], [37], essentially stubbing the toe in order to quickly correct the pitch. This stubbing effect was also anecdotally demonstrated as a leaping task in [17, Section IV-A.2]—here we present additional data from that experiment.

To show the possibilities of this strategy in an extreme case, the apex state for two leaps, one with a strong toe stub and one without, are included in Table 4. This confirms that rapidly slowing down the rear leg at the end of a jump can induce a large pitch correction, greatly affecting the pitch velocity, though at the cost of 17% of the forward velocity.

\(^{19}\)Calculated from (20) with $Y = [1, 0, 0, 1]$ as given and the appropriate $A$ (i.e. (17) where $G_1$ is simply (81) and $J_b$ is simply (85)).

\(^{20}\)Calculated from (35)–(37) with (88)–(92) and $H$ as shown in (62).

4) IMPLICATIONS OF THE TAKEOFF CONDITION

As in the prior subsection, consider the case where the front leg has taken off and only the rear leg remains in contact. The friction cone (7) sets up an implicit bound on torque based on the contact forces (39),\(^{21}\)

$$\tau \geq \begin{bmatrix} I_b \dot{\theta}_2 \cos(\theta_2 - \phi) \rho_2 (\dot{\theta}_2 - \dot{\phi})^2 + \ell_2 \sin(\theta_2) \phi - g \cos(\theta_2 - \phi) \end{bmatrix} \begin{bmatrix} \ell_2 m_b \cos(\theta_2) \rho_2 \cos(\theta_2 - \phi) \\ + \ell_2 \sin(\phi) - I_b \sin(\theta_2 - \phi) \end{bmatrix}$$ \hspace{1cm} (64)

(here shown for a stick leg). Thus the faster the robot is moving ($\dot{\theta}_2$), the more torque is required to maintain contact, and so with any limited-torque actuator, the system speed exhibits a corresponding upper bound. However in general the faster an actuator is moving, the less torque it can produce. Applying a motor model of $\tau \leq \kappa P G \rho (1 - \kappa G \dot{\theta})$ [50] (where $\kappa P$ is proportional to peak motor power and $\kappa G$ is proportional to gear ratio), we can substitute for $\tau$ in (64) to get an equality that imposes a necessary condition for liftoff on the robot’s state. This equality constraint can now be solved with a variety of different implicit functions to gain insight into the manner in which different design choices—either entailing physical parameters, or various behaviors entailing controllers which aim for different state space trajectories—can potentially influence the resulting conditions at liftoff. The simplest of these obtains by considering the equality to be a quadratic form in $\dot{\theta}$, so that the resulting root functions can be passed through the infinitesimal kinematics (62) yielding a closed-form expression for the maximum forward velocity at liftoff, here shown for mid-stance ($\theta_2 = 0$ and $\phi = 0$) with stick legs,

$$\dot{x} = -\kappa_b + \sqrt{\kappa_b^2 + 4I_b \kappa_c} / 2I_b$$

$$\kappa_b := \ell_2 m_b \rho_2 \frac{1}{\kappa G} - \rho_2 g I_b.$$ \hspace{1cm} (65)

For rounded legs, the same maximum forward velocity is,

$$\dot{x} = -\kappa_b + \sqrt{\kappa_b^2 + 4I_b \kappa_c} / 2I_b$$ \hspace{1cm} (66)

Fig. 11 shows this forward velocity bound for a robot with a stick leg and for a rounded leg across a range of leg and body angles typically found for stance. We conclude that,

\(^{21}\)Calculated from (39)–(40) with (88)–(92) and $H$ as shown in (62). Note that the denominator of (64) is positive under Assumption D.1, i.e. normal standing/running ranges, for values listed in Table 2.

### Table 4. Comparison of forward leaps: the toe stub (Result C.3) has significant control authority over body pitch, in this extreme example changing the pitch by $24^\circ$.

| Toe Stub? | $\phi_1$ | $\phi_2$ | $|\dot{\phi}|$ | $T$ |
|-----------|-----------|-----------|----------------|-----|
| Without   | 10.4°     | 57°/s     | 1704 mm/s      | 11.4 J |
| With      | $-13.5^\circ$ | $-145°/s$ | 1414 mm/s      | 8.3 J |
The dynamical bound on the forward speed is higher for rounded legs (66) than for stick legs (65), as is true analytically for mid-stance and therefore by smoothness for some neighborhood around that point, and is numerically shown to be at least 0.6 m/s faster over the entire range of Fig. 11 (all assuming $\dot{\phi} \approx 0$).

This bound is very rough as it does not consider compliance or damping (or in this figure, pitching velocity), but does shed considerable light on the new considerations that emerge in the dynamical regime whereby the rounded legs of RHex when running forward afford a higher speed limit, even at the point of mid-stance whereas, in contrast, quasi-statically the two morphological variants are equivalent (as described in Section II-F).

In contrast, the liftoff constraint (64) can be used in a more conventional numerical manner, for example determining the

$$\max \quad \dot{x}$$

$$\text{s.t.} \quad (U^\lambda)_{2n} \geq 0$$

$$\lambda = -D_\lambda \dot{y} + \tilde{E}_\lambda$$

$$\tau \leq \kappa P G (1 - \kappa G \dot{\phi})$$

allowing $\dot{y}$ to vary. Specifically, note that for each $\phi$ in Fig. 11, there is a unique $\theta_2$ that maximizes $\dot{x}$. As shown in Fig. 12 (shown for $\phi \approx 0$), the optimal gear ratio lies somewhere around 25:1 depending on the pitch at takeoff, only slightly above the actual value of 23:1 for this robot.

Similarly, using this maximal takeoff point to eliminate $\theta_2$ from (64) we can now test the desirability of $\dot{\phi} \approx 0$. Calculated numerically and shown in Fig. 13, we see that a positive pitch $\phi$ but small pitching velocity $\dot{\phi}$ (as results from a splayed posture) results in the highest maximum forward speed (numerically calculated based on Table 2).

Result C.4: The dynamical bound on the forward speed is higher for rounded legs (66) than for stick legs (65). These maximal speed points are specific positions and velocities ($y, \dot{y}$) that may or may not be reachable—in fact as shown in Fig. 10 all of the jumps with double support in this way still had a non-zero positive pitching moment [49].

5) INERTIAL LEG EFFECTS

In the air, the dynamics of the robot are not restricted by the closed-loop constraint (12), as there are no contact forces, however conservation of angular momentum in the absence of external forces now imposes a nonholonomic constraint. Solving the dynamics as in Section II-I takes these conservation laws into account, ensuring that the time derivative of momentum is equal to the applied force, which here is zero. Recall that up until this point under Assumption C.6 we have used massless legs (as $m_1/m_b < 1\%$), but here we drop that assumption to test what effect the very light legs do have. The leg effectiveness [40] is defined as the body velocity per differential leg velocity, which can be calculated by placing the world reference frame, $W$, at the system center of mass and solving the conservation of angular momentum equation as follows (for a single leg)\(^{23}\)

$$\pi_\phi \left( \frac{\partial L}{\partial \phi} \right) = \pi_\phi \left( \mathbf{Ad}^T_{\phi w} \mathbf{M} \left[ \dot{\phi} \mathbf{V}_{\phi w} \right] \right) = 0 \Rightarrow$$

$$\dot{\phi} = -\left( \frac{I_b + m_r \left( \frac{\ell_1}{2} \right)^2 - \ell_1 \frac{\ell_1}{2} \sin \theta_1}{I_b + I_l + m_r \left( \frac{\ell_1}{2} \right)^2 - 2 \ell_1 \left( \frac{\ell_1}{2} \right) \sin \theta_1} \right)$$

$$m_r := \frac{m_l m_1}{m_b + m_l} \left( \frac{m_l m_1}{m_b + m_l} \right) \pi_{\phi_{\text{roll}}}$$

\(^{22}\)Compare (65)–(66) with the quadratic formula and note that the factor of 2 in the inertia ($\ell^2$ term when written out in quadratic form) makes (65) smaller than (66) for any set of positive parameter values and $\phi < 1/\kappa G$; i.e. the motor no-load speed.

\(^{23}\)Calculated with $\pi_{\phi_{\text{roll}}} := \frac{m_l m_1}{m_b + m_l} \pi_{\phi_{\text{roll}}}$. Note that the conservation of momentum laws apply only to body velocities at the center of mass, as used here.
which is the same as [40], up to notation. Note that the effectiveness is shape dependent—the relationship between leg velocity and body velocity is not constant over all \( \theta \). However an interesting simplification arises when two legs equally spaced from the robot COM (Assumption D.2) are controlled so as to be locked in parallel, \( \theta_1(t) \equiv \theta_2(t) \),

\[
\varepsilon_n := \frac{\dot{\phi}}{\dot{\theta}_1} = -\frac{2I_l + m_r(\ell_2/2)^2}{I_b + 2I_l + 2m_l\ell_1 + m_r^2\ell_1^2/2},
\]

\[
m_r := \frac{m_b(2m_l)}{m_b + 2(m_l)} \quad (68)
\]

i.e. the configuration dependence goes away. With all six legs included, the effectiveness for RHex is only \( \varepsilon_n = 0.035 \), though it is almost twice a naïve estimate of \( \frac{\varepsilon_b}{\varepsilon_b} = 0.018 \) (see Table 2).

To verify the inertial effects of the legs, the robot performed a single jump (as in [17]) and the legs were sent to a given position, but in one case they were controlled to recirculate completely, while in the other they were not. Thus between the two tests the leg angles \( \theta \) differ by exactly 360°. This resulted in a difference in body pitch of 14° (median taken across 5 jumps of each type), as shown in Fig. 14. Thus the measured effectiveness is \( \varepsilon_m = 14/360 = 0.039 \), quite close to what the model predicts, and critically,

**Result C.5:** The legs act as inertial tails and, when locked in phase, can produce a body rotation of \( \varepsilon_n \) (68) for each complete rotation they execute during flight (Fig. 14).

### V. CONCLUSION

We have presented a formal framework for the generation of quasi-static and dynamic equations of motion for legged robots across multiple contact conditions. This framework matches as closely as possible the modeling decisions typically used for the analysis of multi-fingered hands, thereby highlighting the similarities and differences between the two classes of problems. The resulting systematic, unified and general methodology for modeling all contact conditions of a jointed, partly actuated, rigid-bodied mechanism that can move by generating ground reaction forces against a lumped parameter substrate promotes analysis of a rich variety of behaviors for common place scenarios that arise as legged robots leave the laboratory and enter the real world.

After the tutorial leading up to a “master equation” (33), we illustrate the utility of this methodology for behavior design in the context of a specifically imagined episode in an autonomous mission of a RHex robot. Along the way to this equation we recall from the manipulation literature the notion of a grasp map, \( G \) (13) whose null space introduces a transparent account of internal and external forces, yielding a simple, general, and provably correct algorithm (53)–(54) for standing still over unknown terrain with minimal power draw (Table 3). Reducing the influence of the highly varied mechanics across the multiplicity of contact conditions to the appearance of one term (17) promotes a straightforward comparison of the quality of exteroception (55)–(59) across all quasi-statically reachable configurations, enabling the robot to act as an active laser wall/cliff detector (Fig. 8). Formal reasoning about the intrinsic conflict between leg torques and liftoff speed (60)–(66) sheds new light on the problem of how to leap forward most energetically without pitching. Finally, in each of these task settings, the formal nature of the results affords clear answers to morphological design questions such as the consequences of leg shape. Tracing back more systematically similar threads between the various design parameter values (Table 2) and the behavioral results they promote or constrain ought to be useful in future redesign of this and other legged robots.

Looking ahead, it is useful to compare the results of this paper with those resulting from a more combinatorially focused study of how to use ground reaction forces in legged self-manipulation. In [17] we examined the dynamic effects different strings of multiple contact modes could impose assuming very simple controls. In contrast, this paper has focused on the effects of much more finely varied control policies implemented within individual (and, sometimes, pairwise sequences of) contact modes. Legged self-manipulation will require a combination of both styles of planning and control: the combinatorial challenge of planning sequences of modes, and the refinement of control strategy within each mode, will both be critical in general for extracting the most capability legged robots can achieve in a complex world.

### APPENDIX

#### A. KINEMATIC CONSTRAINTS

The base constraint (8) may be expanded as,

\[
x_{ck} = \pi_{ck}(g_{ck,w}(q)) = \pi_{ck}(g_{ck,l}(q) \cdot g_{ck,w}(x))
\]

where,

\[
g_{ck,l}(q) := g_{ck,l}(q) \cdot g_{ck,w}(\theta)
\]

\[
g_{ck,w}(x) := g_{ck,l} \cdot g_{ck,w}(x)
\]

however note that while \( g_{ck,l} \) depends on \( q \), the projection \( \pi_{ck}(g_{ck,l}) \) does not vary over time (i.e. it may be thought of as being parameterized by the initial value, \( q_0(0) \), but not a time varying function of \( q \)). This frictional restriction is much clearer when expressed as a velocity constraint, \( B_{ck}^T V_{ck,l} = 0 \),

---

FIGURE 14. Robot state just prior to touchdown after two jumps: (Left) the legs completely recirculated clockwise. (Right) the legs went directly to the final angle counterclockwise.
which is one reason most multi-fingered manipulation texts often skip the base constraint and directly apply a velocity or force constraint (including refs. [1]–[6]).

The velocity constraint equation (11) is proven in twist coordinates by using the following identities (see [1, p. 59] for proof of (69) and Section F, below, for a proof of (70)),

\[
V_{ac} = V_{ab}^s + Ad_{sab} V_{bc}^s
\]  

(69)

\[
V_{ab} = -V_{ba}^s
\]  

(70)

[1, Prop. 2.14, Lemma 2.16], and the friction constraint,

\[
B_c^T V_{c,fs} \equiv 0
\]  

(71)

[1, Eqn. 5.9] (ensuring no motion in the constrained directions) and so,

\[
0 = \frac{d}{dt} a_k \circ q_a = B_c^T \dot{g}_{s,ak} \cdot g_{s,cw} \cdot g_{s,ak} \Rightarrow 0 = B_c^T \dot{V}_{cw}^s
\]

\[
= B_c^T (V_{c,fs} + Ad_{s,fs} V_{fs}^s)
\]

\[
V_{s,ks}^s = V_{s,kp}^s + Ad_{s,kp} V_{wp}^s = 0 = Ad_{s,kp} V_{wp}^s
\]

\[
0 = B_c^T (-Ad_{s,fs} V_{fs}^s - Ad_{s,fs} V_{s,fs}^s - Ad_{s,fs} V_{s,fs}^s)
\]

\[
= -B_c^T Ad_{s,fs} (q)^{\beta}_{c,fs} (\theta) \dot{q} - B_c^T Ad_{s,fs} (q) R_{wp}^T \dot{x}
\]

An alternate proof of (11) is given here using a homogeneous representations of velocity, \( \dot{\mathbf{V}} := (\mathbf{V})^\wedge \), by using the following identities (in addition to (70), proven below),

\[
\dot{V}_{ab} := \dot{g}_{ab} g_{ab}^{-1}
\]

\[
\left(Ad_{s,ab} V_{bc}^s\right)^\wedge = g_{ab} \dot{V}_{bc}^s g_{ab}^{-1}
\]

[1, Eqn. 2.53, Lemma 2.13] (and indeed all of [1, Section 2.4]), and the friction constraint (71) so,

\[
0 = \frac{d}{dt} a_k \circ q_a
\]

\[
= B_c^T \left( g_{c,fs} g_{s,fs} + g_{c,fs} g_{s,fs} g_{s,fs} \right) \dot{q} + g_{c,fs} g_{s,fs} g_{s,fs} \dot{g}_{s,fs}
\]

\[
= B_c^T \left( g_{c,fs} g_{s,fs} + g_{c,fs} g_{s,fs} g_{s,fs} \right) \dot{q} + g_{c,fs} g_{s,fs} g_{s,fs} \dot{g}_{s,fs}
\]

\[
\Rightarrow 0 = B_c^T (V_{c,fs} + Ad_{s,fs} V_{fs}^s + Ad_{s,fs} V_{fs}^s)
\]

\[
= 0 - B_c^T Ad_{s,fs} (q)^{\beta}_{c,fs} (\theta) \dot{q} - B_c^T Ad_{s,fs} (q) R_{wp}^T \dot{x} = A_k (q) \dot{q}
\]

B. GRASP MAP AND HAND JACOBIAN

The definition of the grasp map (13) is often given in terms of body velocity, \( V_{bp}^b \), but computation of the dynamics is easiest in some local coordinates, \( \dot{x} \). Here we have also defined a grasp map in terms of the more relevant body velocity \( V_{wp}^b \). These various versions of the grasp map are, of course, equivalent as summarized here:

\[
G_s^T R_{wp}^T \dot{x} = G_s^T V_{wp}^b = G_s^T V_{bp}^b = G_s^T V_{bp}^b
\]

For further notes on this see [1, Eqn. 6.18] as opposed to [1, Eqn. 5.15], discussion on [1, pp. 279, 283], and paragraph surrounding [6, Eqn. 28.1].

As for the hand Jacobian, note that the contact wrenches (or twists) at one toe have no direct effect on joints on a different leg (hence the block diagonal structure of \( J_h \)). The indirect effects are captured by the closed-loop constraint (11,12). That the legs can be decoupled in this way is less obvious than in the multi-fingered manipulation case, where each finger is rigidly attached to a fixed inertial frame. This decoupling comes from Newton’s third law of motion, that every action has an equal and opposite reaction, and thus we may calculate the joint torques equally well by adding up the effects on either side of the joint.

C. ROLLING CONTACT

As noted in Sec. II-F, the fact that RHx’s legs are not simple sticks does not change any of the analysis thus far, it simply makes the \( G_c \) and \( J_h \) matrices more complicated. The free motion of the hip at any given moment under rolling contact is identical to the motion of an equivalent stick leg connecting the hip to the contact point.

To prove this, consider a point on the circumference of a circle as it rolls—it follows a cycloidal path. If we first assume that the world frame \( W \) is attached to the ground at the toe when the robot is standing (\( \theta = 0 \)), with the \( z \) axis pointing into the ground, then the hip location is \( (x_c, z_c) := \pi g_{wp}(0) = (0, -2 \rho_h) \) (where \( \rho_h := \rho_1 / 2 \) is the leg radius) then the position of the hip as a function of the motor angle \( \theta \) is,

\[
x_c = \rho_h \theta + \rho_h \sin(\theta), \quad z_c = -\rho_h - \rho_h \cos(\theta).
\]

Note that the definition of \( \theta \) used in this paper has the leg in contact on the rounded half of the leg when \( \theta < 0 \), i.e. the leg is on the rounded half before mid-stance, and on the toe afterwards (as opposed to other robots whose half-circle legs are used in the opposite direction [42]).

Now consider a virtual leg extending from the hip to the contact point. Let the angle that this virtual leg makes with the \( \theta \) and the leg length be \( \rho_1 \). The hip position relative to the true world frame at that moment, \( (x_f, z_f) := \pi g_{wp}(\theta) \), for a fixed leg of length \( \rho_1 \) and angle \( \theta \) is,

\[
x_f = \rho_1 \sin(\theta), \quad z_f = -\rho_1 - \rho_1 \cos(\theta).
\]

First note that the triangle consisting of the center of the leg, the hip, and the contact point must be isosceles, as two of the sides are length \( \rho_h \), and the third length \( \rho_1 \). The angle at the center of the circle must then be \( \pi - \theta \) as the supplementary angle is \( \theta \). Therefore by noting that the equal angles in that triangle are \( \theta \), we find that \( \theta = \theta / 2 \).

Now the infinitesimal direction of free motion for each case (i.e. the tangent of the trajectory),

\[
\frac{\partial x_c}{\partial \theta} = \rho_h + \rho_h \cos(\theta), \quad \frac{\partial z_c}{\partial \theta} = \rho_h \sin(\theta)
\]

\[
\frac{\partial x_f}{\partial \theta} = \rho_1 \sin(\theta), \quad \frac{\partial z_f}{\partial \theta} = \rho_1 \sin(\theta)
\]

\[
\frac{\partial x_c}{\partial \theta} = 1 + \cos(\theta), \quad \frac{\partial x_f}{\partial \theta} = \sin(\theta)
\]
can be compared using the double angle identity to find,
\[
\frac{\partial c}{\partial x_e} = \frac{2 \sin(\theta) \cos(\theta)}{1 + 2 \cos^2(\theta) - 1} = \sin(\theta) = \frac{\partial z_f}{\partial x_f}.
\]
Therefore the twist direction is the same whether you follow the full cycloid curve, or at each instant follow a virtual leg. This property is much more general than just for half circle legs, as shown in [1, Chapter 5.6].

**D. DYNAMICS**

The generalized mass matrix, \(\tilde{M}\), is derived by summing the contribution of each link,
\[
T = \frac{1}{2} m_b ||v_{wp}||^2 + \frac{1}{2} J_b ||w_{wp}||^2 + \sum_{i=1}^{n} \left( \frac{1}{2} m_l ||w_i||^2 + \frac{1}{2} J_l ||w_{i||}||^2 \right)
\]
\[
= \frac{1}{2} (v_{wp})^T M_b v_{wp} + \sum_{i=1}^{n} \frac{1}{2} (v_{w_i})^T M_i v_{w_i},
\]
\[
\dot{v}_{w_i} = \dot{A} d_{\phi_i}^{-1} v_{wp} + \dot{v}_{pl_i}
\]
\[
= \frac{1}{2} (v_{wp})^T M_b v_{wp} + \sum_{i=1}^{n} \left( \frac{1}{2} (A d_{\phi_i}^{-1} v_{wp} + v_{pl_i})^T M_i (A d_{\phi_i}^{-1} v_{wp} + v_{pl_i}) \right)
\]
\[
\quad + \left( v_{pl_i}^T M_i \dot{A} d_{\phi_i}^{-1} v_{wp} + \frac{1}{2} v_{pl_i}^T M_i A d_{\phi_i}^{-1} v_{wp} \right)
\]
where both \(A d_{\phi_i}^{-1}\) and \(J_b\) both depend only on \(\theta\). Substituting \(v_{wp} = R_{wp}^T r\) results in the inertia tensor given in (26).

The accelerations and constraint forces may be solved for as follows (33),
\[
\tilde{M} = \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} 0 & M \end{bmatrix} \begin{bmatrix} A & 0 \end{bmatrix} = \begin{bmatrix} \gamma - C \end{bmatrix}
\]
\[
\begin{bmatrix} \dot{\lambda} \\ \lambda \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \dot{\dot{\lambda}}
\]
\[
= \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} 0 & M \end{bmatrix} \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} \gamma - C \end{bmatrix} \dot{\dot{\lambda}}
\]

Some dimensional analysis—the dynamics provides q equations, and the constraint equation provides c. Total, there are \(2q + c\) unknowns, so in non-singular configurations we can solve for the \(q + c\) unknowns \(\dot{\tilde{q}}\) and \(\lambda\) in terms of the \(q\) remaining variables, \(\tilde{q}\), as shown above. Note that this does not require \(\tilde{M}\) be invertible, which is not the case with massless legs, or require \(J_b\) be invertible (as with [1, Eqn. 6.22]), which never is the case with “simple” legs like on RHex.

Instead this method solves for both the system accelerations and constraint forces at the same time by inverting a block matrix that includes \(\tilde{M}\) and \(A\) that in general is non-singular so long as \(\text{rank} (\tilde{M}) + \text{rank} (A) \geq q + c\), or \(\text{rank} (\tilde{M}) \geq c = q - c\) and the constraints are non-singular (this is equivalent to the requirement that the mass matrix of the reduced dynamics, \(\tilde{M}\), to be defined in the next section, be full rank).

Now putting the whole system together we arrive at the differential equation (and splitting up the rows of \(E\) and \(D\) as suggested by the subscript),
\[
\frac{d}{dt} \begin{bmatrix} \dot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -D_{\lambda} & 0 \\ 0 & D_{\dot{\lambda}} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} E_{\dot{q}} \\ 0 \end{bmatrix}
\]
\[
\lambda = -D_{\lambda} \dot{q} + E_{\lambda}.
\]

As used in [1, Eqn. 6.23], and [6, Eqn. 28.20], the dynamics may re-written in twist coordinates (though only after we have derived them in local coordinates), where (33) becomes,
\[
\tilde{M}(\theta) \dot{\tilde{\mathbf{c}}}(\theta, \nu) \nu + \tilde{N}(\theta, \phi) + \begin{bmatrix} -J_h \\ G_i \end{bmatrix} \lambda = \dot{\gamma}(\tau),
\]
\[
\nu := \begin{bmatrix} \dot{\theta} \\ V_{wp} \end{bmatrix},
\]
where \(\tilde{M}\) is the combined body inertia tensor as given in (25), while \(\tilde{C}\) and \(\tilde{N}\) have been suitably rotated by \(R_{wp}\). However the lower line of (72) must reflect \(\tilde{x} = R_{wp} V_{wp}\).

If \(\tilde{M}\) is invertible, the Lagrange multipliers may be solved for first and then used to calculate \(\tilde{q}\).
\[
\lambda = (\tilde{M}^{-1} A^T)^{-1} (\tilde{M}^{-1} (A - \mathbf{C}q - \mathbf{N} - \alpha A^T)\dot{A}),
\]
\[
\tilde{q} = \tilde{M}^{-1} (\tilde{\gamma} - \mathbf{C}q - \mathbf{N} - \alpha A^T),
\]
[1, Eqn. 6.5, 6.6].

**E. REDUCED DYNAMICS**

The reduced dynamics can be found by using the Lagrangian written in the reduced coordinates,
\[
\tilde{T}(\mathbf{y}, \dot{\mathbf{y}}) := \frac{1}{2} \dot{\mathbf{y}}^T H(h) (\mathbf{y}) (\mathbf{y}) (\mathbf{h}) \dot{\mathbf{y}} = \frac{1}{2} \dot{\mathbf{y}}^T \tilde{M} \dot{\mathbf{y}}
\]
\[
\tilde{V}(\mathbf{y}) := V(h(\mathbf{y}))
\]
\[
\tilde{L}(\mathbf{y}, \dot{\mathbf{y}}) = \frac{1}{2} \dot{\mathbf{y}}^T \tilde{M}(h(\mathbf{y})) \dot{\mathbf{y}} - \tilde{V}(h(\mathbf{y}))
\]
and following along as above, or by noting that \(H^T A^T = 0\) and working from (32),
\[
\tilde{M}(\theta) \dot{\tilde{q}} + \tilde{C}(\theta, \tilde{\theta}) \dot{\tilde{q}} + \tilde{N}(\theta, \phi) + A^T (\theta, \phi) \lambda = \gamma
\]
\[
= H^T \tilde{M} \dot{\mathbf{q}} + H^T \tilde{C} \dot{\mathbf{q}} + H^T \mathbf{N} + H^T A^T \lambda = H^T \gamma
\]
\[
= H^T \tilde{M} (\mathbf{h} + \dot{\mathbf{h}}) + H^T \tilde{C} \dot{\mathbf{h}} + H^T \mathbf{N} + 0 = H^T \gamma
\]
\[
\tilde{M} \dot{\mathbf{q}} + \tilde{C} \dot{\mathbf{q}} + \tilde{N} = \gamma
\]
leading to (35) and summarized as,
\[
\frac{d}{dt} \begin{bmatrix} \dot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -D_{\lambda} & 0 \\ 0 & D_{\dot{\lambda}} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} E_{\dot{q}} \\ 0 \end{bmatrix}
\]
(74)
To recover the Lagrange multipliers, using the pseudoinverse $A^*A^T = \mathbf{I}_{c}$ as chosen in (40),

$$
\lambda = A^*(Y - \tilde{M}\tilde{q} - \tilde{C}\tilde{y} - \tilde{N})
= A^*(\tilde{y} - (\tilde{M}\tilde{H})\tilde{y} - (\tilde{M}\tilde{H} + \tilde{C}\tilde{H})\tilde{y} - \tilde{N})
= A^*(\tilde{y} - (\tilde{M}\tilde{H} + \tilde{C}\tilde{H})\tilde{y} - \tilde{N})

\times(\tilde{y} - (\tilde{M}\tilde{H} + \tilde{C}\tilde{H})\tilde{y} - \tilde{N})
= A^*(\tilde{y} - (\tilde{M}\tilde{H} + \tilde{C}\tilde{H})\tilde{y} - \tilde{N}).
$$

Note that the other term in (40) has an interesting (though here unused) interpretation, as $(\tilde{M}\tilde{H})^*A^T = \mathbf{0}$ implies that $\tilde{q}^T \in (\tilde{M}\tilde{H})^*$, $A\mathbf{q} = \mathbf{0}$, i.e. its rows are the state velocities that imparts no contact force whatsoever. Furthermore since $(\tilde{M}\tilde{H})^*(\tilde{M}\tilde{H}) = \mathbf{I}_{c}$, such velocities also impart unit compliment momentum in the reduced system, i.e. $\mathbf{H}^T\tilde{M}\tilde{q}$ is a unit basis vector.

**F. PROOFS OF LEMMAS**

The proof of [1, Lemma 2.16],

$$
V^b_{ab} = -V^b_{ab},
$$

$$
V^h_{ab} = -A_d^b_{ab}V^h_{ba},
$$

uses the following properties of rigid transformations and skew symmetric matrices,

\begin{align*}
R_{ab} &= R_{ba}^{-1} = R_{ba}^T, \\
R^T(p)^*R &= (R^T)p^*, \quad (p)^*R = R(R^T)p^*, \\
RR^T &= -(RR^T)^T, \\
R(S)^{\lor} &= (RSR)^{\lor}, \\
(p)^*q &= -(q)^*p, \\
g^{-1}_{ab} &= (-R^T_{ab}p_{ab}, R^T_{ab}) = (p_{ba}, R_{ba}) = g_{ba}.
\end{align*}

[1, Sec. 2.2.1, 2.4] and the definitions

$$
V^a_{ab} = \begin{bmatrix} -\tilde{R}_{ab}R^T_{ab}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ R^T_{ab}\mathbf{p}_{ab} \end{bmatrix},
$$

$$
V^h_{ab} = \begin{bmatrix} R^T_{ab}\mathbf{p}_{ab} \\ (R^T_{ab}\mathbf{p}_{ab})^{\lor} \end{bmatrix},
$$

$$
A_d^b_{ab} = \begin{bmatrix} R_{ab}(p_{ab})^{\lor}R_{ab} \\ 0 \end{bmatrix},
$$

$$
V^l_{ab} = A_d^b_{ab}V^h_{ba}.
$$

[1, Eqn. 2.58–2.61] and so the second part of the Lemma is proven as,

$$
A_d^b_{ab}V^h_{ba} = -\begin{bmatrix} \mathbf{R}_{ba} & \mathbf{R}_{ba}^T \end{bmatrix} \begin{bmatrix} \mathbf{R}_{ba}^{-1}\tilde{p}_{ba} + \mathbf{p}_{ba}^T \mathbf{R}_{ba}^{-1}\tilde{R}_{ba} \\ \mathbf{R}_{ba}^{-1}\tilde{R}_{ba} \end{bmatrix}
= -\tilde{p}_{ba} + \mathbf{R}_{ba}(\mathbf{R}_{ba}^T\mathbf{p}_{ba})^{\lor}(\mathbf{R}_{ba}^T\mathbf{R}_{ba})^{\lor}
= -\left(\tilde{p}_{ba} - \tilde{R}_{ba}R^T_{ba}\mathbf{p}_{ba}\right),
$$

and thus the first part is also proven by combining this with the definition of $A_d^b_{ba}$,

$$
V^b_{ba} = A_d^b_{ba}V^h_{ba} = -V^b_{ab}.
$$

**G. VALUES FOR RHex**

Based on the specification of Section II-A.

From Section II-C, the point contacts with friction at the toes implies a projection $\pi_c := \pi$ down to the linear components, $x$ and $z$, which thus leads to a planar wrench basis of,

$$
B_{ck} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k \in \{1, 2\}
$$

(75)

corresponding to tangential and normal forces in the contact frame. The corresponding friction cone (7) is (where recall that $C_k$ is defined with the $z$ axis pointing into the ground),

$$
U_k\lambda_c := \begin{bmatrix} \pm 1 & \mu_k \end{bmatrix}\begin{bmatrix} \lambda_{ks} \\ 0 \end{bmatrix} \geq 0, \quad k \in \{1, 2\}
$$

(76)

where the sign of the coefficient on the tangent components is selected to be the opposite of the sign of $\lambda$, or alternatively both signs may be included in separate rows, and $\mu_k$ is the usual static friction coefficient.

When the body contacts the ground, the sliding contact implies a projection $\pi_s := \pi_z$ down to only the normal component, $z$, and thus has a wrench basis of,

$$
B_{bk} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad k \in \{3, 4\}
$$

(77)

and the friction cone is,

$$
U_k\lambda_c := \begin{bmatrix} -1 \end{bmatrix}\begin{bmatrix} \lambda_{ks} \end{bmatrix} \geq 0, \quad k \in \{3, 4\}
$$

(78)

The combined $U$ for all $k$ contacts is then defined as (7),

$$
U := \begin{bmatrix} U_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & U_k \end{bmatrix} \begin{bmatrix} \lambda_{c1} \\ \vdots \\ \lambda_{ck} \end{bmatrix} \geq 0.
$$

(79)

From Section II-D, the active components $a_k$ of the base kinematic constraint are shown in (41)–(43), which combine to form $a$ in each contact mode,

$$
a := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}.
$$

(80)

From Section II-E, the component of the self-manipulation grasp map for each toe contact (0 \leq 0 - \phi < \pi, k \in \{1, 2\}) is
thus,
\[ G_{s,k} = \begin{bmatrix} -\cos \phi & \sin \phi \\ -\sin \phi & -\cos \phi \end{bmatrix} \ell_k \sin \phi - p_0 \cos (\theta_k - \phi) \] (81)
while the component of the grasp map for rolling contact \((-\pi < \theta_k - \phi < 0, k \in 1, 2)\) is,
\[ G_{s,k} = \begin{bmatrix} -\cos \phi & \sin \phi \\ -\sin \phi & -\cos \phi \end{bmatrix} \ell_k \sin \phi - p_0 \cos (\theta_k - \phi) \ell_k \cos \phi - p_0 \sin (\theta_k - \phi) \] (82)
The component of the grasp map for each body contact \(k \in 3, 4\) is thus,
\[ G_{s,k} = \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix} \ell_k \cos \phi. \] (83)
A complete grasp map is then the concatenation of all active components (14),
\[ G_{s} := \left[ G_{s,1} \ldots G_{s,k} \right]. \] (84)
The hand Jacobian for legs in toe contact \((0 < \theta_k - \phi < \pi, k \in 1, 2)\) is thus,
\[ J_{h,k} = \begin{bmatrix} -\rho_k \cos (\theta_k - \phi) \\ -\rho_k \sin (\theta_k - \phi) \end{bmatrix} \] (85)
while the hand Jacobian for rolling contact \((-\pi < \theta_k - \phi < 0, k \in 1, 2)\),
\[ J_{h,k} = \begin{bmatrix} \frac{-\pi}{2} (1 + \cos (\theta_k - \phi)) \\ -\frac{\pi}{2} \sin (\theta_k - \phi) \end{bmatrix} \] (86)
and the hand Jacobian is zero for \(k \in 3, 4\). A complete hand Jacobian is then a block diagonal of all active components,
\[ J_{h} := \left[ J_{h,1} \begin{array}{cc} 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_{h,k} \end{array} \right]. \] (87)
The combined velocity constraint matrix \(A\) for each contact mode is defined from these components (17). For example the combined constraint \(A\) in the mode shown in Fig. 3 (front leg rolling, rear leg toe contact, no body contact) is,
\[ A = \left[ -J_{h} \ G_{s}^T \ R_{p2}^T \right]. \]
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REFERENCES


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