On the Coordinated Navigation of Multiple Independent Disk-Shaped Robots

C. Serkan Karagöz[†], H. Işıl Bozma[†] and Daniel E. Koditschek[‡]

[†] Intelligent Systems Laboratory
Electrical & Electronics Engineering Department
Boğaziçi University, Istanbul 34342 Turkey
http://www.isl.ee.boun.edu.tr

[‡] Electrical & Systems Engineering
School of Engineering & Applied Science
University of Pennsylvania, Philadelphia, PA 19104 USA

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Abstract

This paper addresses the coordinated navigation of multiple independently actuated disk-shaped robots - all placed within the same disk-shaped workspace. Assuming perfect sensing, shared centralized communications and computation, as well as perfect actuation, we encode complete information about the goal, obstacles and workspace boundary using an artificial potential function over the configuration space of the robots simultaneous non-overlapping positions. The closed-loop dynamics governing the motion of each (velocity-controlled) robot take the form of the appropriate projection of the gradient of this function. We impose (conservative) restrictions on the allowable goal positions, that yield sufficient conditions for convergence: we prove that this construction is an essential navigation function that guarantees collision-free motion of each robot to its destination from almost all initial free placements. The results of an extensive simulation study investigate practical issues such as average resulting trajectory length and robustness against simulated sensor noise.

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Contents

1 Introduction

This paper addresses a geometrically simplified version of coordinated motion planning [23]. A collection of disk-like robots inhabits a two-dimensional disk-shaped workspace. Each velocity-controlled[1] robot can move simultaneously with and independently of the other robots. Moreover, each has a specified goal location in which it needs to end up. The ensemble of these locations encodes the overall task. Departing from the classical coordinated motion planning paradigm in the manner of [66, 67], we further require that each robot's control strategy be reactive. By this, we mean that all motion is generated by a vector field — a function of the instantaneous ensemble of locations, parametrized (in part) by the fixed ensemble of goals that returns at each instant a direction of motion for each robot. In this reactive setting, each robot must start from its arbitrary initial placement, confront the other robots as required dynamically and eventually end up in its goal position. Reactive planners offer the usual benefits of feedback relative to the traditional open-loop planners in their sensitivity to execution time disturbances and thus promise more efficient and robust performance. Of course, improperly designed feedback schemes can cause instability, hence the central problem is to demonstrate convergence.

This paper presents a formulation of the problem nearly identical to that of [66, 67] and proposes a similarly close solution[2]. As before, we assume complete centralized information about all the robots' instantaneous positions as well as a fixed goal location assigned to each one. Again, we use this information to construct an artificial potential function and apply its gradient as a centralized controller communicated accurately and instantaneously to the fully actuated robot ensemble. However, now we offer the missing convergence proof, guaranteeing from almost every initial condition within the connected component the movement of all robots to their destinations without any collisions along the way. The coupled closed loop gradient dynamics governing the motion of the robot ensemble projects onto the coordinate slice corresponding to each individual robot a vector field sensitive to its own position as well as those of all the other robots. Although this approach is in principle applicable with complete generality to any navigation problem over a known configuration space [36, 54], and the construction for this very specific class of problems has essentially been in place for over two decades [67, 27], the present paper offers the first formal demonstration of its correctness. Analogous constructions have been shown to be correct in simpler, related versions of the problem [66, 8, 35]. But despite favorable simulation experience, the possibility of spurious local minima on which the system might get stuck has remained an open question. In summary, this paper shows for the first time that the line of reasoning and strategy originating in [54] can be extended constructively to coordinated navigation of disk-shaped robots in a disk-shaped workspace with complete information. Provided certain constraints on the allowed goal positions are satisfied, obstacle-free navigation to the goal placements from almost every initial placement of the robots lying in the connected component of the configuration space is guaranteed.

1.1 Coordinated Motion Planning

Traditionally, the coordinated motion problem has been viewed as a special case of the general open-loop motion planning problem. In this tradition, the kinematics of planning are separated from the dynamics of execution [44]. A geometric planner produces a trajectory in the joint configuration space of the ensemble of robots connecting a pre-specified initial condition to the fixed goal configuration (the total degrees of freedom are given by the sum of the individual machines') [58]. This plan is then "guarded" in real time execution by a local tracking controller. In these open-loop approaches, the focus is on developing computational geometric means that are assured of finding a path in the configuration space that does not violate any of the hypersurfaces encoding the constraints on the robots' degrees of freedom [16, 57]. Most geometric

[1]The extension of our control solution beyond this "quasi-static" or "generalized damper" [45] to the dynamical setting of a second order mechanical system (a motion where controlled forces generate changes in velocity) is almost immediate, according to the procedures discussed in [34, 33].

[2]The major advance beyond [66, 67] in this present formulation is that we can now handle a compact workspace via the imposition of an additional outer boundary as defined by Eq. 2 that makes the problem more broadly applicable but considerably harder.

approaches are based on roadmaps or cell decomposition [10, 39]. Furthermore, depending on how the planning is achieved, they are either classified as being centralized or decentralized [69]. Unfortunately, the computational complexity of the coordinated motion planning has proved to be PSPACE-hard even in two dimensional environments where only translations are allowed and when the final configuration specifying the final positions of all movable objects is known [23, 68]. This result has been viewed as a guide to calibration of problem difficulty and has led researchers to consider the more tractable, but restricted classes of the problem [58, 50, 49]. Against this backdrop, researchers have approached the problem by proposing heuristic or approximate schemes [58]. Centralized approaches propose various solutions such as transforming the problem into a series of subproblems [16], reducing the search dimensionality [61] or introducing additional constraints [59, 6]. Alternatively, in decentralized approaches, the path planner is distributed among the robots that possibly communicate [69, 5]. Intermediate problem formulations (mixing elements of centralized and distributed planning) have also been considered [62, 40, 11]. For all of these feedforward problem formulations, when there is any change in the robots' objectives or the environment, complete recalculation of paths is required. Moreover, in obvious consequence of the heuristic nature of these schemes, there is no guarantee of completeness.

We take an approach within the extreme opposite paradigm: purely feedback-based motion planning. Despite the long established guaranteed existence of such planners in general [36, 54, 8], specific algorithms with provable properties for specific problem settings have been slow to appear. A good summary account of the many heuristic vector field planners that appeared in that decade (e.g., [29, 65, 38, 30, 54, 51]) can be found in [24], and a tutorial account of the following decade's work in this vein (all of which is heuristic and suffers from possibility of local minima) can be found in [10, 39]. A major boost to the theoretical foundations of reactive planning has been contributed by the definition and formal toolbox of topological complexity [18] (which has been determined for this problem in [20, 19]).

In recent years, the construction of provably correct vector field planners has progressed along two major axes. First, a variety of general algorithmic approaches have been recently advanced by assuming the availability of a convex (e.g., cubical [55], or simplicial [42]) cellular decomposition. Notably, in [42], a smooth (C^r) global vector field is achieved by interpolating local vector fields defined over each simplex, ensuring asymptotic convergence to the goal position while guaranteeing collision avoidance. The forbidding complexity of even algebraic [57, 9] much less convex cellular decomposition in the setting of general motion planning problems must give some pause in pursuing this direction. Some preliminary work [2] suggests that the regularity of multi-body configuration spaces such as arise in this problem may render convex cellular decompositions viable for low numbers of cooperating robots - but such computations must inevitably increase geometrically with the degrees of freedom. In contrast, that same regularity permits the use of the closed form expressions we study here, entailing merely quotients of quadratic functions and their gradients - a major benefit of the global analytical approach of this paper[3].

A second direction of recent work on reactive planning has re-examined versions of the multiple disk navigation problem we treat here in response to the two decade old extension [66, 67] of the original navigation function solution to the single disk problem[54]. An excellent review of this more contemporary literature is provided in the most recent of these papers [63] and in [14] which also come the closest in their aims and methods to those of this paper. The chief difference of our work from [14] (and its extension to nonholonomically constrained disks [64]) is their focus on a partially decentralized problem version: all agents have global, instantaneous knowledge of all others' positions, but an agent's ultimate destination is known only to itself. Their navigation function has much greater complexity, apparently in consequence. Both this paper and [63] follow the original construction [67] and analysis [54] in their concern to exhibit a provably correct navigation function for multiple, fully actuated first order disk navigation under the assumption of noise-free global sensing and inter-agent communication, affording recourse to a completely centralized computation and exact, deterministic implementation of the associated gradient field as a control law. In [63], the construction departs in significant ways from that of [67], most notably by recourse to a continuous but non-differentiable navigation function, yet the pattern of analysis introduced in [54] is presented in nearly identical form, modulo the introduction of methods from nonsmooth analysis [12]. In this paper, our construction is similar to [67] with the addition that the workspace is bounded by an a priori specified radius

[3]This tradeoff between analytically intricate, computationally simple vs. analytically simple, computationally intensive global representations in reactive motion planning seems to echo a more general pattern in computational topology [15].

in which all the robots are required to remain. Furthermore, notwithstanding the major overlap with the mode of analysis introduced in [54], we are forced to depart from that pattern at certain essential junctures as explained throughout the paper. In contrast to [63], our construction is smooth on the interior of the free space but of course cannot be smooth on the (non-smooth, sub-analytic) boundary[4]. Beyond the intrinsic interest in smooth controllers articulated originally in [36], a parallel literature initiated around the same time [33] employs the lift of a navigation function as a key component of obstacle-avoiding controllers for second order plants: in some important application settings this lift will require the jacobian of the original gradient field —- for example see [1] for a very nice recent example of this approach[5] applied to the dynamical version of the present setting of multiple coordinated vehicles[6].

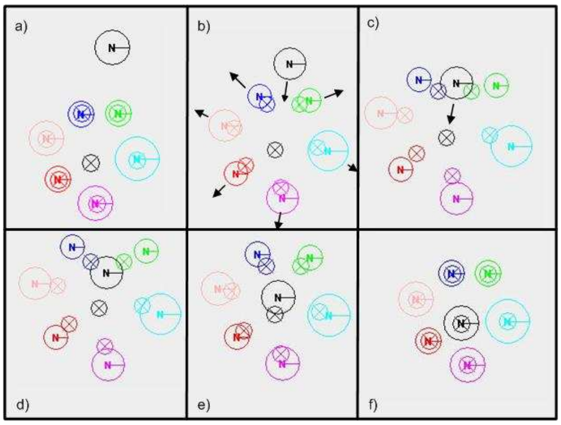

Figure 1: (a) A coordinated navigation scenario; (b)-(f) Snapshots from a task.

1.2 Motivation

Consider the scenario depicted in Figure 1a where larger circles represent individual robots and each circle with a cross represents the goal position of its specified robot counterpart. In this illustration, all robots except the top one are initially located very close to their goal positions[7]. The robots are very closely packed and need to move away from their goal positions in order to let the top robot pass through. Our feedback-based planner leads to emergent cooperation: all the robots nudge slightly away from the center enough to allow the top robot to pass through as seen in Figures 1b-d, and then move back as shown in Figures 1e-f while the top robot also homes to its goal as well. It is important to emphasize that these motions were not

[4]Please see the discussion in Section 1.4 where we address this issue by relaxing the requirement for nondegeneracy over the closed freespace to merely over the interior.

[5]Note that simple potential-dissipative controllers [34, 33] can lift an unmodified gradient field to achieve an asymptotically stable second order system with no need for further derivatives. However these simpler constructions do not achieve the same performance as required in applications such as [1], which follows a more aggressive approach originally proposed in [31], and developed in the subsequent literature [13, 52]. Intuitively, the difference in performance is akin to that between an underdamped vs. a critically damped LTI system, and the ability to regulate the transients in this manner is often quite important in practical settings.

[6]In these cases, when performance considerations motivate controlling the graph error [1]-[52], we know of no alternative to the sort of smooth construction we pursue here since even Lipschitz continuous non-smooth gradients yield unacceptable discontinuous lifts.

[7]The outer boundary which encloses an area four times that illustrated is not shown so as not to lose the desired detail of visualization.

"planned" a priori in the conventional sense. Rather, at each instant of time, each of the robots is given a velocity vector that is a function of its present position as well as the positions of all the others. The detailed path followed by the ensemble of robots emerges from their "reactive" integration of this set of cooperative vector fields. Our proof guarantees that all the robots will reach the specified ensemble of goals from any arbitrary initial configuration in the goal-connected component (excepting some set of measure zero) with the guarantee of no collisions along the way.

1.3 The Problem Statement

Consider a collection of p disk shaped robots lying on the same two dimensional workspace bounded by an outer disk. Each robot has two completely actuated degrees of freedom in this workspace, is assigned to a goal position vector and can move independently of the others. Thus each robot becomes an obstacle – possibly moving – for the remaining other robots. We assume[8] that:

(i) Each robot has a "perfect" velocity controller that can achieve exactly and instantaneously any desired bounded planar velocity command vector;

(ii) At every instant, each robot has perfect real time knowledge of its own position; and

(iii) At every instant, each robot knows exactly the sizes and the locations of all the other robots at that instant.

(iv) For all time, each robot knows exactly its own goal location as well as that of all the other robots.

If there are p individual planar robots, then let $b \in \mathbb{R}^{2p}$ denote the augmented state vector of all robots and $g \in \mathbb{R}^{2p}$ denote the augmented state vector of all goal positions. As assumed in (i), above, we consider the simplest control setting and model their change of state $\dot{b}$ according to control law: $\dot{b} = u$. As discussed above, we will set the control input, u, to be the gradient vector of an appropriate smooth map, $\varphi : \mathcal{F} \to [0, 1]$ on the *free robot configuration space* $\mathcal{F} \subset \mathbb{R}^{2p}$ (to be formally defined below) so that $u = -\nabla\varphi$. The equilibria $b(\infty)$ of this system constitute its fixed points. This task is successfully completed if $b(\infty) = g$ or successfully terminated if $b(\infty) \neq g$ (i.e., the system cannot cycle but must eventually converge to some critical point - the wrong one only from an initial condition set of measure zero [36, 54]).

1.4 Navigation Functions

Since the basin of a point attractor is a topological ball [7], and the free space is not contractible [17] there clearly cannot exist vector fields that take every point $b \in \mathcal{F}$ to the goal g. However, there is no such obstruction to smooth vector fields with a point attractor whose basin includes the connected component of the goal in $\mathcal{F}$ excluding a set of zero measure. We believe that the disadvantage of "losing the way" on an "invisible" subset of freespace is offset by the many considerable advantages that dynamical systems based motion planning enjoy, as reviewed, for example in [8], hence our interest in the following class of scalar valued functions, originally defined in [36]. A map $\varphi : \mathcal{F} \to [0, 1]$ is a *navigation function* if it is[9]:

1. Analytic on $\mathcal{F}$;
2. Admissible on $\mathcal{F}$ — that is, it attains its maximum on the boundary $\partial\mathcal{F}$.
3. Polar on $\mathcal{F}$ — that is, its unique minimum occurs at the goal configuration $g \in \mathring{\mathcal{F}}$;
4. Morse on $\mathcal{F}$ — that is, all critical points are non-degenerate;

[8] While these assumptions do not require that any information about future positions or motion be available in a given instant (beyond knowledge of the final goals), they do embody the most extreme version of centralized control with perfect information. We are pursuing in ongoing work the prospects for weakening these strong control and communications requirements without losing the theoretical convergence guarantees.

[9] Here and in the sequel we use notation from the standard literature in real analysis and point-set topology, e.g., [56].

If the negative gradient of φ is transverse on the boundary and directed inwards, all solutions of the gradient system approach the critical points where the gradient vanishes. If φ is a Morse function (critical points are non-degenerate), then critical points are isolated, and the unstable equilibria attract a set of points whose measure is zero. In particular, if g is a unique minimum of φ, then almost all points in the connected component of the goal, g, move toward it and asymptotically achieve it. Thus, an appropriately constructed φ solves the geometric path planning problem. Moreover, if φ is interpreted as an artificial potential function, then the gradient vector field leads to the automated generation of robots' control velocities. Furthermore, within certain constraints, the robots' limiting behavior is identical to that of the vector field.
We will find it convenient to relax point 4) of the definition above, and introduce the notion of an *essential navigation function*, by stipulating instead that φ be:

4) Morse on $\mathring{\mathcal{F}}$ — All interior critical points are non-degenerate;

While the freespace interior is smooth, its boundary cannot be — there arises the familiar problem of "corner points" [41] over which set the Hessian is undefined. Rather than introducing the machinery of non-smooth analysis as in [63], we simply relax the condition because it confers no advantage on the boundary. In other words, while degeneracy might possibly occur on $\partial\mathcal{F}$, no open set of initial conditions can be attracted to such critical points since φ cannot increase along the motion of $-\nabla\varphi$.

1.5 Contribution of the Paper

The main contribution of the paper is to show that our construction (Eqs. 4-5) is indeed an essential navigation function. For in the present case of disk-shaped robots all moving independently in a disk-shaped workspace, this guarantees an exact coordinated navigation algorithm that employs feedback to drive all robots to their respective goals with no collisions along the way from almost every initial configuration in the connected component of the goal. More precisely, we show that with some conservative but readily computed restrictions on the goal positions, the constructed artificial potential function can be made to be an essential navigation function – by suitable assignment of the parameters that we prescribe exactly in Theorem 1 as a function of the known problem geometry.

2 The Candidate Potential Function

2.1 Notation

We will index the collection of $p \in \mathbb{Z}^+$ robots by the set $P = \{1, \ldots, p\}$. Each robot $i \in P$ is located by its center point $b_i \in \mathbb{R}^2$, parametrized by its radius $\rho_i \in \mathbb{R}^+$ and assigned a goal position $g_i \in \mathbb{R}^2$. The state $b \in \mathbb{R}^{2p}$ of all the robots is defined as[10] $b \triangleq \sum_{i\in P} b_i \otimes e_i$, where $e_1, e_2, \ldots, e_p \in \mathbb{R}^p$ are the unit base vectors in $\mathbb{R}^p$. The aggregate goal vector $g \in \mathbb{R}^{2p}$ is defined by $g \triangleq \sum_{i\in P} g_i \otimes e_i$.

Now, define the index set of robot pairs $Q = \{(i,j)\,|i,j \in P, i < j\}$. The cardinality of Q is denoted by $q \triangleq |Q| = \binom{p}{2} = p(p-1)/2$. For all robot pairs $(i,j) \in Q$, define their distance difference $d_{ij} \in \mathbb{R}^2$ as $d_{ij} \triangleq b_i - b_j$. Note that by definition $d_{ij} = \left(I_2 \otimes c_{ij}^T\right) b$, where I_n is the n dimensional identity matrix and $c_{ij} \triangleq e_i - e_j$. The robots' pairwise relative distance is $\delta_{ij} \triangleq \|d_{ij}\|$. Similarly, their relative pairwise distance difference at the goal is $g_{ij} \in \mathbb{R}^2$ defined by $g_{ij} \triangleq g_i - g_j$. Again, by definition $g_{ij} = \left(I_2 \otimes c_{ij}^T\right) g$. Let Q^0 denote the index set of robot pairs including the workspace boundary as a *zeroth* disk, that is, $Q^0 \triangleq Q \cup \{(0,i)\,|\forall i \in P\}$.

The robots cannot overlap, so we require that:

$$\delta_{ij} \geq \rho_{ij} \triangleq \rho_i + \rho_j \qquad \forall (i,j) \in Q \tag{1}$$

[10]Here, $\otimes$ denotes the Kronecker product, where, if $A \in R^{n\times m}, B \in R^{p\times q}$, then $A \otimes B \in R^{np\times mq}$ with an ij^{th} block of size $p \times q$ specified by $a_{ij}B$.

Differing from the original construction [67], the workspace is bounded by radius $\rho_0 \in \mathbb{R}^+$, hence each robot i must remain inside a disk of radius $\rho_{0i} \triangleq \rho_0 - \rho_i$, that is:

$$\|b_i\| \le \rho_{0i} \qquad \forall i \in P \tag{2}$$

The free robot configuration space $\mathcal{F}$, is defined as the subset of robot positions in $\mathbb{R}^{2p}$ which satisfy (1) and (2).

$$\mathcal{F} \triangleq \left\{ b \in \mathbb{R}^{2p} \,|(\forall i \in P, \|b_i\| \le \rho_{0i}) \wedge (\forall (i,j) \in Q, \delta_{ij} \ge \rho_{ij}) \right\} \tag{3}$$

In other words, we are concerned with the closure of non-contacting placements. For the reader's convenience, we have included in Appendix A a summary table of the principal notation introduced in this section as well as in Section 3.

2.2 Construction

Following the recipes in [54] and [67], the candidate function $\varphi : \mathcal{F} \to [0,1]$ is constructed as the composition:

$$\varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) \tag{4}$$

The function $\hat{\varphi} : \mathcal{F} \to [0,\infty)$ encodes the goal point and the obstacles of all the robots using the quotient of two functions $\gamma : \mathcal{F} \to [0,\infty)$ and $\beta : \mathcal{F} \to [0,\infty)$:

$$\hat{\varphi}(b) \triangleq \frac{\gamma^k(b)}{\beta(b)} \quad k \in \mathbb{Z}^+ \tag{5}$$

The numerator $\gamma(b) \triangleq (b-g)^T(b-g)$ encodes the Euclidean distance from the goal. The denominator encodes the distance from freespace boundary and is defined as $\beta(b) \triangleq \prod_{(i,j)\in Q^0} \beta_{ij}(b)$, where $\forall (i,j) \in Q$, $\beta_{ij}(b) = \delta_{ij}^2 - \rho_{ij}^2$ and $\forall i \in P$, $\beta_{0i}(b) = \rho_{0i}^2 - \|b_i\|^2$. The freespace boundary $\partial\mathcal{F}$ is the zero level set of $\beta^{-1}(0)$ and entails robots touching each other or the workspace boundary. The parameter k is a design parameter that determines the relative weight of these two terms. As will be seen in the sequel, k plays a critical role in ensuring that the function φ is an essential navigation function.

Since $\hat{\varphi}$ blows up on $\partial\mathcal{F}$, it is not admissible. In order to make $\hat{\varphi}$ admissible, it is squashed by the function $\sigma : [0,\infty] \to [0,1]$, defined by $\sigma(x) = \frac{x}{1+x}$. The resulting function becomes admissible but the goal point g is a degenerate critical point. In order to restore the goal point's non-degeneracy, the sharpening function $\sigma_d : [0,1] \to [0,1]$ is applied, given by $\sigma_d(x) = x^{1/k}$. Thus, the resulting function φ becomes admissible and has non-degenerate minimum at $b = g$.

2.3 Restriction on Goal Locus - g

Our proof requires a few natural restrictions on allowable goal positions g. Similar constraints have been introduced for the different, but related versions of the problem in earlier studies. For example, to retain the geometry as well as the topology of a "sphere world" in the freespace, the robot is defined as a point mass object in [54]. In [53], the minimal gap between any pair of obstacles is restricted to be larger than the diameter of the robot and the mated object. Our assumptions constrain how closely the robots may be commanded to locate finally with respect to each other and to the outer boundary in their goal positions. The goal g is allowed to be chosen from a subset of $\mathcal{F}$ subject to two assumptions given in the sequel.

First, it is helpful to introduce a classification of the freespace that is ϵ away from the boundary by defining a notion of robot neighborhoods and their associated "clusters". Much past research on the coordination of multiple robots has encountered the need to decompose a neighborhood of the configuration space boundary into a hierarchy of variously arranged clusters, the earliest mention of this idea known to us having been contributed in [43]. Most closely related to our present formulation of robot neighborhoods and their associated "robot clusters" is the introduction in [14] of a family of "relation verification" functions whose members roughly correspond to each of these different possible "clusters" and, like ours, are indexed over all possible partitions of the set of agents. The cardinality of the collection of partitions grows super-exponentially in the cardinality of the base set. Fortunately, in our problem formulation, these clusters do not enter into the controller itself but only play a role in the analysis of correctness, specifically in Prop. 3.6. In contrast, likely because of their focus on the more challenging decentralized version of the problem, the obstacle term in the navigation functions that generate the controllers of [60] explicitly include each of these super-exponentially many factors.

Robot Neighborhoods: Let $\varepsilon \in \mathbb{R}^+$ be an arbitrarily small design parameter that determines robot neighborhoods. In particular, its value is set as to ensure that

$$0 < \varepsilon < \rho'' \text{ where } \rho'' = \min_{i \in P} \{\rho_{0i}\} \tag{6}$$

$\forall i \in P$, define an ε-neighbor set $N_\varepsilon(b,i) \subseteq P$ to be the indices of its closest neighbors – namely $N_\varepsilon(b,i) \triangleq \{j \in P \,|\, 0 < \beta_{ij}(b) \leq \varepsilon\}$ [11]. Now, recursively define the nth ε-neighbor sets $N_\varepsilon^n(b,i) \subseteq P$ as $N_\varepsilon^0(b,i) := \{i\}$ and

$$N_\varepsilon^{n+1}(b,i) \quad := \quad \left(\bigcup_{j \in N_\varepsilon^n(b,i)} N_\varepsilon(b,j) \right) \bigcap_{l \leq n} \bar{N}_\varepsilon^{\,l}(b,i)$$

According to this definition, each $(n+1)^{st}$ neighbor of robot i is ε close to some n^{th} neighbor of robot i, but no closer - i.e. it is not ε close to any $(n-1)^{st}$ neighbor. The process is stopped when $N_\varepsilon^{n+1}(b,i) = \emptyset$.

Robot Clusters: Specify a partition $\{P_1(b), ..., P_{s(b)}(b)\}$ where $P_i(b) \in 2^P$ and $s(b)$ is the number of cells in this partition using a recursively defined function $P_i(b)$ and its complementary function $\bar{P}_i(b)$ as follows: The base step is given by

$$r_1 := 1, \qquad P_1(b) := \bigcup_{j=0}^{p-1} N_\varepsilon^j(b, r_1)$$

and the recursive step is given by

$$r_{n+1} := \min \left(\bigcap_{j \leq n} \bar{P}_j(b) \right), \qquad P_{n+1}(b) := \bigcup_{j=0}^{p-1} N_\varepsilon^j(b, r_{n+1})$$

stopping when $\bigcap_{j \leq n} \bar{P}_j(b) = \emptyset$. At each configuration this partition divides up the robots into distinctive clusters of "closest neighbors". For convenience, we wish to keep track of the partition cell index set $S(b) \triangleq \{i \in P \,|\, i \leq s(b)\}$. We verify that $\coprod_{i \in S(b)} P_i(b)$ [12] is a partition over the robot index set in Lemma B.3.

Next, consider an arbitrary cluster $P' \subseteq P$ containing at least two elements $|P'| \geq 2$. Associate with it $\mathcal{F}' \subseteq \mathcal{F}$

$$\mathcal{F}' \triangleq \{b \in \mathcal{F} \,|\, \exists i \in S(b), P_i(b) = P'\}$$

Let $Q' \subseteq Q$ be the corresponding pair index set defined as:

$$Q' \triangleq \{(i,j) \in Q \,|\, i,j \in P'\} \tag{7}$$

Finally define two derived problem parameters Λ' and Λ'' defined as follows:

$$\Lambda' \triangleq \max_{b \in \mathcal{F}'} \left\{ \sum_{(i,j) \in Q'} \delta_{ij} + \frac{2|P'| - 2}{\rho'} \left\| \sum_{n \in P'} J\left(b_n - \bar{g}'\right) \otimes e_n \right\|^2 \right\} \tag{8}$$

and

$$\Lambda'' \triangleq \max_{b \in \mathcal{F}', i = \arg\max_{n \in P'} \|b_n\|} \left\{ \sum_{j \in P'} \delta_{ij} \right\} \tag{9}$$

[11] We will denote by an overbar the complementary index set so that, for example, $\bar{N}_\varepsilon(b,i) = P - N_\varepsilon(b,i)$.

[12] The symbol $\coprod$ denotes disjoint union [41].

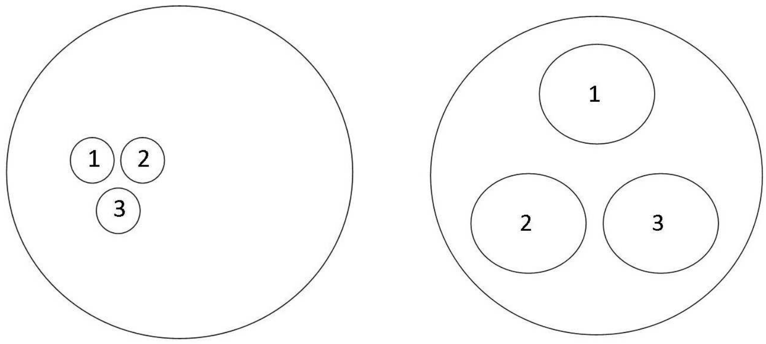

Figure 2: Sample goal configurations. Left: A goal configuration failing Assumption 1. Right: A goal configuration failing Assumption 2.

where $\rho' \triangleq \min_{(i,j)\in Q}\{\rho_{ij}\}$, $J \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the 90° planar rotation matrix and $\bar{g}' \triangleq \frac{1}{|P'|}\sum_{i\in P'} b_i$ is the centroid of the robots in the cell P'.

With these definitions in place we are now ready to introduce the assumptions that restrict the allowable goal configurations. The first states that for any robot cluster, the goal positions of the robots in this group are separated from each other by a value of Λ'. This term is the maximum value of a function of the pairwise distances between the robots and their centroid.This maximization is over any cell containing these robots. Figure 2(Left) shows a workspace configuration containing three robots (big circles) which might block the way of each other while navigating to their goal positions (dark points) since the goal points are not separated enough according to Assumption 1.

Assumption 1 $\forall P' \in 2^P$ *where* $|P'| \geq 2$

$$\sum_{(i,j)\in Q'} \|g_{ij}\| > \Lambda'$$

where Q' and Λ' are calculated according to the Eq. 7-8. □

The second assumption states that for any robot group, each goal position is not allowed to be located closer to the workspace boundary more than a value of Λ''. This term is the maximum value of the sum of the distances between the closest robot to the workspace boundary and the other robots. This maximization is over any cell containing these robots. Figure 2(Right) illustrates a disconnected free space as the robot radii are too large with respect to that of the workspace which is an infeasible goal position according to Assumption 2.

Assumption 2 $\forall P' \in 2^P$ *where* $\|P'\| \geq 2$

$$|P'|\sqrt{\rho''^2 - \varepsilon} - \Lambda'' - \sum_{i\in P'} \|g_i\| > 0$$

where Λ'' is calculated according to the Eq. 9. □

These assumptions, introduced to facilitate the proof as remarked above, are sufficient for the desired result, but involve bounds that have proven to be conservative in the simulations. For example, it seems clear that they guarantee a completely connected freespace, but the dependence of the homotopy type of $\mathcal{F}$ (including the conditions for its connectedness) on the disk radii is a delicate issue of great importance — indeed touching on such longstanding questions as the ancient sphere packing problem[13] [3] — whose

[13]For example, authors of [3] point out that determining conditions on the disk radii yielding a non-empty free space (e.g.

characterization goes far beyond the scope of the present paper. Nonetheless, for formal guarantees to hold, the goals would need to satisfy the two assumptions and the tuning parameter k, would indeed need to be set as a function of these bounds.

3 The Candidate is an Essential Navigation Function

3.1 Statement of Main Theorem

If φ is a navigation function, then its associated gradient field automatically generates velocity control policies for each of the robots under whose joint influence they all achieve the desired goal, g, from almost all initial conditions in its connected component of the freespace with the guarantee of no collisions along the way [25].

Theorem 1 *For any goal g satisfying assumptions 1 and 2, there exists a positive integer $K^* \in \mathbb{Z}^+$ such that for every $k > K^*$, the real-valued function,*

$$\varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left(\frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)} \right)^{1/k} \tag{10}$$

is an essential navigation function.

Proof: By definition, φ is analytic and admissible on $\mathcal{F}$. By Proposition 3.1, assumptions 1 and 2 imply that there exists a positive integer $\mathrm{K} \in \mathbb{Z}^+$ such that for every $k > \mathrm{K}$, φ is polar in $\mathcal{F}$. By Proposition 3.2, assumptions 1 and 2 imply that there exists a positive integer $\mathrm{N} \in \mathbb{Z}^+$ such that for every $k > \mathrm{N}$, φ is Morse on $\mathring{\mathcal{F}}$. Taking $K^* = \max\{\mathrm{K}, \mathrm{N}\}$, the result thus follows. $\square$

3.2 Proof of Correctness

Consider the partition of the free configuration space $\mathcal{F}$ into five disjoint subsets - following a line of reasoning inspired by that of [54]:

1. the goal point $\{g\}$
2. the boundary of the free space $\partial\mathcal{F} = \beta^{-1}(0)$
3. the set near the outer boundary $\mathcal{F}_0(\varepsilon) = \{b \in \mathcal{F} \,|\, \exists i \in S(b), \exists j \in P_i(b), 0 < \beta_{0j}(b) \le \varepsilon\} - (\{g\} \cup \partial\mathcal{F})$
4. the set near the internal obstacles $\mathcal{F}_1(\varepsilon) = \{b \in \mathcal{F} \,|\, \exists i \in S(b), |P_i(b)| \ge 2\} - (\{g\} \cup \partial\mathcal{F} \cup \mathcal{F}_0(\varepsilon))$
5. the set away from the obstacles $\mathcal{F}_2(\varepsilon) = \mathcal{F} - (\{g\} \cup \partial\mathcal{F} \cup \mathcal{F}_0(\varepsilon) \cup \mathcal{F}_1(\varepsilon))$

Note that because the goal is held away from $\partial\mathcal{F}$, ε is a design parameter as stated in Section 2.3. Let $\mathcal{C}_\varphi \stackrel{\triangle}{=} \{b \in \mathcal{F} \,|\, \|D\varphi(b)\| = 0\}$ denote the set of critical points of the function φ. Let $T : \mathcal{F} \to 2^{Q^o}$ denote the pair touching function – that is

$$T(b) \stackrel{\triangle}{=} \{(i,j) \in Q \,|\, \delta_{ij} = \rho_{ij}\} \bigcup \{(0,i), i \in P \,|\, \|b_i\| = \rho_{0i}\}$$

The following proposition shows the absence of the local minima of function φ.

the smallest radius precluding any free placement of movable uniform disks) restates the sphere packing problem in a bounded region. See [22] for a nice overview of the history of this problem which, as he shows, stretches back at least a millennium prior to Kepler's famous conjecture in 1611 [28].

Proposition 3.1 *For any free robot configuration space $\mathcal{F}$ constrained by Assumptions 1 and 2, there exists a positive integer $\mathrm{K} \in \mathbb{Z}^+$ such that for every $k > \mathrm{K}$, the real-valued function,*

$$\varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left(\frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)}\right)^{1/k} \tag{11}$$

has unique minimum point at g, that is, φ is polar on $\mathcal{F}$.

Proof: The polarity of φ is analyzed in each subset of $\mathcal{F}$. Note that the functions φ and $\hat{\varphi}$ have the same critical points with the same type (minimum, maximum or a saddle) except at $\partial\mathcal{F}$.

1. By definition, $\varphi(g) = \frac{\gamma(g)}{(\gamma^k(g)+\beta(g))^{1/k}}$. Taking the gradient $\nabla\gamma(b) = 2(b-g)$ and noting that $\gamma(g) = 0$ and $\nabla\gamma(g) = 0$,
$$\begin{aligned}\nabla\varphi(g) &= \tfrac{1}{(\gamma^k(g)+\beta(g))^{2/k}} \times \\ &\left(\left(\gamma^k(g) + \beta(g)\right)^{1/k} \nabla\gamma(g) - \gamma(g)\nabla\left(\gamma^k(g) + \beta(g)\right)^{1/k}\right) \\ &= 0\end{aligned}$$
Then g is a critical point of φ. Since $\gamma(g) = 0$, $\varphi(g) = 0$. Furthermore, by construction, $\varphi : \mathcal{F} \to [0, 1]$, then g is a minimum point of φ.

2. Next, consider φ on $\partial\mathcal{F}$. By definition, at least two robots must touch to each other or one robot must touch to the workspace boundary. Partition $\partial\mathcal{F} = \{b \in \partial\mathcal{F} : |T(b)| = 1\} \cup \{b \in \partial\mathcal{F} : |T(b)| > 1\}$. There are no critical points in $\{b \in \partial\mathcal{F} : |T(b)| = 1\}$ by Proposition C.1 given in Section C.1. The critical points in $\{b \in \partial\mathcal{F} : |T(b)| > 1\}$ are maxima by Proposition C.2.

3. $\hat{\varphi}$ has no critical points in $\mathcal{F}_0(\varepsilon)$ by Proposition C.3 - which asserts that for a given design parameter ε, there exists a lower bound on the parameter k, $\mathrm{K}_3(\varepsilon) > 0$, such that, if $k > \mathrm{K}_3(\varepsilon)$, then $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_0(\varepsilon) = \emptyset$.

4. The critical points in $\mathcal{F}_1(\varepsilon)$ are not minima by Proposition C.4 – which asserts the following: For a given design parameter ε, there exists a lower bound on the parameter k, $\mathrm{K}_2(\varepsilon) > 0$, such that, if $k > \mathrm{K}_2(\varepsilon)$ then $\hat{\varphi}$ has no minimum in any set $\mathcal{F}_1(\varepsilon)$.

5. $\hat{\varphi}$ has no critical points in $\mathcal{F}_2(\varepsilon)$ by Proposition C.5 – which asserts that for a given design parameter ε there exists a lower bound on the parameter k, $\mathrm{K}_1(\varepsilon) > 0$, such that if $k \geq \mathrm{K}_1(\varepsilon)$ then $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_2(\varepsilon) = \emptyset$.

The proof of Proposition 3.1 is completed by choosing lower bound $\mathrm{K} > 0$ on the parameter k as follows,

$$\mathrm{K} = \max\{\mathrm{K}_1(\varepsilon), \mathrm{K}_2(\varepsilon), \mathrm{K}_3(\varepsilon)\} \tag{12}$$

□

Non-degeneracy, the Morse property, is established by the next result, Proposition 3.2.

Proposition 3.2 *For any free robot configuration space $\mathring{\mathcal{F}}$ subject to Assumptions 1 and 2 and for a given design parameter ε, there exists a positive integer $\mathrm{N}(\varepsilon) \in \mathbb{Z}^+$ such that for every $k > \mathrm{N}(\varepsilon)$, the real-valued function,*

$$\varphi(b) = \sigma_d \circ \sigma \circ \hat{\varphi}(b) = \left(\frac{\gamma^k(b)}{\gamma^k(b) + \beta(b)}\right)^{1/k} \tag{13}$$

has non-degenerate critical points, that is, φ is Morse in $\mathring{\mathcal{F}}$.

Proof: The function φ is analyzed in each disjoint region of $\mathring{\mathcal{F}}$.

1. The goal point g is a non-degenerate minimum point by Proposition C.6.

2. There are no critical points in $\mathcal{F}_0(\varepsilon)$ by Proposition C.3

3. By Proposition C.7, there exists a lower bound $\mathrm{N}(\varepsilon) > 0$ on the parameter k such that if $k > \mathrm{N}(\varepsilon)$, then $D^2\hat{\varphi}$ restricted to $\mathcal{F}_1(\varepsilon)$ is non-singular.

4. There are no critical points in $\mathcal{F}_2(\varepsilon)$ by Proposition C.5.

If the parameter k is chosen accordingly, the result follows. □

Proposition 3.1 and Proposition 3.2 follow mostly a line of reasoning similar to their counterparts in [54]. However, they also depart from the respective analysis. First, in Proposition 3.1, we define partitions over the robot index set and use the robot clusters to "find" the unstable tangent direction. Second, Proposition 3.2 invokes Proposition C.7 wherein we depart necessarily from the approach taken in [54]. In that problem setting, every saddle is associated with two complementary subspaces where the Hessian matrix is sign-definite with the corresponding negative and positive cones explicitly revealed by computation [36]. In contrast, the present problem introduces a configuration space of dimension $2p$ (with $p > 1$), which is known to have nonzero Betti numbers [46] for every intermediate dimension [3]. Hence, according to the Morse inequalities [48], there must now be saddles of every index and the hope of explicitly revealing the corresponding positive and negative cones of each different type seems hopeless. Instead, here we abandon that geometric approach and instead focus directly on satisfying algebraic conditions for nonsingularity by appeal to notions of diagonal dominance. Specifically, we use a theorem to this effect by Sherman, Morrison and Woodbury [21] along with some related results in linear algebra [60].

4 Simulations

We now report on simulations of the flows associated with the construction to suggest the nature and quality of the motion planning resulting from the artificial potential function φ. A workspace tightness measure *tight* is defined as:

$$tight = \frac{100}{\log_{10}\left(\prod_{(i,j)\in Q} \|g_{ij}\|^2 - \rho_{ij}^2\right)}$$

Note that this measure of tightness captures the difficulty of the task. The closer the robots need to be packed together the more careful and precise the robots have to be in their movements. We will summarize performance by means of the measures originally introduced in [67]. The first performance measure is the normalized robot path length measure *nrl* which is the total distance traveled by the robots normalized by the sum of the Euclidean distances between initial and final positions of the robots,

$$nrl = \frac{\sum_{i\in P} \int_0^{t_f} \|\dot{b}_i(t)\| dt}{\sum_{i\in P} \|b_i(0) - g_i\|}$$

Here, t_f denotes the duration of a simulation, $b_i(t)$ denotes the position vector of robot i at time t and $b_i(0)$ denotes the initial position of robot i. The second measure is the design parameter k of function φ. Recall in case of accurate positional data, the robots are ensured of moving without any collisions along the way.

4.1 Circular Formations

We first study a problem involving six robots and five different randomly chosen goal configurations of circular formations with increasing tightness as shown in Figure 3. Figure 4(left) shows the variation of *nrl* as a function of goal tightness measure *tight*. In this graphic, each bar represents the mean and the standard deviation of 30-40 sample runs with random initial configurations. k is taken to be 60. The effect of k is discussed in the following section. Unlike [67], we observe that the general trend and the deviation of *nrl* values increase with increasing workspace tightness. This result is expected since the closer the robots need to pack together, the more times will encounter each other, thus requiring longer paths that move around each other in order to reach their goal positions. It is seen that in the most complex workspace, path length is on average 1.25 - 25 percent longer than the (typically infeasible) Euclidean straight line between initial and final configurations. In the easiest workspace, this value decreases to 1.08.

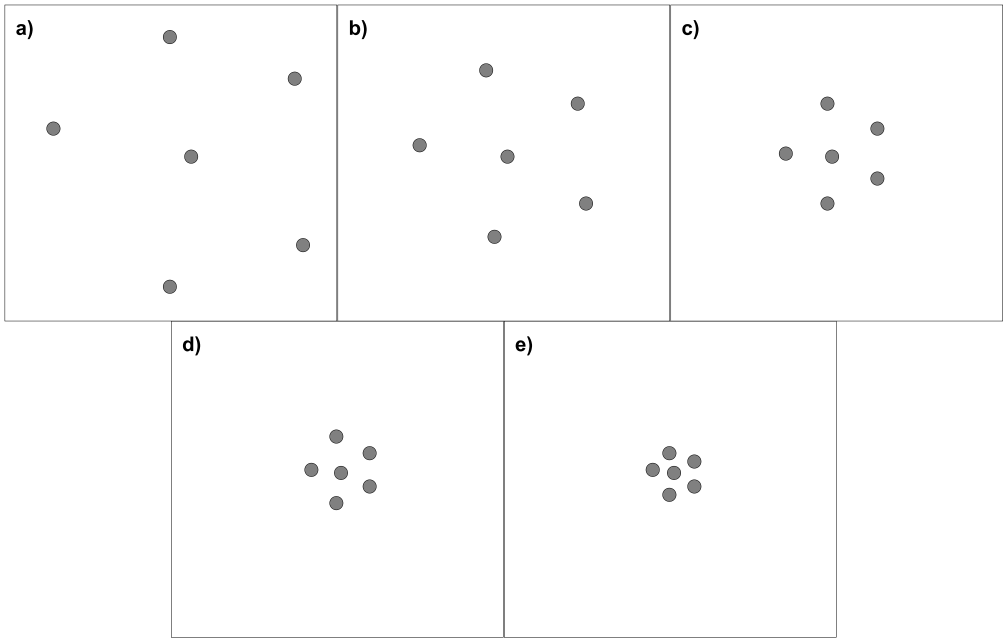

Figure 3: Circular formations of increasing tightness: a) $tight = 2.44$, b) $tight = 2.63$, c) $tight = 2.87$, d) $tight = 3.30$ and e) $tight = 3.45$.

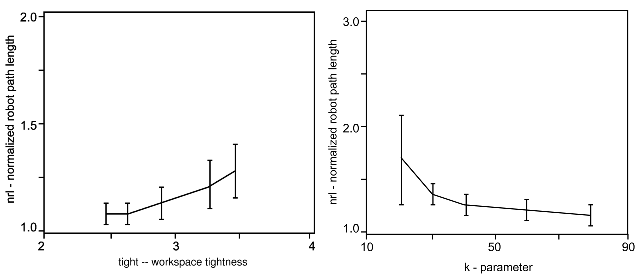

Figure 4: Left:Normalized robot path length vs workspace tightness for circular formations; Right: Normalized robot path length vs. k.

Figure 4(right) shows the dependence of nrl values on k parameter. The graphic presents the mean and the standard deviation values of 30-40 sample runs for the goal configuration given in Figure 3 and starting from random initial configurations. It is observed that the general trend of nrl values agree with those presented in [67] and decreases with the increasing k parameter.

This result can be attributed to these facts:

1. For small k values, in the constructed potential function, the term for obstacle avoidance dominates. The robots attempt to increase their proximity to nearby robots as much as possible. Consequently, the paths taken by the robots get longer. Still, the maximum mean nrl value is 1.68 when $k = 20$. Furthermore, the moving task is not accomplished for k values smaller than 20 in the simulations starting from some initial configurations. This fact is expected since there is a lower bound on k for convergence to the goal positions.

2. For large k values, the robots are concerned with pointing towards their goal positions rather than avoiding each other. In this case, a robot may try to pass through the spaces between the other robots which are only 1-2 cm larger than its diameter. Therefore, the paths taken by the robots become shorter.

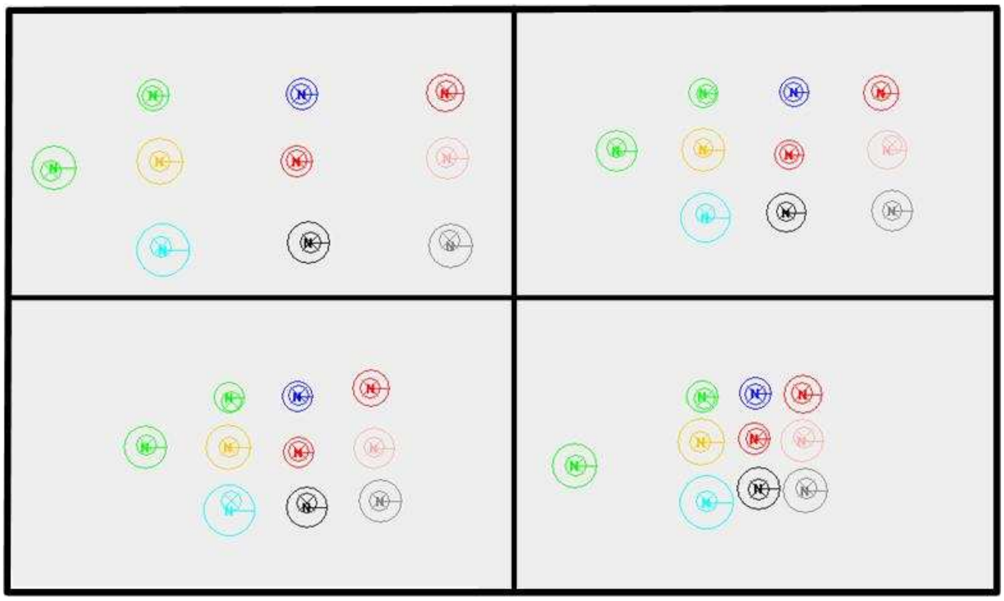

Figure 5: Array-like formations of increasing tightness: a) $tight = 0.504$, b) $tight = 0.538$, c) $tight = 0.569$, d) $tight = 0.611$.

4.2 Array-like Formations

We next study a problem involving ten robots and four different randomly chosen goal configurations of array-like formations with increasing tightness as shown in Figure 5. The variation of nrl with respect to goal tightness is as shown in Figure 6. Again, it is observed that with increased tightness, there is a tendency for the path lengths traveled by the robots to increase as well. In this simulation, we then consider the tightest goal and assume that sensor measurements are subject to noise[14]. The noisy state observations $\hat{b}_i$ are generated as

$$\hat{b}_i = b_i + \eta_s$$

where η_s represents the position measurement noise. It is assumed to be Gaussian $\eta_s \sim N(0, \Sigma_s)$ where the covariance Σ_s are known. In our simulations, different noise levels are considered: Low($\sigma = 0.1$), moderate ($\sigma = 0.5$) and high ($\sigma = 1$). Figure 7 shows nrl vs $\sigma-$ where it is observed that although nrl increases dramatically, the tasks still can be completed. However, it should be noted that with higher levels of noise, the probability of collisions between the robots increase as expected since there is a discrepancy between where each robot is actually and where it thinks it is. Let us note that in this case, the performance of robots can be improved by resorting to state estimation methods as has been shown in a different, but related task of parts' moving [4].

4.3 Random Goal Positions

Finally, we consider randomly positioned goal locations of varying tightness for robot populations of 20, 30 and 40 as seen in Fig. 8. The variation of nrl with respect to the number of robots is as given in Fig. 9

[14]With a strict, smooth (essentially) global Lyapunov function in place, standard results immediately yield local (in this setting: away from interior saddles and boundary) persistence: e.g., small sensor or model noise results in controllably small errors [32, 46, 48], or alternatively, for statistical disturbance models, integral formulations yield analogous persistence results [25]. Of course, these standard arguments are generally very conservative, and it is of interest to see how well such formal disturbance immunity properties translate practically in particular instances. Note that we we have not addressed formally the global version of this question (e.g., just how "close" one can come to the interior saddles or obstacles while maintaining guarantees), but these numerical results give the reassuring suggestion that the controller remains reasonably robust relative to small disturbances over large volumes in the freespace, including regions close to the boundary.

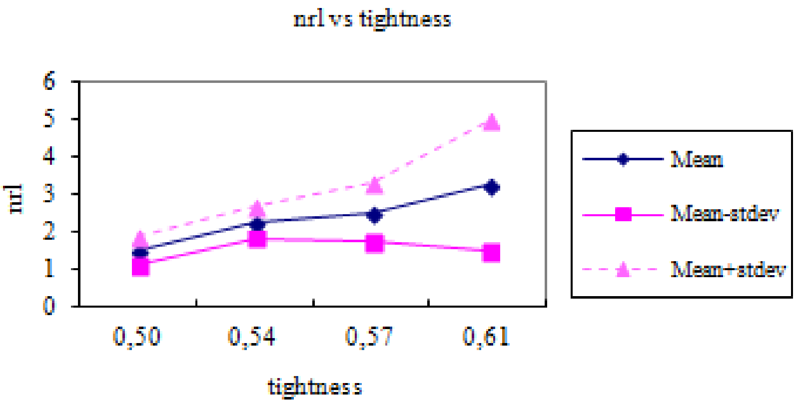

Figure 6: Normalized robot path length vs. workspace tightness in array-like formations.

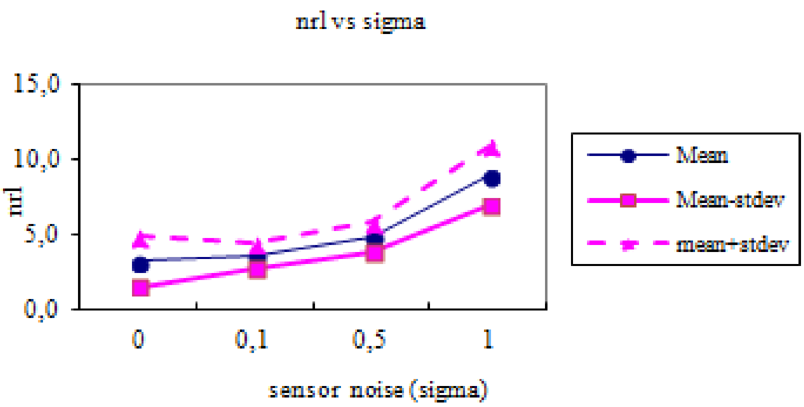

Figure 7: Normalized robot path length vs. noise σ in array-like formations.

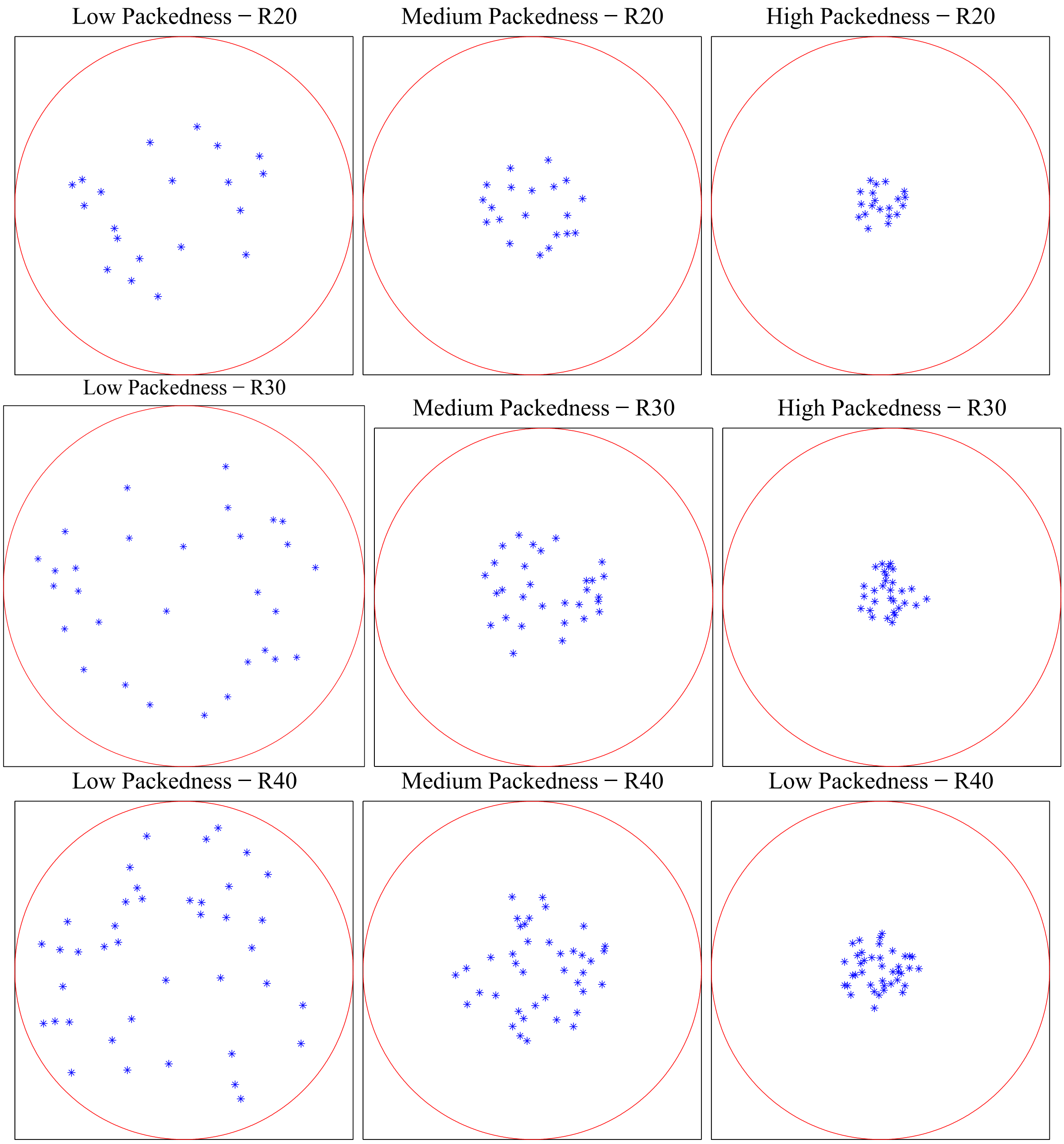

Figure 8: Random goal positions for varying packedness (low, medium, high) and for varying population (20,30,40) robot teams.

where the results are average values for 20 runs with random initial positions. It is observed that increase in the number of robots does not affect nrl much.

Finally, despite a large number of numerical experiments with goals, $g \in \mathcal{F}$ that violate Assumptions 1) and 2) of Section 2.3 conditions we have not been able to find goal configurations that are not attainable, bolstering our strong sense that these assumptions, while convenient to our proof, are pessimistically conservative and not necessary for the desired result. Numerous successful simulations run on very "tight" goal configurations certainly belie their difficulty and we suspect that only very specific goal "shapes" may give trouble. Provided that k is set high enough and numerical overflow/underflow problems are eliminated, the goals have always proven to be attainable. However, even in the worst case, if some goals "tight" enough to

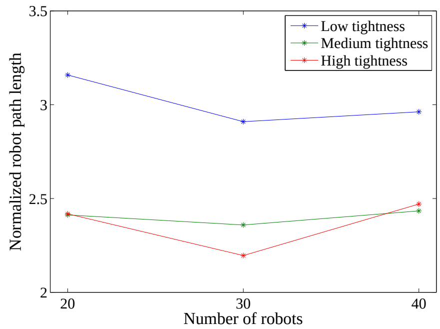

Figure 9: Normalized robot path length vs. number of robots in random goal positions.

violate Assumptions 1) and 2) do not yield a successful navigation function, our construction (4) gives rise to safe (guaranteed no collisions) non-degenerate gradient systems which have only isolated point attractors. Hence "blocked" initial conditions would reach unacceptable equilibrium positions rather than exhibiting oscillatory (some more exotic, undetectable) behavior.

5 Conclusion

This paper extends the navigation function methodology [36] to the coordinated navigation of independent disk-shaped robots moving in a disk-shaped planar workspace as first proposed over two decades ago [67]. Intuitively, the source of difficulty that characterizes this problem arises because each robot becomes a dynamic obstacle for the remaining robots. Since this is a real time dynamical systems based planner, there can be no a priori knowledge of robots' trajectories. However, by making assumptions i) - iv) in Section 1.3, we adopt the framework of encoding complete information about the goal, dynamic obstacles and workspace boundary. The main contribution is to establish that our proposed construction is indeed an essential navigation function - namely it satisfies the properties 1) - 4) listed in Section 1.4. The analysis yields closed-form expressions that depend on the goal configuration and the k parameter of this construction. First, lower bounds constrain the allowable goal proximity of among robot pairs as well as to the workspace boundary to be "reasonable". Next, suitable parameter values are found sufficient to ensure the construction indeed holds the required properties. As a consequence of its defining properties, the gradient field resulting from an essential navigation function yields a flow guaranteed to bring almost every initial condition in the connected component to the goal with no collision along the way. The recourse to an online feedback based planner lends robustness against the unanticipated changes in workspace configuration (state stability) and inevitable sensor and actuator inaccuracies (structural stability). Even if disk-shaped robots treated here constitute a very small portion of the general coordinated navigation problem of arbitrary robots in arbitrary workspaces, we expect that this construction will advance the design of artificial potential functions for scenarios that are progressively more realistic respecting geometry, actuation, sensing and distributed information.

Appendices

A Definitions

This Section presents a summary of the most commonly used the definitions in the paper. The third column indicates place of first introduction.

$p \subset \mathbb{Z}^+$	The number of robots	Section 2.1

$P = \{1, \ldots, p\}$	Robot index set	Section 2.1
$b_i \in \mathbb{R}^2$	Center of robot i	Section 2.1
$\rho_i \in \mathbb{R}^+$	Radius of robot i	Section 2.1
$g_i \in \mathbb{R}^2$	Goal of robot i	Section 2.1
$e_1, e_2, \ldots, e_p$	Canonical orthonormal basis vectors in $\mathbb{R}^p$	Section 2.1
$b \in \mathbb{R}^{2p}$	$\triangleq \sum_{i \in P} b_i \otimes e_i$	Section 2.1
$g \in \mathbb{R}^{2p}$	$\triangleq \sum_{i \in P} g_i \otimes e_i$	Section 2.1
ρ_{ij}	$\triangleq \rho_i + \rho_j$	Section 2.1
Q	$\triangleq \{(i,j) \mid i,j \in P, i < j\}$	Section 2.1
Q^0	$\triangleq Q \cup \{(0,i) \mid \forall i \in P\}$	Section 2.1
I_n	n dimensional identity matrix	Section 2.1
c_{ij}	$\triangleq e_i - e_j$	Section 2.1
$d_{ij} \in \mathbb{R}^2$	$\triangleq= b_i - b_j \ = \ \left(I_2 \otimes c_{ij}^T\right) b$	Section 2.1
δ_{ij}	$\triangleq \lVert d_{ij} \rVert$	
$g_{ij} \in \mathbb{R}^2$	$\triangleq g_i - g_j \ = \ \left(I_2 \otimes c_{ij}^T\right) g$	Section 2.1
$\gamma(\mathrm{b})$	$\triangleq (b-g)^T(b-g)$	Section 2.2
$\beta_{ij}(b)$	$\triangleq \delta_{ij}^2 - \rho_{ij}^2 \ = \ \left\lVert \left(I_2 \otimes c_{ij}^T\right) b \right\rVert^2 - \rho_{ij}^2$	Section 2.2
$\beta_{0i}(b)$	$\triangleq \rho_{0i}^2 - \lVert b_i \rVert^2 \ = \ \rho_{0i}^2 - \left\lVert \left(I_2 \otimes e_i^T\right) b \right\rVert^2$	Section 2.2
$\bar{\beta}_{ln}$	$\triangleq \prod_{(i,j) \in Q^0}^{(i,j) \neq (l,n)} \beta_{ij}$	Section 2.2
$\beta(b)$	$\triangleq \prod_{(i,j) \in Q^0} \beta_{ij}(b)$	Section 2.2
$\hat{\varphi}$	$\triangleq \frac{\gamma^k}{\beta}$	Section 2.2
$\mathcal{C}_\psi$	The set of critical points of ψ	Section 3.2
L_{0i}	$\triangleq -\frac{1}{\sqrt{\beta_{0i}}}\left(I_2 \otimes e_i^T\right), \forall i \in P$	Appendix C
L_{ij}	$\triangleq \frac{1}{\sqrt{\beta_{ij}}}\left(I_2 \otimes c_{ij}^T\right), \forall (i,j) \in Q$	Appendix C
$L_0 \in \mathbb{R}^{2p} \times \mathbb{R}^{2p}$	$\triangleq \left[L_{01}^T \ldots L_{0p}^T\right]$	Appendix C
$L_1 \in \mathbb{R}^{2p} \times \mathbb{R}^{2q}$	$\triangleq \left[L_{12}^T \ldots L_{p-1,p}^T\right]$.	
$L \in \mathbb{R}^{2p} \times \mathbb{R}^{2(p+q)}$	$\triangleq [L_0 L_1]$	Appendix C
$o \in \mathbb{R}^{q+p}$	$\triangleq \left[\underbrace{-1 \ \ldots \ -1}_{p}, \underbrace{1 \ \ldots \ 1}_{q}\right]^T$	Appendix C
$M \in \mathbb{R}^{2(p+q)} \times \mathbb{R}^{p+q}$	$\triangleq \begin{bmatrix} L_{01}b & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & L_{p-1,p}b \end{bmatrix}$	Appendix C
$\coprod_{i \in S(b)} P_i(b)$	A partition defined on P as a function of b	Section 2.3
P_z	A cell in the partition where $\lvert P_z \rvert \geq 2$	Appendix C.1.2
Q_z	$\triangleq \{(i,j) \in Q \mid i,j \in P_z\}$	Appendix C.1.2
Q_z^*	$\triangleq Q \setminus Q_z$	Appendix C.1.2
α_{ij}	$\triangleq \frac{\gamma}{k\beta_{ij}}, \forall (i,j) \in Q$	Appendix C.1.2
α_{0j}	$\triangleq \frac{\gamma}{k\beta_{0j}}, \forall j \in P.$	Appendix C.1.2
Q_z'	$\triangleq \left\{(l,n) \in Q_z \,\middle\vert\, \alpha_{ln} \geq \frac{1}{2}\right\}$	Lemma C.15
Q_z''	$\triangleq \left\{(l,n) \in Q_z \,\middle\vert\, \alpha_{ln} < \frac{1}{2}\right\}$.	Lemma C.15

B Partition Over Robot Index Set

In this Section, Lemma B.3 shows that $\coprod_{i\in S(b)} P_i(b)$ as defined in Section 2.3 is a partition over the robot index set. Lemmas B.1 and B.2 present two statements used in this proof. For simplicity of the notation, b argument is omitted in the rest of the paper.

First, let us introduce the algebra of strings of the robot labels. First, let $P^1 \triangleq P$, for $n \in \mathbb{Z}^+$ and $1 \le n \le p-1$, $P^{n+1} \triangleq \{xw \,|x \in P, w \in P^n\}$. Following, define $\forall w \in P^{n+1}$, $w \triangleq w_0 w_1 \dots w_n$ where $\forall j \in \{0, \dots, n\}$, $w_j \in P$. Now, recursively define the robot string sets $A^n(i) \subseteq P^{n+1}$ as $A^0(i) := \{i\}$ and

$$A^{n+1}(i) \quad := \quad \left\{ wx \,\middle|\, w \in A^n(i), x \in N_\varepsilon^{n+1}(i), x \in N_\varepsilon(w_n) \right\}$$

The robot string set $A^{n+1}(i)$, consists of all strings from robot i to each of its $(n+1)^{st}$ neighbors having the property that contiguous robot numbers denote the ε-neighbor relationship. Note that $\forall w \in A^n$, the length of w is $n+1$. For convenience, we wish to keep track of the index set $R \triangleq \{r_i \in P \,|\forall i \in S\}$ where r_i is as defined in 2.3. The index set R is the set of all seed robots for all the cells in the partion.

Lemma B.1 *For $0 \le n < p$, $\forall i \in R$, $\forall x \in N_\varepsilon^n(i)$ and $\exists w \in A^n(i)$ such that $w_n = x$.*

Proof: Mathematical induction method will be used.

Base step: $n = 0$, $\forall i \in R$, $\forall x \in N_\varepsilon^0(i) = \{i\}$, $\exists w \in A^0(i) = \{i\}$ which means $w_0 = i$.

Induction: Assume statement $\forall x \in N_\varepsilon^n(i)$, $\exists w \in A^n(i)$ such that $w_n = x$. By definition,

$$N_\varepsilon^{n+1}(i) = \left(\bigcup_{x \in N_\varepsilon^n(i)} N_\varepsilon(x) \right) \bigcap_{l \le n} \bar{N}_\varepsilon{}^l(i)$$

Thus $N_\varepsilon^{n+1}(i) \subseteq \bigcup_{x\in N_\varepsilon^n(i)} N_\varepsilon(x)$. Let $y' \in N_\varepsilon^{n+1}(i)$ then $y' \in \bigcup_{x\in N_\varepsilon^n(i)} N_\varepsilon(x)$. Then $\exists x' \in N_\varepsilon^n(i)$ such that $y' \in N_\varepsilon(x')$. By assumption, $\forall x \in N_\varepsilon^n(i)$, $\exists w \in A^n(i)$ such that $w_n = x$. So, $\exists w' \in A^n(i)$ such that $w'_n = x'$. We find that $w'y' \in A^{n+1}(i)$, since

$$\begin{aligned} A^{n+1}(i) \quad &= \quad \{wy \,|w \in A^n(i), y \in N_\varepsilon^{n+1}(i), y \in N_\varepsilon(w_n)\} \\ &\supseteq \quad \{wy \,|w = w', y = y', y' \in N_\varepsilon(x')\} \\ &\supseteq \quad \{w'y'\} \end{aligned}$$

Then $\forall y \in N_\varepsilon^{n+1}(i)$, $\exists w \in A^n(i)$ such that $v = wy \in A^{n+1}(i)$ and finally, $v_{n+1} = y$. □

Lemma B.2 *$\forall i, j \in R$, $\forall n_i, n_j \in \{0, \dots, p-1\}$, $\forall v \in A^{n_i}(i)$, $\forall w \in A^{n_j}(j)$ if $v_{n_i} = w_{n_j}$ then $\exists l \in \{0, \dots, p-1\}$, $\exists u \in A^l(i)$ such that $u_l = j$.*

Proof: Let $v = v_0 v_1 \dots v_{n_i} = i v_1 \dots v_{n_{i-1}} r$ and $w = w_0 w_1 \dots w_{n_j} = j w_1 \dots w_{n_{j-1}} r$. $\exists k_i, k_j \in \{0, \dots, p-1\}$ such that $v_{k_i} \in N_\varepsilon(w_{k_j})$ since for $k_i = n_i$, $k_j = n_j$, $v_{k_i} = v_{k_j} = r$ and $r \in N_\varepsilon(r)$. Let $k'_i = \min\left\{k_i \le n_i \,\middle|v_{k_i} \in N_\varepsilon(w_{k_j}), k_j \le n_j \right\}$ and $k'_j = \min\left\{k_j \le n_j \,\middle|v_{k'_i} \in N_\varepsilon(w_{k_j}) \right\}$ then construct a string u' with the length l' as follows,

$$u' = \begin{cases} i v_1 \dots v_{k'_i} w_{k'_j} \dots w_1 j & \text{if } v_{k'_i} \ne w_{k'_j} \\ i v_1 \dots v_{k'_i} w_{k'_{j-1}} \dots w_1 j & \text{if } v_{k'_i} = w_{k'_j} \end{cases}$$

Note that if $v_{k'_i} \ne w_{k'_j}$ then the length l' of u' is $l' = k'_i + k'_j + 1$. Otherwise, $l' = k'_i + k'_j$. u' denotes a string from the robot i to the robot j in which adjacent robot numbers indicate an ε-neighbor relationship. However, this string may not not be the shortest string containing robots i and j. Then, choose the string with the minimum length such that $l = \min\left\{l \le l' \,\middle|j \in N_\varepsilon^l(i) \right\}$. By Lemma B.1, $\exists u \in A^l(i)$ such that $w_l = j$. (Note that $u = u'$ if $l = l'$). □

Lemma B.3 *$\coprod_{i\in S} P_i$ is a partition over the robot index set P.*

Proof: By definition, if $\coprod_{i \in S} P_i$ is a partition, the following must hold:

1. $\forall i \in S$, $P_i \neq \emptyset$,
2. $P = \bigcup_{i \in S} P_i$,
3. $\forall i, j \in S$, $i \neq j$, $P_i \cap P_j = \emptyset$.

To establish (1), note that, by construction $P_i = \bigcup_{l=0}^{p-1} N_\varepsilon^l(r_i)$ contains at least $N_\varepsilon^0(r_i) = \{r_i\} \neq \emptyset$ as long as $i \leq s$.

To establish (2), use the termination condition in the definition, $\bigcap_{l \leq s} \bar{P}_l = \emptyset$, and take the complement of both sides to get $\bigcup_{l \leq s} P_l = P$.

Finally, to establish (3), we use proof by contradiction. Suppose $\exists x, y \in S$, $x < y$ such that $P_x \cap P_y \neq \emptyset$. Let $r \in P_x \cap P_y$. Define $i = r_x$ and $j = r_y$. Using definitions of P_x and P_y,

$$r \in \left(\bigcup_{n=0}^{p-1} N_\varepsilon^n(i) \right) \cap \left(\bigcup_{n=0}^{p-1} N_\varepsilon^n(j) \right)$$

Then, $\exists n_i, n_j \in \{1, \ldots, p-1\}$ such that,

$$r \in N_\varepsilon^{n_i}(i) \cap N_\varepsilon^{n_j}(j)$$

By Lemma B.1, $\exists v \in A^{n_i}$ and $\exists w \in A^{n_j}$ such that, $v_0 = i$, $w_0 = j$ and $v_{n_i} = w_{n_j} = r$. By Lemma B.2, $\exists l \in \{1, \ldots, p-1\}$ and $\exists u \in A^l(i)$ such that, $u_l = j$. Then, $j \in N_\varepsilon^l(i) \subseteq P_x$. But, this is a contradiction since $j \neq i$ and j is chosen from a set that is intersected with $\bar{P}_x$, therefore $j \notin P_x$. Then the proof is completed. $\square$

C Computational Lemmas

This Section presents several lemmas that are used in the polarity and nondegeneracy analyses. The reader is referred to Appendix A for a summary of all the symbols used.
The following lemmas specify certain properties of e_i and c_{ij} vectors.

Lemma C.1

$$e_i^T e_j = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$

Proof: If $i = j$, then $e_i^T e_j = e_i^T e_i = \|e_i\|^2$. But, by definition $e_i \in \mathbb{R}^p$ is a unit vector. Then, $\|e_i\|^2 = 1$. By definition, e_i and e_j are base vectors. If $i \neq j$ then these vectors turn out to be orthogonal. Then $e_i^T e_j = 0$. $\square$

Lemma C.2 *If $i \neq j$,*

$$c_{ij}^T e_n = \begin{cases} 1 & \textit{if } n = i \\ -1 & \textit{if } n = j \\ 0 & \textit{if } n \neq i \textit{ and } n \neq j \end{cases}$$

Proof: By definition,

$$\begin{aligned} c_{ij}^T e_n &= (e_i - e_j)^T e_n \\ &= e_i^T e_n - e_j^T e_n \end{aligned}$$

In case of $n = i$, $c_{ij}^T e_n = e_i^T e_i - e_j^T e_i = 1$, by Lemma C.1. In case of $n = j$, $c_{ij}^T e_n = e_i^T e_j - e_j^T e_j = -1$, by Lemma C.1. In case of $n \neq i$ and $n \neq j$, $c_{ij}^T e_n = 0$, by Lemma C.1. $\square$
The following lemma provides formulas for computing the gradient and Hessian matrix of a function of the form $\psi = \frac{u}{w}$.

Lemma C.3 *Let u and w be smooth real-valued maps defined on $\mathbb{R}^n$, and let $\psi = \frac{u}{w}$. Then $\forall b \in \mathcal{C}_\psi$,*

$$\begin{aligned} w\nabla u &= u\nabla w \\ D^2\psi\big|_{\mathcal{C}_\psi} &= \frac{1}{w^2}\left(wD^2u - uD^2w\right) \end{aligned} \tag{14}$$

Proof: Using rules of differentiation, the gradient of ψ is:

$$\nabla\psi = \frac{1}{w^2}\left(w\nabla u - u\nabla w\right) \tag{15}$$

Similarly, the Hessian is:

$$D^2\psi = \frac{1}{w^2}\left(wD^2u + \nabla u\nabla w^T - \nabla w\nabla u^T - uD^2w\right) + w^2\nabla\psi\left(\nabla\frac{1}{w^2}\right)^T$$

At a critical point $\nabla\psi = 0$ which implies that $w\nabla u = u\nabla w$ and thus the first result holds. Next note that this implies that $\nabla u = \psi\nabla w$. Hence

$$D^2\psi\big|_{\mathcal{C}_\psi} = \frac{1}{w^2}\left(wD^2u - uD^2w\right)$$

□

The following lemma gives a formula for the gradient on $\partial\mathcal{F}$.

Lemma C.4

$$\nabla\varphi|_{\partial\mathcal{F}} = -\frac{1}{k\gamma^k}\nabla\beta = -\frac{1}{k\gamma^k}\sum_{(i,j)\in Q^0}\bar{\beta}_{ij}\nabla\beta_{ij}$$

Proof: Using rules of differentiation, the gradient of φ is,

$$\nabla\varphi = \frac{1}{(\gamma^k+\beta)^{2/k}}\left((\gamma^k+\beta)^{1/k}\nabla\gamma - \gamma\nabla(\gamma^k+\beta)^{1/k}\right)$$

Substituting $\nabla(\gamma^k+\beta)^{1/k} = \frac{1}{k}(\gamma^k+\beta)^{\frac{1-k}{k}}\nabla(\gamma^k+\beta)$ and noting that $\beta|_{\partial\mathcal{F}} = 0$ on the right-hand side of $\nabla\varphi$,

$$\begin{aligned} \nabla\varphi|_{\partial\mathcal{F}} &= \frac{1}{\gamma^2}\left(\gamma\nabla\gamma - \gamma\left[\frac{1}{k}\gamma^{1-k}\left(\nabla\gamma^k + \nabla\beta\right)\right]\right) \\ &= \frac{1}{\gamma^2}\left(\gamma\nabla\gamma - \frac{1}{k}\gamma^{2-k}k\gamma^{k-1}\nabla\gamma - \frac{\gamma^{2-k}}{k}\nabla\beta\right) \\ &= -\frac{1}{k\gamma^k}\nabla\beta \\ &= -\frac{1}{k\gamma^k}\sum_{(i,j)\in Q^0}\bar{\beta}_{ij}\nabla\beta_{ij} \end{aligned} \tag{16}$$

Lemma C.5 $\hat{\varphi} = \frac{\gamma^k}{\beta}$, $\forall b \in \mathcal{C}_{\hat{\varphi}}$, $k\beta\nabla\gamma = \gamma\nabla\beta$

Proof: By Lemma C.3,

$$\beta\nabla\gamma^k = \gamma^k\nabla\beta$$

Expanding the lhs and simplifying

$$\begin{aligned} k\beta\gamma^{k-1}\nabla\gamma &= \gamma^k\nabla\beta \\ k\beta\nabla\gamma &= \gamma\nabla\beta \end{aligned} \tag{17}$$

□

Lemma C.6 $\hat{\varphi} = \frac{\gamma^k}{\beta}$, $\forall b \in \mathcal{C}_{\hat{\varphi}}$,

$$D^2\hat{\varphi} \;=\; \frac{\gamma^{k-2}}{\beta^2}\left(k\beta\left(\gamma D^2\gamma + (k-1)\nabla\gamma\nabla\gamma^T\right) - \gamma^2 D^2\beta\right)$$

Proof: Using Lemma C.3, at a critical point, $D^2\hat{\varphi}$ is computed to be

$$D^2\hat{\varphi} \;=\; \frac{1}{\beta^2}\left(\beta D^2\gamma^k - \gamma^k D^2\beta\right) \tag{18}$$

Substituting $D^2\gamma^k = k\beta\left(\gamma D^2\gamma + (k-1)\nabla\gamma\nabla\gamma^T\right)$ in the rhs of eq. 18

$$D^2\hat{\varphi} \;=\; \frac{\gamma^{k-2}}{\beta^2}\left(k\beta\left(\gamma D^2\gamma + (k-1)\nabla\gamma\nabla\gamma^T\right) - \gamma^2 D^2\beta\right) \tag{19}$$

□

Lemma C.7

$$D^2\varphi\big|_{\mathcal{C}_\varphi} \;=\; \frac{1}{(\gamma^k+\beta)^{2/k}}\left((\gamma^k+\beta)^{1/k} 2I_{2p} - \gamma D^2(\gamma^k+\beta)^{1/k}\right)$$

Proof: By definition (eqn. 10), $\varphi = \frac{\gamma}{(\gamma^k+\beta)^{1/k}}$. Using Lemma C.3 and noting that $D^2\gamma = 2I_{2p}$,

$$D^2\varphi\big|_{\mathcal{C}_\varphi} \;=\; \frac{1}{(\gamma^k+\beta)^{2/k}}\left((\gamma^k+\beta)^{1/k} 2I_{2p} - \gamma D^2(\gamma^k+\beta)^{1/k}\right)$$

□

Define $L_{0i} \triangleq -\frac{1}{\sqrt{\beta_{0i}}}\left(I_2 \otimes e_i^T\right)$, $\forall i \in P$ and $L_{ij} \triangleq \frac{1}{\sqrt{\beta_{ij}}}\left(I_2 \otimes c_{ij}^T\right)$, $\forall (i,j) \in Q$ Let L_0 be the $2p \times 2p$ matrix $L_0 \triangleq \left[L_{01}^T \ldots L_{0p}^T\right]$ and L_1 be the $2p \times 2q$ matrix $L_1 \triangleq \left[L_{12}^T \ldots L_{p-1,p}^T\right]$. Let L be the $2p \times 2(p+q)$ matrix $L \triangleq [L_0 L_1]$ and o be the $(q+p) \times 1$ vector $o \triangleq \left[\underbrace{-1\ \ldots\ -1}_{p}, \underbrace{1\ \ldots\ 1}_{q}\right]^T$. Let M be the $2(p+q) \times (p+q)$ matrix,

$$M \triangleq \begin{bmatrix} L_{01}b & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & L_{p-1,p}b \end{bmatrix}$$

Lemma C.8 *L has rank 2p.*

Proof: By definition, $L \triangleq \left[L_{01}^T \ldots L_{0p}^T L_{12}^T \ldots L_{p-1,p}^T\right]$ has at most rank 2p. Moreover, note that rank $L \geq$ rank L_0. Furthermore observe that by definition $L_0 = I_2 \otimes \begin{bmatrix} -\frac{1}{\sqrt{\beta_{01}}} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & -\frac{1}{\sqrt{\beta_{0p}}} \end{bmatrix}$ where I_2 is rank 2 and the second matrix is of rank p. Hence, from definition, L_0 is of rank 2p. Hence the result. □

Next, Lemmas C.9-C.12 presented. These lemmas are used in lemma C.13. Note the following: By definition, $\beta_{ij} = \left\|\left(I_2 \otimes c_{ij}^T\right) b\right\|^2 - \rho_{ij}^2$ and $\beta_{0i}(b) = \rho_{0i}^2 - \left\|\left(I_2 \otimes e_i^T\right) b\right\|^2$. Then, $\nabla\beta_{ij} = 2\left(I_2 \otimes c_{ij}\right)\left(I_2 \otimes c_{ij}^T\right) b$ and $D^2\beta_{ij} = 2\left(I_2 \otimes c_{ij}\right)\left(I_2 \otimes c_{ij}^T\right)$, $\forall (i,j) \in Q$; $\nabla\beta_{0i} = -2\left(I_2 \otimes e_i\right)\left(I_2 \otimes e_i^T\right) b$ and $D^2\beta_{0i} = -2\left(I_2 \otimes e_i\right)\left(I_2 \otimes e_i^T\right)$, $\forall i \in P$.

Lemma C.9

$$\sum_{(i,j)\in Q^0} \frac{\nabla \beta_{ij}}{\beta_{ij}} \quad = \quad 2LMo$$

Proof:

$$\begin{aligned}
\sum_{(i,j)\in Q^0} \frac{\nabla \beta_{ij}}{\beta_{ij}} &= \sum_{i\in P} \frac{\nabla \beta_{0i}}{\beta_{0i}} + \sum_{(i,j)\in Q} \frac{\nabla \beta_{ij}}{\beta_{ij}} \\
&= \sum_{i\in P} \frac{-2\left(I_2 \otimes e_i\right)\left(I_2 \otimes e_i^T\right) b}{\sqrt{\beta_{0i}}\sqrt{\beta_{0i}}} + \sum_{(i,j)\in Q} \frac{2\left(I_2 \otimes c_{ij}\right)\left(I_2 \otimes c_{ij}^T\right) b}{\sqrt{\beta_{ij}}\sqrt{\beta_{ij}}} \\
&= -2\sum_{i\in P} L_{0i}^T L_{0i} b + 2 \sum_{(i,j)\in Q} L_{ij}^T L_{ij} b \\
&= 2LMo
\end{aligned}$$

□

Lemma C.10

$$\sum_{(i,j)\in Q^0} \frac{D^2 \beta_{ij}}{\beta_{ij}} \quad = \quad 2L_1L_1^T - 2L_0L_0^T$$

Proof:

$$\begin{aligned}
\sum_{(i,j)\in Q^0} \frac{D^2 \beta_{ij}}{\beta_{ij}} &= \sum_{i\in P} \frac{D^2 \beta_{0i}}{\beta_{0i}} + \sum_{(i,j)\in Q} \frac{D^2 \beta_{ij}}{\beta_{ij}} \\
&= \sum_{i\in P} \frac{-2\left(I_2 \otimes e_i\right)\left(I_2 \otimes e_i^T\right)}{\sqrt{\beta_{0i}}\sqrt{\beta_{0i}}} + \sum_{(i,j)\in Q} \frac{2\left(I_2 \otimes c_{ij}\right)\left(I_2 \otimes c_{ij}^T\right)}{\sqrt{\beta_{ij}}\sqrt{\beta_{ij}}} \\
&= -2\sum_{i\in P} L_{0i}^T L_{0i} + 2 \sum_{(i,j)\in Q} L_{ij}^T L_{ij} \\
&= -2L_0L_0^T + 2L_1L_1^T
\end{aligned}$$

□

Lemma C.11

$$\sum_{(i,j)\in Q^0} \frac{\nabla \beta_{ij} \nabla \beta_{ij}^T}{\beta_{ij}^2} \quad = \quad 4LMM^TL^T$$

Proof:

$$
\begin{aligned}
\sum_{(i,j)\in Q^0} \frac{\nabla\beta_{ij}\nabla\beta_{ij}^T}{\beta_{ij}^2} &= \sum_{i\in P} \frac{\nabla\beta_{0i}\nabla\beta_{0i}^T}{\beta_{0i}^2} + \sum_{(i,j)\in Q} \frac{\nabla\beta_{ij}\nabla\beta_{ij}^T}{\beta_{ij}^2} \\
&= \sum_{i\in P} \left(\frac{-2\left(I_2\otimes e_i\right)\left(I_2\otimes e_i^T\right)b}{\sqrt{\beta_{0i}}\sqrt{\beta_{0i}}} \right) \left(\frac{-2\left(I_2\otimes e_i\right)\left(I_2\otimes e_i^T\right)b}{\sqrt{\beta_{0i}}\sqrt{\beta_{0i}}} \right)^T \\
&\quad + \sum_{(i,j)\in Q} \left(\frac{2\left(I_2\otimes c_{ij}\right)\left(I_2\otimes c_{ij}^T\right)b}{\sqrt{\beta_{ij}}\sqrt{\beta_{ij}}} \right) \left(\frac{2\left(I_2\otimes c_{ij}\right)\left(I_2\otimes c_{ij}^T\right)b}{\sqrt{\beta_{ij}}\sqrt{\beta_{ij}}} \right)^T \\
&= 4\sum_{i\in P} L_{0i}^T L_{0i} b (L_{0i}^T L_{0i} b)^T + 4\sum_{(i,j)\in Q} L_{ij}^T L_{ij} b (L_{ij}^T L_{ij} b)^T \\
&= 4LMM^TL^T
\end{aligned}
$$

□

Lemma C.12

$$
\sum_{(i,j)\in Q^0} \sum_{(l,n)\in Q^0}^{(l,n)\neq(i,j)} \frac{\nabla\beta_{ij}\nabla\beta_{ln}^T}{\beta_{ij}\beta_{ln}} = 4LMoo^TM^TL^T - 4LMM^TL^T
$$

Proof:

$$
\sum_{(i,j)\in Q^0} \sum_{(l,n)\in Q^0}^{(l,n)\neq(i,j)} \frac{\nabla\beta_{ij}\nabla\beta_{ln}^T}{\beta_{ij}\beta_{ln}} = \sum_{(i,j)\in Q^0} \frac{\nabla\beta_{ij}}{\beta_{ij}} \sum_{(i,j)\in Q^0} \frac{\nabla\beta_{ij}^T}{\beta_{ij}} - \sum_{(i,j)\in Q^0} \frac{\nabla\beta_{ij}\nabla\beta_{ij}^T}{\beta_{ij}^2}
$$

By lemmas C.10-C.11,

$$
\begin{aligned}
\sum_{(i,j)\in Q^0} \sum_{(l,n)\in Q^0}^{(l,n)\neq(i,j)} \frac{\nabla\beta_{ij}\nabla\beta_{ln}^T}{\beta_{ij}\beta_{ln}} &= 2LMo(2LMo)^T - 4LMM^TL^T \\
&= 4LMoo^TM^TL^T - 4LMM^TL^T
\end{aligned}
$$

□

The following lemma is used to derive the hessian of $D^2\hat{\varphi}$ restricted to $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_1(\varepsilon)$. It is used in Propositions C.7 and C.4.

Lemma C.13 $\forall b \in \mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_1(\varepsilon)$,

$$
\frac{\beta}{2\gamma^k} D^2\hat{\varphi} = \frac{k}{\gamma} I_{2p} + 2LMM^TL^T - \frac{2}{k} LMoo^TM^TL^T - L_1L_1^T + L_0L_0^T
$$

Proof: By Lemma C.6, $D^2\hat{\varphi}$ computed at a critical point, is equal to:

$$
D^2\hat{\varphi} = \frac{\gamma^{k-2}}{\beta^2} \left(k\beta \left(\gamma D^2\gamma + (k-1)\nabla\gamma\nabla\gamma^T \right) - \gamma^2 D^2\beta \right) \tag{20}
$$

Using Lemma C.5, $k\beta\nabla\gamma = \gamma\nabla\beta$. Take the outer-product of the both sides, $(k\beta)^2\nabla\gamma\nabla\gamma^T = \gamma^2\nabla\beta\nabla\beta^T$. Assuming $b \neq g$, substitute this on the right-hand side of eqn. 20,

$$
D^2\hat{\varphi} = \frac{\gamma^{k-1}}{\beta^2} \left(k\beta D^2\gamma + \left(1 - \frac{1}{k}\right) \frac{\gamma}{\beta} \nabla\beta\nabla\beta^T - \gamma D^2\beta \right)
$$

Note $D^2\gamma = 2I_{2p}$ and write the equivalent expanded terms for $\nabla\beta$ and $D^2\beta$ as:

$$\begin{aligned}\frac{\beta^2}{\gamma^{k-1}}D^2\hat{\varphi} &= 2k\beta I_{2p} + \left(1-\frac{1}{k}\right)\frac{\gamma}{\beta}\left(\sum_{(i,j)\in Q^0}\frac{\beta}{\beta_{ij}}\nabla\beta_{ij}\right)\left(\sum_{(l,n)\in Q^0}\frac{\beta}{\beta_{ln}}\nabla\beta_{ln}^T\right)\\ &\quad -\gamma\left(\sum_{(i,j)\in Q^0}\frac{\beta}{\beta_{ij}}D^2\beta_{ij} + \sum_{(i,j)\in Q^0}\sum_{(l,n)\in Q^0}^{(l,n)\neq(i,j)}\frac{\beta}{\beta_{ij}\beta_{ln}}\nabla\beta_{ij}\nabla\beta_{ln}^T\right)\end{aligned}$$

Note that $\forall b \in \mathcal{F}_1(\varepsilon)$, $\gamma \neq 0$ as $g \notin \mathcal{F}_1(\varepsilon)$. By Lemma C.9,

$$\sum_{(i,j)\in Q^0}\frac{\nabla\beta_{ij}}{\beta_{ij}} = 2LMo$$

By Lemma C.10,

$$\sum_{(i,j)\in Q^0}\frac{D^2\beta_{ij}}{\beta_{ij}} = 2L_1L_1^T - 2L_0L_0^T$$

By Lemma C.12,

$$\sum_{(i,j)\in Q^0}\sum_{(l,n)\in Q^0}^{(l,n)\neq(i,j)}\frac{\nabla\beta_{ij}\nabla\beta_{ln}^T}{\beta_{ij}\beta_{ln}} = 4LMoo^TM^TL^T - 4LMM^TL^T$$

Using these equalities and the definitions of L_1, L_0, L and M, with some simplifications,

$$\begin{aligned}\frac{\beta^2}{\gamma^{k-1}}D^2\hat{\varphi} &= 2k\beta I_{2p} + \left(1-\frac{1}{k}\right)\frac{\gamma}{\beta}(2\beta LMo)(2\beta LMo)^T\\ &\quad -\gamma\left[\beta(2L_1L_1^T - 2L_0L_0^T) + \beta(4LMoo^TM^TL^T - 4LMM^TL^T)\right]\\ &= 2k\beta I_{2p} + \left(1-\frac{1}{k}\right)4\gamma\beta LMoo^TM^TL^T + 2\gamma\beta L_0L_0^T\\ &\quad -2\gamma\beta L_1L_1^T - 4\gamma\beta LMoo^TM^TL^T + 4\gamma\beta LMM^TL^T\\ &= 2k\beta I_{2p} - \frac{4}{k}\gamma\beta LMoo^TM^TL^T + 2\gamma\beta L_0L_0^T\\ &\quad -2\gamma\beta L_1L_1^T + 4\gamma\beta LMM^TL^T\end{aligned}$$

Divide both sides by $2\gamma\beta$ and collect terms together,

$$\frac{\beta}{2\gamma^k}D^2\hat{\varphi} = \frac{k}{\gamma}I_{2p} + 2LMM^TL^T - \frac{2}{k}LMoo^TM^TL^T - L_1L_1^T + L_0L_0^T$$

□

The following lemma is used in Lemma C.15.

Lemma C.14

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il} + \sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li} + \sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in} + \sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right) = (2p_z-4)\sum_{(l,n)\in Q_z}\alpha_{ln}\delta_{ln} + (p_z-1)\sum_{(i,j)\in Q_z^*}\alpha_{ij}\delta_{ij}$$

Proof: Recalling that $P = P_z \cup P_z'$, we can expand the summations on the rhs as:

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il} + \sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li} + \sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in} + \sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right) =$$

$$\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P_z}^{i<l}\alpha_{il}\delta_{il}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P_z}^{i>l,i\neq n}\alpha_{li}\delta_{li}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P_z}^{i<n,i\neq l}\alpha_{in}\delta_{in}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P_z}^{i>n}\alpha_{ni}\delta_{ni}$$
$$+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P'_z}^{i<l}\alpha_{il}\delta_{il}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P'_z}^{i>l}\alpha_{li}\delta_{li}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P'_z}^{i<n}\alpha_{in}\delta_{in}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{i\in P'_z}^{i>n}\alpha_{ni}\delta_{ni}$$

Next change the order of summations in the rhs consecutively,

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il}+\sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li}+\sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in}+\sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right)=$$

$$\sum_{i\in P_z}\sum_{l\in P_z}^{l>i}\sum_{n\in P_z}^{n>l}\alpha_{il}\delta_{il}+\sum_{l\in P_z}\sum_{i\in P_z}^{i>l}\sum_{n\in P_z}^{n>l,n\neq i}\alpha_{li}\delta_{li}+\sum_{i\in P_z}\sum_{n\in P_z}^{n>i}\sum_{l\in P_z}^{l<n,l\neq i}\alpha_{in}\delta_{in}+\sum_{n\in P_z}\sum_{i\in P_z}^{i>n}\sum_{l\in P_z}^{l<n}\alpha_{ni}\delta_{ni}$$
$$+\sum_{i\in P'_z}\sum_{l\in P_z}^{l>i}\sum_{n\in P_z}^{n>l}\alpha_{il}\delta_{il}+\sum_{l\in P_z}\sum_{i\in P'_z}^{i>l}\sum_{n\in P_z}^{n>l}\alpha_{li}\delta_{li}+\sum_{i\in P'_z}\sum_{n\in P_z}^{n>i}\sum_{l\in P_z}^{l<n}\alpha_{in}\delta_{in}+\sum_{n\in P_z}\sum_{i\in P'_z}^{i>n}\sum_{l\in P_z}^{l<n}\alpha_{ni}\delta_{ni}$$

Next change the indices of the summations in the rhs,

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il}+\sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li}+\sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in}+\sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right)=$$

$$\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{m\in P_z}^{m>n}\alpha_{ln}\delta_{ln}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{m\in P_z}^{m>l,m\neq n}\alpha_{ln}\delta_{ln}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{m\in P_z}^{m<n,m\neq l}\alpha_{ln}\delta_{ln}+\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}\sum_{m\in P_z}^{m<l}\alpha_{ln}\delta_{ln}$$
$$+\sum_{i\in P'_z}\sum_{j\in P_z}^{j>i}\sum_{m\in P_z}^{m>j}\alpha_{ij}\delta_{ij}+\sum_{i\in P_z}\sum_{j\in P'_z}^{j>i}\sum_{m\in P_z}^{m>i}\alpha_{ij}\delta_{ij}+\sum_{i\in P'_z}\sum_{j\in P_z}^{j>i}\sum_{m\in P_z}^{m<j}\alpha_{ij}\delta_{ij}+\sum_{i\in P_z}\sum_{j\in P'_z}^{j>i}\sum_{m\in P_z}^{m<i}\alpha_{ij}\delta_{ij}$$

Collecting similar terms into one summation,

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il}+\sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li}+\sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in}+\sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right)=$$

$$\sum_{l\in P_z}\sum_{n\in P_z}^{n>l}2\sum_{m\in P_z}^{m\neq l,n}\alpha_{ln}\delta_{ln}+\sum_{i\in P'_z}\sum_{j\in P_z}^{j>i}\sum_{m\in P_z}^{m\neq j}\alpha_{ij}\delta_{ij}+\sum_{i\in P_z}\sum_{j\in P'_z}^{j>i}\sum_{m\in P_z}^{m\neq i}\alpha_{ij}\delta_{ij}$$

Grouping the summations and simplifying rhs,

$$\sum_{(l,n)\in Q_z}\left(\sum_{i\in P}^{i<l}\alpha_{il}\delta_{il}+\sum_{i\in P}^{i>l,i\neq n}\alpha_{li}\delta_{li}+\sum_{i\in P}^{i<n,i\neq l}\alpha_{in}\delta_{in}+\sum_{i\in P}^{i>n}\alpha_{ni}\delta_{ni}\right)=\sum_{(l,n)\in Q_z}2(p_z-2)\alpha_{ln}\delta_{ln}+\sum_{(i,j)\in Q_z^*}(p_z-1)\alpha_{ij}\delta_{ij}$$

□

The following lemma is used in Proposition C.4.

Lemma C.15

$$\frac{\rho'}{2p_z-2}\left(\sum_{(l,n)\in Q_z}\|g_{ln}\|-\sum_{(l,n)\in Q_z}\delta_{ln}-\frac{2p_z-2}{\rho'}\|v_z\|^2\right)-\frac{\gamma\rho'}{2k\varepsilon}\left(\rho_0 p_z+\sum_{(i,j)\in Q_z^*}\sqrt{\rho_{ij}^2+\varepsilon}\right)\leq\sum_{(l,n)\in Q_z}\alpha_{ln}\delta_{ln}^2-\|v_z\|^2$$

where $\alpha_{ij} \triangleq \frac{\gamma}{k\beta_{ij}}$, $\forall (i,j) \in Q$, *and* $\alpha_{0j} \triangleq \frac{\gamma}{k\beta_{0j}}$, $\forall j \in P$.

Proof: By Lemma C.5, $k\beta\nabla\gamma = \gamma\nabla\beta$. Expanding the terms $\nabla\gamma$ and $\nabla\beta$ respectively,

$$2k\beta(b-g) = \gamma \sum_{(i,j)\in Q} \frac{2\beta}{\beta_{ij}} \left(I_2 \otimes c_{ij}\right) d_{ij} - \gamma \sum_{j\in P} \frac{2\beta}{\beta_{0j}} \left(I_2 \otimes e_j\right) b_j$$

Now let $\alpha_{ij} \triangleq \frac{\gamma}{k\beta_{ij}}$, $\forall (i,j) \in Q$, and $\alpha_{0j} \triangleq \frac{\gamma}{k\beta_{0j}}$, $\forall j \in P$. Manipulating the β and k terms and replacing the $\frac{\gamma}{k\beta_{ij}}$ and $\frac{\gamma}{k\beta_{0j}}$ terms by α_{ij} and α_{0j} respectively,

$$\sum_{i\in P}(b_i - g_i) \otimes e_i = \sum_{(i,j)\in Q} \alpha_{ij} \left(I_2 \otimes c_{ij}\right) d_{ij} - \sum_{j\in P} \alpha_{0j} \left(I_2 \otimes e_j\right) b_j$$

Both sides are multiplied by $\left(I_2 \otimes c_{ln}^T\right)$ where $l < n$ and simplified as:

$$\begin{aligned}
\left(I_2 \otimes c_{ln}^T\right) \sum_{i\in P}(b_i - g_i) \otimes e_i &= \left(I_2 \otimes c_{ln}^T\right) \sum_{(i,j)\in Q} \alpha_{ij} \left(I_2 \otimes c_{ij}\right) d_{ij} \\
&\quad - \left(I_2 \otimes c_{ln}^T\right) \sum_{j\in P} \alpha_{0j} \left(I_2 \otimes e_j\right) b_j \\
\sum_{i\in P}(b_i - g_i) \otimes c_{ln}^T e_i &= \sum_{(i,j)\in Q} \alpha_{ij} \left(I_2 \otimes c_{ln}^T c_{ij}\right) d_{ij} - \sum_{j\in P} \alpha_{0j} \left(I_2 \otimes c_{ln}^T e_j\right) b_j
\end{aligned}$$

Using Lemmas C.1 and C.2, both sides are simplified as:

$$\begin{aligned}
d_{ln} - g_{ln} &= 2\alpha_{ln} d_{ln} + \sum_{i\in P}^{i<l} \alpha_{il} d_{il} + \sum_{i\in P}^{i>l, i\neq n} \alpha_{li} d_{li} + \sum_{i\in P}^{i<n, i\neq l} \alpha_{in} d_{in} + \sum_{i\in P}^{i>n} \alpha_{ni} d_{ni} - \alpha_{0l} b_l + \alpha_{0n} b_n \\
-g_{ln} &= (2\alpha_{ln} - 1) d_{ln} + \sum_{i\in P}^{i<l} \alpha_{il} d_{il} + \sum_{i\in P}^{i>l, i\neq n} \alpha_{li} d_{li} + \sum_{i\in P}^{i<n, i\neq l} \alpha_{in} d_{in} + \sum_{i\in P}^{i>n} \alpha_{ni} d_{ni} - \alpha_{0l} b_l + \alpha_{0n} b_n
\end{aligned}$$

Summing g_{ln} terms over Q_z and using triangular inequality,

$$\begin{aligned}
\sum_{(l,n)\in Q_z} \|g_{ln}\| &\leq \sum_{(l,n)\in Q_z} |2\alpha_{ln} - 1| \delta_{ln} \\
&\quad + \sum_{(l,n)\in Q_z} \left(\sum_{i\in P}^{i<l} \alpha_{il}\delta_{il} + \sum_{i\in P}^{i>l, i\neq n} \alpha_{li}\delta_{li} + \sum_{i\in P}^{i<n, i\neq l} \alpha_{in}\delta_{in} + \sum_{i\in P}^{i>n} \alpha_{ni}\delta_{ni} \right) \\
&\quad + \sum_{(l,n)\in Q_z} \left(\frac{\gamma}{k\beta_{0l}} \|b_l\| + \frac{\gamma}{k\beta_{0n}} \|b_n\| \right)
\end{aligned}$$

Let $p_z \triangleq |P_z|$. Using Lemma C.14 and noting that for $\forall b \in \mathcal{F}_1(\varepsilon)$, $\beta_{0i} > \varepsilon$, $\forall i \in P$ and $\|b_i\| < \rho_0$. ,

$$\begin{aligned}
\sum_{(l,n)\in Q_z} \|g_{ln}\| &\leq \sum_{(l,n)\in Q_z} |2\alpha_{ln} - 1| \delta_{ln} + (2p_z - 4) \sum_{(l,n)\in Q_z} \alpha_{ln}\delta_{ln} \\
&\quad + (p_z - 1) \sum_{(i,j)\in Q_z^*} \alpha_{ij}\delta_{ij} + \frac{\gamma\rho_0}{k\varepsilon} p_z (p_z - 1)
\end{aligned}$$

Let $Q_z = Q_z' \cup Q_z''$ where Q_z' and Q_z'' are defined as: $Q_z' \triangleq \left\{(l,n) \in Q_z \,\middle|\, \alpha_{ln} \geq \frac{1}{2}\right\}$ and $Q_z'' \triangleq \left\{(l,n) \in Q_z \,\middle|\, \alpha_{ln} < \frac{1}{2}\right\}$.

The summation terms are then decomposed with respect to Q'_z and Q''_z:

$$\begin{aligned}\sum_{(l,n)\in Q_z}\|g_{ln}\| &\le \sum_{(l,n)\in Q'_z}(2\alpha_{ln}-1)\delta_{ln}+\sum_{(l,n)\in Q''_z}(1-2\alpha_{ln})\delta_{ln}+(2p_z-4)\sum_{(l,n)\in Q'_z}\alpha_{ln}\delta_{ln}\\ &+ (2p_z-4)\sum_{(l,n)\in Q''_z}\alpha_{ln}\delta_{ln}+(p_z-1)\sum_{(i,j)\in Q^*_z}\alpha_{ij}\delta_{ij}+\frac{\gamma\rho_0}{k\varepsilon}p_z(p_z-1)\end{aligned}$$

Next note that $\forall(i,j)\in Q^*_z$, $\beta_{ij}>\varepsilon$ and $\frac{\delta_{ij}}{\beta_{ij}}\le\frac{2\rho_0}{\varepsilon}$. Using these bounds and simplifying,

$$\begin{aligned}\sum_{(l,n)\in Q_z}\|g_{ln}\|+\sum_{(l,n)\in Q'_z}\delta_{ln}-\sum_{(l,n)\in Q''_z}\delta_{ln} &\le (2p_z-2)\sum_{(l,n)\in Q'_z}\alpha_{ln}\delta_{ln}+(2p_z-6)\sum_{(l,n)\in Q''_z}\alpha_{ln}\delta_{ln}\\ &+ \frac{\gamma(p_z-1)}{k\varepsilon}\sum_{(i,j)\in Q^*_z}2\rho_0+\frac{\gamma\rho_0}{k\varepsilon}p_z(p_z-1)\end{aligned}$$

Let $\rho'\triangleq\min_{(i,j)\in Q}\{\rho_{ij}\}$. Multiply both sides by $\frac{\rho'}{2p_z-2}$ and collecting terms together,

$$\begin{aligned}\frac{\rho'}{2p_z-2}\left(\sum_{(l,n)\in Q_z}\|g_{ln}\|-\sum_{(l,n)\in Q_z}\delta_{ln}\right) &\le \sum_{(l,n)\in Q'_z}\alpha_{ln}\delta_{ln}\rho'+\sum_{(l,n)\in Q''_z}\frac{p_z-3}{p_z-1}\alpha_{ln}\delta_{ln}\rho'\\ &+ \frac{\gamma\rho'}{2k\varepsilon}\left(\rho_0p_z+\sum_{(i,j)\in Q^*_z}2\rho_0\right)\end{aligned}$$

Following note $\forall(l,n)\in Q$, $\rho'\le\delta_{ln}$. Using the lower bound in the rhs,

$$\begin{aligned}\frac{\rho'}{2p_z-2}\left(\sum_{(l,n)\in Q_z}\|g_{ln}\|-\sum_{(l,n)\in Q_z}\delta_{ln}\right) &\le \sum_{(l,n)\in Q'_z}\alpha_{ln}\delta^2_{ln}+\sum_{(l,n)\in Q''_z}\alpha_{ln}\delta^2_{ln}\\ &+ \frac{\gamma\rho'}{2k\varepsilon}\left(\rho_0p_z+\sum_{(i,j)\in Q^*_z}2\rho_0\right)\end{aligned}$$

Subtracting the term $\|v_z\|^2$ from both sides and re-group terms, hence the result:

$$\begin{aligned}\frac{\rho'}{2p_z-2}\left(\sum_{(l,n)\in Q_z}\|g_{ln}\|-\sum_{(l,n)\in Q_z}\delta_{ln}-\frac{2p_z-2}{\rho'}\|v_z\|^2\right) &- \frac{\gamma\rho'}{2k\varepsilon}\left(\rho_0p_z+\sum_{(i,j)\in Q^*_z}2\rho_0\right)\\ &\le \sum_{(l,n)\in Q_z}\alpha_{ln}\delta^2_{ln}-\|v_z\|^2\end{aligned}$$

□

Lemma C.16 is used in Proposition C.5.

Lemma C.16 *If $b\in\mathcal{F}_2(\varepsilon)$, then*

$$\frac{\sqrt{\gamma}\|\nabla\beta\|}{2\beta}\le\frac{\max_{\forall b\in\mathcal{F}_2(\varepsilon)}\{\sqrt{\gamma}\}}{\varepsilon}\left(\sum_{(i,j)\in Q}\sqrt{2}\sqrt{\rho^2_{ij}+\varepsilon}+\sum_{i\in P}\sqrt{\rho^2_{0i}-\varepsilon}\right)$$

Proof: Expanding the term $\nabla\beta$ and using triangular inequality,

$$\frac{\sqrt{\gamma}\|\nabla\beta\|}{2\beta} \leq \frac{\sqrt{\gamma}}{2}\sum_{(i,j)\in Q^0}\frac{\|\nabla\beta_{ij}\|}{\beta_{ij}}$$

Using $\nabla\beta_{ij} = 2\left(I_2\otimes c_{ij}\right)\left(I_2\otimes c_{ij}^T\right)b$ and $\nabla\beta_{0i} = -2\left(I_2\otimes e_i\right)\left(I_2\otimes e_i^T\right)b$,

$$\frac{\sqrt{\gamma}\|\nabla\beta\|}{2\beta} \leq \frac{\sqrt{\gamma}}{2}\left(\sum_{(i,j)\in Q}\frac{\|2\left(I_2\otimes c_{ij}\right)d_{ij}\|}{\beta_{ij}} + \sum_{i\in P}\frac{\|2\left(I_2\otimes e_i\right)b_i\|}{\beta_{0i}}\right)$$

Taking the norm of the vectors and using $\frac{\|d_{ij}\|}{\beta_{ij}} \leq \frac{\sqrt{\rho_{ij}^2+\varepsilon}}{\varepsilon}$ and $\frac{\|b_i\|}{\beta_{0i}} \leq \frac{\sqrt{\rho_{0i}^2-\varepsilon}}{\varepsilon}$ in $\mathcal{F}_2(\varepsilon)$,

$$\begin{aligned}\frac{\sqrt{\gamma}\|\nabla\beta\|}{2\beta} &\leq \max_{\forall b\in\mathcal{F}_2(\varepsilon)}\{\sqrt{\gamma}\}\left(\sum_{(i,j)\in Q}\sqrt{2}\frac{\sqrt{\rho_{ij}^2+\varepsilon}}{\varepsilon} + \sum_{i\in P}\frac{\sqrt{\rho_{0i}^2-\varepsilon}}{\varepsilon}\right)\\ &\leq \frac{\max_{\forall b\in\mathcal{F}_2(\varepsilon)}\{\sqrt{\gamma}\}}{\varepsilon}\left(\sum_{(i,j)\in Q}\sqrt{2}\sqrt{\rho_{ij}^2+\varepsilon} + \sum_{i\in P}\sqrt{\rho_{0i}^2-\varepsilon}\right)\end{aligned}$$

□

Lemmas C.17, C.18, C.19 are used in Proposition C.4.

Lemma C.17

$$\|L_1^T v_z\|^2 = \sum_{(i,j)\in Q_z}\frac{\delta_{ij}^2}{\beta_{ij}} + \sum_{i\in P_z'}\left[\sum_{j\in P_z}^{j<i}\frac{\|b_j-\bar{g}_z\|^2}{\beta_{ji}} + \sum_{j\in P_z}^{j>i}\frac{\|b_j-\bar{g}_z\|^2}{\beta_{ij}}\right]$$

Proof: By definition,

$$\|L_1^T v_z\|^2 = \sum_{(i,j)\in Q}\|L_{ij}v_z\|^2$$

Expanding right-hand side by using the definitions of L_{ij} and v_z,

$$\|L_1^T v_z\|^2 = \sum_{(i,j)\in Q}\left\|\frac{1}{\sqrt{\beta_{ij}}}\left(I_2\otimes c_{ij}^T\right)\sum_{n\in P_z}J(b_n-\bar{g}_z)\otimes e_n\right\|^2$$

Using the Kronecker product property that $(a\otimes b)(c\otimes d) = ac\otimes bd$,

$$\|L_1^T v_z\|^2 = \sum_{(i,j)\in Q}\frac{1}{\beta_{ij}}\left\|\sum_{n\in P_z}J(b_n-\bar{g}_z)\otimes c_{ij}^T e_n\right\|^2$$

Collecting the terms of Q_z and P_z' separately, and using Lemma C.2,

$$\begin{aligned}\|L_1^T v_z\|^2 &= \sum_{(i,j)\in Q_z}\frac{1}{\beta_{ij}}\|J(b_i-\bar{g}_z)-J(b_j-\bar{g}_z)\|^2\\ &\quad+\sum_{i\in P_z'}\left[\sum_{j\in P_z}^{j<i}\frac{1}{\beta_{ji}}\|J(b_j-\bar{g}_z)\|^2 + \sum_{j\in P_z}^{j>i}\frac{1}{\beta_{ij}}\|J(b_j-\bar{g}_z)\|^2\right]\end{aligned}$$

Simplifying the terms and using the definition of δ_{ij},

$$\|L_1^T v_z\|^2 = \sum_{(i,j)\in Q_z} \frac{\delta_{ij}^2}{\beta_{ij}} + \sum_{i\in P_z'} \left[\sum_{j\in P_z}^{j<i} \frac{\|b_j - \bar{g}_z\|^2}{\beta_{ji}} + \sum_{j\in P_z}^{j>i} \frac{\|b_j - \bar{g}_z\|^2}{\beta_{ij}} \right]$$

□

Lemma C.18

$$\|L_0^T v_z\|^2 = \sum_{j\in P_z} \frac{\|b_j - \bar{g}_z\|^2}{\beta_{0j}}$$

Proof: By definition,

$$\|L_0^T v_z\|^2 = \sum_{j\in P} \|L_{0j} v_z\|^2$$

Expanding right-hand side by using the definitions of L_{0j} and v_z,

$$\|L_0^T v_z\|^2 = \sum_{j\in P} \left\| \frac{1}{\sqrt{\beta_{0j}}} \left(I_2 \otimes e_j^T\right) \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes e_n \right\|^2$$

Using the Kronecker product property,

$$\|L_0^T v_z\|^2 = \sum_{j\in P} \frac{1}{\beta_{0j}} \left\| \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes e_j^T e_n \right\|^2$$

Using Lemma C.1,

$$\begin{aligned}\|L_0^T v_z\|^2 &= \sum_{j\in P_z} \frac{1}{\beta_{0j}} \|J(b_j - \bar{g}_z)\|^2 \\ &= \sum_{j\in P_z} \frac{\|b_j - \bar{g}_z\|^2}{\beta_{0j}}\end{aligned}$$

□

Lemma C.19

$$\|M^T L^T v_z\|^2 = \sum_{j\in P_z} \frac{1}{\beta_{0j}^2} \left[b_j^T J \bar{g}_z\right]^2 + \sum_{i\in P_z'} \left[\sum_{j\in P_z}^{j<i} \frac{1}{\beta_{ji}^2} \left[d_{ji}^T J(b_j - \bar{g}_z)\right]^2 + \sum_{j\in P_z}^{j>i} \frac{1}{\beta_{ij}^2} \left[d_{ij}^T J(b_j - \bar{g}_z)\right]^2 \right]$$

Proof: By definition,

$$\|M^T L^T v_z\|^2 = \sum_{(i,j)\in Q^0} \left(b^T L_{ij}^T L_{ij} v_z\right)^2$$

Expanding right-hand side by using the definitions of L_{ij} and v_z,

$$\begin{aligned}\|M^T L^T v_z\|^2 &= \sum_{(i,j)\in Q} \left[\frac{1}{\beta_{ij}} b^T \left(I_2 \otimes c_{ij}\right) \left(I_2 \otimes c_{ij}^T\right) \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes e_n \right]^2 \\ &+ \sum_{j\in P} \left[\frac{1}{\beta_{0j}} b^T \left(I_2 \otimes e_j\right) \left(I_2 \otimes e_j^T\right) \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes e_n \right]^2\end{aligned}$$

Using the definition of d_{ij} and Kronecker product property,

$$\begin{aligned}\|M^T L^T v_z\|^2 &= \sum_{(i,j)\in Q} \frac{1}{\beta_{ij}^2} \left[d_{ij}^T \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes c_{ij}^T e_n \right]^2 \\ &\quad + \sum_{j\in P} \frac{1}{\beta_{0j}^2} \left[b_j^T \sum_{n\in P_z} J(b_n - \bar{g}_z) \otimes e_j^T e_n \right]^2\end{aligned}$$

Collecting the terms of Q_z and P_z' separately, and using Lemmas C.1 and C.2,

$$\begin{aligned}\|M^T L^T v_z\|^2 &= \sum_{(i,j)\in Q_z} \frac{1}{\beta_{ij}^2} \left(d_{ij}^T J \left[(b_i - \bar{g}_z) - (b_j - \bar{g}_z) \right] \right)^2 \\ &\quad + \sum_{i\in P_z'} \left[\sum_{j\in P_z}^{j<i} \frac{1}{\beta_{ji}^2} \left[d_{ji}^T J(b_j - \bar{g}_z) \right]^2 + \sum_{j\in P_z}^{j>i} \frac{1}{\beta_{ij}^2} \left[d_{ij}^T J(b_j - \bar{g}_z) \right]^2 \right] \\ &\quad + \sum_{j\in P_z} \frac{1}{\beta_{0j}^2} \left[b_j^T J(b_j - \bar{g}_z) \right]^2\end{aligned}$$

Simplifying the terms,

$$\begin{aligned}\|M^T L^T v_z\|^2 &= \sum_{(i,j)\in Q_z} \frac{1}{\beta_{ij}^2} \left(d_{ij}^T J d_{ij} \right)^2 + \sum_{j\in P_z} \frac{1}{\beta_{0j}^2} \left[b_j^T J \bar{g}_z \right]^2 \\ &\quad + \sum_{i\in P_z'} \left[\sum_{j\in P_z}^{j<i} \frac{1}{\beta_{ji}^2} \left[d_{ji}^T J(b_j - \bar{g}_z) \right]^2 + \sum_{j\in P_z}^{j>i} \frac{1}{\beta_{ij}^2} \left[d_{ij}^T J(b_j - \bar{g}_z) \right]^2 \right] \\ &= \sum_{j\in P_z} \frac{1}{\beta_{0j}^2} \left[b_j^T J \bar{g}_z \right]^2 + \sum_{i\in P_z'} \left[\sum_{j\in P_z}^{j<i} \frac{1}{\beta_{ji}^2} \left[d_{ji}^T J(b_j - \bar{g}_z) \right]^2 + \sum_{j\in P_z}^{j>i} \frac{1}{\beta_{ij}^2} \left[d_{ij}^T J(b_j - \bar{g}_z) \right]^2 \right]\end{aligned}$$

□

C.1 Polarity

The details of proof of Proposition 3.1 are presented in this section. Due to space restrictions, some of the very most detailed computations supporting the proofs of some of the constituent lemmas cannot be included in this paper. However they are available in [26].

C.1.1 The Free Space Boundary: $\partial\mathcal{F} = \beta^{-1}(0)$

Referring to the definition of the pair-touching function T (defined in Section 3.2), $|T(b)| = 0$ means no robots are touching each other and none of them is touching the workspace boundary. The free space boundary $\partial\mathcal{F}$ will be investigated for two cases: (i) Case 1: $|T(b)| = 1$, (ii) Case 2: $|T(b)| \geq 2$. The following proposition proves that there are no critical points on $\partial\mathcal{F}$ for Case 1.

Proposition C.1 *If $|T(b)| = 1$, then $\mathcal{C}_\varphi \cap \partial\mathcal{F} = \emptyset$.*

Proof: If $|T(b)| = 1$, then only one of the terms of β is zero. Call this term β_{ln}, $(l,n) \in Q^0$. Then, all the summation terms of $\nabla\varphi$ vanish except the ones that containing $\bar{\beta}_{ln} \neq 0$ and $\nabla\beta_{ln} \neq 0$. Hence, $\nabla\varphi|_{\partial\mathcal{F}} = -\frac{1}{k\gamma^k}\left(\bar{\beta}_{ln}\nabla\beta_{ln}\right) \neq 0$. □

The following proposition proves that φ admits maximum valued critical points on ∂F for Case 2.

Proposition C.2 *If $|T(b)| \geq 2$, then $\mathcal{C}_\varphi \cap \partial\mathcal{F}$ contains only maximum valued critical points.*

Proof: Since $|T(b)| \geq 2$, $\exists (i,j),(l,n) \in T$, such that $\beta_{ij} = \beta_{ln} = 0$. Then, all the summation terms of $\nabla\varphi$ vanish except the ones containing $\bar{\beta}_{ij} \neq 0$ or $\bar{\beta}_{ln} \neq 0$, resulting in $\nabla\varphi|_{\partial\mathcal{F}} = -\frac{1}{k\gamma^k}\left(\bar{\beta}_{ij}\nabla\beta_{ij} + \bar{\beta}_{ln}\nabla\beta_{ln}\right) = 0$. But $\varphi : \mathcal{F} \to [0,1]$ and $\varphi|_{\mathcal{C}_\varphi \cap \partial\mathcal{F}} = \frac{\gamma}{(\gamma^k+\beta)^{1/k}} = 1$, which means that those critical points achieve the maximum value of φ. □

C.1.2 The Set Near the Outer Boundary: $\mathcal{F}_0(\varepsilon)$

The following proposition shows that there are no critical points in $\mathcal{F}_0(\varepsilon)$ - the subspace of $\mathcal{F}$ that is close to the outer boundary.

Proposition C.3 *For a given design parameter ε, there exists a lower bound on the parameter k, $\mathrm{K}_1(\varepsilon) > 0$, such that, if $k > \mathrm{K}_1(\varepsilon)$, then $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_0(\varepsilon) = \emptyset$.*

Proof: (By contradiction) By definition, $\forall b \in \mathcal{F}_0(\varepsilon)$ if $\phi(b) = \coprod_{i\in S(b)} P_i(b)$ is the corresponding partition then $\exists i \in S(b)$ such that $\exists j \in P_i$, $\beta_{0j} \leq \varepsilon$. In other words, there exists at least one cell consisting of at least one robot close to the workspace boundary.

First, denote the cell which is arbitrarily chosen from the cells consisting of at least one robot close to the boundary by P_z. Let z' refer to the index of the closest robot to the boundary in the cell P_z, that is, $z' \triangleq \arg\max_{i\in P_z, \beta_{0i}\leq\varepsilon}\{\|b_i\|\}$. If b is a critical point, then $k\beta\nabla\gamma = \gamma\nabla\beta$. After expanding the terms $\nabla\gamma$ and $\nabla\beta$, using the definitions b and g, letting $\alpha_{ij} \triangleq \frac{\gamma}{k\beta_{ij}}$ $\forall (i,j) \in Q^0$, $Q_z \triangleq \{(i,j) \in Q\,|i,j \in P_z\}$ and $Q_z^* \triangleq Q \setminus Q_z$, decompose the summation over Q and P respectively and simplify as:

$$\sum_{n\in P_z}(1+\alpha_{0n})b_n = \sum_{n\in P_z} g_n + \sum_{(i,j)\in Q_z^*}\alpha_{ij}d_{ij}$$

After taking the magnitude of both sides and applying the triangle inequality, using $b_n = b_{z'} + d_{nz'}$ on the left-hand side and maximizing δ_{ij}/β_{ij}, taking minimum of left-hand side and finally using $\forall n \in P$, $\alpha_{0n} > 0$

$$\sum_{n\in P_z}(\|b_{z'}\| - \delta_{nz'}) \leq \sum_{n\in P_z}\|g_n\| + \frac{\gamma}{k\varepsilon}\sum_{(i,j)\in Q_z^*}\sqrt{\rho_{ij}^2+\varepsilon}$$

Recall that $\rho'' = \min_{i\in P}\{\rho_{0i}\}$. Using $\min_{b\in\mathcal{F}_0(\varepsilon)}\{\|b_{z'}\|\} = \sqrt{\rho''^2 - \varepsilon}$ and minimizing left-hand side,

$$|P_z|\sqrt{\rho''^2-\varepsilon} - \textstyle\sum_{n\in P_z}\delta_{nz'} - \sum_{n\in P_z}\|g_n\| \leq \frac{\gamma}{k\varepsilon}\sum_{(i,j)\in Q_z^*}\sqrt{\rho_{ij}^2+\varepsilon}$$

Using Assumption 2, if g is chosen appropriately the left-hand side of the above inequality will be positive. If k is chosen as,

$$k > \max_{\forall b\in\mathcal{F}_0(\varepsilon)}\left\{\frac{\gamma\sum_{(i,j)\in Q_z^*}\sqrt{\rho_{ij}^2+\varepsilon}}{\left[|P_z|\sqrt{\rho''^2-\varepsilon}-\sum_{n\in P_z}\delta_{nz'}-\sum_{n\in P_z}\|g_n\|\right]\varepsilon}\right\} \triangleq \mathrm{K}_1(\varepsilon)$$

then b cannot be a critical point. Thus, $\hat{\varphi}$ has no critical points in $\mathcal{F}_0(\varepsilon)$. Further details can be found in [25] or [26].

□

C.1.3 The Set Near the Internal Obstacles: $\mathcal{F}_1(\varepsilon)$

The following proposition shows that $\hat{\varphi}$ has no minimum in $\mathcal{F}_1(\varepsilon)$ - the subset of $\mathcal{F}$ that is close to the internal obstacles.

Proposition C.4 *For a given design parameter ε, there exists a lower bound on the parameter k, $\mathrm{K}_2(\varepsilon) > 0$, such that, if $k > \mathrm{K}_2(\varepsilon)$ then $\hat{\varphi}$ has no minimum in any set $\mathcal{F}_1(\varepsilon)$.*

Proof: It is sufficient to show that for $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_1(\varepsilon)$, $\exists v \in \mathbb{R}^{2p}$ such that $v^T D^2 \hat{\varphi} v < 0$. By definition, $\forall b \in \mathcal{F}_1(\varepsilon)$, there is a partition $\coprod_{i \in S(b)} P_i(b)$ such that $\exists i \in S(b)$ where $|P_i(b)| \geq 2$. Pick arbitrarily a cell consisting of at least two robots and denote it by P_z – that is $|P_z| \geq 2$. Now consider the following vector, $v_z \triangleq \sum_{n \in P_z} J(b_n - \bar{g}_z) \otimes e_n$ where $\bar{g}_z$ denotes the centroid of the robots in the cell P_z, $\bar{g}_z \triangleq \frac{1}{|P_z|} \sum_{n \in P_z} b_n$. We have chosen this vector based on our following observation in the simulations: When the robots are getting close to each other, each starts moving in a direction perpendicular to line between their center and the cell centroid. Recall that $Q_z \triangleq \{(i,j) \in Q \,|\, i,j \in P_z\}$ and $P_z' \triangleq P \setminus P_z$. Let

$$A \triangleq \left(\sum_{(l,n) \in Q_z} \|g_{ln}\| - \sum_{(l,n) \in Q_z} \delta_{ln} - \frac{2p_z - 2}{\rho'} \|v_z\|^2 \right)$$

Doing some manipulations and grouping the terms on the right-hand side as follows,

$$\begin{aligned}
&\tfrac{\beta}{2\gamma^k} v_z^T D^2 \hat{\varphi} v_z \leq \\
&-\tfrac{k}{\gamma} \left(\underbrace{\frac{\rho'}{2p_z - 2} A - \frac{\gamma \rho_0 \rho'}{2k\varepsilon} \left[(p - p_z)(p - p_z - 1) + p_z \right]}_{\sigma_1} \right) \\
&\underbrace{+ \sum_{j \in P_z} \left(\frac{2}{\beta_{0j}^2} \left[b_j^T J \bar{g}_z \right]^2 + \frac{1}{\beta_{0j}} \|b_j - \bar{g}_z\|^2 \right)}_{\sigma_3} \\
&\underbrace{+ \sum_{i \in P_z'} \sum_{j \in P_z}^{j<i} \left(\frac{2}{\beta_{ji}^2} \left[d_{ji}^T J (b_j - \bar{g}_z) \right]^2 - \frac{1}{\beta_{ji}} \|(b_j - \bar{g}_z)\|^2 \right)}_{\sigma_2'} \\
&\underbrace{+ \sum_{i \in P_z'} \sum_{j \in P_z}^{j>i} \left(\frac{2}{\beta_{ij}^2} \left[d_{ij}^T J (b_j - \bar{g}_z) \right]^2 - \frac{1}{\beta_{ij}} \|(b_j - \bar{g}_z)\|^2 \right)}_{\sigma_2''}
\end{aligned}$$

Let $\sigma_2 \triangleq \sigma_2' + \sigma_2''$. Note that $Q_z^* \triangleq Q \setminus Q_z$. If g is chosen according to Assumption 1, then term $A > 0$. If k is chosen as,

$$\begin{aligned}
&k > \max_{\forall b \in \mathcal{F}_1(\varepsilon)} \\
&\left\{ \frac{\gamma (p_z - 1) \rho_0 [p_z + (p - p_z)(p - p_z - 1)]}{\left[\sum_{(l,n) \in Q_z} \|g_{ln}\| - \sum_{(l,n) \in Q_z} \delta_{ln} - \frac{2p_z - 2}{\rho'} \|v_z\|^2 \right] \varepsilon} \right\} \\
&\triangleq \mathrm{K}_{21}(\varepsilon)
\end{aligned}$$

then $\sigma_1 > 0$. Thus, a sufficient condition to make $v_z^T D^2 \hat{\varphi} v_z < 0$, is

$$k > \max_{\forall b \in \mathcal{F}_1(\varepsilon)} \left\{ \frac{(\sigma_2 + \sigma_3)\gamma}{\sigma_1} \right\} \triangleq \mathrm{K}_{22}(\varepsilon)$$

Finally, the proof is completed by choosing, $\mathrm{K}_2(\varepsilon) = \max\{\mathrm{K}_{21}(\varepsilon), \mathrm{K}_{22}(\varepsilon)\}$. Please refer to [25] or [26] for further details.

□

C.1.4 The Set Away From the Obstacles: $\mathcal{F}_2(\varepsilon)$

The following proposition shows that for sufficiently large k values, there are no critical points in $\mathcal{F}_2(\varepsilon)$.

Proposition C.5 *For a given design parameter ε there exists a lower bound on the parameter k, $\mathrm{K}_3(\varepsilon) > 0$, such that if $k \geq \mathrm{K}_3(\varepsilon)$ then $\mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_2(\varepsilon) = \emptyset$.*

Proof: $\forall b \in \mathcal{C}_{\hat{\varphi}}$, $k\beta\nabla\gamma = \gamma\nabla\beta$. Taking the norm of the both sides and re-arranging terms in $2k\beta = \sqrt{\gamma}\|\nabla\beta$,

$$k \;=\; \frac{\sqrt{\gamma}\|\nabla\beta\|}{2\beta} \tag{21}$$

If k is selected to have value,

$$\begin{array}{l} k > \max_{\forall b \in \mathcal{F}_2(\varepsilon)} \frac{\sqrt{\gamma}}{\varepsilon} \times \\ \left(\sum_{(i,j)\in Q} \sqrt{2}\sqrt{\rho_{ij}^2 + \varepsilon} + \sum_{i\in P} \sqrt{\rho_{0i}^2 - \varepsilon} \right) \\ \triangleq \mathrm{K}_3(\varepsilon) \end{array}$$

then, Eq. 21 does not hold which in turn implies that there are no critical points in $\mathcal{F}_2(\varepsilon)$. □

C.2 Nondegeneracy

The details of Proposition 3.2 are given in this section.

C.2.1 Goal point $\{g\}$

Proposition C.6 *The goal point, g is a non-degenerate minimum of φ.*

Proof: It can be shown that

$$\begin{array}{c} D^2\varphi\big|_{\mathcal{C}_\varphi} = \\ \frac{1}{(\gamma^k+\beta)^{2/k}} \left((\gamma^k + \beta)^{1/k} 2I_{2p} - \gamma D^2 (\gamma^k + \beta)^{1/k} \right) \end{array}$$

Noting that $\gamma\big|_g = 0$ and $\nabla\gamma\big|_g = 2(b-g) = 0$;

$$D^2\varphi\big|_g \;=\; \frac{2}{\beta^{1/k}} I_{2p}$$

implies that g is a non-degenerate minimum of φ. □

C.2.2 The Set Near the Internal Obstacles: $\mathcal{F}_1(\varepsilon)$

There are no critical points in $\{b \in \partial\mathcal{F} : |T(b)| = 1\}$ by Proposition C.1 given in Section C.1. The critical points in $\{b \in \partial\mathcal{F} : |T(b)| > 1\}$ are maxima by Proposition C.2. $\hat{\varphi}$ has no critical points in $\mathcal{F}_0(\varepsilon)$, $\mathcal{F}_2(\varepsilon)$ by Proposition C.3 and Proposition C.5 respectively. Now let us consider the critical points of $\hat{\varphi}$ that are in $\mathcal{F}_1(\varepsilon)$.

Proposition C.7 *$\exists \mathrm{N}(\varepsilon)$ such that for $k > \mathrm{N}(\varepsilon)$, $D^2\hat{\varphi}$ restricted to $\mathcal{F}_1(\varepsilon)$ is non-singular.*

Proof: Define $L_{0i} \triangleq -\frac{1}{\sqrt{\beta_{0i}}}\left(I_2 \otimes e_i^T\right)$, $\forall i \in P$ and $L_{ij} \triangleq \frac{1}{\sqrt{\beta_{ij}}}\left(I_2 \otimes c_{ij}^T\right)$, $\forall (i,j) \in Q$.

Let L_0 be the $2p \times 2p$ matrix $L_0 \triangleq \left[L_{01}^T \dots L_{0p}^T\right]$ and L_1 be the $2p \times 2q$ matrix $L_1 \triangleq \left[L_{12}^T \dots L_{p-1,p}^T\right]$. Let L be the $2p \times 2(p+q)$ matrix $L \triangleq [L_0 L_1]$ and o be the $(q+p) \times 1$ vector $o \triangleq \left[\underbrace{-1 \;\dots\; -1}_{p}, \underbrace{1 \;\dots\; 1}_{q}\right]^T$. Note that L has rank $2p$ in $\mathcal{F}_1(\varepsilon)$. Let M be the $2(p+q) \times (p+q)$ block diagonal matrix,

$$M \triangleq \begin{bmatrix} L_{01}b & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & L_{p-1,p}b \end{bmatrix}$$

It can be shown that (Lemma E.13 [26]) $\forall b \in \mathcal{C}_{\hat{\varphi}} \cap \mathcal{F}_1(\varepsilon)$

$$\frac{\beta}{2\gamma^k} D^2 \hat{\varphi} =$$
$$\frac{k}{\gamma} I_{2p} + 2LMM^T L^T - \frac{2}{k} LMoo^T M^T L^T - L_1 L_1^T + L_0 L_0^T$$

Letting $A = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, we may re-write the previous equation as:

$$\frac{\beta}{\gamma^k} D^2 \hat{\varphi} =$$
$$\frac{k}{\gamma} I_{2p} + \underbrace{L\left(2M\left(I_{q+p} - \frac{1}{k} oo^T\right) M^T + A\right) L^T}_{B}$$

Next consider $I_{q+p} - \frac{1}{k} oo^T$. By construction,

$$I_{q+p} - \frac{1}{k} oo^T = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} - \frac{1}{k} \begin{bmatrix} 1_{p \times p} & -1_{p \times q} \\ -1_{q \times p} & 1_{q \times q} \end{bmatrix}$$

Now let $V = 2\left(I_{q+p} - \frac{1}{k} oo^T\right)$. We show that V is full rank via considering its elements:

$$|v_{ij}| = \begin{cases} 2\left(1 - \frac{1}{k}\right) & \text{if } i = j \\ \frac{2}{k} & \text{otherwise} \end{cases}$$

Note that for each row of V,

$$\sum_{j \neq i} |v_{ij}| = 2\frac{p+q-1}{k} \tag{22}$$

Hence, if $k > p + q$, then for every i, if every diagonal element

$$\begin{aligned} |v_{ii}| &= 2\left(1 - \frac{1}{k}\right) \\ &> 2\frac{p+q-1}{k} \\ &= \sum_{j \neq i} |c_{ij}| \end{aligned}$$

Hence, since V is strictly diagonally dominant, it follows that V is of full rank by Levy-Desplanques theorem [37] and we have,

$$rank(2\left(I_{q+p} - \frac{1}{k} oo^T\right)) = p + q$$

Hence the result holds for its inverse. Using Lemma C.20, each v_{ij}^{-1} entry of V^{-1} has the following form:

$$v_{ij}^{-1} = \begin{cases} \frac{1}{2} \frac{k-(p+q)+1}{k-(p+q)} & \text{if } i = j \\ \frac{1}{2} \frac{1}{k-(p+q)} & \text{if } i, j \leq p \text{ and } i \neq j \\ \frac{1}{2} \frac{1}{k-(p+q)} & \text{if } i, j > p \text{ and } i \neq j \\ -\frac{1}{2} \frac{1}{k-(p+q)} & \text{otherwise} \end{cases}$$

Now consider $rank(MVM^T + A)$. According to a theorem by Sherman, Morrison and Woodbury, $MVM^T + A$ is invertible iff $M^T A^{-1} M + V^{-1}$ is invertible [21]. Namely,

$$rank(MVM^T + A) = 2(p + q)$$
iff
$$rank\left(M^T A^{-1} M + V^{-1}\right) = p + q$$

Now, consider $M^T A^{-1} M$. Note that by construction $A^{-1} = A$ and $rank(A) = 2(p+q)$. Let M be represented as a block matrix as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_22 \end{bmatrix}$$

where M_{11} is a $2p \times p$, M_{12} is a $2p \times q$, M_{21} is a $2q \times p$ and M_{22} is a $2q \times q$ matrix respectively. Hence, it can be shown that

$$M^T A M = \begin{bmatrix} M_{11}^T M_{11} & 0_{p\times q} \\ 0_{q\times p} & -M_{22}^T M_{22} \end{bmatrix}$$

By construction, both $M_{11}^T M_{11}$ and $-M_{22}^T M_{22}$ are diagonal matrices. Furthermore, if $\mathcal{S}$ is the ordered set of permutations of $\mathcal{P}$ and ι denotes the lexicographic order of a given permutation ln, each diagonal entry $\breve{m}_{ii}$ of $M^T AM$ is defined as

$$\breve{m}_{ii} = \begin{cases} \|L_{0i}b\|^2 & \text{if } i \le p \\ -\|L_{ln}b\|^2 & \text{if } i = p + \iota(ln) \end{cases}$$

where it should be recalled that $\|L_{0i}b\|^2 = \frac{b_i^T b_i}{b_i^T b_i - \rho_{0i}^2}$ and $\|L_{ln}b\| = \frac{(b_l-b_n)^T(b_l-b_n)}{(b_l-b_n)^T(b_l-b_n)-\rho_{ln}^2}$.

Let $X = M^T A^{-1} M + V^{-1}$ where x_{ij} denote the elements of x. Next, we show that X is a nonsingular matrix via diagonal dominance. First, note that each diagonal element $x_{ii}, l = 1, \ldots, p+q$ is equal to:

$$x_{ii} = \begin{cases} \frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)} + \frac{\beta_{0i}+\rho_{0i}^2}{\beta_{0i}} & \text{if } i \le p \\ \frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)} - \frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} & \text{if } i > p \text{ and } i = p + \iota(ln) \end{cases}$$

On the other hand, each off-diagonal element x_{ij}, $i \neq j$ is equal to:

$$x_{ij} = \begin{cases} \frac{1}{2}\frac{1}{k-(p+q)} & \text{if } i,j \le p \text{ and } i \neq j \\ \frac{1}{2}\frac{1}{k-(p+q)} & \text{if } i,j > p \text{ and } i \neq j \\ -\frac{1}{2}\frac{1}{k-(p+q)} & \text{otherwise} \end{cases}$$

Consider $\sum_{j\neq i} |x_{ij}|$.

$$\sum_{j\neq i} |x_{ij}| = \frac{1}{2}\frac{p+q-1}{k-(p+q)}$$

First consider $i \le p$. Using Lemma C.21,

$$\begin{aligned} |x_{ii}| - \sum_{j\neq i} |x_{ij}| &= \frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)} + \frac{\beta_{0i}+\rho_{0i}^2}{\beta_{0i}} \\ &\quad -\frac{1}{2}\frac{p+q-1}{k-(p+q)} \\ &= \frac{-p^2-p+k+2}{-p^2-p+2k} + \frac{\beta_{0i}+\rho_{0i}^2}{\beta_{0i}} \end{aligned}$$

Since the first term on the rhs is is an increasing function of k and $\frac{\beta_{0i}+\rho_{0i}^2}{\beta_{0i}} > 1$, $\exists$ $\mathrm{K}_4(\varepsilon)$, such that for $k > \mathrm{K}_4(\varepsilon)$

$$|x_{ii}| > \sum_{j\neq i} |x_{ij}|$$

Now consider $i > p$. First note that either $0 < \beta_{ln} \le \varepsilon$ (Case 1) or $\varepsilon < \beta_{ln} \le (2\rho_0 - \rho_{ln})^2 - \rho_{ln}^2$ (Case 2). The first case holds for all robot pairs that are within ε neighborhood of each other while the second case holds for all the other remaining pairs since the workspace is bounded. Of course, by assumption, as we considering $\mathcal{F}_1(\varepsilon)$, there exists at least one $(l,n) \in Q$ such that $0 < \beta_{ln} \le \varepsilon$. Hence

$$-\infty < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \le -\frac{\rho_{ln}^2}{\varepsilon}$$

Using Lemma C.22, $|x_{ii}|$ is bounded as:

$$\left|\frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)}-\frac{\rho_{ln}^2}{\varepsilon}\right| \leq |x_{ii}| < \infty$$

Now let us consider $|x_{ii}| - \sum_{j\neq i}|x_{ij}|$ with the lower bound on $|x_{ii}|$ which is equal to

$$\left|\frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)}-\frac{\rho_{ln}^2}{\varepsilon}\right| - \frac{1}{2}\frac{p+q-1}{k-(p+q)} \tag{23}$$

Since ε is an arbitrarily small design parameter as discussed in Section 2.3, the term $\frac{\rho_{ln}^2}{\varepsilon}$ will dominate in Eq. 23 and hence $|x_{ii}| > \sum_{j\neq i}|x_{ij}|$. Now consider the second case where using Lemma C.23, the bound on $-\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}}$ is as:

$$-\frac{\varepsilon+\rho_{ln}^2}{\varepsilon} < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \leq -\frac{\rho_{ln}^2}{(2\rho-\rho_{ln})^2}$$

Hence, $|x_{ii}|$ is bounded as:

$$\left|\frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)}-\frac{\rho_{ln}^2}{(2\rho_0-\rho_{ln})^2}\right| \quad \leq \quad |x_{ii}| \leq \infty$$

Let us now consider with $|x_{ii}| - \sum_{j\neq i}|x_{ij}|$ with $|x_{ii}|$ at its smallest value as:

$$\left|\frac{1}{2}\frac{k-(p+q)+1}{k-(p+q)}-\frac{\rho_{ln}^2}{(2\rho_0-\rho_{ln})^2}\right| - \frac{1}{2}\frac{p+q-1}{k-(p+q)}$$

This is an increasing function of k. Hence for $k \geq \mathrm{K}_{5_i}(\varepsilon) > 0$, $|x_{ii}| - \sum_{j\neq i}|x_{ij}| > 0$ which implies that $|x_{ii}| > \sum_{j\neq i}|x_{ij}|$. Now let $\mathrm{K}_5(\varepsilon) = max_{i\geq p}\mathrm{K}_{5_i}(\varepsilon)$. Hence, since X is strictly diagonally dominant, hence according to Levy-Desplanques theorem [37]:

$$rank\,(X) = rank\left(M^TA^{-1}M + V^{-1}\right) = p+q$$

This in turn implies that

$$rank\left(MVM^T + A\right) = 2(p+q)$$

Recalling that $B = L\left(MVM^T + A\right)L^T$, since $rank(L) = rank(L^T) = 2p$, according to lower and upper bounds on the rank of product of matrices [60], the following holds true :

$$2p \leq rank(B) \leq 2p$$

Hence, B is ensured of being full rank and hence non-singular. If $B = D\Lambda D^{-1}$ be an eigendecomposition of B where the Λ is a diagonal matrix with eigenvalues λ_i and U is the matrix of eigenvectors, then

$$\begin{aligned}
\left|B+\frac{k}{\gamma}I_{2p}\right| &= \left|UDU^{-1}+\frac{k}{\gamma}I_{2p}\right| \\
&= |U|\left|D+\frac{k}{\gamma}I_{2p}\right|\left|U^{-1}\right| \\
&= \left|D+\frac{k}{\gamma}I_{2p}\right| \\
&= \prod_{l=1}^{2p}(\lambda_l+\frac{k}{\gamma})
\end{aligned} \tag{24}$$

Recall that by Prop. C.4, there exists at least one negative eigenvalue. If $b \in \mathcal{C}_\varphi$, then by definition

$$k\beta\nabla\gamma \quad = \quad \gamma\nabla\beta \tag{25}$$

Thus, Eq. 25 is equivalently expressed as:

$$\nabla\gamma \quad = \quad \frac{\gamma}{k\beta}\nabla\beta$$

Recall that since $\gamma(b) \triangleq (b-g)^T(b-g)$ by definition, equivalently $\gamma(b) = \frac{1}{4}\nabla\gamma^T\nabla\gamma$. Hence, at a critical point, γ is equal to:

$$\gamma = \frac{\gamma^2}{4k^2\beta^2}\nabla\beta^T\nabla\beta = \frac{\gamma^2}{k^2}\Omega \tag{26}$$

with $\Omega \triangleq \frac{\nabla\beta^T\nabla\beta}{4\beta^2}$ which implies that

$$\gamma = \frac{k^2}{\Omega} \tag{27}$$

Rewriting $\frac{k}{\gamma}$ after substituting for γ using Eq. 27 and simplifying

$$\frac{k}{\gamma} = \frac{\Omega}{k}$$

Thus, for $b \in \mathcal{C}_\varphi$, $B + \frac{k}{\gamma}I_{2p}$ is equal to $B + \frac{\Omega}{k}I_{2p}$. Using Eq. 24

$$\left|B + \frac{k}{\gamma}I_{2p}\right| \quad = \quad \prod_{l=1}^{2p}(\lambda_l + \frac{\Omega}{k})$$

By Lemma C.24, $\mathcal{F}_1(\varepsilon)$ can be partitioned into two subsets - $\mathcal{F}_{0\eta}$ and $\mathcal{F}_{0\varepsilon} = \mathcal{F}_1(\varepsilon) - \mathcal{F}_{0\eta}$. Now consider the negative eigenvalue of B having smallest magnitude and denote it by $\lambda'(B)$. Consider the closure of $\mathcal{F}_{0\varepsilon}$ - namely $\bar{\mathcal{F}}_{0\varepsilon}$. As $\bar{\mathcal{F}}_{0\varepsilon}$ is compact, let

$$\lambda^* = \inf_{b\in\bar{\mathcal{F}}_{0\varepsilon}} |\lambda'(B)|$$

Finally, choose

$$k > \sup_{b\in\bar{\mathcal{F}}_{0\varepsilon}} \frac{\Omega}{\lambda^*} \triangleq \mathrm{K}_6(\varepsilon)$$

Thus, if $b \in \mathcal{C}_\varphi \cap \mathcal{F}_1(\varepsilon)$, then $\frac{\gamma}{k}I_{2p} + B$ is nonsingular. The proof is completed by choosing

$$\mathrm{N}(\varepsilon) = \max\{\mathrm{K}_4(\varepsilon), \mathrm{K}_5(\varepsilon), \mathrm{K}_6(\varepsilon)\} \qquad \square$$

Lemma C.20

$$V^{-1} \quad = \quad \frac{1}{2}\left\{I_{q+p} + \frac{1}{k}\frac{1}{1 + tr(-\frac{1}{k}oo^T)}oo^T)\right\}$$

Proof: According to a lemma as presented in [47], if $V = 2\left(I_{q+p} - \frac{1}{k}oo^T\right)$, then V^{-1} is equal to:

$$\begin{aligned} V^{-1} \quad &= \quad \frac{1}{2}(I_{q+p} - \frac{1}{k}oo^T)^{-1} \\ &= \quad \frac{1}{2}\left\{I_{q+p} + \frac{1}{1 + tr(-\frac{1}{k}oo^T)}I_{q+p}\frac{1}{k}oo^T I_{q+p}\right\} \end{aligned}$$

Noting that $tr(\frac{1}{k}oo^T) = \frac{p+q}{k}$ and simplifying

$$\frac{1}{1 + tr(-\frac{1}{k}oo^T)} = -\frac{p+q}{k}$$

the result follows as:

$$V^{-1} \quad = \quad \frac{1}{2}\left\{I_{q+p} + \frac{1}{k}\frac{1}{1 + tr(-\frac{1}{k}oo^T)}oo^T)\right\} \square$$

Lemma C.21

$$\frac{k-(p+q)+1}{k-(p+q)} - \frac{p+q-1}{k-(p+q)} \quad = \quad 2\frac{-p^2-p+k+2}{-p^2-p+2k}$$

Proof: Simplifying

$$\frac{k-(p+q)+1}{k-(p+q)} - \frac{p+q-1}{k-(p+q)} \quad = \quad \frac{k-2(p+q)+2}{k-(p+q)}$$

Recalling that $q = \frac{p(p-1)}{2}$ and substituting,

$$\frac{k-(p+q)+1}{k-(p+q)} - \frac{p+q-1}{k-(p+q)} \quad = \quad 2\frac{-p^2-p+k+2}{-p^2-p+2k} \,\Box$$

Lemma C.22

$$-\infty < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \leq -\frac{1}{\varepsilon}.$$

Proof: By assumption, $0 < \beta_{ln} \leq \varepsilon$. Hence, $-\infty < -\frac{1}{\beta_{ln}} \leq -\frac{1}{\varepsilon}$. The inequality will continue to hold after multiplying all terms by ρ_{ln}^2 and subtracting 1 from only the first two terms as $-\infty < -\frac{\rho_{ln}^2}{\beta_{ln}} - 1 \leq -\frac{\rho_{ln}^2}{\varepsilon}$. Simplifying $-\infty < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \leq -\frac{1}{\varepsilon}$. $\Box$

Lemma C.23

$$-\frac{\varepsilon+\rho_{ln}^2}{\varepsilon} < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \leq -\frac{(2\rho_0-2\rho_{ln})^2+\rho_{ln}^2}{(2\rho-2\rho_{ln})^2}$$

Proof: By assumption, $\varepsilon < \beta_{ln} \leq (2\rho_0-\rho_{ln})^2 - \rho_{ln}^2$ which implies that

$$-\frac{1}{\varepsilon} < -\frac{1}{\beta_{ln}} \leq -\frac{1}{(2\rho_0-\rho_{ln})^2-\rho_{ln}^2\}}$$

The inequality will continue to hold after multiplying all terms by ρ_{ln}^2 and subtracting 1 from only the first two terms as

$$-\frac{\rho_{ln}^2}{\varepsilon} - 1 < -\frac{\rho_{ln}^2}{\beta_{ln}} - 1 \leq -\frac{\rho_{ln}^2}{(2\rho_0-\rho_{ln})^2-\rho_{ln}^2\}}$$

Simplifying

$$-\frac{\varepsilon+\rho_{ln}^2}{\varepsilon} < -\frac{\beta_{ln}+\rho_{ln}^2}{\beta_{ln}} \leq -\frac{\rho_{ln}^2}{(2\rho_0-\rho_{ln})^2-\rho_{ln}^2}$$

$\Box$

For the following lemma, let $N_\eta(\partial\mathcal{F})$ denote $\eta-$neighborhood of $\partial\mathcal{F}$ as:

$$N_\eta(\partial\mathcal{F}) = \{b \in \partial\mathcal{F} \cup \mathcal{F}_1(\varepsilon)\,|0 \leq \beta < \eta, \eta < \varepsilon\}$$

Lemma C.24 $\exists\eta > 0$ *such that* $\mathcal{C}_\varphi \cap N_\eta(\partial\mathcal{F}) = \emptyset$

Proof: (By contradiction) Let $b \in \partial\mathcal{F} \cap \mathcal{C}_\varphi$. Suppose that the interior of $\mathcal{F}$ - $\overset{\circ}{\mathcal{F}}$ contains infinitely many points close b. By Milnor's Curve Selection Lemma, there exists a real analytic function $\zeta : [0,1] \to \mathcal{F}$ such that $\zeta(0) = b$ and $\zeta((0,1)) \subset \overset{\circ}{\mathcal{F}}$ and every point in $Im(\zeta)$ is a critical point of φ. This implies that $\forall b \in Im(\zeta)$, $\varphi(b) = 1$. Since $b \in \overset{\circ}{\mathcal{F}}$, this is not possible. Hence, contradiction.$\Box$

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