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Essays on Consumer Default

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Essays on Consumer Default

Abstract
Legislation dealing with consumer default has consistently struggled with an important trade-off: more debt forgiveness directly benefits households but indirectly makes credit more expensive. This dissertation assesses this trade-off in two ways and provides a tool for future research into this and other topics. Specifically, the first chapter analyzes how business cycles affect the positive and normative consequences of eliminating or restricting default, including the default restrictions put in place by the recent Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (BAPCPA). The second chapter examines the implications of eliminating bankruptcy protection while still allowing households access to informal default. Lastly, the third chapter provides a novel tool for computing equilibrium in dynamic heterogeneous-agent economies, such as the economy in the first chapter.

There are four main findings. First, accounting for business cycles substantially reduces the welfare benefit of eliminating default. With or without business cycles, eliminating default greatly expands credit availability; however, when a protracted recession is possible, households use less credit unless they have a default option. Second, while business cycles reduce the welfare gain of eliminating default, this is not necessarily true for restricting default: BAPCPA significantly improves welfare whether or not aggregate risk is taken into account. Because the policy makes default more costly only for earnings-rich households, the reform improves credit markets while still preserving most of the insurance value of default. Third, eliminating bankruptcy protection leads to an increase in total defaults, debt, and welfare. Without bankruptcy protection, creditors can collect on defaulted debt to the extent permitted by wage garnishment laws. The elimination lowers the default premium on unsecured debt and permits low-net-worth individuals suffering bad earnings shocks to smooth consumption by borrowing. Last, the proposed computational method is capable of delivering a more accurate solution than the most widely used method and can be as efficient.

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DEDICATION

Dedicated to my loving wife Joanne, my encouraging parents Duff and Laine, and my faithful God, to whom be all the glory.
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This dissertation has benefited immensely from the supervision of Dirk Krueger who always was willing to help. Without the opportunity to work with Satyajit Chatterjee and have his guidance, the first two chapters of this dissertation would probably not have been written. Every suggestion Jesús Fernández-Villaverde has made regarding this dissertation has strengthened it.

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Legislation dealing with consumer default has consistently struggled with an important trade-off: more debt forgiveness directly benefits households but indirectly makes credit more expensive. This dissertation assesses this trade-off in two ways and provides a tool for future research into this and other topics. Specifically, the first chapter analyzes how business cycles affect the positive and normative consequences of eliminating or restricting default, including the default restrictions put in place by the recent Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (BAPCPA). The second chapter examines the implications of eliminating bankruptcy protection while still allowing households access to informal default. Lastly, the third chapter provides a novel tool for computing equilibrium in dynamic heterogeneous-agent economies, such as the economy in the first chapter.

There are four main findings. First, accounting for business cycles substantially reduces the welfare benefit of eliminating default. With or without business cycles, eliminating default greatly expands credit availability; however, when a protracted recession is possible, households use less credit unless they have a default option. Second, while business cycles reduce the welfare gain of eliminating default, this is not necessarily true for restricting default: BAPCPA significantly improves welfare whether or not aggregate risk is taken into account. Because the policy makes default more costly only for earnings-rich households, the reform improves credit markets while still preserving most of the insurance value of default. Third, eliminating bankruptcy protection leads to an increase in total defaults, debt, and welfare. Without bankruptcy protection, creditors can collect on defaulted debt to the extent permitted by wage garnishment laws. The elimination lowers the default premium on unsecured debt and permits low-net-worth individuals suffering bad earnings shocks to smooth consumption by borrowing. Last, the proposed computational method is capable of delivering a more accurate solution than the most widely used method and can be as efficient.
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Chapter 1

Evaluating Default Policy: The Business Cycle Matters

Grey Gordon

Summary

Legislation dealing with consumer default has consistently struggled with an important trade-off: more debt forgiveness directly benefits households but indirectly makes credit more expensive. Complicating the issue is that part of the risk households face is aggregate risk. This paper asks, “How does aggregate risk affect the consequences of eliminating or restricting default?” I find aggregate risk substantially reduces the welfare benefit of eliminating default, but its effect on restricting default depends crucially on the restrictions in place. In a calibrated general equilibrium life-cycle model, eliminating default results in an ex-ante welfare gain of 1.8% of lifetime consumption in steady state. Once the business cycle—the type of aggregate risk considered in this paper—is added, this gain drops to .5%. With or without aggregate risk, eliminating default greatly expands credit availability; however, when a protracted recession is possible, households use less credit unless they have a default option. While aggregate risk reduces the welfare gain of eliminating default this is not necessarily true for restricting default. A policy that pushes earnings-rich households into partial debt repayment (like a major 2005 reform) generates a gain of 2% with or with-
out the business cycle. Moreover, with the policy instituted, eliminating default produces a welfare loss of .1%, which aggregate risk deepens to 1.5%. The reform improves credit markets while still preserving most of the insurance value of default. A different type of policy that restricts default to only be in recessions or expansions sharply reduces welfare relative to always allowing default (a loss of 1.4%) or never allowing it (a loss of 1.9%). The policy introduces uncertainty that makes credit expensive and keeps households from relying on the default option.

1.1 Introduction

In both recent and past history, the merits and flaws of legislation dealing with consumer default have been subject to intense debate.1 Central to the arguments has been the trade-off between debt relief and credit.2 More debt relief directly benefits households but indirectly harms them through reduced access to credit: to cover losses due to default, creditors must charge a premium. Complicating the issue is that the circumstances of unfortunate debtors are often caused by events outside their control. Panics, financial crises, stock market crashes, and housing busts can leave otherwise prosperous, stable households in financial ruin.

The interaction between aggregate risk and consumer default is particularly relevant in light of recent US history. Preceded by twenty years of stable economic growth, the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 made debt-forgiveness significantly more difficult.3 Since then the economy has slipped into the most severe recession since the Great Depression. Far from being just coincidence, this sequence of events fits into a pattern where legislation responds to negative aggregate shocks with more debt

---

1 Bankruptcy laws changed frequently prior to 1900 and even since then have had major revisions in 1938, 1978, and 2005. Bankruptcy in its present form did not take shape until 1898, and even then it was highly controversial with efforts to repeal it made in 1903, 1909, and 1910 (Warren, 1935, p. 143).

2 For instance, one Senator argued for a bill he introduced in 1885 saying, “At present interest rates … are from 8 to 20%, because of the doubt whether the creditor will have his fair share of the estate …” (quoted in Warren, 1935, p. 133). Another Senator argued against the bill saying, “it is no time to pass bills of this character—when every man in trade … is suffering from the depression …” (quoted in Warren, 1935, p. 132).

3 The key provision of the reform is that households with above-median income may no longer file for Chapter 7 bankruptcy. Because Chapter 7 bankruptcy typically offers much more debt forgiveness than the only other widely-used form of consumer bankruptcy, Chapter 13, the reform severely restricted debt forgiveness for these households. The reform made a number of other significant changes (see White, 2007).
relief and to positive shocks with less. Was the 2005 reform beneficial in light of the last recession? Would it have been if the economy had remained stable? How does law changing in response to aggregate shocks impact households? More generally, how does aggregate risk affect the welfare consequences of eliminating or restricting default? This paper seeks to answer these questions.

I find aggregate risk substantially decreases the welfare benefit of eliminating default. In a calibrated general equilibrium life-cycle model and absent aggregate risk, eliminating default improves welfare by 1.8% of lifetime consumption for newborn households. Once a business cycle—the type of aggregate risk considered in this paper—is added, this gain drops to .5%. Eliminating default in either the steady state or business cycle environment means that creditors offer any amount of debt at a risk-free rate. For their part, households avoid taking out debt beyond what they can repay with probability one. This amount is much smaller in the business cycle because of the possibility, albeit remote, of a life-long recession which would severely depress aggregate capital and wages. With a default option, credit is more expensive and less abundant, but households can safely use all of it by defaulting if a protracted recession hits.

While including aggregate risk reduces the welfare gain of eliminating default, I find this need not be the case when restricting default. I consider two cases of default being restricted. First is a policy change designed to mimic the 2005 reform by forcing households with above-median earnings to repay a substantial fraction (but not all) of their debt. This reform increases welfare by around 2% with or without the business cycle. Further, with the reform instituted, households experience a welfare loss from eliminating default: .1% absent aggregate uncertainty and 1.5% with it. The reform increases repayment rates of the earnings rich, those with the greatest ability to repay, resulting in a large expansion of credit. At the same time, most of the insurance value of default is preserved: all households have access to some amount of default and earnings-poor households can easily default. This

---

4I thank Satyajit Chatterjee for pointing this out to me. A recent example of this pattern, besides the 2005 reform, is the Emergency Economic Stabilization Act of 2008—passed in response to the latest recession—that authorized an ongoing mortgage-modification program (the Home Affordable Modification Program). There are also many older examples where “prosperity muted demands for relief” until a panic, drought, or war “eased . . . opposition to a bankruptcy law” (Coleman, 1974, p. 24, 28). Robe, Steiger, and Michel (2006) argue from a broad historical perspective that default penalties become lighter as risk becomes “more important” (p. 3).
result suggests that “default” is not a problem but rather the amount of default allowed.

The second type of default restriction I consider is meant to capture aggregate risk’s
effect on legislation by only allowing default in recessions or expansions. Surprisingly, the
outcome of either policy is inferior to always having default (a loss of 1.4%) or never having
it (a loss of 1.9%). The reason for this is that uncertainty about whether or not default will
be allowed has two negative effects. First, it causes creditors to charge a default premium
which households must always pay.\footnote{Consistent with the rest of the literature default premiums are paid upfront. If the premiums were charged only \textit{after} the aggregate state was revealed, this effect would not be present.} Second, households must be prepared to never have the
default option by limiting the amount of debt they take on. Consequently debt is expensive
\textit{and} households cannot rely on the default option.

Most research suggests that eliminating default would substantially improve welfare in
the absence of uninsured “disaster” states.\footnote{Examples include Athreya (2002), Livshits, MacGee, and Tertilt (2007), Athreya (2008), Athreya, Tam, and Young (2009a), Athreya, Tam, and Young (2009b), and Chatterjee and Gordon (2011).} The result has proven surprisingly robust. In
particular, it has been found in both partial and general equilibrium environments, for
finitely-lived and infinitely-lived households, under different informational settings, and un-
der alternative debt-pricing formulations.\footnote{Throughout the paper “partial equilibrium” refers to allowing default-risk premia to adjust while holding fixed default-risk-free prices at the equilibrium values in the default economy. In the business cycle model this requires using the law of motion from the default economy to forecast prices.} In addition, Athreya et al. (2009b) find the result
holds for numerous specifications of earnings risk and preferences. Further, Chatterjee and
Gordon (2011) consider multiple alternatives to bankruptcy law and find one that com-
pletely eliminates default is optimal. To my knowledge, the only exception in the literature
is the work by Li and Sarte (2006) that shows accounting for general equilibrium effects can
reduce the welfare benefit of eliminating default even to the point of making it a loss.\footnote{One potentially important caveat is that none of these papers have computed the transition. Chatterjee and Gordon (2011) compute the transition for eliminating bankruptcy—the focus of their paper—but not for eliminating default.} A
central contribution of the present paper is to show that aggregate risk is also an important
determinant of the welfare consequences of eliminating default.

Research on the welfare gains of restricting access to default is more mixed and has
have found modest changes in welfare and allocations from the reform.\footnote{Athreya (2002) studied the Bankruptcy Reform Act of 1999, the main provisions of which would even-}
Nakajima, and Ríos-Rull (2007) and Mitman (2011), on the other hand, have found sizable welfare gains. None of these papers have allowed for aggregate risk, and a contribution of the present paper is to take aggregate risk into account in studying how the 2005 reform—passed just before the Great Recession—affected households. I show that the reform likely made them better off. Another contribution of the paper is to examine default policy that restricts or permits default in response to aggregate shocks. I show that such a policy can be very harmful to households.

Little quantitative work has been done on default and business cycles. Nakajima and Ríos-Rull (2004) and Nakajima and Ríos-Rull (2005) examine whether bankruptcy amplifies or smooths aggregate shocks. In the first paper, the authors use a production economy where creditors make profits or losses and distribute them via a dividend. In the second, the authors use a storage economy but impose a special timing on the model to ensure creditors make zero profits loan-by-loan. While these papers examine default’s effect on aggregate dynamics, the present paper examines the effect of aggregate dynamics on default. Consequently, they are complementary. A technical contribution of the paper is to model the economy with aggregate uncertainty in a way that ensures creditors make zero profits loan-by-loan without imposing a special timing. This is done by allowing some households to have adjustable portfolios and offset losses from charge-offs.

The quantitative framework I use is an extension of Chatterjee et al. (2007) and Livshits et al. (2007) to include aggregate uncertainty. Recessions are modeled as a negative shock to total factor productivity, a large increase in earnings variance a la Storesletten, Telmer, and Yaron (2004), and a decline in exogenous labor supply (which is calibrated to match the standard deviation of hours worked). General equilibrium and the life-cycle are both included as the work of Li and Sarte (2006) and Livshits et al. (2007) suggests these are very important for evaluating default policy. Following Athreya et al. (2009b), I abstract from expenditure shocks (large negative shocks to asset positions).

---

10 Specifically, households must make their default decisions before the aggregate shock is realized.

11 Livshits et al. (2007) have argued that expenditure shocks are quantitatively important for the evaluation of default policy. I abstract from expenditure shocks for three reasons. First, as Athreya et al. (2009b) argue, while catastrophic events (like large expenditure shocks) are sufficient to warrant loose default penalties, an important and practical question is whether they are also necessary. It is a practical question because debt forgiveness has often been made contingent on the hardships faced by households. Second,
In a series of robustness exercises, I find the result that aggregate risk substantially reduces the welfare gain of eliminating default is quantitatively robust. Specifically, it is true for different earnings processes, mortality risk profiles, and economies of scale in household size. It is also true when a substantial portion of labor income is guaranteed. Further, the result holds even if default is not completely eliminated but just made very costly. While the reduction in the welfare gain is robust, the level of the welfare gain is not robust and can vary greatly. This is especially true when a substantial portion of earnings is guaranteed. In this case, the welfare gain from eliminating default can be significantly lower after including aggregate risk but still be very high in absolute terms.

The rest of the paper is organized as follows. Section 1.2 describes the models with and without aggregate uncertainty. Section 1.3 discusses the calibration and baseline model properties. Section 1.4 examines the consequences of eliminating default. Section 1.5 examines the consequences of restricting default. Section 1.6 concludes. Appendices include extended model and data descriptions, notes on computation, and robustness exercises.

1.2 Model

I first lay out the model without aggregate uncertainty and then modify it to include aggregate uncertainty. Time is discrete in both models.

Steady State Model

The steady state model is a completely standard model of consumer default with general equilibrium and the life cycle.

Demographics, Endowments, Technology, and Preferences

The economy is populated by a unit mass of households who die with certainty after $T$ years. Households are endowed with one unit of time but differ in the productive efficiency $e$ of while households may not purchase insurance for expenditure shocks (which in part represent large medical bills), insurance is often available. Hence assuming these shocks are uninsurable, and that this would remain the case were default to be eliminated, is not entirely reasonable. Third, as Livshits et al. (2007) point out, expenditure shocks of a sufficiently large size necessitate default. That said, expenditure shocks would likely increase the welfare gains of low default-cost regimes relative to high (but not infinite) cost regimes, and I plan to explore this.
their time endowment and in certain other characteristics \( s \), including age. Characteristics lie in a finite set \( S \) and evolve according to a Markov chain \( F(s'|s) \).\(^{12}\) Efficiency is distributed iid conditional on characteristics according to a density function \( f(e|s) \) which has support in \( \mathbb{R}^{++} \) for all \( s \).

Households face an age-dependent conditional probability of survival \( \rho_s \). Households who die are replaced by “newborn” households having zero assets, efficiency distributed according to \( \hat{f}(e|s) \), and characteristics distributed according to \( \hat{F}(s) \). The utility from death is normalized to zero.

Preferences over consumption are given by

\[
\sum_{t=1}^{T} \beta^{t-1} U(c_t, s_t)
\]

(1.1)

where \( \beta > 0 \) is the discount factor and \( c \) is consumption. The period utility function is

\[
U(c, s) = \frac{(c/\theta_s)^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \sigma \neq 1.
\]

(1.2)

where \( \theta_s \) is the age-dependent effective number of household members.\(^{13}\) When \( \sigma = 1 \), \( U(c, s) = \log(c/\theta_s) \). Households do not value leisure.

A neoclassical production firm operates the production technology \( K^\alpha N^{1-\alpha} \) with \( \alpha \in (0,1) \) that uses as inputs capital \( K \) rented at rate \( r \) and labor \( N \) hired at wage \( w \). Capital depreciates at a constant rate \( \delta \in (0,1) \).

**Legal Environment**

The legal environment is designed to resemble Chapter 7 bankruptcy in US law. Households have a credit history \( h \in \{0, 1\} \). Households in good standing, \( h = 0 \), have the right to file for bankruptcy. If they do, three things happen in the filing period: their debts are discharged in exchange for all their assets, they may not save and may not borrow, and a fraction \( \chi \in (0,1) \) of their income is given to creditors.\(^{14}\) In the period after filing, a household’s credit history records that they filed for bankruptcy in the past, \( h = 1 \). This record is

\(^{12}\)Age evolves deterministically so that, if \( t(s) \) denotes the age of a household, \( F(s'|s) = 0 \) whenever \( t(s') \neq t(s) + 1 \) (except if \( t(s) = T \) in which case the transition is immaterial).

\(^{13}\)Since \( \sigma \) will be 2 in the calibration, \( \theta_s \) will shift marginal utilities in the same way as in, for instance, Attanasio and Weber (1995) and Gourinchas and Parker (2002).

\(^{14}\)There are a number of reasons to believe some income is transferred to creditors in the period of (but not after) default. First, as discussed in Livshits, MacGee, and Tertilt (2010, p. 174, footnote 12),
removed and their history shows $h = 0$ with probability $1 - \lambda$. For as long as a household is in bad standing, $h = 1$, a household is not allowed to borrow but may save.\footnote{Musto (2004) finds credit opportunities are severely restricted while the record of a bankruptcy remains.} Households begin life with $h = 0$.

**Asset Markets**

Households do not directly hold claims to capital, but rather enter into debt or savings contracts with a financial intermediary who owns the capital stock and rents it to the production firm. I now describe these contracts.

From a household’s perspective, a debt/savings contract looks just like a risk-free discount bond. The face value $a'$ lies in a finite set $A$ that includes zero and both positive and negative elements. I use the convention $a' \geq 0$ denotes savings and $a' < 0$ denotes borrowing. Because of default, each contract has a potentially different yield and so has a distinct price $q(a', s)$ that varies with all factors that can potentially influence next period’s default decision. Since $e$ is iid conditional on $s$, these are entirely summarized in characteristics $s$.\footnote{The contract can be made contingent on $(a, e, s, h)$. However since the default decision next period is a function of $(a', e', s', h')$, only $a'$ and $s$ matter. The credit history doesn’t appear in $q(a', s)$ because debt, $a' < 0$, is only consistent with $h = 0$ (and $h' = 0$) and savings will not be defaulted upon regardless of $h$.}

The prices $q(a', s)$ for $a' \in A, s \in S$ define a “price schedule.”

From the intermediary’s perspective, a debt/savings contract is a repayment agreement that through pooling gives a certain return. In exchange for $q(a', s)a'$ of the consumption good, the intermediary expects a yield $\rho_s p(a', s)a'$ next period comprised of two parts. First, from households surviving to the next period, he expects to recover only $p(a', s) \in [0, 1]$ fraction of the debt because some may default. Second, he expects that only $\rho_s$ fraction of households will survive to even have a chance of repaying.

In addition to contracts, the intermediary (but not households) has access to two other assets, capital and a risk-free discount bond. The bond $B'$ has price $\bar{q}_B$ and capital $K'$ has return $1 + r - \delta$. Capital cannot be short sold and the bond is in zero net supply. Note that no arbitrage necessitates the bond’s return be equated to the return on contracts. This pins the bankruptcy code incorporates “good faith” requirements that are not explicitly modeled here. Second, earnings are sometimes garnished before households file for bankruptcy with one estimate in Chatterjee and Gordon (2011) putting the net recovery rate on defaulted revolving debt at around 15%. Third, some households initially file for Chapter 13 (which results in a debt repayment plan) but subsequently file for Chapter 7.
down the price schedule as

\[ q(a', s) = \tilde{q} B \rho_s p(a', s) \]  

(1.3)

for \( a' \neq 0 \). Without loss of generality, \( q(0, s) \) is taken to be zero.

**The Household Problems**

The household problems are as follows.\(^{17}\) Let \( V(a, e, s, h) \) denote the value function of a household. A household in good standing \( h = 0 \) that can repay their debt solves

\[ V(a, e, s, 0) = \max_{d \in \{0, 1\}} (1 - d) \cdot V^R(a, e, s) + d \cdot V^D(e, s) \]  

(1.4)

where the value of repaying is

\[ V^R(a, e, s) = \max_{c, a'} U(c, s) + \beta \rho_s \mathbb{E} V(a', e', s', 0) \]

\[ c + q(a', s) a' = we + a \]  

\[ c \geq 0, a' \in A \]  

(1.5)

and the value of defaulting is

\[ V^D(e, s) = \max_{c, a'} U(c, s) + \beta \rho_s \mathbb{E} V(0, e', s', 1) \]

\[ c = we(1 - \chi) \]  

\[ c \geq 0, a' = 0. \]  

(1.6)

A household in good standing that cannot repay their debt must default.

A household in bad standing \( h = 1 \) solves

\[ V(a, e, s, 1) = \max_{c, a'} U(c, s) + \beta \rho_s \lambda \mathbb{E} V(a', e', s', 1) + \beta \rho_s (1 - \lambda) \mathbb{E} V(a', e', s', 0) \]

\[ c + q(a', s) a' = we + a \]  

\[ c, a' \geq 0, a' \in A. \]  

(1.7)

Let the policy functions be denoted \( d(a, e, s, h) \), \( a'(a, e, s, h) \), \( c(a, e, s, h) \), where a household in bad standing is said to not default, i.e. \( d(a, e, s, 1) = 0. \)

\(^{17}\)When there is no risk of confusion I omit the conditional expectation’s information set.
The Intermediary’s Problem

Details of the intermediary’s problem are in Appendix A.1. The intermediary maximizes the net present value of financial income using contracts, capital, and the bond. He is indifferent over all feasible allocations if contracts are priced according to (1.3) and if capital’s return equals the bond’s return.

Equilibrium

I now give a simplified definition of equilibrium. An unsimplified definition, along with the characterizations that lead to this definition, are given in Appendix A.1.

A steady state equilibrium is a collection of prices \(r, w, q\), recovery rates \(p\), policy functions \(c, a', d\), a value function \(V\), a strictly positive capital stock \(K\) and labor supply \(N\), and a distribution of households \(\mu\) such that all of the following hold:

1. The policies and value function solve the household problems.

2. The capital stock and aggregate labor supply are given by the distribution:\(^{18}\)

\[
K = f(a + d(a, e, s, h)(-a - \chi we))d\mu/(1 + r - \delta) \quad (1.8)
\]

\[
N = \int ed\mu. \quad (1.9)
\]

3. Prices satisfy

\[
q(a', s) = \bar{q}_B \rho_s p(a', s) \quad (1.10)
\]

\[
\bar{q}_B = 1/(1 + r - \delta) \quad (1.11)
\]

\[
r = \alpha(K/N)\alpha - 1 \quad (1.12)
\]

\[
w = (1 - \alpha)(K/N)^\alpha. \quad (1.13)
\]

4. Repayment probabilities are consistent: for all \(a, s_{-1}\),

\[
p(a, s_{-1}) = \sum_s \int (1 - d(a, e, s, 0) + d(a, e, s, 0)\chi we/(-a)) f(e|s)deF(s|s_{-1}). \quad (1.14)
\]

5. The distribution is invariant to household policies and stochastic transitions.

\(^{18}\)A proof that this is the aggregate capital stock is given in Appendix A.1.
Business Cycle Model

I now modify the steady state model to include aggregate uncertainty. The model is setup recursively using \( S = (z, \mu) \) as the aggregate state where \( \mu \) is a distribution of households and \( z \) is a productivity shock. The aggregate state evolves according to a law of motion \( \Gamma \) with \( S' = \Gamma(z', S) \) denoting next period’s aggregate state conditional on a \( z' \) realization. Further, I use \( \mathcal{G} = \Gamma(g, S) \) and \( \mathcal{B} = \Gamma(b, S) \) to denote the states that will arise conditional on a \( g \) or \( b \) realization.

Demographics, Endowments, Technology, and Preferences

The production technology is now \( zK^\alpha N^{1-\alpha} \). The productivity shock \( z \) takes on one of two possible values in \( Z = \{g, b\} \) with \( g > b \) and evolves according to a Markov chain \( F(z'|z) \). Capital is rented at rate \( r(S) \) and labor is hired at wage \( w(S) \). As before, capital depreciates at a constant rate \( \delta \).

The household efficiency process is now allowed to vary with the aggregate state. Specifically, \( e \) is drawn from \( f(e|s, z) \) and \( s \) evolves according to \( F(s'|s, z') \). Similarly the distributions for newborn households are \( \hat{f}(e|s, z) \) and \( \hat{F}(s|z) \).

Household preferences over consumption are the same as before and leisure is still not valued. The assumptions on preferences, mortality risk, and endowments imply an exogenous stochastic process for aggregate labor supply \( N \). It is assumed that the labor supply conditional on \( z' = g \) is always weakly larger than the labor supply conditional on \( z' = b \).

Legal Environment

The legal environment is the same as in steady state.

Asset Markets

As before households enter into debt or savings contracts with a financial intermediary, but now contracts are separated into two types: \( g \)-contingent and \( b \)-contingent. From the household’s perspective, these contracts look like two Arrow securities \( a'_g \) and \( a'_b \). The face value \( a'_z \) is to be delivered (if positive) or repaid (if negative) if and only if next period’s
productivity shock is $z'$. The price of a $z'$-contingent contract is denoted $q_{z'}(a', s; S)$. Hence there are two price schedules, $q_g(a', s; S)$ and $q_b(a', s; S)$.

From the intermediary’s perspective, a contract costs $q_{z'}(a', s; S)a'$ and gives a certain yield $\rho_s p(a', s; S')a'$ contingent on a $z'$ realization. The recovery rate $p(a', s; S')$ reflects that not only may households default, but that their decision depends on the aggregate state $S'_{z'}$.

As before, the intermediary has access to capital $K'$ and a risk-free discount bond $B'$ with price $\bar{q}_B(S)$. Additionally, and following Krusell, Mukoyama, and Şahin (2010), the intermediary has access to an “aggregate complete” set of Arrow securities $A'_g$ and $A'_b$ with prices $\bar{q}_g(S)$ and $\bar{q}_b(S)$. Because capital’s return is risky and the bond’s return is risk-free, these Arrow securities are redundant when the short-sale constraint on capital is not binding.\(^{19}\) All assets except capital are in zero net supply.

With the aggregate-complete set of Arrow securities, contract pricing is simple. Because a $z'$-contingent contract can be replicated by a $z'$-contingent Arrow security, no arbitrage dictates that

$$q_{z'}(a', s; S) = \bar{q}_{z'}(S)\rho_s p(a', s; S'_{z'})$$

(1.15)

for $a' \neq 0$. Without loss of generality $q_{z'}(0, s; S) = 0$. This shows the price of $z'$-contingent contract reflects the cost of transferring resources to that state $\bar{q}_{z'}(S)$, the probability of survival $\rho_s$, and the recovery rate $p(a', s; S'_{z'})$ conditional on reaching that state. This contract pricing ensures that there is no cross-subsidization across different loan types and is therefore consistent with free entry by intermediaries.

While contracts are priced in this independent fashion, household access to these contracts is restricted. Specifically, a household of type $s$ must choose $(a'_g, a'_b)$ from a set $P(s)$.\(^{20}\) In the calibrated model, there will be two groups of households separated, essentially, into the bottom 80% of labor income earners and the top 20%. The bottom 80% will have access to a fixed portfolio which for the sake of simplicity and clarity is a bond.\(^{21}\) For

\(^{19}\)Recall next period’s labor supply is assumed to be weakly larger if $g$ is realized than $b$ is realized. This, together with $g > b$, ensures $r(G) > r(B)$ so that capital is indeed risky.

\(^{20}\)I assume $A^+ \times A^+ \subset P(s)$ for some $s$ having $F(s|z) > 0$ for each $z$ and $T \geq 2$. This technical assumption ensures some positive measure of households can always be incentivized to increase $a'_g$ (in the aggregate) or $a'_b$ by varying $\bar{q}_g$ and $\bar{q}_b$.

\(^{21}\)The restriction to a bond, $a'_{g} = a'_{b}$, rather than some other fixed portfolio such as capital, $a'_{z'} =
them \( P(s) = \{(a'_g, a'_b) \in A \times A | a'_g = a'_b \}\). The top 20\% on the other hand will have access to a bond for borrowing but can save using any \((a'_g, a'_b)\) combination. For them \( P(s) = \{(a'_g, a'_b) \in A \times A | a'_g = a'_b \text{ if } a'_g < 0 \text{ or } a'_b < 0 \}\). These portfolio restrictions are similar to the ones in Chien, Cole, and Lustig (2011) where 10\% of the population has freely adjustable portfolios, 20\% use a fixed, weighted portfolio of bonds and equity, and the remaining 70\% use a bond.

The portfolio restrictions are meant to capture that while a large menu of tradeable assets is available, only rich households seem to use them.\(^{22}\) In particular, Kennickell (2009) demonstrates that the top 20\% of the income distribution (and especially the top 5\%) hold a disproportionate share of their portfolio in businesses and other non-housing wealth while the bottom 80\% hold primarily housing wealth.\(^{23}\) Moreover, Campbell (2006) shows the portfolios of households with the least assets contain virtually only safe ones.\(^{24}\) Together with the observation that interest on consumer credit is almost always tied to the prime rate (and not, for instance, the return on equity), the assumed portfolio restrictions seem plausible. That said, there will in general be large welfare gains from removing these portfolio restrictions. In Appendix A.4, I show one of the main results, that aggregate risk reduces the welfare gain of eliminating default, holds when there are no portfolio restrictions.

The Household Problems

Let \( V(a, e, s, h; \mathcal{S}) \) denote the value function of a household. Taking the law of motion and prices as given, households solve the following problems.

A household in good standing \( h = 0 \) that can repay its debt solves

\[
V(a, e, s, 0; \mathcal{S}) = \max_{d \in \{0, 1\}} \left( 1 - d \right) \cdot V^R(a, e, s; \mathcal{S}) + d \cdot V^D(e, s; \mathcal{S})
\]

\((1 + r(S'_{\mathcal{S}}) - \delta)k'\) for some \( k' \) and each \( z' \), has three advantages. First, it is consistent with the theoretical structure of the model (where \( A \) is a fixed set). Second, in the computation it avoids interpolating the value function and price schedules in the \( a \) direction. Third, it allows for the natural borrowing limit to be written down in a straightforward fashion.

\(^{22}\)While imposing exogenous portfolio restrictions to capture this endogenous outcome is not ideal, it seems reasonable given that the model abstracts from informational costs and other potential barriers to entry in financial markets.

\(^{23}\)See p. 25 and Figure 27 of his paper. Carroll (2000) documents a similar fact for the wealth distribution.

\(^{24}\)See Figure 3 of his paper. He defines safe assets as “checking, saving, money market and call accounts, CDs, and U.S. savings bonds” (p. 1563).
where the value of repaying is

\[ V^R(a, e, s; S) = \max_{c, a'_g, a'_b} U(c, s) + \beta \rho_s \mathbb{E}V(a'_{z'}, e', s', 0; S'_{z'}) \]

\[ c + q_g(a'_g, s; S)a'_g + q_b(a'_b, s; S)a'_b = w(S)e + a \]

(1.17)

and the value of defaulting is

\[ V^D(e, s; S) = \max_{c, a'_g, a'_b} U(c, s) + \beta \rho_s \mathbb{E}V(0, e', s', 1; S'_{z'}) \]

\[ c = w(S)e(1 - \chi) \]

(1.18)

A household in good standing that cannot repay its debt must default.

A household in bad standing that cannot repay its debt must default.

A household in bad standing that cannot repay its debt must default.

Let the associated policy functions be denoted \( d(a, e, s, h; S) \), \( a'_g(a, e, s, h; S) \), \( a'_b(a, e, s, h; S) \), \( c(a, e, s, h; S) \)

where a household in bad standing is said to not default \( d(a, e, s, 1; S) = 0 \).

The Intermediary’s Problem

Details of the intermediary’s problem may be found in Appendix A.1. Essentially, the intermediary maximizes the net present value of financial income discounted by Arrow security prices \( \bar{q}_i(S) \). He does so using contracts, capital, a risk-free bond, and the two Arrow securities. He is indifferent over all feasible allocations if contract prices satisfy (1.15) and if

\[ 1 = \bar{q}_g(S)(1 + r(G) - \delta) + \bar{q}_b(S)(1 + r(B) - \delta) \]

\[ \bar{q}_B(S) = \bar{q}_g(S) + \bar{q}_b(S). \]

(1.20)

These are equivalent to

\[ \bar{q}_g(S) = (\bar{q}_B(S)(1 + r(G) - \delta) - 1)/(r(G) - r(B)) \]

\[ \bar{q}_b(S) = (\bar{q}_B(S)(1 + r(G) - \delta)) / (r(G) - r(B)). \]

(1.21)

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While not immediately apparent, the presence of adjustable portfolios permits the financial intermediary to make zero profits in every state (not just in expectation). A proof of this fact is given in Appendix A.1. Because the intermediary makes zero profits, a theory of how any gains or losses are distributed across households does not need to be developed.

Equilibrium

I now give a definition of equilibrium that has been substantially simplified. The unsimplified definition, as well as the characterizations leading to it, are in Appendix A.1. A recursive competitive equilibrium is a collection of price functions $r, w, \bar{q}_g, \bar{q}_b, q_g, q_b$, recovery rates $p$, policy functions, $c, a'_g, a'_b, d$, a value function $V$, a capital stock $K$ and labor supply $N$ as functions of the aggregate state, and a law of motion $\Gamma$ such that the following hold:

1. The policies and value function solve the household problems.

2. The aggregate capital stock and labor supply are given by the distribution and the capital stock is strictly positive:

$$K(S) = f(a + d(a, e, s, h; \mathcal{S})(-a - \chi w(S)e))d\mu/(1 + r(S) - \delta) > 0 \quad (1.22)$$

$$N(S) = f e d\mu. \quad (1.23)$$

3. Prices satisfy

$$q_{z'}(a', s; \mathcal{S}) = \bar{q}_{z'}(\mathcal{S})\rho_s p(a', s; \mathcal{S}') \quad (1.24)$$

$$\bar{q}_g(\mathcal{S}) = (1 - \bar{q}_B(\mathcal{S})(1 + r(B) - \delta))/(r(G) - r(B)) \quad (1.25)$$

$$\bar{q}_b(\mathcal{S}) = (\bar{q}_B(\mathcal{S})(1 + r(G) - \delta) - 1)/(r(G) - r(B)) \quad (1.26)$$

$$r(S) = z\alpha(K(S)/N(S))^{\alpha-1} \quad (1.27)$$

$$w(S) = z(1 - \alpha)(K(S)/N(S))^{\alpha}. \quad (1.28)$$

---

25Intuitively, for zero profits to obtain, the intermediary’s capital income must exactly offset his contract obligations. This implies contract obligations must have an identical return structure to capital. In general, there will not exist a fixed portfolio of assets that will deliver this because the presence of default makes contract obligations, and in particular charge-offs, vary in a non-trivial way. By allowing flexible portfolios, some households can be induced to change the ratio $a'_g/a'_b$ in the aggregate and so make contract obligations and capital have the same return structure. This is accomplished by varying the bond price $\bar{q}_B$, which controls the relative price $\bar{q}_b/\bar{q}_g$. 

15
4. Repayment probabilities are consistent: for all \(a, s_{-1}\) and \(S\),

\[
p(a, s_{-1}; S) = \sum_{s} \int \left( 1 - d(a, e, s, 0; S) + \frac{d(a, e, s, 0; S)\chi w(S) e/(-a)}{f(e|s,z)} \right) d\mu(e|s,z) \int f(e|s,z) d\mu(e|s,z). \tag{1.29}
\]

5. All asset markets clear and the intermediary makes zero profits which is equivalent to

\[
(1 + r(B) - \delta) \sum_{a'} \int p_s(a', s; B) a' 1[a' = a'_g(a, e, s, h; S)] d\mu = (1 + r(G) - \delta) \sum_{a'} \int p_s(a', s; B) a' 1[a' = a'_b(a, e, s, h; S)] d\mu. \tag{1.30}
\]

6. The law of motion is consistent with stochastic transitions and household policies.

The least obvious equation is (1.30). This ensures the only asset the intermediary needs to use (besides contracts) is capital. To see this, suppose there was no mortality risk, no default, and that portfolios mimicked capital, \(a'_{z'} = (1 + r(S'_{z'}) - \delta)k'\) for some \(k'\) and each \(z'\). In this case, (1.30) becomes

\[
(1 + r(B) - \delta) f(1 + r(B) - \delta)k'd\mu = (1 + r(G) - \delta) f(1 + r(B) - \delta)k'd\mu \tag{1.31}
\]

which always holds. To carry household savings into the next period, all the intermediary must do is choose capital holdings \(K'\) equal to \(\int k'd\mu\) which makes the bond and Arrow securities unnecessary. With default, mortality risk, and flexible portfolios, some adjustments must be made to reflect that savings are contingent and that only the aggregate portfolio must resemble capital.

The only “deep” prices in the model are the factor prices, \(r\) and \(w\), and the risk-free discount bond price \(\overline{q_B}\). Essentially by no arbitrage, the Arrow security prices \(\overline{q_g}, \overline{q_b}\) and price schedules \(q_g, q_b\) can be written as functions of these, as is evident when looking at equations (1.24),(1.25), and (1.26). The bond price is used to clear the asset markets, as summarized in equation (1.30), by controlling the relative cost of saving using \(a'_g\) versus using \(a'_b\).

\[\text{Importantly, as } \overline{q_B} \text{ approaches } 1/(1 + r(B) - \delta) \text{ from above, the Arrow price } \overline{q_g} \text{ and hence the price schedule } q_g \text{ approach zero. This makes saving using } a'_g \text{ become arbitrarily cheap driving up the left hand side of (1.30). At the same time, the Arrow price } \overline{q_b} \text{ approaches } 1/(1 + r(B) - \delta) \text{ meaning saving using } a'_b \text{ does not become arbitrarily cheap. This keeps the right hand side of (1.30) bounded from above. As } \overline{q_B} \text{ approaches } 1/(1 + r(G) - \delta), \text{ the reverse is true.}\]
1.3 Calibration and Baseline Properties

This section discusses the calibration and baseline model properties. The model period is a year.

Functional Forms

I first describe the functional form for the efficiency process and then for portfolio availability.

Efficiency

I select the efficiency process to capture three potentially important features of the data. First is that earnings persistence and variance change over the life cycle. This feature is demonstrated in Karahan and Ozkan (2010) where it is shown earnings shocks are less persistent early in life. In the model, persistence and contract pricing are tightly linked, so this could prove important. Second is that the variance of persistent earnings shocks increases in recessions and decreases in expansions as demonstrated in Storesletten et al. (2004). As default provides a way of intratemporally smoothing consumption, this fluctuation in intratemporal dispersion could also prove important. Third is that the earnings distribution has a thick right tail. By selecting an efficiency process that generates this right tail, the model will also be able to match the concentration of wealth in the data. This is important because the wealthiest households will not be directly affected by default policy. Consequently, general equilibrium effects of changes in default policy would likely be overstated were these households missing.

To account for these features of the data, I use two efficiency processes with working households stochastically transitioning between them, as well as a separate process for retirement. The efficiency process for the majority of working households and all newborns is governed by

\[
\begin{align*}
    e_{h,z} &= \psi_{z} \phi_{h} \exp(u_{h} + \varepsilon_{h}) \\
    u_{h} &= \gamma_{h} u_{h-1} + \eta_{h,z}, \quad u_0 = 0 \tag{1.32} \\
    \eta_{h,z} &\sim N(0, \sigma_{\eta_{h,z}}), \quad \varepsilon_{h} \sim N(0, \sigma_{\varepsilon_{h}})
\end{align*}
\]
where $h$ denotes age. This process has a deterministic component $\phi_h$, a persistent component $u_h$, a transitory component $\varepsilon_h$, and an “aggregate labor supply shifter” $\psi_z$. As labor is supplied inelastically, the supply shifter is used to match the cyclical volatility of hours worked.\textsuperscript{27} Note that the persistence is governed by $\gamma_h$ which is age-dependent, as are the variances of the persistent and transitory shocks. Also note the variance of the persistent shock $\sigma^2_{\eta,h,z}$ is a function of $z$ (the steady state variance is $\sigma^2_{\eta,h,1}$).

While most working households use this “log process,” they have a probability $\pi_{bw}$ (bw for “blue to white” collar) of transiting to the process

$$e_{h,z} = \psi_z \phi_h v$$

$$v \sim \left( \frac{v - \bar{v}}{\bar{v} - \underline{v}} \right) \xi \quad \text{with support } [\underline{v}, \bar{v}].$$

This “super rich” process also has an aggregate, deterministic, and transitory component, however the transitory component is drawn from a different distribution governed by three parameters, $\underline{v}$, $\bar{v}$, and $\xi$. The functional form for the $v$ distribution is taken from Chatterjee et al. (2007) who in the spirit of Castañeda, Díaz-Giménez, and Ríos-Rull (2003) posit a low-probability state where earnings are very high but transitory. As Chatterjee et al. (2007) argue, such a state provides households with the “opportunity and incentive” to accumulate large amounts of wealth.\textsuperscript{28} Households revert to the log process with probability $\pi_{wb}$ and draw $u_h$ from $N(0, \sigma^2_{\eta,1,z})$ and $\varepsilon_h$ from $N(0, \sigma^2_{\varepsilon,1})$.\textsuperscript{29} The transition is deterministic (i.e. $\pi_{wb} = 1$) in the last period of working life.

When households retire, their efficiency is\textsuperscript{30}

$$e_{h,z} = \kappa_F \psi_z \phi_J \exp(u_J) + \kappa_G \psi_z$$

where $J$ is the first period of retirement. This retirement process is very similar to the one in Livshits et al. (2007) and Athreya et al. (2009a). The proportional component $\kappa_F$ reflects

\textsuperscript{27}Note that the literature tends to focus on residual earnings, i.e. what is left after running a first-stage regression on log earnings using a complete set of time dummies. The model process is consistent with this in that running such a regression would remove $\psi_z$ (and the wage $w$) which will be common across all households.

\textsuperscript{28}The only way to equate marginal utilities, given the high income and transitory nature of the state, is to save a large amount. In the calibration, the iid assumption on $v$, together with a large value for $\bar{v}$ and a small value for $\xi$, delivers this.

\textsuperscript{29}They draw from the distribution for newborn households both for simplicity and because the persistent variance is large (reflecting a large cross-sectional variance of the persistent shock).

\textsuperscript{30}I follow Krusell, Mukoyama, Şahin, and Smith (2009) and model retirement income as home production to avoid explicitly modeling a government.
earnings made over a household’s lifetime, and the guaranteed component \( \kappa_G \) provides a fraction of average earnings. Mean efficiency is normalized to one in steady state. By assumption all households are under the log process in the last period of working life \( J - 1 \) and take one last draw of \( \eta \) to arrive at \( u_J \).

**Portfolio Availability**

Recall that the portfolio \( P(s) \) available to households is allowed to vary with their characteristics \( s \). I now let \( P(s) = \{(a'_g, a'_b) \in A \times A | a'_g = a'_b \text{ if } a'_g < 0 \text{ or } a'_b < 0\} \) for “super-rich” households and \( P(s) = \{(a'_g, a'_b) \in A \times A | a'_g = a'_b \} \) for all others.\(^{31}\) This means the typical household will use only a risk-free bond, but a few earnings-rich households will, in addition to a risk-free bond, have access to freely adjustable portfolios for savings.

**Parameter Values and Baseline Properties**

For the steady state calibration, I adopt most targets and many parameters from Chatterjee et al. (2007). The coefficient of relative risk aversion \( \sigma \) is 2. The capital share of income \( \alpha \) is 0.36 and the depreciation rate of capital \( \delta \) is 0.10. The average duration of a bad credit record \( 1/(1 - \lambda) \) is taken to be ten years implying \( \lambda = 0.9 \). Wealth and earnings targets are reported in Table 1.1. I target a debt-output ratio of .0067 and a fraction in debt of 6.7. For the filing rate, I target .50%.\(^{32}\)

Households begin life at age 20, begin retirement at 65, and live to at most 85. The profiles of residual earnings persistence and variance \( (\gamma_h, \sigma^2_{\eta, h, 1}, \sigma^2_{\mu, h, 1}) \) are the non-parametric estimates of Karahan and Ozkan (2011).\(^{33}\) The persistence \( \gamma_h \) begins low at around .7 before increasing to unity by age 40 and mostly staying there until retirement. For countercyclical earnings variance, there are no age-specific estimates available, so I use estimates from

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\(^{31}\)The technical assumption on \( P(s) \) dictates some \( \epsilon > 0 \) measure of households is born super rich.

\(^{32}\)This is the average number of filings per household from 1999-2003, .93%, prorated to only account for the number of filings due to earnings shocks which Chatterjee et al. (2007) put at 53.5% based on Chakravarty and Rhee (1999). The literature has used a wide range of targets from .29% in Chatterjee et al. (2007) to 1.2% in Athreya et al. (2009a).

\(^{33}\)The sample in Karahan and Ozkan (2011) is for ages 24 to 60 and I use nearest-neighbor extrapolation to recover the values for households aged 23 or less and 60 or older. I view this extrapolation procedure as conservative because the authors report in their earlier working paper, Karahan and Ozkan (2010), that the shape of the profiles is even stronger for ages 20 to 65. A last detail, the authors allow for time loading factors on the persistent and transitory shock variance, and I use the average of the loadings from 1988 to 1992.
Storesletten et al. (2004) assuming the ratio \( \sigma_{\eta,h,g}/\sigma_{\eta,h,b} \) is age-independent and equal to .59 and that \( .5\sigma_{\eta,h,b} + .5\sigma_{\eta,h,g} = \sigma_{\eta,h,1} \). The retirement parameters \((\kappa_F, \kappa_G)\) are set to (.35, .15) giving an average replacement rate of roughly 50%. The deterministic component of efficiency \( \phi_h \) and mortality risk profile \( \rho_s \) are calibrated using estimates from Hubbard, Skinner, and Zeldes (1994). The earnings profile \( \phi_h \) follows a hump-shape over the life cycle, nearly doubling from age 20 to its peak at age 48 before almost returning to its age-20 value by retirement. The effective household-size profile \( \theta_s \) is calibrated using the “mean” equivalence scale in Fernández-Villaverde and Krueger (2007) and the profile of household size calculated from the CPS by Bick and Choi (2011). All the profiles are reported in Appendix A.2.

The productivity process is assumed to be symmetric with an expected duration of expansions and recessions at 3 years. This implies \( F(g|g) = F(b|b) = 2/3 \). A standard deviation of 2.24% is targeted for the Solow residual implying the technology shock takes on \( g = 1.0224 \) or \( b = 0.9776 \). The aggregate labor supply shifter \( \psi_z \) is calibrated, once the rest of the calibration is set, to match the 1.74% standard deviation of log hours worked reported in Castañeda, Díaz-Giménez, and Ríos-Rull (1998). This results in \((\psi_g, \psi_b) = (1.025, .975)\) with the steady state value \( \psi_1 \) normalized to 1.

There are 7 remaining parameters: 1 preference parameter \( \beta \), 1 technology parameter \( \chi \), and 5 efficiency-process parameters \((\upsilon, \bar{\upsilon}, \xi, \pi_{bw}, \pi_{wb})\). These are used to minimize the weighted distance between the steady state model and target statistics. The steady state model is computed with grid search and backward induction in Fortran. For details of the

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34The value of \( \sigma_{\eta,g}/\sigma_{\eta,b} \) is .59 in Storesletten et al. (2004).
35Although Athreya et al. (2009a) and Livshits et al. (2010) set \((\kappa_F, \kappa_G) = (.35, .2)\), they do not have the earnings rich. The lower value for \( \kappa_G \) (the guaranteed portion of mean earnings) of .15 reflects a mean/median income that is approximately 75% of its value when including the income-rich (roughly 1.2 compared to 1.6).
36Hubbard et al. (1994) estimate separate deterministic profiles for household heads with less than 12 years of education (NHS), 12-15 years (HS), and 16+ years (COL) of education. I average the profiles of these three types for the year 1986 assuming 13% of the population is NHS, 48% is HS, and 39% is COL. This breakdown of educational attainment is from the 2010 Current Population Survey for ages 30-34 (see Table 1 at http://www.census.gov/hhes/socdemo/education/data/cps/2010/tables.html).
37Specifically, I assume \( \theta_s = f(N_s) \) where \( N_s \) is the average number of household members and \( f \) is the mean equivalence scale in Fernández-Villaverde and Krueger (2007) linearly-interpolated to be continuous. The values for \( \theta_s \) are very similar to the ones in Livshits et al. (2007). The benchmark calibration over-predicts the hump in consumption, but in a robustness exercise I show \( \theta_s = 1 \) makes the results even stronger (see Appendix A.4 Table A.8). This suggests estimating \( \theta_s \) to match the consumption profile would not significantly alter the results.
38This is the unconditional standard deviation estimated in Cooley (1995) (who uses a labor share of .4).
computation see Appendix A.3.

The results from the calibration are listed in Table 1.1. The model does fairly well at reproducing the targets. Specifically the right tails of both the earnings and wealth distributions are matched closely as is the capital-output ratio. As in virtually all the bankruptcy literature, the calibration has difficulty jointly matching the filing and debt statistics. Having some income transferred to creditors in the period of default (via the default cost $\chi$) helps a lot in matching these statistics, as does some flexibility with the efficiency process.

The model with aggregate uncertainty is computed using the method of Krusell and Smith (1998). The distribution is summarized with two of its moments, aggregate wealth and labor supply, and an equity premium is included as a state variable. The household problem is solved with grid search, backward induction, and linear interpolation of the aggregate moments. For results to be comparable between the steady state and business cycle versions, the same number and placement of grid points are used in both models (in both the asset and efficiency direction). The model is simulated non-stochastically as in Young (2010), and the asset markets are cleared in each period of the simulation. The resulting approximate law of motion makes accurate price forecasts 1-step ahead with $R^2$ values above .997 and maximum errors below .13% as well as 50-steps ahead with $R^2$ values above .995 and maximum errors below .37%. For additional details, the reader is referred to Appendix A.3.

Table 1.2 reports the cyclical properties of the model and US economies with a discussion of the levels of model aggregates deferred to the next section. The data are described in Appendix A.2. Filings are too volatile and too countercyclical; however, default in the model is based entirely on earnings shocks. Other reasons for default, such as divorce, health bills, and lawsuits, are probably not as correlated with the business cycle nor as volatile. The volatility of consumption is too low. The labor supply volatility is exactly as targeted, and

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39 Three approaches have been used to try to overcome this problem. First is the approach used by Chatterjee et al. (2007) and Sánchez (2007) who calibrate the earnings process to match some earnings statistics. Second is the approach used in Athreya et al. (2009b) who take estimates from the literature but use a stochastic punishment for default. Third is the approach used in Livshits et al. (2007) who take estimates from the literature but make debt partially secured through a “garnishment” technology and posit transaction costs for debt. The approach I use here is a mixture of the first and the third.

40 Including an equity premium as a state variable is fairly common in the literature. For instance, Storesletten, Telmer, and Yaron (2007) include $E_S [R(S_t)] = \bar{q}_u (S)^{-1}$. The benefit of doing this is that it makes market clearing during the simulation much faster.
<table>
<thead>
<tr>
<th>Statistic</th>
<th>Target</th>
<th>Model</th>
<th>Parameter*</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Targets Determined Independently</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient of Relative Risk Aversion</td>
<td>2</td>
<td>2</td>
<td>σ</td>
<td>2</td>
</tr>
<tr>
<td>Capital Share of Income</td>
<td>.36</td>
<td>.36</td>
<td>α</td>
<td>.36</td>
</tr>
<tr>
<td>Depreciation Rate of Capital</td>
<td>.10</td>
<td>.10</td>
<td>δ</td>
<td>.10</td>
</tr>
<tr>
<td>Avg. Duration of a Bad Credit History</td>
<td>10</td>
<td>10</td>
<td>λ</td>
<td>.9</td>
</tr>
<tr>
<td>Avg. Duration of a Business Cycle</td>
<td>3</td>
<td>3</td>
<td>( F(z</td>
<td>z) )</td>
</tr>
<tr>
<td>Solow Residual Standard Deviation (%)</td>
<td>2.24</td>
<td>2.24</td>
<td>( g − 1, 1 − b )</td>
<td>0.0224</td>
</tr>
<tr>
<td><strong>Targets Determined Jointly</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital-Output Ratio</td>
<td>3.08</td>
<td>3.08</td>
<td>ξ</td>
<td>1.59e-3</td>
</tr>
<tr>
<td>Debt-Output Ratio × 100</td>
<td>.67</td>
<td>.69</td>
<td>χ</td>
<td>.12</td>
</tr>
<tr>
<td>Population Filing (%)</td>
<td>.50</td>
<td>.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Population in Debt (%)</td>
<td>6.7</td>
<td>10.5</td>
<td>β</td>
<td>.940</td>
</tr>
<tr>
<td>Wealth Gini</td>
<td>.80</td>
<td>.83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth Mean/Median</td>
<td>4.03</td>
<td>4.13</td>
<td>( \pi_{wb} )</td>
<td>.588</td>
</tr>
<tr>
<td>Percentage of Wealth held by Top 5%</td>
<td>57.8</td>
<td>69.5</td>
<td>( \bar{v} )</td>
<td>1429.3</td>
</tr>
<tr>
<td>Percentage of Wealth held by Top 20%</td>
<td>81.7</td>
<td>83.6</td>
<td>( \bar{v} )</td>
<td>2.67</td>
</tr>
<tr>
<td>Percentage of Wealth held by 40-20%</td>
<td>12.2</td>
<td>9.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of Wealth held by 60-40%</td>
<td>5.0</td>
<td>4.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of Wealth held by 80-60%</td>
<td>1.3</td>
<td>1.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Earnings Gini</td>
<td>.61</td>
<td>.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Earnings Mean/Median</td>
<td>1.57</td>
<td>1.59</td>
<td>( \pi_{bw} )</td>
<td>.153</td>
</tr>
<tr>
<td>Percentage of Earnings held by Top 5%</td>
<td>31.1</td>
<td>30.5</td>
<td></td>
<td></td>
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<tr>
<td>Percentage of Earnings held by Top 20%</td>
<td>60.2</td>
<td>55.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of Earnings held by 40-20%</td>
<td>22.9</td>
<td>19.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of Earnings held by 60-40%</td>
<td>13.0</td>
<td>12.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of Earnings held by 80-60%</td>
<td>4.0</td>
<td>7.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Parameters are listed beside statistics they strongly influence.

Table 1.1: Model Targets, Statistics, and Parameters

Output and investment are close to their US counterparts. Output is persistent but not as persistent as in the data.

### 1.4 Eliminating Default

Overall, the calibrated model is broadly consistent with US data. I now explore the consequences of eliminating default both in terms of welfare and allocations, and see how these change in the presence of aggregate risk.\(^{41}\) The benchmark economy, that is the one with

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\(^{41}\)I do not account for the transition, and there are two reasons for this. First is that when a policy change results in capital decumulation, accounting for the transition tends to improve welfare (this is the case, for instance, in Chatterjee and Gordon, 2011). As will be seen shortly, eliminating default does result in capital decumulation.
<table>
<thead>
<tr>
<th>Variable (x)</th>
<th>Stdev x in (%)</th>
<th>Stdev y</th>
<th>Correlation of lagged x with y</th>
<th>Correlation of lagged x with y</th>
</tr>
</thead>
<tbody>
<tr>
<td>US (1960-2004)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output (y)</td>
<td>2.44</td>
<td>1.00</td>
<td>0.01</td>
<td>0.58</td>
</tr>
<tr>
<td>Consumption</td>
<td>1.69</td>
<td>0.69</td>
<td>0.02</td>
<td>0.59</td>
</tr>
<tr>
<td>Investment</td>
<td>7.19</td>
<td>2.94</td>
<td>0.04</td>
<td>0.53</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>1.60</td>
<td>0.66</td>
<td>-0.23</td>
<td>0.26</td>
</tr>
<tr>
<td>Defaulting Pop</td>
<td>10.25</td>
<td>4.20</td>
<td>0.17</td>
<td>0.00</td>
</tr>
<tr>
<td>Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output (y)</td>
<td>2.96</td>
<td>1.00</td>
<td>-0.10</td>
<td>0.08</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.82</td>
<td>0.28</td>
<td>-0.28</td>
<td>-0.11</td>
</tr>
<tr>
<td>Investment</td>
<td>8.14</td>
<td>2.75</td>
<td>-0.05</td>
<td>0.12</td>
</tr>
<tr>
<td>Labor Supply</td>
<td>1.74</td>
<td>0.59</td>
<td>0.03</td>
<td>0.18</td>
</tr>
<tr>
<td>Defaulting Pop</td>
<td>12.34</td>
<td>4.17</td>
<td>0.27</td>
<td>0.15</td>
</tr>
<tr>
<td>Debt*</td>
<td>6.54</td>
<td>2.21</td>
<td>0.28</td>
<td>0.25</td>
</tr>
<tr>
<td>Population in Debt*</td>
<td>4.50</td>
<td>4.52</td>
<td>0.29</td>
<td>0.25</td>
</tr>
</tbody>
</table>

*Debt is negative net worth. US counterparts to my knowledge are not available.

**Table 1.2: Business Cycle Properties: Model and Data**

default, is referred to as the CD economy for “consumer default.” The economy where default has been eliminated is referred to as the ND economy for “no default.” It is the limit of default economies as default becomes infinitely costly. The ND economy is just a “natural borrowing limit” economy, as in Aiyagari (1994), where households are able to borrow at a risk-free rate (because \( p = 1 \)) as much as they can repay with certainty. This limit is the net present value of the worst possible labor income stream.

While in most studies the use of a natural borrowing limit rather than some tighter exogenously fixed limit is of secondary importance, here it is the consequence of eliminating default. Consequently, the natural limit is extremely important. There are then three issues to consider, namely, what is the limit in the theory, in the computation, and in the data.

In the theory, the use of a log-efficiency process implies efficiency, and hence labor income, decumulation. However, the capital decumulation is larger in the business cycle than in the steady state environment. Hence, I conjecture that accounting for the transition will only the strengthen the main result that aggregate risk reduces the welfare gain of eliminating default. Second is that it is not conceptually clear how to account for the transition from one ergodic distribution to another, and I am unaware of any paper that has done so.

\[42\] One could think of this as \( \chi \) converging up to 1 or some dead-weight cost (not modeled) levied on households. The actual form of the cost does not matter as the ND economy will not have default in equilibrium. Table 1.6 confirms computationally that the CD economy converges to the ND economy as \( \chi \uparrow 1 \).
can be arbitrarily close to zero prior to retirement. However, because some labor income is guaranteed in retirement, the natural limit will not be zero (as long as $\kappa_G > 0$). In the computation, the log process is discretized with the method of Tauchen (1986) using a large “coverage.” This makes the lowest efficiency realization prior to retirement, .0043, very close to zero. In the data, Carroll (1992) documents that non-capital household income, including transfer income, falls to (or very close to) zero between .30% and .65% of the time for working-age households. Importantly, this sample does not appear to reflect measurement error. At the same time, part of Social Security income in retirement seems to indeed be guaranteed. Consistent with the data, the process I use has near-zero-earnings events but allows for truly guaranteed earnings in retirement in both the theory and computation.

### Steady State

In steady state, the elimination of default results in a large increase in debt and wealth inequality and a significantly lower capital-output ratio. This is borne out in Table 1.3 which lists key wealth and debt statistics for both models. Most striking are the debt statistics: the population in debt increases by 60% from 11% to 17% and the debt-output ratio increases by 500% from .007 to .044. The increased indebtedness translates into a 3.5% decline in the capital-output ratio and is paired with a sharp increase in wealth inequality.

Why is there so much more debt once default has been eliminated? Households in both economies have incentive to borrow because of earnings uncertainty, impatience, and a hump-shaped earnings profile. However, the ND economy gives households the opportunity

---

43 The method specifies a way of approximating an AR1 process with a (finite state) Markov chain given bounds for the states, i.e. the coverage. The coverage I use is $\pm 6.25\bar{\sigma}_\eta, 1 / \sqrt{1 - \bar{\gamma}^2}$ for the persistent shock and $\pm 3\bar{\sigma}_\epsilon$ for the transitory shock where a bar denotes the numerical average across ages. The average values are $\bar{\gamma} = .946$, $\bar{\sigma}_\eta, 1 = .200$, and $\bar{\sigma}_\epsilon = .241$. The coverage of $\pm 6.25\bar{\sigma}_\eta, 1 / \sqrt{1 - \bar{\gamma}^2}$ for the persistent shock reflects a coverage of $\pm 5\bar{\sigma}_\eta, b / \sqrt{1 - \bar{\gamma}^2}$ when using the standard deviation in recessions.

44 Carroll (1992) argues that these observations are not the result of measurement error for two reasons. First, when not-self-employed households report zero (non-capital) income, typically they experienced unemployment, injury, or health problems in the same period or just prior. Second, Duncan and Hill (1985) find outliers for annual income generally correspond to actual experience, not measurement error.

45 The most secure part may be Supplemental Security Income. Under this program an individual is eligible for benefits if they are 65 or older, legally reside in the US, have income that is not too high, and apply for benefits. Certain other individuals are also eligible. See [http://www.ssa.gov/ssi/text-eligibility-ussi.htm](http://www.ssa.gov/ssi/text-eligibility-ussi.htm).

46 That said, robustness tests are conducted with respect to the lower bound on efficiency. As mentioned in the introduction, it is found the result that aggregate risk reduces the welfare gain of eliminating default is robust. However, the levels of the gains are not robust. See Table A.6 in Appendix A.4.
<table>
<thead>
<tr>
<th>Statistic</th>
<th>CD</th>
<th>ND</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital-Output Ratio</td>
<td>3.08</td>
<td>3.00</td>
</tr>
<tr>
<td>Debt-Output Ratio × 100</td>
<td>.69</td>
<td>4.37</td>
</tr>
<tr>
<td>Percentage in Debt</td>
<td>10.5</td>
<td>17.2</td>
</tr>
<tr>
<td>Wealth Gini</td>
<td>.83</td>
<td>.88</td>
</tr>
<tr>
<td>Wealth Mean/Median</td>
<td>4.13</td>
<td>4.95</td>
</tr>
<tr>
<td>Percentage of Wealth held by Top 5%</td>
<td>69.5</td>
<td>72.6</td>
</tr>
<tr>
<td>Percentage of Wealth held by Top 20%</td>
<td>83.6</td>
<td>86.9</td>
</tr>
<tr>
<td>Percentage of Wealth held by 40-20%</td>
<td>9.8</td>
<td>9.4</td>
</tr>
<tr>
<td>Percentage of Wealth held by 60-40%</td>
<td>4.9</td>
<td>4.1</td>
</tr>
<tr>
<td>Percentage of Wealth held by 80-60%</td>
<td>1.8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 1.3: CD and ND Steady State Comparison

to borrow large amounts while the CD economy does not. One way to see this is to consider borrowing limits in the two economies. While there is no “borrowing limit” in the CD economy, there is a maximum loan size that can be taken out by a household of type $s$: $\max_{a'} q(a', s)(-a')$. Similarly, no household in the ND economy would ever take out a loan worth more than $q(\bar{A}(s), s)(-\bar{A}(s))$ where $\bar{A}(s)$ is the natural borrowing limit (for type $s$).

Figure 1.1 compares these borrowing limits by age (averaging out the other components of $s$). As is clear, the maximum loan size in the ND economy is uniformly and typically much higher than in the CD economy. Because of this, and because households have incentive to use debt, the ND economy is much more indebted. Figure 1.1 also reveals that the elimination of default has a differential effect on the life-cycle profile of borrowing limits. While there are increases for each age, the largest occurs late in life and the smallest when young. Moreover, the shapes are very different with the CD limit tracking the earnings profile and the ND limit sharply increasing until retirement and thereafter sharply decreasing.

Three effects give the natural borrowing limit its shape. The first effect drives the borrowing limit up and is that as retirement approaches, the guaranteed earnings in retirement are discounted less, increasing their net present value. This increase in the net present value translates into an increase in the borrowing limit because the lowest efficiency realization prior to retirement is close to zero. The second effect drives the net present value downward. This is that once retirement begins, each year reduces the number of periods left of retirement income. The last effect causes the jump at around age 65 and is that once households
enter retirement, they face no more uncertainty. This drastically increases the natural limit for all but the unluckiest households.

The borrowing limits in the default economy are shaped by two effects that are quite different from the effects shaping the natural limit. First is that retired households have little incentive to borrow because, by assumption, there is no uncertainty. Consequently, the punishment from default, which to a large extent is exclusion from credit markets, is very low. Because households would readily default, creditors are unwilling to extend much credit. Second is the composition effect caused by a hump-shaped earnings profile. \textit{Conditional} on a level of debt and earnings, the incentives to default and repay do not vary much over the life cycle. However, else equal, earnings-rich households are less likely to default. Because average earnings follow a hump shape prior to retirement, so does the average limit.

The differences in credit, prices, and the availability of a default option have important ramifications for consumption smoothing. To see this, it is useful to consider how the mean and variance of log consumption evolve over the life cycle. Figure 1.2 plots these for the CD and ND economies. Note that the mean is higher and the variance lower in the ND

Figure 1.1: Borrowing Limits in the Default and No Default Economies
I now turn attention to welfare. To measure the welfare gain of eliminating default, I use the percent increase in consumption that a newborn household would need in every state.
of the CD economy to be indifferent between living in the CD economy and moving to the ND economy.\textsuperscript{47} This consumption-equivalent variation measure is computed as

$$\omega = \left( \frac{\sum_s \hat{F}(s) \int V_{ND}(0, e, s, 0) \hat{f}(e|s)de}{\sum_s \hat{F}(s) \int V_{CD}(0, e, s, 0) \hat{f}(e|s)de} \right)^{1/(1-\sigma)} - 1$$

(1.35)

where $V_X$ denotes the value function from the $X$ economy. If $\omega > 0$, then eliminating default is welfare improving. For a more complete picture of the welfare effects, I also report the percentage of the population in favor of the policy change and the welfare gains for various subsets of the population using consumption-equivalent variation.\textsuperscript{48}

The increased ability to borrow once default is eliminated results in large welfare gains. In general equilibrium, the gain is 1.82% of lifetime consumption. In partial equilibrium, i.e. holding fixed $r, w, \text{ and } \bar{q_B}$, it is in fact much larger at 3.94%. Moreover, in partial equilibrium 100% of the population prefer the move. However, taking general equilibrium effects into account results in substantial disagreement over the policy change: only 56.8% of households now favor it.

To understand who gains and loses from the policy change, it is useful to break out the welfare gains by age both in partial and general equilibrium. This is done in Figure 1.3. In partial equilibrium, the gains begin high for young households, decline monotonically, and approach zero in retirement. The decline is due to diminishing incentives for borrowing: young households have life-cycle reasons to borrow and face uncertainty, middle-aged households have only uncertainty, and retired households have neither. Once general equilibrium effects are accounted for, the consumption gains drop sharply for almost all households. This is especially true for young households. Further, now the decline is not monotonic. Because the CD economy has a higher capital-output ratio than the ND economy, households at every age lose labor income from the price changes. At the same time, households with savings gain capital income. Because the primary source of income for most households is

\textsuperscript{47}This is the same welfare measure used in Livshits et al. (2007), Athreya et al. (2009b), and many others.

\textsuperscript{48}When calculating welfare gains for different groups, I assume the policy is changed after households have made their default decisions but before they have made their savings decisions. Defining $\omega(a, e, s, h) = ((V_{ND}(a, e, s, h)/V_{CD}(a, e, s, h = 0))^{1/(1-\sigma)} - 1)$, the welfare gain for a subset $I$ of the population is calculated as $\int_I \omega(a, e, s, h)d\mu)/\mu(I)$. Here, $\mu$ is the steady state distribution from the CD economy but with households who would have defaulted transited to $(a, e, s, h) = (0, e, s, 0)$. The population in favor is $\mu(\{\omega(a, e, s, h) > 0\})$. The timing convention, from Chatterjee and Gordon (2011), ensures no unanticipated losses or gains are inflicted on the intermediary.
derived from labor (by necessity of $\alpha = .36$), the first effect dominates and welfare tends to be lower in general equilibrium.

![Figure 1.3: Welfare Gains of Eliminating Default By Age in Steady State](image)

**Business Cycle**

Consistent with findings from the literature, the results from the steady-state model suggest eliminating default expands credit drastically and results in large welfare gains for young households (with mixed effects for the rest of the population due to changes in factor prices). Now I turn to the model with aggregate risk to see whether the same conclusions are reached.

Implications of aggregate risk for aggregates in the CD and ND economies are substantially different. Table 1.4 lists how the levels of key aggregates change after including aggregate risk. As one would expect, the capital-output ratio increases for the ND economy. This is expected because of precautionary savings: the increased risk causes households to increase their buffer stock of capital. Interestingly, this is *not* the case in the CD economy despite having tighter borrowing limits. In fact, the capital-output ratio only increases by .02%. The debt statistics tell a similar story: the debt-output ratio falls by 38% in the ND economy but only 24% in the CD economy, and the percentage in debt declines 20% in the
ND economy but only 14% in the CD economy. At the same time, the number of households filing for bankruptcy increases by 47%.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Business Cycle</th>
<th>% Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CD</td>
<td>ND</td>
</tr>
<tr>
<td>Capital-Output Ratio</td>
<td>3.08</td>
<td>3.03</td>
</tr>
<tr>
<td>Debt-Output Ratio × 100</td>
<td>0.53</td>
<td>2.72</td>
</tr>
<tr>
<td>Percentage in Debt</td>
<td>9.00</td>
<td>16.53</td>
</tr>
<tr>
<td>Percentage Filing</td>
<td>0.26</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1.4: CD and ND Aggregates and Change from Steady State

Why does debt contract sharply in the ND economy but much less so in the CD economy? Figure 1.4, which plots borrowing limits by age, gives part of the answer. At every age, the natural limit contracts, and this is especially true for the youngest households. The limit for newborns decreases by 40% from .250 to .149. At the same time, the amount households can borrow in the CD economy declines only slightly. For newborns, the limit decreases 8% from .146 to .135.

What drives the decline in the natural borrowing limit? To answer this, it is useful to consider the natural borrowing limit definitions for a newborn household. These are

\[
\text{Steady State} \quad \min \left\{ s_t \right\}_{t=2}^T \sum_{t=2}^{T-1} \left( \prod_{j=2}^{T-1} \bar{q}_B \rho_{s_t} \right) w_{s_t} e_{s_t} \quad \text{s.t.} \quad F(s_{t+1}|s_t) > 0 \quad \text{s}_1 \text{ given}
\]

\[
\text{Business Cycle} \quad \min \left\{ s_t, z_t \right\}_{t=2}^T \sum_{t=2}^{T-1} \left( \prod_{j=2}^{T-1} q_{j|s_t} \right) w_{s_t} e_{s_t, z_t} \quad \text{s.t.} \quad F(s_{t+1}|s_t, z_{t+1}) > 0 \quad \text{(1.36)}
\]

\[
\begin{align*}
&w_t = w(S_t), q_t = \bar{q}_B(S_t) \\
&S_{t+1} = \Gamma(z_{t+1}, S_t) \\
&s_1, S_1 \text{ given}
\end{align*}
\]

The borrowing limit definitions are analogous to the steady state ones. I calculate “average” price schedules by integrating out the aggregate state using an invariant distribution over nodes from the Krusell and Smith (1998) method (because linear interpolation between moments is used, the law of motion can be thought of as a probabilistic transition matrix with an invariant distribution). With these average price schedules in hand, I compute the CD economy’s borrowing limit as \(\max_{a'}(q_B(a', s) + q_B(a', s)(-a'))\) and the ND economy’s limit as \(q_B(\bar{A}(s), s) + q_B(\bar{A}(s), s)(-\bar{A}(s))\). Here, \(\bar{A}(s)\) denotes the smallest value of \(a'\) that results in a household’s continuation value (with the aggregate state integrated out) being above a very negative number.

---

49 The borrowing limit definitions are analogous to the steady state ones. I calculate “average” price schedules by integrating out the aggregate state using an invariant distribution over nodes from the Krusell and Smith (1998) method (because linear interpolation between moments is used, the law of motion can be thought of as a probabilistic transition matrix with an invariant distribution). With these average price schedules in hand, I compute the CD economy’s borrowing limit as \(\max_{a'}(q_B(a', s) + q_B(a', s)(-a'))\) and the ND economy’s limit as \(q_B(\bar{A}(s), s) + q_B(\bar{A}(s), s)(-\bar{A}(s))\). Here, \(\bar{A}(s)\) denotes the smallest value of \(a'\) that results in a household’s continuation value (with the aggregate state integrated out) being above a very negative number.
where $e_s$ denotes the lowest efficiency conditional on $s$ and similarly for $e_{s,z}$. As these definitions make clear, three factors can make the limit decrease: changes in the support of $e$, changes in the support of $s$, and changes in the wage and bond prices. The support of $e$ changes in the business cycle because of the supply shifter $\psi_z$. Specifically, $\psi_b = .975$ causes the limit to decrease by 2.5%. The support of $s$ also changes slightly due to numerical precision.\footnote{The erf function in Intel Fortran (and Matlab) which I use to compute the normal cdf returns a value of zero for large negative numbers. As a consequence, the normal cdf used in the method of Tauchen (1986) essentially draws from a bounded normal with a support of around 8 standard deviations. Because of this and countercyclical earnings variance, the support effectively increases in recessions and households can reach the lowest efficiency state (slightly) faster in the business cycle.} However, by process of elimination, the factor prices must account for most of the change.

Is it possible that the principal reason for the decline in the natural borrowing limit is price changes? To check this, consider how the net present value of guaranteed retirement income, a close proxy for the natural limit, changes for young households in the deepest possible recession:

\[
\sum_{t=46}^{T}(\prod_{j=2}^{t} q_j \rho_{s_t}) w_t \kappa G = .66 \tag{1.37}
\]

where $w_t$ and $q_t$ reflect the lowest $K$, $N$ and $z$ possible for all $t$. Clearly, the decline is large:
A key feature here is that in a protracted recession, capital decumulation causes both $w$ and $q$ to decline. This lowers the natural limit for any age but especially so for young households because the decline in $q$ is compounded.

It is worth briefly interpreting this result. When default is eliminated, creditors extend any amount of debt at a risk-free rate. While creditors offer any amount, households avoid taking on debt beyond what they can repay in the worst case scenario. In the steady state economy, the worst case scenario is bad efficiency shocks forever. In the business cycle, the possibility of a protracted recession makes the worst case scenario worse: households must now be prepared for the worst efficiency shocks and the worst wages and the highest interest rates (lowest bond prices). Hence they must further limit their debt exposure. This is the why the natural limit contracts.

While the ND economy’s limit contracts sharply after including the business cycle, credit in the CD economy is roughly the same. To understand why, consider Figure 1.5, which plots the value from default $V^D$ and repayment $V^R$ for a typical newborn household before and after the inclusion of aggregate uncertainty. The presence of aggregate risk makes the utility from repaying fall. Else equal, this makes households more prone to default. However, the value from default also falls, making households less prone to default. What matters for credit is the difference between how much these fall. If the value of default falls less, as is the case here, then households default more and credit contracts. If however the value of repayment falls less, then households default less and credit expands. The slight credit contraction observed in Figure 1.4 indicates that the value of default falls less on average than the value of repayment.

The changes in credit markets, and specifically the sharp contraction in the ND economy, affect consumption smoothing. This is made clear in Figure 1.6 which plots the variance of log consumption for young households. Both with and without aggregate uncertainty, the ND economy tends to have lower consumption variance. However, including the business cycle substantially increases the variance in the ND economy while leaving the CD economy’s

---

51 If one does the same exercise for the lowest $K/N$ and $z$, one finds 49%. This is a less intuitive (because labor is not high in recessions) but perhaps better number to look at. The interpolation procedure ends up placing some weight on this state because the grid for $N$ only has two points. This would make it seem that the grid matters a lot. However, I show in a robustness exercise that this is not the case. See Table A.5.
variance roughly unchanged.

These results suggest eliminating default may not be as welfare-improving in the business cycle context as it appeared in steady state. To assess this, I now measure the welfare gain of eliminating default. As before, I ask what increase in consumption in the CD economy would be required to make a newborn household indifferent between the CD and ND economies; however, this is now slightly more involved to compute. In the computational work, the method of Krusell and Smith (1998) is used with linear interpolation of the aggregate moments (i.e. summary statistics of the distribution). Because of this, the law of motion can be thought of as a transition matrix with an implied invariant distribution over aggregate states. I use this long-run distribution over states to “integrate out” the aggregate state from the value functions and calculate welfare in the following way:

$$\omega = \left( \frac{\sum_{z,s,m} F(z) \hat{F}(s|z) \int V_{ND}(0, e, s, 0; z, m) \Pi_{ND}(m|z) \hat{f}(e|s, z)de}{\sum_{z,s,m} F(z) \hat{F}(s|z) \int V_{CD}(0, e, s, 0; z, m) \Pi_{CD}(m|z) \hat{f}(e|s, z)de} \right)^{1/(1-\sigma)} - 1$$

where $m$ denotes a moment and $\Pi_X$ denotes the invariant distribution in economy $X$. As before, I also report the population in favor of the policy change and break out the welfare
Table 1.5 reports the welfare gains of eliminating default. Aggregate risk reduces the welfare gain of eliminating default to 0.49% of lifetime consumption, a decline of 1.33% from the steady state counterpart 1.82%. In partial equilibrium, i.e. holding fixed the law of motion and prices \( r, w \) and \( \bar{q}_B \), the drop is larger at 1.69% with the gain going from 3.94% to 2.25%. Support for eliminating default increases slightly due to general equilibrium effects (recall the ND economy’s average capital is about 1% higher in the business cycle). In partial equilibrium the support drops from 100% to 80.3%.

To aid understanding of how these gains are distributed over the life cycle, Figure 1.7 plots average gains by age. Consider first the gains in partial equilibrium. Note that aggregate risk causes a uniform decline in the gains with the largest drops occurring for

---

52Because the mapping from aggregate moments to distributions is not one-to-one, calculating welfare for households who are not newborn—and hence for whom the joint distribution over wealth, efficiency, and credit history is not known—is not straightforward. To get a rough idea of the changes in welfare, I do the following: define \( \tilde{V}_N(a, e, s, h) = \sum_{z,m} F(z) \Pi_X(m|z) V_N(a, e, s, h; z, m) \); define \( \tilde{d}(a, e, s, h) = \max_{z,m} d(a, e, s, h; z, m) \); define \( \omega(a, e, s, h) = ((V_{ND}(a, e, s, h)/V_{CD}(a, e, s, h = 0)^{1/(1-\sigma)} - 1) \); and compute the welfare gain for a subset \( I \) of the population as \( \int_I ((1 - \tilde{d}(a, e, s, h)) \omega(a, e, s, h) + \tilde{d}(a, e, s, h) \omega(0, e, s, 0)) d\mu \) where \( \mu \) is the average distribution along the simulated path of the CD economy. The population in favor is \( \int (1 - \tilde{d}(a, e, s, h)) 1[\omega(a, e, s, h) > 0] + \tilde{d}(a, e, s, h) 1[\omega(0, e, s, 0) > 0]) d\mu \).
<table>
<thead>
<tr>
<th>Aggregate Risk</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General Eq.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare Gain (%)</td>
<td>0.49</td>
<td>1.82</td>
</tr>
<tr>
<td>Population in Favor (%)</td>
<td>57.0</td>
<td>56.8</td>
</tr>
<tr>
<td><strong>Partial Eq.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare Gain (%)</td>
<td>2.25</td>
<td>3.94</td>
</tr>
<tr>
<td>Population in Favor (%)</td>
<td>80.3</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Table 1.5: Welfare Gains of Eliminating Default

young households. This shows the effect of credit becoming less effective in the ND economy. Now consider general equilibrium. As before, young households experience a significant welfare decline from eliminating default. However some households, specifically middle-aged households and very old retired households, have the opposite experience.

Figure 1.7: Welfare Gain By Age with and without the Business Cycle

Figure 1.8 confirms and extends these findings. Using circles it plots those households whose views of default have improved—that is their welfare gain of eliminating default has decreased—by more than 5%, and using dots it plots households whose views have deterio-
rated by more than 5%.

Those who view default less favorably are concentrated among old households who are earnings and asset poor. These are households whose earnings are no longer mean-reverting, thus making credit ineffective for self-insurance, and for whom the value of labor income in retirement is especially important; since the no-default economy has higher wages in the business cycle, they like default less. Those who view default more favorably are concentrated among the young to middle aged, those with below-median earnings, and the asset poor. These are households for whom credit, and a lot of it, is especially important: credit is important because their future labor income is high in expectation, and a lot of credit is important because they may already be near the CD borrowing limit or have highly-persistent shocks. Absent aggregate risk, they were able to (and did) borrow large amounts in the ND economy to insure themselves. With aggregate risk, this is no longer an option.

![Figure 1.8: Business Cycle Effect on Welfare Gain of Eliminating Default](image)

One might be tempted to conclude the welfare loss and contraction of credit in the ND economy occurs because the natural borrowing limit is a special case. It definitely is a special case. However, aggregate risk causes economies with high levels of commitment to

---

53 Regions of the state space are only plotted if they are visited with positive probability.
experience similar contractions of credit and reductions in welfare. To show this, I computed equilibrium for different levels of the default cost \( \chi \). The welfare gain of moving from the CD economy to each of these is reported in Table 1.6.

<table>
<thead>
<tr>
<th>CD ( \rightarrow )</th>
<th>( \chi = )</th>
<th>.00</th>
<th>.25</th>
<th>.50</th>
<th>.75</th>
<th>.90</th>
<th>.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welfare</td>
<td>Welfare Gain (%) S( S )</td>
<td>-0.9</td>
<td>1.9</td>
<td>5.3</td>
<td>7.4</td>
<td>7.5</td>
<td>7.0</td>
</tr>
<tr>
<td>Welfare</td>
<td>Welfare Gain (%) B( C )</td>
<td>-0.7</td>
<td>1.8</td>
<td>4.6</td>
<td>5.6</td>
<td>4.2</td>
<td>4.1</td>
</tr>
<tr>
<td>Pop in Favor (%) S( S )</td>
<td></td>
<td>47.1</td>
<td>50.5</td>
<td>49.1</td>
<td>46.6</td>
<td>45.1</td>
<td>46.7</td>
</tr>
<tr>
<td>Pop in Favor (%) B( C )</td>
<td></td>
<td>44.7</td>
<td>49.6</td>
<td>48.9</td>
<td>44.6</td>
<td>39.7</td>
<td>41.3</td>
</tr>
<tr>
<td>Allocations</td>
<td>Debt/( Y \times 100 ) S( S )</td>
<td>0.4</td>
<td>1.9</td>
<td>7.2</td>
<td>16.3</td>
<td>23.0</td>
<td>19.5</td>
</tr>
<tr>
<td>Allocations</td>
<td>Debt/( Y \times 100 ) B( C )</td>
<td>0.4</td>
<td>1.5</td>
<td>5.6</td>
<td>12.0</td>
<td>14.4</td>
<td>11.5</td>
</tr>
<tr>
<td>Allocations</td>
<td>Pop in Debt (%) S( S )</td>
<td>4.3</td>
<td>16.3</td>
<td>24.7</td>
<td>32.3</td>
<td>35.7</td>
<td>34.3</td>
</tr>
<tr>
<td>Allocations</td>
<td>Pop in Debt (%) B( C )</td>
<td>3.5</td>
<td>14.9</td>
<td>22.0</td>
<td>28.2</td>
<td>29.5</td>
<td>27.8</td>
</tr>
<tr>
<td>Allocations</td>
<td>Pop Filing (%) S( S )</td>
<td>1.04</td>
<td>0.07</td>
<td>0.04</td>
<td>0.03</td>
<td>8e-4</td>
<td>1e-5</td>
</tr>
<tr>
<td>Allocations</td>
<td>Pop Filing (%) B( C )</td>
<td>1.03</td>
<td>0.10</td>
<td>0.07</td>
<td>0.04</td>
<td>6e-4</td>
<td>1e-5</td>
</tr>
<tr>
<td>Allocations</td>
<td>K/( Y ) S( S )</td>
<td>3.10</td>
<td>3.06</td>
<td>2.98</td>
<td>2.87</td>
<td>2.81</td>
<td>2.84</td>
</tr>
<tr>
<td>Allocations</td>
<td>K/( Y ) B( C )</td>
<td>3.09</td>
<td>3.06</td>
<td>3.00</td>
<td>2.93</td>
<td>2.91</td>
<td>2.93</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \chi = 1 - 10^x ), ( x = )</th>
<th>-4</th>
<th>-8</th>
<th>-20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welfare</td>
<td>6.5</td>
<td>4.9</td>
<td>1.8</td>
</tr>
<tr>
<td>Welfare</td>
<td>3.9</td>
<td>3.6</td>
<td>0.5</td>
</tr>
<tr>
<td>Pop in Favor (%) S( S )</td>
<td>49.0</td>
<td>51.9</td>
<td>56.8</td>
</tr>
<tr>
<td>Pop in Favor (%) B( C )</td>
<td>44.1</td>
<td>48.9</td>
<td>57.0</td>
</tr>
</tbody>
</table>

Note: S\( S \) means “steady state” and B\( C \) means “business cycle.”

Table 1.6: Welfare Gains of Making Default More Costly

The business cycle significantly reduces the welfare gain of making default more costly for each level of \( \chi \) greater than its benchmark value \( \chi = .12 \). Whereas the steady state gain from high default-cost regimes reaches as much as 7.5%, the gain in the business cycle never crosses 5.6%. Moreover, the regime with the highest gain in steady state is \( \chi = .90 \) while in the business cycle it is lower at .75. Paired with the reduced welfare gain in high default-cost regimes is a reduction in the amount of debt and an increase in the capital output-ratio. These results show that the qualitative features from making default infinitely costly (as it is in the ND economy) extend to the case of just making it more costly.

### 1.5 Restricting Default

The previous section analyzed how aggregate risk affected the consequences of eliminating default. I now consider how aggregate risk affects the consequences of restricting default. I consider two restrictions on default. The first is modeled after the 2005 reform, and the
second only allows default in recessions or expansions.

The BAPCPA Reform

The Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (BAPCPA) is an important example of restricting default. The reform has made it typically impossible for households to file for Chapter 7 bankruptcy if they make above median income in their state. At the same time, it did not eliminate debt forgiveness for these households because they may still file for Chapter 13. This chapter of the bankruptcy code allows households to obtain partial debt forgiveness by forfeiting future income for a number of years (with the benefit being that households may keep their assets).

This legal environment is mapped into the model in the following way. First, households with efficiency less than $\psi \tilde{e}$, where $\tilde{e}$ is the median efficiency in steady state, may default as before with a fraction $\chi$ of their earnings transferred to creditors. Second, households with efficiency above $\psi \tilde{e}$ may forfeit a fraction $\chi_{13}$ of their earnings to creditors in the period of default in exchange for all their debt. They subsequently enter into bad standing ($h = 1$) with a zero net asset position. These assumptions imply contract pricing in steady state is

$$q(a', s) = \bar{q}_B \rho_s \mathbb{E}_s (1 - d(a', e', s', 0) + d(a', e', s', 0)(1_{e' \leq \tilde{e}} \chi + 1_{e' > \tilde{e}} \chi_{13}) we') / (-a'),$$  

with similar pricing in the business cycle. I set $\chi_{13} = .75$ to loosely represent a 15% annual contribution of earnings over the course of 5 years.

---

54 There are ways to get around this, especially through quitting your job, since the law treats “income” as average income over the six months prior to filing. The law also introduced other changes (increased filing costs of Chapter 7 bankruptcy, changed how much of disposable income must be used for Chapter 13 repayments, and others) which I do not consider. For a comprehensive list of changes, see White (2007).

55 This formulation uses a below-median earnings test rather than a below-median income test. However, because $a$ in the model is net worth, $a < 0$ can be thought of zero assets and $-a$ debts implying households have no capital income. A satisfactory modeling of Chapter 13 discharge would require separating assets from debts, allowing for endogenous labor choice, and having debt-level contingent repayment plans as is done in Li and Sarte (2006). This would take the present paper too far afield. The approach I use captures an important feature of Chapter 13, namely that debt is forgiven in exchange for a substantial portion of income, while maintaining tractability.

56 In the law, households in this situation would have to contribute 100% of “disposable income,” income exceeding allowances defined by the IRS, for 5 years. In terms of what creditors get, $\chi_{13}$ might be too low as earnings will tend to mean-revert. However it is not clear this is too low as labor supply might be reduced in the 5 year period or households might temporarily lower their income and file for Chapter 7. In terms of the utility cost to defaulters, $\chi_{13}$ might be too high as the default period is typically one of low consumption.
What are the positive effects of BAPCPA? Table 1.7 reports the capital-output ratio and debt and filing statistics after the policy change alongside their CD and ND counterparts. While BAPCPA roughly doubles total filings, it lowers filings of above-median earnings households by around 80%. Somewhat surprisingly, once aggregate risk has been accounted for, the BAPCPA economy has the most households in debt, the lowest capital-output ratio, and the most filings.\footnote{Clearly these are testable long-run implications. While the most recent recession has likely contributed to the filing numbers, it does appear Chapter 7 filings are returning to their pre-2005 levels. See Figure A.1 in Appendix A.2. Note that although the filings nearly double, filings in the model are only caused by earnings shocks. If shocks resulting in large amounts of debt, such as uninsured hospital bills or lawsuits, where included in the model, the filings would likely not rise as much: a household’s only option might be to file regardless of how costly it is.} However, the debt-output ratio is less than in the ND economy because it contracts more in the presence of aggregate risk.

<table>
<thead>
<tr>
<th>Economy</th>
<th>K/Y</th>
<th>Debt/Y</th>
<th>% in Debt</th>
<th>% Filing</th>
<th>% Filing $e \leq \psi_2 \tilde{e}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CD</td>
<td>3.08</td>
<td>.0069</td>
<td>10.5</td>
<td>0.18</td>
<td>0.09</td>
</tr>
<tr>
<td>ND</td>
<td>3.00</td>
<td>.0437</td>
<td>20.8</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>BAPCPA</td>
<td>3.00</td>
<td>.0581</td>
<td>20.4</td>
<td>0.39</td>
<td>0.37</td>
</tr>
<tr>
<td>Business Cycle</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CD</td>
<td>3.08</td>
<td>.0053</td>
<td>9.0</td>
<td>0.26</td>
<td>0.16</td>
</tr>
<tr>
<td>ND</td>
<td>3.03</td>
<td>.0272</td>
<td>16.5</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>BAPCPA</td>
<td>3.01</td>
<td>.0197</td>
<td>18.2</td>
<td>0.45</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Table 1.7: Effects of Policy Changes on Allocations

To understand these effects, it is useful to look again at the average borrowing limits. These are plotted in Figure 1.9 for the BAPCPA, CD and ND economies. The most striking feature is that the BAPCPA limit is hump-shaped and, until around age 55, larger than the limit in both the CD and ND economy. In fact once the business cycle is accounted for, newborn households in the BAPCPA economy can borrow more than twice the natural limit. It is also clear that the BAPCPA reform causes credit to expand more sharply for middle-aged households than for other households.

Why does the reform cause credit to expand, and why is this effect strongest for middle-aged households? The key insight for answering these questions is that an increase in default costs has both a direct and indirect effect. The direct effect is that when default is more...
costly, households are naturally less likely to default, and this expands credit. The indirect effect is that, when credit opportunities are better in the future, households are less likely to default now because they do not want to lose access to this expanded credit: by defaulting, a household would lose the ability to borrow for 10 years on average. At every age, the direct effect will be present for some households, specifically those making above-median earnings who now must pay $\chi_{13}$ rather than $\chi$ when they default. However, because of a hump-shaped earnings profile, the direct effect will be present for most middle-aged households. This causes a large expansion in credit for these households. At the same time, the indirect effect is strongest for young households. These households look forward and see a large expansion in credit in middle age. This makes them much less prone to default.

This argument also suggests a reason for why the BAPCPA reform increases total filings but decreases filings of above-median earnings households. The direct effect of higher default costs makes households less likely to default. However, this direct effect only applies to some households. At the same time, all households enjoy expanded credit, and, consequently, become more indebted. Because the benefit from default is increasing in indebtedness, households whose default costs have not increased end up defaulting more often.

An important but less obvious feature in Figure 1.9 is how aggregate risk causes the debt limit to contract in the BAPCPA economy. The largest contractions are seen for middle-aged households, for whom the median-earnings restriction tends to bind, while for younger households or those approaching retirement the contraction is virtually nonexistent. Consistent with the findings for the high default-cost regimes in Table 1.6, the business cycle causes credit to contract when default is costly. However, here, default is only more costly for the earnings rich. These households experience worse credit in the business cycle while credit for the earnings poor is roughly unchanged.

Was restricting access to default through the 2005 reform a good idea from a welfare perspective? The models with and without aggregate uncertainty suggest so. The top half of Table 1.8 lists the welfare gains of moving from the CD economy to the BAPCPA one. With or without aggregate uncertainty there is a large gain of around 2%. In fact, the gain is slightly higher after accounting for aggregate risk. Despite these seemingly large improvements in welfare, there is still substantial disagreement with only a slight majority
favoring the change.

<table>
<thead>
<tr>
<th></th>
<th>Aggregate Risk</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td><strong>CD to BAPCPA</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare Gain (%)</td>
<td>2.00</td>
<td>1.95</td>
</tr>
<tr>
<td>Population in Favor (%)</td>
<td>52.7</td>
<td>51.2</td>
</tr>
<tr>
<td><strong>BAPCPA to ND</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare Gain (%)</td>
<td>-1.49</td>
<td>-0.13</td>
</tr>
<tr>
<td>Population in Favor (%)</td>
<td>45.8</td>
<td>78.1</td>
</tr>
</tbody>
</table>

Table 1.8: Welfare Gains of the 2005 Reform

One might ask, now that bankruptcy has been restricted, should the law go further and eliminate default altogether? The model suggests not. The bottom portion of Table 1.8 reports the welfare gains of moving from the BAPCPA economy to the ND economy. Absent aggregate uncertainty, this policy change results in a welfare loss of 0.1%. Despite this, around 80% of households prefer the move. Once the business cycle is included, this majority disappears and the welfare loss deepens to 1.5% of lifetime consumption.

The key difference between eliminating default and restricting default through the 2005
reform is that the reform is targeted: it gives those with little earnings, from whom creditors would not receive much anyway, a fresh start. At the same time, those with more ability to pay back must repay. In this way, the BAPCPA reform preserves most of the insurance value of default while substantially improving credit.

Aggregate-State Contingent Default Policy

I now consider a different type of default restriction that allows default only in recessions. As discussed in the introduction, this is meant to capture the pattern of legislation responding to negative aggregate shocks with more debt forgiveness and to positive shocks with less. For completeness, I also consider the case where default is only allowed in expansions. The policy that allows default in recessions \( z = b \) is referred to as CDR; the policy where default is only allowed in expansions \( z = g \) is referred to as CDE.

\textit{A priori}, one might expect that either economy would look like a convex combination of the CD and ND economies. Surprisingly, the aggregate statistics reported in Table 1.9 do not bear this out. In particular, the CDR and CDE economies have the highest capital-output ratio and the least amount of debt.

<table>
<thead>
<tr>
<th>Economy</th>
<th>K/Y</th>
<th>Debt/Y</th>
<th>% in Debt</th>
<th>% Filing</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>3.08</td>
<td>.0053</td>
<td>9.0</td>
<td>0.26</td>
</tr>
<tr>
<td>ND</td>
<td>3.03</td>
<td>.0272</td>
<td>16.5</td>
<td>0.00</td>
</tr>
<tr>
<td>CDR</td>
<td>3.09</td>
<td>.0036</td>
<td>7.8</td>
<td>0.12</td>
</tr>
<tr>
<td>CDE</td>
<td>3.09</td>
<td>.0017</td>
<td>6.3</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Table 1.9: Effects of Policy Changes on Allocations with Aggregate Risk

The proximate cause of this lack of debt is revealed in Figure 1.10 which plots borrowing limits for the CD, ND, and CDR economies (the CDE and CDR limits look very similar). In the CDR economy, the borrowing limit for newborn households is virtually zero and until age 63 lies below the CD economy’s limit. In retirement, the limit is higher than in the CD economy, but it is still much smaller than in the ND economy.

To understand why the CDR limit is so tight, first note that in one possible scenario, expansion for a lifetime, households will never be allowed to default. Consequently, a natural borrowing limit is in effect because households will not borrow more than the net present
value of the lowest possible earnings stream conditional on this sequence of aggregate shocks. Now note that households with the lowest possible earnings stream are very prone to default if a recession occurs. This fact, together with \( \tilde{q}_g(S) \approx \tilde{q}_B(S)F(g|g) \) in the calibrated model, implies the price of a bond for these households is approximately \( \frac{2}{3} \) of the risk-free price \( \tilde{q}_B(S) \). Because they must borrow at this low price, the net present value of the worst earnings stream is drastically lowered: a unit of labor income from age 65 is worth roughly \( 0.45 \) at age 20 when discounted by \( \tilde{q}_B(S) \); this drops to only \( 0.5 \times 10^{-8} = (0.98 \times 2/3)^{45} \)

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58 Let \( \tilde{s} \) denote the persistent state associated with the worst earnings stream. Then \( p(a', \tilde{s}; B) \approx 0 \) for virtually any \( a' < 0 \) because households are very likely to default. Using this and \( \tilde{q}_b(s) \approx \tilde{q}_B(S)F(g|g) \), one has

\[
q_g(a', \tilde{s}; S) + q_b(a', \tilde{s}; S) = \tilde{q}_g(S) + \tilde{q}_b(S)p(a', \tilde{s}; B) \approx \tilde{q}_g(S) \approx \tilde{q}_B(S)F(g|g) = 2/3 \tilde{q}_B(S).
\]
when discounted by $\bar{q}_B(S) \cdot 2/3$. Because future earnings are discounted heavily, and because hardly any earnings are guaranteed until retirement, the limit in the CDR economy is very tight and especially so for young households.

Given the tight borrowing constraints, one would expect that both the CDR and CDE economies are worse than either the CD or ND economy in terms of welfare. This is confirmed in Table 1.10 which reports the welfare gains of allowing default in recessions or expansions only. Relative to the CD economy, either policy change results in a welfare loss of around 1.4%; relative to the ND economy, the loss is higher at 1.9%. The move to default only in recessions is viewed more favorably than a move to default only in expansions, but the difference is slight.

One caveat to these results deserves mention. In this model (and, to my knowledge, in all the quantitative literature), the default-risk premium is paid entirely upfront. This modeling assumption matters here because each period households pay the premium even if they never default. If losses due to default were recouped within each period, the results would likely be different.

1.6 Conclusion

It was found that aggregate risk substantially lowers the welfare benefit of eliminating default. Without the ability to discharge debt, households limit their debt to what they can repay in the worst of circumstances. A business cycle makes the worst circumstances worse by introducing the possibility of a protracted recession. This possibility causes households to limit their debt more than in steady state unless they have a default option, reducing their ability to self-insure. When default is allowed, households can safely use all available credit, even with the possibility of a protracted recession, by defaulting if such a recession occurs. This benefit of having a default option is, by necessity, missing in a steady state environment.

While aggregate risk reduces the welfare benefit of eliminating default, aggregate risk’s effect on restricting default depends on the restrictions in place. A policy change resembling the 2005 reform substantially increased welfare, and, unlike eliminating default, improved
welfare more in the business cycle environment. A key aspect of the policy is that it only makes default more costly for some households, those with above-median earnings. This substantially improves credit while preserving most of the insurance value of default by allowing earnings-poor households to easily default.

In contrast, a different type of default restriction, one that allows default only in recessions or expansions, introduced uncertainty that resulted in substantially lower welfare. Specifically, it introduced a small probability that households would never be able to default. This kept households from borrowing more than what they could repay with probability one. At the same time, it introduced a fairly high probability that households would be able to default. This made credit expensive. Consequently, the economy was worse than always having default, in which case households can rely on the default option, and worse than never having default, in which case credit is cheap.

The welfare gains from restrictive default regimes in this paper should be viewed as upper bounds for a number of reasons. First, this paper treated aggregate risk as a productivity shock, countercyclical earnings variance, and a shock to aggregate labor supply. This is a narrow view. More broadly, aggregate risk could include risk to entitlement programs as well as deflationary risk. These types of aggregate risk would likely reduce the welfare gains of harsh default regimes. Second, as experience has shown, regardless of legislation some households will always default. Severely punishing these households can entail significant social and fiscal costs which are not modeled here. Third, the only idiosyncratic risk in this paper came from earnings shocks. Including expenditure shocks, such as uninsured medical expenses, would also likely reduce the welfare gains from harsh regimes.
Chapter 2

Dealing with Consumer Default: Bankruptcy vs Garnishment

Satyajit Chatterjee and Grey Gordon

Summary

What are the positive and normative implications of eliminating bankruptcy protection for indebted individuals? Without bankruptcy protection, creditors can collect on defaulted debt to the extent permitted by wage garnishment laws. The elimination lowers the default premium on unsecured debt and permits low-net-worth individuals suffering bad earnings shocks to smooth consumption by borrowing. There is a large increase in consumer debt financed essentially by super-wealthy individuals, a modest drop in capital per worker, and a higher frequency of consumer default. Average welfare rises by 1 percent of consumption in perpetuity, with about 90 percent of households favoring the change.

2.1 Introduction

Unlike most other industrialized countries, default on consumer debt is a very common occurrence in the United States. Fundamentally, this feature of the US consumer credit market derives from the institution of personal bankruptcy: An indebted individual has the
legal right to petition a bankruptcy court to have his or her financial obligations discharged, following which creditors must cease all efforts to collect on the debt. The option to declare bankruptcy limits how vigorously creditors can pursue delinquent debtors and, knowing this, debtors choose to default on their debt more readily. Since the start of the crisis-induced downturn in September 2008, the outstanding stock of revolving consumer debt has declined by more than 15 percent. It declined, in part, because debtors stopped making payments on their obligations and, as required by regulation and law, the defaulted debt was charged off and removed from the balance sheets of creditor banks.

Given this recent experience, it is tempting to ask whether policies designed to discourage consumer bankruptcy are desirable. The answer is not obvious. On the one hand, discouraging bankruptcy makes it harder for over-extended households to escape the consequences of bad luck. On the other hand, by making default less likely it makes credit cheaper and permits better consumption-smoothing. Exactly how much cheaper, though, depends on the constraints imposed on creditors by garnishment laws. These laws allow households some measure of protection against creditors and serve somewhat the same function as the “safety valve” of personal bankruptcy. The goal of this paper is to answer the following specific question using quantitative theoretic methods: What are the positive and normative implications of eliminating the personal bankruptcy option and letting current garnishment laws be the sole operative law dealing with consumer default?

The implications of eliminating the bankruptcy option, or discharge, have been studied by previous authors in a quantitative setting. However, these studies uniformly equate the “no-bankruptcy” world to an environment with an infinite cost of defaulting on consumer debt.\(^1\) This is problematic for two reasons. First, there is ample historical evidence to suggest that there will be default on consumer debt even in the absence of bankruptcy protection. Indeed, it was the plight of delinquent debtors caught in the grip of unrelenting creditors that provided the impetus and motivation for discharge.\(^2\) Second, the assumption has unpalatable consequences for the theory: It implies that the consumer can borrow, at the risk-free rate, as much as the present discounted value of the stream of the lowest

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\(^1\)Examples are Athreya (2002, 2008), Li and Sarte (2006), Chatterjee, Corbae, Nakajima and Ríos-Rull (2007) and Athreya, Tam and Young (2009b).

\(^2\)See, for instance, the discussion in Coleman (1999) and Warren (1935).
earnings realization possible. Even for very low earnings realizations this bound (the so-called “natural borrowing limit”) can be quite large relative to average income. Unrestricted ability to borrow such large sums at the risk-free rate is patently unrealistic and distorts the assessment of the welfare gain from eliminating bankruptcy protection.

In contrast, the “no-bankruptcy” world in this paper features a realistic alternative to the bankruptcy option, one that is based on garnishment laws actually in existence. The elimination of bankruptcy protection does not eliminate consumer default – individuals in dire straits can default and repay their debt gradually over time by subjecting themselves to wage garnishment. The possibility of default, and subsequent slow repayment on defaulted debt, makes consumer loans expensive even in the absence of bankruptcy protection. Thus the approach taken in this paper features a more plausible counterfactual loan supply schedule and is consistent with the historical experience of consumer default during the pre-discharge era.

Elimination of bankruptcy protection results in only moderate changes in factor prices and a 1 percent increase in average welfare. However, it results in a two orders of magnitude increase in unsecured debt. This large increase in debt results from the fact that the added commitment to honor debt contracts lowers interest rates and encourages low-net-worth individuals suffering bad earnings shocks to borrow more in order to smooth consumption. The additional borrowing results in a small rise in the risk-free interest rate – and a correspondingly small decline in real wages – because the very wealthy have a high interest elasticity of savings. Since low-net-worth individuals borrow from high-net-worth individuals, the expansion in debt results in a correspondingly large increase in wealth inequality. The large increase in debt is accompanied by an increase in the frequency of consumer default. Despite lower wages and higher default rates, the majority of individuals prefer the “no-bankruptcy” world: the wealthy benefit from the higher risk-free interest rate and the poor from a decrease in borrowing costs.

Although delinquent debtors are permitted to lower their labor supply in response to the “garnishment tax,” this channel is essentially inoperative in our model. There are two reasons for this: First, the elasticity of labor supply is chosen to produce a factor of 2 difference in the hours worked between high and low efficiency workers. Since these workers earn
very different wages, the implied labor supply elasticity – consistent with microeconomic studies – is quite low.\(^3\) Second, when people go into garnishment they often earn less than the threshold above which garnishment is operative, so their labor supply choice is not distorted. Thus, the findings reported in this paper do not support the notion that bankruptcy provides superior labor supply incentives.\(^4\)

The prediction that elimination of bankruptcy protection will result in a large increase in consumer debt (and the associated large increase in wealth inequality) seems at variance with the experience of continental European countries. These countries, which historically have not permitted discharge of debt, do not display the high wealth inequality predicted by our garnishment-only (i.e., “no-bankruptcy”) model. On the other hand, European countries display much less idiosyncratic earnings risk than the US, which could also account for their less extreme wealth distributions. Taking Sweden (which did not permit discharge until 2005) as a test case, we show that when we simulate our garnishment-only US model with the Swedish earnings process, the model generates a wealth distribution that is close to Sweden’s actual wealth distribution. Thus less risky earnings processes may explain why these countries do not display the extreme wealth distribution predicted by our model despite having not permitted discharge historically.

A policy change as dramatic as elimination of bankruptcy protection compels a rethinking of garnishment law as well. With this in mind, the paper also investigates the optimal garnishment regime in the absence of discharge. It finds that welfare is higher if elimination of discharge is also accompanied by less liberal (from the viewpoint of debtors) garnishment laws. How much less depends on the details of the earnings process: The possibility of a low probability “disaster state” (in which earnings are very low) pushes policy in the direction of a more liberal garnishment law. Still, results suggest that current garnishment laws are too liberal: Welfare would be higher if less income is protected from garnishment.

Among existing quantitative studies, two come closest to the spirit of this study. The first is Livshits, MacGee and Tertilt’s (2007) comparison of the discharge option (or “Fresh

\(^3\) Increasing the difference in hours worked to a factor of 4 leads to essentially the same results.

\(^4\) This finding is in line with the results reported in Li and Han (2007). Chen (2010) argues that the labor supply effect of bankruptcy is positive but small because of off-setting wealth and substitution effects.
Start”) with something akin to garnishment (which they label the “European System”). But there are some important differences between their study and this one. First, our goal is to compare a regime in which both bankruptcy and garnishment are active in equilibrium (as it is in reality) with a regime in which only garnishment can be active. Second, we model the production side of the economy, whereas Livshits, MacGee and Tertilt work with an endowment economy. The first difference is important because the co-existence of bankruptcy and garnishment bounds the costs of garnishment (these costs must be much lower than those for bankruptcy; otherwise, individuals will always opt to discharge their debts) which, in turn, has consequences for the deadweight costs of default in the garnishment-only economy. The second difference is important because strong general equilibrium effects can potentially emerge from elimination of bankruptcy protection.

Second is Li and Sarte (2006), who examine the welfare effects of means-testing for obtaining a discharge (so-called Chapter 7 filing), when the alternative to discharge is partial debt repayment (so-called Chapter 13 filing). If the qualifications for a discharge are made so stringent as to leave Chapter 13 as the only default option, the institutional arrangement may seem to resemble one in which defaulters can either repay their debts or subject themselves to “garnishment.” But this resemblance is more apparent than real. In practice (as well in the theory presented in Li and Sarte, 2006), a Chapter 13 filing will involve substantial forgiveness of debt. Thus, permitting Chapter 13 filings only is not the same as eliminating discharge altogether. Li and Sarte do consider the case where bankruptcy protection is eliminated completely (so neither Chapter 7 or 13 filings are permitted) but they do not consider the possibility that debtors may still default and repay their debts gradually in accordance with the debtor protection offered under garnishment laws.

There is also an important substantive difference in the environment analyzed in this

\[\text{5}\] In their “European System,” an individual cannot discharge his debt, so default leads to some portion of his future wage earnings being taken in satisfaction of the creditors’ claim. In the working paper version of the paper, Livshits, McGee, and Tertilt (2003), the authors also considered the case where the garnishment process allowed households to work less and featured a constant exemption level.

\[\text{6}\] This point was stressed in Li and Sarte (2006), who showed that taking into account general equilibrium effects can overturn welfare results obtained in partial equilibrium settings.

\[\text{7}\] “Chapter choice” is a decision about protecting current assets versus future earnings. In a Chapter 7 filing, the individual relinquishes all his non-exempt assets but in return gets to keep his future wage earnings; in a Chapter 13 filing, the individual gets to keep his non-exempt assets but agrees to repay his outstanding debts over time but only up to the value of the non-exempt assets that would have been relinquished under a Chapter 7 filing.
paper and the one in Li and Sarte (2006). The latter assume that people borrow (subject to an exogenously given borrowing limit) at an undifferentiated interest rate that fetches zero profits when returns are averaged across all borrowers. Since the interest rate charged on loans is independent of the size of the loan, large borrowers (who are worse risks) are subsidized by small borrowers. In contrast, the approach followed in this paper eschews any form of cross-subsidization across borrowers: Every loan makes zero profits, in expectation. Since default risk varies with individual characteristics (in particular, earnings) as well as the amount borrowed, individuals borrow at differentiated interest rates. This is consistent with the evidence: For the US, Edelberg (2006) has shown that interest rates on unsecured consumer credit vary positively with perceived default risk - i.e., pricing of consumer loans is risk-based. Because bigger loans require higher interest rates, how much individuals can borrow to smooth consumption is limited. Consequently, the elimination of discharge – and the resulting shift out in individual loan supply schedules – has a more dramatic effect on an individual’s capacity to borrow. This expansion in credit, combined with the riskiness of the US earnings process, underlies both the predicted welfare gain as well as the predicted increase in wealth inequality.

In related work, Athreya (2008) presents a careful study of the implications of eliminating bankruptcy protection in a partial equilibrium life-cycle setting with earnings risk. Elimination is taken to mean that default becomes infinitely costly, which, in turn, means that the maximum level of borrowing that can be supported in the no-bankruptcy equilibrium is determined by the natural borrowing limit. This introduces a tight link between

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8Conceptually, this formulation is problematic for the following reason: A lender can come in to serve small borrowers at a slightly lower interest rate and make positive profits. Thus, this formulation implicitly assumes restriction on entry.

9In practice, the interest rate paid by a borrower is negatively related to the individual’s credit score – which is an index of the probability of repayment of a loan. Credit scores are lower for observably higher risk individuals and they tend to fall with increased borrowing. Thus higher risk individuals pay higher interest rates and interest rates paid tend to rise with the amount of outstanding debt. Chatterjee, Corbae and Ríos-Rull (2011) provide more details on the relationship between repayment behavior, borrowing and the evolution of credit scores.

10In the case where Li and Sarte (2006) allow only Chapter 13 filings, the increase in the debt-to-income ratio is about 4 percent and the increase in steady state welfare -1 percent (Table 4, p. 628). It is worth noting that their calibration is different in 3 important respects. First, their debt-to-income ratio is 6 percent, much higher than the 0.09 percent used in this study. In this paper, only negative net-worth individuals are viewed as borrowing unsecured so the debt-to-income ratio is much lower. Second, the efficiency process used by Li and Sarte has an unconditional standard deviation of 0.56, while the process used in this paper has an unconditional standard deviation of 7.92 (mean of both processes is 1). Third, Li and Sarte allow for proportional transactions costs on loans, while transactions costs are ignored in this study.
“default policy” and “social insurance provision,” which the paper explores. In Athreya, Tam and Young (2009b), the focus is on understanding (again, in a partial equilibrium context) the merits of harsh default penalties (in effect, making the cost of default infinite) versus keeping penalties low but providing loan guarantees to lenders so as to lower the price of credit to households.

2.2 The Model Economy

This section discusses the model economy. For a more extensive model description, the reader is referred to the supplementary appendix.\(^{11}\)

Preferences and Technology

At any given time, there is a unit mass of people in the economy. Each person has a probability of survival given by \(\rho \in (0, 1)\) so that a fraction \(1 - \rho\) of the population dies each period and is replaced by newborns.

Each person has a unit of time endowment. People differ in terms of the productive efficiency of their time endowment, which varies stochastically over time. These efficiencies are denoted by \(e\). In any period, an individual’s \(e\) is drawn from a discrete probability distribution with compact support \(E \subset \mathbb{R}^{+\infty}\) and probability mass function \(\phi_s(e)\).\(^{12}\) Here, \(s\) is a finite-state Markov chain taking values in a set \(S\) with transition probabilities \(\pi_{s,s'}\). Draws from this process, as well as from \(\phi_s(e)\), are independent across people. Thus, the efficiency process has a persistent component controlled by \(s\) and a transitory component controlled by \(\phi_s\). A person’s anticipated lifetime utility from a sequence \(\{c_t, n_t, e_t\}\) of consumption, effort and efficiency levels is given by

\[
\sum_{t=0}^{\infty} (\beta \rho)^t u(c_t, n_t, e_t)
\]  

\(^{11}\)Supplementary materials are available at http://www.sciencedirect.com/science/journal/03043932.

\(^{12}\)In Chatterjee et al. (2007), these probability distributions were assumed to be continuous, not discrete (continuous \(e\) is necessary to prove the existence of a competitive equilibrium). Provided the number of grid points on \(e\) is large, reasonably accurate solutions to equilibrium prices can be found. See Chatterjee and Eyigungor (2010) for discussion of the computational challenges involved in computing debt/default models of this type and their Appendix B for a comparison of the numerical accuracy of the solution when \(e\) is taken to be discrete vs continuous.
where $\beta$ is the discount factor, $\rho$ is the probability of survival and the momentary utility function $u(c, n, e) : [0, \infty) \times [0, 1] \times E \rightarrow R$ is strictly increasing and concave in $c$, strictly decreasing and convex in $n$, and differentiable in the first two arguments. For technical reasons, the efficiency level is allowed to affect period utility.

There is an aggregate production function $F(K, N) : R_+ \times R_+ \rightarrow R_+$, which gives the total quantity of the single good produced in this economy as a function of the aggregate capital stock $K$ and aggregate efficiency units of labor $N$. We assume that $F$ is CRS, differentiable, increasing and displays diminishing marginal products with respect to each input. The capital stock depreciates at the rate $\delta \in (0, 1)$.

**Market Arrangement**

In each period, there is a market for efficiency units of labor where people and the representative firm in charge of the (aggregate) production technology transact in labor services: people can sell any portion of their efficiency endowments to the firm at the wage $w_t$ per efficiency unit, where the wage is expressed in terms of the period-$t$ consumption good.

There is a market for the services of physical capital. The representative firm can rent physical capital from an intermediary sector at the rate of $r_t$ units of consumption good per unit of capital.

Most crucially, there is a market in which people can borrow and lend. When a person borrows, the option to default implies that the interest rate at which he borrows will depend on his likelihood of default. The latter, in turn, will depend on all observable factors that potentially influence that likelihood. In the context of this model, these factors are (i) the size of the liability (or promise), (ii) the person’s current efficiency status and (iii) all current and future factor prices. It is notationally convenient to denote assets by positive numbers and liabilities by negative numbers. We use $p$ to denote the sequence of current and future factor prices $\{w_j, r_j\}_{j=0}^{\infty}$. Then, the unit price of a promise to deliver $y$ (if $y < 0$) units of the consumption good next period by a person with current persistent state $s$ is $q(y, s, p) > 0$.

By making this promise, the person receives $q(y, s, p)(-y)$ units of the consumption good

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13 The price depends only on the persistent component of efficiency because this is the component that helps predict the future efficiency level $e'$ and persistent state $s'$. In particular, current $e$ does not affect the price because it is a purely transitory draw that does not predict future earnings.
in the current period. If \( y \geq 0 \), the person obtains a promise to receive \( y \) next period and gives up \( \tilde{q}(p)y \) in the current period. Thus, people borrow at interest rates that vary with the loan size but lend at an interest rate that is independent of the amount lent and the person’s efficiency level. These prices depend on the current and future trajectory of factor prices because our analysis allows for transition dynamics. In addition to the market for new loans and deposits, there is also a market where intermediaries may trade debt that is in default. The market price of an unpaid obligation of the amount \( y < 0 \) belonging to an individual with current persistent efficiency level \( s \) is denoted \( x(y, s, p) \).

It is assumed that the intermediary sector is the counterparty in all intertemporal trades entered into by people. One implication of this assumption is that if a person dies, his assets or liabilities are absorbed by the intermediary sector.

**Garnishment, Bankruptcy, Collecting and Reporting Laws**

A brief description of US wage garnishment laws is now provided. If a debtor fails to repay a debt, creditors have the legal right to seize the debtor’s property and earnings in satisfaction of their claims. The purpose of wage garnishment laws is to provide some measure of debtor protection against creditor rights. Federal law stipulates that 75 percent of a debtor’s disposable earnings are outside the reach of creditors, with many states choosing to protect even more.\(^{14}\) To garnish a person’s wages, a creditor must obtain a court order and this order is granted for a limited time only. Upon expiration of the order, a new order must be obtained if the garnishment is to continue. Because garnishment is costly, creditors have a strong incentive to pass these costs on to the debtor. Federal law (the Fair Debt Collection Practices Act) stipulates that creditors cannot add additional charges (such as fees and interest charges) to the original obligation unless such additions are permitted explicitly by the contract or by the state (if the contract is silent on it). State practice varies quite a bit in this regard, with some states permitting additional fees and interest charges on the unpaid debt. However, courts generally take a dim view of creditors’ attempts to recover more than reasonable collection costs through this channel. Lastly, a federal statute

\(^{14}\)See Lefgren and McIntyre (2009), Table 2.
of limitations on unpaid debt exists: if a debt has not been paid in over 10 years, the creditor loses the right to garnish wages (or seize property) in satisfaction of the claim.

This institutional setup is mapped into our model in the following way. First, it is assumed that no additional fees or interest charges can be assessed on unpaid debt. Second, there is assumed to be no statute of limitations on unpaid debt and no transactions costs of enforcing wage garnishment. Consequently, if a delinquent debtor chooses not to file for bankruptcy, garnishment continues for as long as there is any unpaid obligation. Third, as long as there is any unpaid obligation, the debtor cannot accumulate assets and must pay some legally determined fraction of disposable income to the creditor. In the model, the garnishment formula is modeled as the assumption that the delinquent debtor must pay at least \( \min\{\max\{0, \gamma(wen-c_{\min})\}, -a\} \) toward reducing his obligation, where \( \gamma \) is the fraction of disposable income that can be garnished, \( wen \) is current period earnings, \( -a \) is the size of the unpaid obligation in the current period, and \( c_{\min} \) is “reasonable living expenses” as determined by law. Importantly, the choice of \( n \) is left to the delinquent debtor and there is no compulsion to earn above \( c_{\min} \). Lastly, it is assumed that delinquency and garnishment have pecuniary costs to the debtor, which are modeled as a consumption loss of proportion \( \chi_g \) of earnings. These costs are paid every period the debtor is under garnishment and, once the garnishment ends, for as long as lenders know that the person was garnished sometime in the past (more on this below).

Turning attention to the modeling of bankruptcy, the following assumptions are made. First, a debtor has the right to have his unpaid obligations discharged. Second, there are no transactions costs of filing for bankruptcy. Third, a debtor filing for bankruptcy must forfeit all his assets to satisfaction of the claim. Fourth, the process of obtaining discharge consumes the entire period so that in the period of bankruptcy, the debtor can neither accumulate assets nor borrow. Lastly, it is assumed that bankruptcy imposes pecuniary costs that result in a loss of consumption equal to a proportion \( \chi_b \) of earnings and that these are paid for as long as lenders know a person declared bankruptcy in the past.

In addition to garnishment and bankruptcy laws, the Fair Credit Reporting Act stipulates how long negative information, such as late payments, bankruptcies, garnishments, and tax liens, may stay on a person’s credit report. By law, bankruptcy information can
stay on a credit report for ten years. Garnishments can stay on the report for twelve years from the date of entry or for seven years from the date they were satisfied. This aspect of US law is relevant for our study because negative information in a person’s credit report appears to impair the person’s access to credit.\textsuperscript{15}

In the model, the following assumptions are made regarding the consequence and duration of negative credit information. First, a person with a record of a past bankruptcy or a past garnishment cannot borrow. Second, the record of a past bankruptcy is removed from a person’s credit history with probability $\lambda_b$. Third, a record of garnishment always appears as long as the individual is under garnishment. Fourth, if a person under garnishment files for bankruptcy, his record of garnishment is replaced by a record of bankruptcy. Finally, a record of past garnishment is removed with probability $\lambda_g$.

**Equilibrium**

The preceding environment maps into decision problems for individuals in the following way. Surviving individuals enter into a period with either assets or debt and with either a clean credit record or an impaired one. A credit record is impaired if it has either a bankruptcy or garnishment “flag,” i.e., a record of past bankruptcy or garnishment that has not been removed. An individual with debt and a clean credit record gets to decide if he wants to default on the debt and, conditional on defaulting, whether to file for bankruptcy or subject himself to wage garnishment. If the person chooses not to default, he decides how much to borrow or save in the current period. An individual with debt and a garnishment flag gets to choose whether to continue on in garnishment or to declare bankruptcy. If the person declares bankruptcy, then he cannot borrow or save in the current period and his garnishment flag is turned into a bankruptcy flag; if he continues on in garnishment, he cannot borrow but he can accumulate assets if he pays off all his unpaid obligations. An individual who enters the period without debt does not have a default decision to make. If his credit record is clean, he chooses how much to borrow or save; if his credit record is impaired, he cannot borrow but he can save. All individuals, no matter what their circumstances, get

\textsuperscript{15}Musto (2004) provides compelling evidence in favor of this assumption. See Chatterjee, Corbae and Ríos-Rull (2011) for the theoretical foundation for this finding.
to choose how hard to work.

The (representative) competitive intermediary’s decision problem is static: it simply
decides how much of each type of loan to make at the going price of each type of loan.
By the law of large numbers, the intermediary’s aggregate return on its loan portfolio is
constant. Thus, it operates like a risk-neutral lender with respect to each individual loan.
In equilibrium, the price of any individual loan adjusts to generate exactly zero net return.
Thus, the intermediary is indifferent about making any particular loan: it simply writes
loans that consumers want. A loan that defaults into bankruptcy pays nothing; a loan that
defaults into garnishment may pay something and, in addition, becomes a defaulted debt
that can be traded in the market at some price. If there is no default on the loan, the loan
pays back what was promised.

2.3 Calibration

This section discusses the calibration of the model economy.

Functional Forms

For $u(\cdot)$ it is assumed that

$$u(c, n, e) = (1 - \sigma)^{-1} \left( c - \zeta \frac{n^{1+\xi}}{1+\xi} + A(e) \right)^{(1-\sigma)}, \text{ with } \sigma > 0, \zeta > 0, \xi > 0. \quad (2.2)$$

Thus, we adopt the Greenwood, Hercowitz and Huffman (1988) specification for preferences,
modified slightly as in Mehlkopf (2010), to allow for a state dependent constant term $A(e)$.
There are two advantages to this specification. First, the level of consumption $c$ does not
affect the MRS between consumption and effort, which is simply given by $\zeta n^\xi$. Second, for
any feasible $c$, the requirement that $c - \zeta n^{1+\xi}/(1+\xi) + A(e) \geq 0$ – which is needed for current
utility to be well-defined – can be effectively reduced to the requirement that $c \geq 0$ by an
appropriate choice of $A(e)$. By the first property, the unconstrained choice of $n$ for a person
in good standing is given by $\bar{n}(e; \bar{p}) = (ew(\bar{p})/\zeta)^{1/\xi}$. In what follows, we set $A(e)$ to be equal
to $\zeta(\bar{n}(e; \bar{p}))^{1+\xi}/(1+\xi)$ where $\bar{p}$ denotes the sequence of (constant) factor prices associated
with the targeted capital output ratio. In the steady state, where $w$ is constant over time,
this term will vary with $e$ only and will generally offset the $-\zeta(n(a, e, h, s; p))^{1+\xi}/(1 + \xi)$ term. Thus, the requirement that $c - \zeta n^{1+\xi}/(1 + \xi) + A(e) \geq 0$ will effectively become the requirement that $c \geq 0$. Furthermore, when the offset is operative, utility is simply given by $c^{1-\sigma}/(1 - \sigma)$.

It is assumed that for the vast majority of the population, the efficiency level $e$ follows the process

$$\ln(e_t) = \omega + z_t + \nu_t \text{ with } z_t = \psi z_{t-1} + \varepsilon_t, \psi \in (0, 1), t \geq 1$$

(2.3)

where $\omega$ is drawn at birth from a Normal distribution with mean 0 and variance $\sigma^2_\omega$, $\nu_t$ and $\varepsilon_t$ are drawn from Normal distributions with mean 0 and variance $\sigma^2_\nu$ and $\sigma^2_\varepsilon$, and $z_0$ is drawn from the invariant distribution of the AR1 process. Thus the efficiency process (and consequently the earnings process) has three components: a permanent component that is determined at the time the person enters the economy, a persistent component that follows an AR1 process and a purely transitory component. However, it is assumed that any individual, regardless of his or her $\omega, z_t$, and $\nu_t$, can draw an extremely high (relative to mean) efficiency level, denoted $E_{\text{max}}$, with a (small) probability $\pi_0$. From this “super-rich” state, he returns with probability $\pi_1$ to an efficiency level drawn according to the invariant distribution of $\omega, z_t$, and $\nu_t$. This super-rich state is added to generate the highly skewed wealth inequality seen in the US. The combined efficiency process can be mapped back to the model’s efficiency process via a suitable choice of the set $S$ and the distributions $\phi_s$.

Lastly, it is assumed that the aggregate production function is given by $K^\alpha N^{1-\alpha}$.

**Data Targets and Parameter Values**

With these functional forms, aside from $A(e)$, there are twenty parameter values to fix. These are four preference parameters ($\beta, \sigma, \zeta, \xi$); one demographic parameter ($\rho$); four technology parameters ($\alpha, \delta, \chi_g, \chi_b$); four legal system parameters ($\lambda_g, \lambda_b, c_{\text{min}}, \gamma$); and seven efficiency parameters ($\sigma^2_\omega, \sigma^2_\nu, \sigma^2_\varepsilon, \psi, \pi_0, \pi_1, E_{\text{max}}$).

Values for $(\sigma^2_\omega, \sigma^2_\nu, \sigma^2_\varepsilon, \psi)$ are chosen to match the wage process estimates in Floden and Linde (2001) for the US. Wages as measured in Floden and Linde correspond to $w(p)e$ in our model. Thus, in the steady state, their estimated wage process can be used to calibrate the
efficiency process and this fixes $(\sigma^2_2, \sigma^2_3, \sigma^2_4, \psi)$ to $(0.1175, 0.0421, 0.0426, 0.9136)$. The parameters $\pi_0$ and $\pi_1$ are taken from Chatterjee et al. (Table III, p. 1550), who also incorporate this state to generate the observed US wealth inequality. This fixes $(\pi_0, \pi_1)=(0.0001, 0.020)$.\textsuperscript{16} The mean of the augmented efficiency process is normalized to 1, with $E_{\text{max}}$ equal to 731.7. The value of $E_{\text{max}}$ was set to essentially match the capital output ratio.\textsuperscript{17} By way of comparison, if mean household income of $60,000 is equated to the mean earnings in the model, $E_{\text{max}}$ results in income of $71$ million.

The capital share of income $\alpha$ is set to 0.36 and the depreciation rate of capital $\delta$ is set to 0.10, values that are standard in quantitative studies. The value of $\rho$ was set to .975 so that the expected lifetime is 40 years. The value of $\sigma$ was set to 2. The value of $\xi$ was constrained by requiring that the highest paid person work twice as long as the lowest paid person. Using the expression for unconstrained labor choice, this restriction requires that $[E_{\text{max}}/E_{\text{min}}] = 2^{\xi}$ where $E_{\text{min}}$ is the lowest value of the discretized efficiency process. This fixes $\xi$ to 11.8, which implies a labor supply elasticity of 0.09, consistent with the generally low values of elasticities found in micro studies.\textsuperscript{18}

The garnishment rate $\gamma$ was chosen to be 0.25, which is the federal limit.\textsuperscript{19} IRS Financial Collection Standards for allowable living expenses were used to estimate the “reasonable cost of living,” $c_{\text{min}}$. This took into account the allowable costs of housing, utilities, food, personal care and services, and miscellaneous expenses for households of different sizes. The distribution of household size in the US was then used to arrive at an average estimate for reasonable living expenses. Normalizing this estimate by average household income gives a value of 0.6103. The value of $c_{\text{min}}$ was set such that the ratio of $c_{\text{min}}$ to average earnings in the model is 0.6103. Thus, roughly speaking, if a person’s earnings are less than 60 percent of mean income, he will not be obligated to make any payments on his defaulted debt. The

\textsuperscript{16}Note $\pi_0$ is calculated from Chatterjee et al.’s parameters as the probability of moving to the super-rich state conditional on being either white-collar or blue-collar.

\textsuperscript{17}Although the capital output ratio is affected by other parameters, $E_{\text{max}}$ largely controls the amount of wealth held by the super-rich and thus has a disproportionately strong effect on the capital output ratio.

\textsuperscript{18}Domeij and Floden (2006) have noted that borrowing constraints could downwardly bias the estimate of labor supply elasticity for certain specifications of utility functions. Our GHH specification does not suffer from this bias because labor supply is independent of consumption and therefore wealth.

\textsuperscript{19}Lefgren and McIntyre (2009) (Table 2, pp. 376-77) report that 23 US states adhere to the federal guideline on the fraction of disposable income that must be protected from garnishment (75 percent). Of the remaining states, 15 allow more than 75 percent to be protected from garnishment and the rest have an absolute minimum level of weekly earnings that are protected from garnishment.
values of $\lambda_b$ and $\lambda_g$ were set to 0.1 and .143 respectively, so that a record of bankruptcy remains on a credit history on average for 10 years and a record of garnishment on average for 7 years.

This leaves four parameters, $\zeta, \beta, \chi_b$ and $\chi_g$, which are set so as to make model moments come close to relevant data moments.\textsuperscript{20} These data moments are (i) the fraction of hours worked, (ii) the fraction of people in debt, (iii) the fraction of people filing for bankruptcy, (iv) the debt-to-output ratio, and (v) the aggregate collection rate on defaulted debt. The last statistic is simply the ratio of the amount paid each period on delinquent debt by people under garnishment to the total debt defaulted upon each period. Data on the first four statistics are easily available. Data on the fifth statistic (the aggregate collections ratio) are not. The target of 20 percent is an estimate by one researcher familiar with the collections industry.\textsuperscript{21}

Table 2.3 gives the values of the parameters and the data targets that determine these parameters (for the four parameters that are jointly determined, the assignment is to the data target that mostly determines that parameter). Since we are calibrating to the wage process from the PSID, the earning inequality is not as high as in the data. This discrepancy presumably reflects the fact that the PSID does not provide accurate information on people with very high incomes. Although the PSID earnings process is augmented with the super-rich state, the fraction of people earning very high incomes is still very low so that the earnings Gini is well below what is observed in the US data. But the addition of the super-rich state does help bring the model Gini on wealth close to the data. The people who

\textsuperscript{20}The model is solved using a collection of mostly standard techniques. The persistent and permanent components of the efficiency process are discretized using the recent Rouwenhorst method introduced by Kopecky and Suen (2010). We use 3 permanent states and 5 persistent states. Conditional on having a permanent and persistent state, the probability of having a particular level of efficiency is calculated using Tauchen’s (1986) method. The levels of efficiency (but not their probabilites) are chosen by discretizing the unconditional distribution using the Rouwenhorst method. We use 5 levels of efficiency (for a total of 75 states). The additional super-rich and/or super-poor state is added as discussed in the main text. The household problem is solved using a grid search accelerated with policy function iteration.

\textsuperscript{21}We thank Robert Hunt of the Federal Reserve Bank of Philadelphia’s Payment Cards Center for this estimate. Hunt reports that according to ACA International, the average gross recovery rate on defaulted revolving debt in 2008 was 22 percent (the sample size is about 50 collection firms). From this amount the collection firm takes its cut (about 30 percent) and returns the remainder to the company that placed the debt for collection. So the net recovery rate to the owner of the debt is about 15 percent. Information from a large credit card lender indicates that the gross recovery rate is typically 20 percent and the net recovery rate is typically 12-14 percent. Our model abstracts from transactions costs, so we target the gross figure of 20 percent in our calibration.
become super-rich have both the opportunity (very high incomes) and the incentive (the state is very transitory) to accumulate large amounts of wealth. The model is able to match the capital output ratio and the aggregate collections ratio fairly well but cannot match the debt statistics exactly. The filing rate and the debt-to-output ratio are too low relative to the data and the fraction of population in debt too high. To get more debt into the model, the interest rates on debt must fall but that increases the fraction in debt further above target and tends to push the filing rate further below target. The existing configuration is about the best the model can do. The average interest rate paid by borrowers is 22 percent and the debt-weighted average is 35 percent.

A novel feature of our model is the choice between bankruptcy and garnishment. In the bottom section of Table 1 some statistics relevant to this choice are reported. In equilibrium, both options are active, with the fraction defaulting and going into garnishment being 0.42 percent each period (as compared to 0.29 percent for bankruptcy). The total stock of people with impaired credit (with either a bankruptcy or a garnishment “flag” in their credit history) is 5.2 percent, with a majority having a garnishment flag. Although the fraction filing for bankruptcy is only 34 percent of the total number of people defaulting, the fraction of debt that is written off in bankruptcy is 72 percent of total defaulted debt. This is intuitive: Bankruptcy is not the optimal choice when the individual wishes to default on a low level of debt because both the flow cost and the duration of punishment are higher for bankruptcy than for garnishment. Consistent with this logic, the average income of debtors filing under bankruptcy is higher than the average income of debtors defaulting into garnishment (0.16 vs 0.13). Because the debt that goes into garnishment is relatively small, garnishment generally does not last long: On average a person is under garnishment for 4 years. Most garnishments end in full repayment of debt, with only 1.2 percent of

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22The difficulty in matching the debt-to-income ratio, the filing rate and percentage in debt seems to be common to this class of models (see, for instance, Athreya, Tam and Young, 2009b, and Chatterjee et al., 2007).

23These numbers seem high relative to reported interest rates. However, lenders often disguise interest charges as fees of various sorts. For instance, if a borrower is paying 18 percent interest on a credit balance of $500 and is charged late fees of $35 twice, the effective annual interest rate is close to 32 percent.

24Since garnishment information appears in people’s credit history for some length of time, it should be possible to determine what fraction of people are carrying a garnishment flag. Unfortunately, given the aggregated form in which credit bureau data are made available to researchers, it is not possible to determine this fraction.
Table 2.1: Model Statistics and Parameter Values

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Target</th>
<th>Model</th>
<th>Param.</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Targets determined independently</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Avg. years of life</td>
<td>40</td>
<td>40</td>
<td>$\rho$</td>
<td>0.975</td>
</tr>
<tr>
<td>Coefficient of risk aversion</td>
<td>2.0</td>
<td>2.0</td>
<td>$\sigma$</td>
<td>2.000</td>
</tr>
<tr>
<td>Capital share of income</td>
<td>0.36</td>
<td>0.36</td>
<td>$\alpha$</td>
<td>0.360</td>
</tr>
<tr>
<td>Depreciation rate of capital</td>
<td>0.10</td>
<td>0.10</td>
<td>$\delta$</td>
<td>0.100</td>
</tr>
<tr>
<td>Avg. years of exclusion following bank.</td>
<td>10</td>
<td>10</td>
<td>$\lambda_b$</td>
<td>0.100</td>
</tr>
<tr>
<td>Avg. years of exclusion following gar.</td>
<td>7</td>
<td>7</td>
<td>$\lambda_g$</td>
<td>0.143</td>
</tr>
<tr>
<td><strong>Targets determined jointly</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average hours worked</td>
<td>0.33</td>
<td>0.33</td>
<td>$\zeta$</td>
<td>4.3x10^5</td>
</tr>
<tr>
<td>Earnings Gini index</td>
<td>0.61</td>
<td>0.46</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth Gini index</td>
<td>0.80</td>
<td>0.83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of filers</td>
<td>0.29</td>
<td>0.22</td>
<td>$\chi_b$</td>
<td>0.01094</td>
</tr>
<tr>
<td>Percentage in debt</td>
<td>3.6</td>
<td>4.9</td>
<td>$\beta$</td>
<td>0.952</td>
</tr>
<tr>
<td>Capital-output ratio</td>
<td>3.08</td>
<td>3.07</td>
<td>$E_{\text{max}}$</td>
<td>731.7</td>
</tr>
<tr>
<td>Debt-output ratio x 100</td>
<td>0.36</td>
<td>0.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aggregate Collection Ratio</td>
<td>0.20</td>
<td>0.24</td>
<td>$\chi_g$</td>
<td>0.00104</td>
</tr>
<tr>
<td><strong>Other Statistics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annual debt-weighted average interest rate</td>
<td></td>
<td></td>
<td></td>
<td>35.50</td>
</tr>
<tr>
<td>Annual average interest rate</td>
<td></td>
<td></td>
<td></td>
<td>16.40</td>
</tr>
<tr>
<td>Wealth share of the top 5 percent</td>
<td>57.8</td>
<td>65.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth share of the 5th quintile</td>
<td>81.7</td>
<td>83.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth share of the 4th quintile</td>
<td>12.2</td>
<td>10.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth share of the 3rd quintile</td>
<td>5.0</td>
<td>4.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth share of the 2nd quintile</td>
<td>1.3</td>
<td>1.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wealth share of the 1st quintile</td>
<td>-0.2</td>
<td>0.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>% of Population With Record of Bank.</td>
<td></td>
<td></td>
<td></td>
<td>1.78</td>
</tr>
<tr>
<td>% of Population With Record of Gar.</td>
<td></td>
<td></td>
<td></td>
<td>3.50</td>
</tr>
<tr>
<td>% of Population Defaulting into Gar.</td>
<td></td>
<td></td>
<td></td>
<td>0.42</td>
</tr>
<tr>
<td>% of Population in Gar. with Debt</td>
<td>1.49</td>
<td>1.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Avg. Inc. in the Economy</td>
<td></td>
<td></td>
<td></td>
<td>0.44</td>
</tr>
<tr>
<td>Avg. Inc. of Debtors Filing for Bank.</td>
<td></td>
<td></td>
<td></td>
<td>0.16</td>
</tr>
<tr>
<td>Avg. Inc. of Debtors Defaulting into Gar.</td>
<td></td>
<td></td>
<td></td>
<td>0.13</td>
</tr>
<tr>
<td>Fraction under Gar. Filing Bank.</td>
<td></td>
<td></td>
<td></td>
<td>0.012</td>
</tr>
<tr>
<td>Defaulted Debt / Total Debt (%)</td>
<td>4.81</td>
<td>19.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discharged Debt / Total Defaulted Debt (%)</td>
<td></td>
<td></td>
<td></td>
<td>72.0</td>
</tr>
</tbody>
</table>
debtors being garnished moving into bankruptcy each period.\textsuperscript{25} The population in debt and being garnished is 1.67 percent. In the data, this number is between 1.5 and 1.7 percent, which is a remarkably good fit.\textsuperscript{26} Finally, the model predicts a much higher charge-off rate than what is observed in the data. In the data, the average gross charge-off rate between 1984 and 2007 is 4.81 percent. The comparable statistic in the model is 19.23 percent. This deviation is to be expected given the fact that we have too little debt in the model but the filing rate is about right.

Lefgren and McIntyre (2009) empirically examine the determinants of the frequency of Chapter 7 bankruptcy filings across US states. Among other findings, they report that states that mandate a higher threshold and protect a larger fraction of earnings above the threshold (restricted garnishment states) experience fewer Chapter 7 filings (Table 3, p. 381). This prediction is checked against the model by confronting two small (more precisely, measure zero) subsets of the model population with alternative garnishment laws: a less-restricted garnishment law in which $c_{\text{min}}$ is set to 0.1 times average earnings and $\gamma = 0.25$ (no state can have $\gamma$ higher than 0.25) and a more-restricted law in which $c_{\text{min}}$ is set to 0.61 times average earnings and $\gamma = 0.10$. The long-run filing frequency is 0.25 for the less restricted garnishment law and 0.22 for the more restricted law. Correspondingly, the frequency of garnishment is 0.13 in the former and 0.51 in the latter. Thus, filing frequencies move in the direction consistent with the evidence, although the movement appears to be more muted in the model than in the data.\textsuperscript{27}

\textsuperscript{25}The probability of filing increases over time for people in garnishment: If a person under garnishment does not exit garnishment, it is because his circumstances have either remained unchanged or deteriorated further. Further deterioration in earnings may trigger a bankruptcy filing.

\textsuperscript{26}The figures are from PSID for 1997, 2002 and 2007. In these years, the survey asked (with minor variations) the following question: Have you had your wages attached or garnished by a creditor in the last 12 months? The percent is the number of respondents who answered yes to the question. It is possible that respondents may have answered no to this question if they made payments to creditors on the threat of garnishment, rather than actual garnishment. To the extent this is true, these figures underestimate the fraction of people under garnishment each period in the sense meant in the model. Also, the phrasing of the question seems to refer to any type of garnishment including those arising from back taxes, child support payments and other judgments. Since non-dischargeable debt is not considered in this paper, these figures are upper bounds on the model-relevant fraction of people being garnished.

\textsuperscript{27}This may be due to the fact that the model does not contain a third category of borrowers: the class of “informal bankrupts,” who have stopped servicing their debts but are not pursued by their creditors because it is costly to do so. Movements in and out of this category could account for the more pronounced effects of differences in garnishment laws (Dawsey and Ausubel, 2004 and Dawsey, Hynes and Ausubel, 2004). Furthermore, state variation in Chapter 7 filing rates may also result from state differences in Chapter 7 homestead exemptions. This is the explanation put forward in Mitman (2011) in the context of a quantitative-theoretic model that distinguishes between housing and non-housing wealth; Li and White (2009) make a
2.4 Eliminating Bankruptcy Protection

This section reports how prices, allocations and welfare are affected if bankruptcy protection (the right to the discharge of debt) is eliminated. Indebted households may still default but creditors have the right to collect on their claims to the extent permitted by wage garnishment laws.

Allocations and Prices

Table 2.4 compares the baseline steady state with the garnishment-only steady state. In the garnishment economy, all parameters that are common between the baseline and garnishment economies are set to the values determined for the bankruptcy economy.

Table 2.2: Comparison of Baseline and Garnishment-Only Economies

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Baseline</th>
<th>Garn. Only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average hours worked</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>Earnings Gini index</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td>Percentage in debt</td>
<td>4.89</td>
<td>29.69</td>
</tr>
<tr>
<td>Debt-output ratio as percentage</td>
<td>0.09</td>
<td>22.30</td>
</tr>
<tr>
<td>Percentage of defaulters</td>
<td>0.65</td>
<td>0.80</td>
</tr>
<tr>
<td>Percentage of people under garnishment with debt</td>
<td>1.67</td>
<td>16.31</td>
</tr>
<tr>
<td>Percentage of pop w/ impaired credit</td>
<td>5.28</td>
<td>18.27</td>
</tr>
<tr>
<td>Capital-output ratio</td>
<td>3.07</td>
<td>2.99</td>
</tr>
<tr>
<td>Wage per efficiency unit</td>
<td>1.20</td>
<td>1.18</td>
</tr>
<tr>
<td>Rental rate on capital (MPK - δ) in %</td>
<td>1.74</td>
<td>2.05</td>
</tr>
<tr>
<td>Annual debt-weighted average interest rate</td>
<td>35.50</td>
<td>21.60</td>
</tr>
<tr>
<td>Annual average interest rate</td>
<td>16.40</td>
<td>13.10</td>
</tr>
<tr>
<td>Wealth Gini index</td>
<td>0.83</td>
<td>1.00</td>
</tr>
<tr>
<td>Wealth share of the top 5 percent</td>
<td>65.6</td>
<td>76.0</td>
</tr>
<tr>
<td>Wealth share of the 5th quintile</td>
<td>83.4</td>
<td>94.3</td>
</tr>
<tr>
<td>Wealth share of the 4th quintile</td>
<td>10.6</td>
<td>10.0</td>
</tr>
<tr>
<td>Wealth share of the 3rd quintile</td>
<td>4.4</td>
<td>2.9</td>
</tr>
<tr>
<td>Wealth share of the 2nd quintile</td>
<td>1.4</td>
<td>-0.1</td>
</tr>
<tr>
<td>Wealth share of the 1st quintile</td>
<td>0.1</td>
<td>-7.0</td>
</tr>
</tbody>
</table>

Comparison reveals some similarities and also some very striking differences. First, the
average labor supply in the two steady states are basically the same – actually, aggregate labor supply is slightly lower in the garnishment economy. There are two reasons for this. First, labor supply is lower because wages are lower (as we will see below). Second, garnishment distorts effort choices downward because of the “tax” element. However, these effects do not amount to much because the elasticity of labor supply is low and an individual will leave himself the possibility of being garnished only if it’s not that distortive for him (either because the individual earns less than $c_{\text{min}}$ or he can pay off the debt quickly). Because labor supply is not that much affected by the garnishment regime, the earnings Gini remains essentially the same.

The most striking difference between the two equilibria is in the debt measures. In the baseline economy the percentage in debt is a little under 5 percent, but in the garnishment economy it is a little under 30 percent – an increase of a factor of 6. Additionally, the debt-to-output ratio goes from 0.09 percent to around 22 percent. The proximate reason for this huge expansion in credit is a shift up of the $q(a, s; p)$ schedule, stemming from a decline in the probability of default.\(^{28}\) The lower interest rates motivate people to borrow more and the expansion in debt continues until the default rate reaches roughly the same level as in the baseline economy. Importantly, the elimination of discharge does not reduce the default – in fact, it increases it.

Even though the default rate is only somewhat above the baseline economy, the fraction of people with impaired credit (i.e., in bad standing) is much higher. The reason is that the duration of garnishment lasts much longer now because when people default they do so on much larger levels of debt and it takes longer to repay those debts and exit garnishment.

The increase in consumer credit can be expected to crowd out fixed capital, and it does, but surprisingly little. The capital to output ratio declines from 3.08 to 2.97. The drop results in a slightly higher risk-free interest rate, which rises from 1.74 percent to 2.05 percent, a rise of about 30 basis points. The decrease in capital per worker results in a decline

\(^{28}\)Athreya (2008, Table 1 p. 762) reports what happens if default is eliminated, so individuals can borrow at the risk-free rate up to their natural borrowing limit. Although the calibration of his model is quite different from ours and he keeps the risk-free interest rate constant, it is interesting to compare his findings to ours. Athreya finds that if default is eliminated, the fraction of indebted households rises to almost 40 percent and the aggregate debt to output ratio rises to 39 percent. The latter increase is almost twice what we find in our paper. The difference in results highlights our contention that what exactly replaces the bankruptcy option matters for the outcome.
in wages of 1.67 percent. The decline in capital stock is muted because of the presence of the super-rich. These individuals have a very elastic supply of savings and expand their savings to accommodate the increased demand for consumer loans. If we eliminate the super-rich along with discharge, the capital output ratio falls to 2.71 and the (net) rental return on capital climbs to 3.29 percent. Thus, getting the baseline wealth distribution to match reality (which necessitated the addition of the super-rich state) has important implications for the counterfactual.

Finally, Table 2.4 shows that there is a massive increase in wealth inequality. This comes about because so many individuals become indebted. The top 5 percent of the population ends up holding 66 percent of total wealth in the garnishment-only economy compared with 56 percent in the baseline economy. The bottom quintile has negative net-worth amounting to 7 percent of total wealth.

Why exactly does the incentive to default change so drastically in the garnishment economy? There are two effects at work. One is the stick effect, which is that default is more costly to the individual, and the other is the carrot effect, which is that maintaining access to markets is more beneficial. Figure 2.1 shows that both effects are at work. The mostly flat dashed line with circles shows for a typical efficiency level the value function of defaulting in the baseline economy, which is the maximum of declaring bankruptcy and entering garnishment. The flat region is where bankruptcy is the best option. The dashed line without circles is the value of defaulting in the garnishment-only economy. Observe that as the debt level rises, default in the garnishment-only economy becomes increasingly worse relative to the value of default in the baseline. This is the stick effect of garnishment: default under garnishment is simply not as beneficial to the individual as default under bankruptcy. The solid line with and the solid line without circles show the value function conditional on repaying debt in the baseline and garnishment-only economies respectively. Notice that the value of repaying debt in the garnishment-only economy lies considerably above the value of repaying debt in the baseline. This is the carrot effect of garnishment: by lowering the costs of borrowing, the garnishment-only economy increases the value of maintaining access to the credit market. The lens-shaped areas trapped between the solid and dashed lines is where households do not default. This area is much larger in the garnishment-only economy.
because of these two effects.

Figure 2.1: Value Functions

Typical Value Function Comparisons

Value of Repaying, Baseline
Value of Defaulting, Baseline
Value of Repaying, Garnishment Only
Value of Defaulting, Garnishment Only

The increased value of maintaining access to the credit market is apparent in the positioning of the price schedules in the baseline and the garnishment-only economies. Figure 2.2 shows the average loan price for the two economies for different levels of debt: credit is available under more generous terms in the garnishment-only economy than in the baseline economy. Competitive lenders are willing to extend loans on more generous terms because debtors do not default as much, and even when they do default, they pay their debts back in due course.

Is the large increase in wealth inequality a credible implication of the lack of discharge? A case in point is Sweden, which until 2005 did not permit discharge of debt and, yet, Sweden does not have the wealth inequality predicted by our model. But Sweden’s income process is different as well. Table 2.3 reports wealth distribution statistics if the US garnishment-only economy is fed Sweden’s income process, also estimated and presented in Floden and Linde (2001).\textsuperscript{29} All other parameters are exactly as in the US economy, except for $\xi$, which is set

\textsuperscript{29}The AR1 coefficient on the persistent process is 0.8139 and the variance of the innovation to this process
to a value for which the highest paid person in Sweden works twice as long as the lowest paid person. The table also reports the wealth distribution statistics for Sweden taken from Domeij and Klein (1998). As is evident, the wealth distribution for our “Swedish” economy is surprisingly close to the actual Swedish wealth distribution. In particular, the bottom 20 percent of the population, on aggregate, holds debt (as opposed to assets) in both the model economy as well as in the data. Indeed, the level of indebtedness in the Swedish data is actually higher than in our “Swedish” model. This limited exercise indicates that the “extreme” distributional implications of the US garnishment-only economy may reflect the much greater degree of income risk in the US compared to European economies. Also, if Sweden were to adopt US-style discharge laws, then, as shown in the final column, the bottom quintile will become net savers; the capital output ratio will rise; and the distribution is 0.0326; the variance of the permanent shock is 0.0467 and the variance of the transitory shock is 0.0251. We should note that there are competing explanations for the large fraction of negative net-worth individuals in Sweden. Domeij and Klein (2002) argue that it is the nature of the Swedish public pension system that accounts for Sweden’s considerable wealth inequality despite Sweden having a relatively low earnings inequality.
of wealth would look less unequal.

Table 2.3: Wealth Distribution for Sweden: Data and Model

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>Garn. Only</th>
<th>Garn. &amp; Bank.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wealth Gini Index</td>
<td>0.79</td>
<td>0.69</td>
<td>0.57</td>
</tr>
<tr>
<td>Wealth share of the top 5 %</td>
<td>33</td>
<td>26</td>
<td>23</td>
</tr>
<tr>
<td>Wealth share of the 5th quintile</td>
<td>72</td>
<td>65</td>
<td>58</td>
</tr>
<tr>
<td>Wealth share of the 4th quintile</td>
<td>25</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>Wealth share of the 3rd quintile</td>
<td>9</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>Wealth share of the 2nd quintile</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Wealth share of the 1st quintile</td>
<td>-7</td>
<td>-5</td>
<td>1</td>
</tr>
</tbody>
</table>

Welfare

The welfare effects of eliminating bankruptcy protection are now presented. The first measure gives the flow consumption a person would give up to go from a regime in which there is bankruptcy to a regime in which bankruptcy is eliminated.\textsuperscript{31} The second measure simply counts the fraction of people who would be in favor of eliminating bankruptcy. The latter measure provides insight into the degree of political support in favor of or against the institution of bankruptcy.

In both cases, it is assumed that the question is posed in an unanticipated manner after people have made their default decision but before they have chosen their new asset positions. This timing ensures that the contemplated switch in regime does not impose unanticipated profits or losses on the intermediary sector.\textsuperscript{32}

The top panel of Table 2.4 reports the consumption equivalent measure from eliminating bankruptcy, taking into account the transition to the new steady state. Each cell gives

\textsuperscript{31}The consumption equivalent measure is computed as follows: Given policies \(c(a, e, h), n(a, e, h)\), and \(d(a, e, h)\) corresponding to a value function \(V(a, e, h)\), the value of using policies \(\tilde{c}(a, e, h) = (1 + \Gamma)c(a, e, h)\), \(n(a, e, h)\), and \(d(a, e, h)\) forever is computed using policy iteration. This is done for 30 values of \(\Gamma\) between -0.9 and 2. To find the \(\Gamma\) for which \(\tilde{V}(a, e, h; \Gamma)\) equals some level of utility \(W\), a nonlinear equation solver (interpolating \(\tilde{V}\) in the \(\Gamma\) dimension) is used.

\textsuperscript{32}We start with the steady state bankruptcy distribution in period 0. Then, all households that file for bankruptcy are transitioned to 0 assets and given a garnishment flag. Next, households make their \(c, n\) and \(a'\) choices conditional on there being no bankruptcy option going forward.
the consumption flow averaged across the cell’s households. Overall, there is a significant gain from eliminating bankruptcy – amounting to about 1.0 percent of consumption in perpetuity. The gain is not uniform: Indebted people gain more than others. This makes sense because borrowing is cheaper in the garnishment-only economy. The income level matters as well: those receiving the lowest persistent or transitory efficiency shocks gain the most and those receiving the highest shocks the least. This pattern reflects the fact that those in need of loans are the ones who gain most from the decrease in borrowing rates. For the permanent shock, the pattern of relative gain is reversed: those with the highest permanent shock gain more than those with the lowest permanent shock. The high permanent shock individuals own a large amount of assets and they prefer the garnishment because of the higher associated interest rate on savings.

The bottom panel of the table reports the fraction of people in each cell in favor of eliminating bankruptcy protection. About 10 percent of the population opposes it and, interestingly, they are drawn mostly from the ranks of indebted people with high persistent and transitory efficiency shocks. Why do these people oppose the elimination of discharge? Because they have high income which they expect to mean revert, their need to borrow is low, and because they are indebted, they are unlikely to have a high level of assets. For
these individuals the main effect on welfare comes from the decline in real wages. The effect is small (because the decline in wages is small) but they oppose the change, nevertheless.

Although factor prices do not change very much from one steady state to the other, Table 2.4 indicates that the steady-state welfare gains are about 0.2 percentage point, or less, lower than those taking the transition into account. Also, ignoring the transition can give a misleading picture of the degree of political support for the elimination of bankruptcy. Without the added benefit of the higher consumption afforded by the de-accumulation of capital, support for elimination of discharge is much lower. Overall, only 67 percent of the population now favors eliminating discharge.

Table 2.5: Steady State Welfare Gains From Elimination of Bankruptcy

<table>
<thead>
<tr>
<th>Average Consumption Gain</th>
<th>All</th>
<th>Permanent</th>
<th>Persistent</th>
<th>Transitory</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>L</td>
<td>H</td>
<td>L</td>
<td>H</td>
</tr>
<tr>
<td>All</td>
<td>0.9</td>
<td>1.2</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>In Debt</td>
<td>4.5</td>
<td>5.8</td>
<td>3.1</td>
<td>-0.4</td>
</tr>
<tr>
<td>No Debt</td>
<td>0.8</td>
<td>1.0</td>
<td>0.6</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Population in Favor</th>
<th>All</th>
<th>Permanent</th>
<th>Persistent</th>
<th>Transitory</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>L</td>
<td>H</td>
<td>L</td>
<td>H</td>
</tr>
<tr>
<td>All</td>
<td>67.4</td>
<td>80.4</td>
<td>62.9</td>
<td>67.0</td>
</tr>
<tr>
<td>In Debt</td>
<td>91.4</td>
<td>88.2</td>
<td>94.8</td>
<td>0.0</td>
</tr>
<tr>
<td>No Debt</td>
<td>66.7</td>
<td>80.2</td>
<td>62.3</td>
<td>67.2</td>
</tr>
</tbody>
</table>

Garnishment increases the value of maintaining access to credit markets and allows for an expansion of credit. It is useful to understand exactly what this expansion allows in terms of consumption profiles. With this in mind, a large number of individuals were simulated who “start life” with \( a = 0, h = 0 \) and \( e \) drawn from the invariant distribution. Their consumption and asset holdings were recorded for the next 80 periods (years) under both the bankruptcy steady state and the garnishment steady state. The following two figures display average consumption and average asset holdings across the two regimes for each of the 80 periods.
Figure 2.3 shows that mean asset holdings rise more slowly in the garnishment-only economy, whereas they increase rapidly in the bankruptcy economy. In the latter, the high cost of loans forces individuals to accumulate assets in order to self-insure. In the garnishment economy, the need to accumulate precautionary savings is much less urgent, since the loan supply schedule is much more attractive. Mean consumption is higher in the garnishment economy because people are saving less. Mean consumption is higher for some time until the accumulated debt burden begins to lower consumption below that in the baseline economy.

The effects of better consumption smoothing can be seen in Figure 2.4, which displays the coefficient of variation of consumption for each age. Observe that the coefficient of variation is initially lower in the garnishment-only economy but then exceeds that of the baseline economy. It is lower initially because of the superior consumption smoothing afforded by the generous loan supply schedules in the garnishment-only economy. But the other side of the same coin is the increased dispersion of asset holdings resulting from enhanced bor-
rowing and lending. Higher wealth inequality eventually translates into higher consumption inequality.

Figure 2.4: Dispersion of Consumption by “Age”

![Sample Coefficient of Variation of Consumption](image)

Finally, the optimal garnishment regime, if discharge were to be eliminated, is investigated. This exercise is motivated by the consideration that elimination of discharge is a large institutional change that, if it were to be instituted, would almost surely result in significant changes in garnishment law as well. The average steady-state consumption gains for a range of $c_{\text{min}}$ and $\gamma$ values were computed. The optimal garnishment regime is one with $\gamma = 1$ and $c_{\text{min}} = 0$. This is a “zero tolerance for delinquency” regime in which the creditors have the right to garnish all of the debtor’s earnings in case of default. Eliminating discharge and instituting this garnishment regime raises average steady-state welfare by 3.71 percent as compared to 0.856 percent for the current garnishment regime. The optimal garnishment economy is essentially a “natural borrowing limit” (a la Aiyagari, 1994) economy with no default. There is no voluntary default because the creditors can garnish earnings fully to

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33We ignored the transition because it is time consuming to compute.
recover the defaulted debt so the defaulter does not gain current consumption but pays the reputation costs associated with a bad credit history. And there is no involuntary default because individuals never find it in their interest to borrow more than the amount that can be rolled over even in the event the debtor has the lowest efficiency level. For the baseline calibration, this natural borrowing limit is $-1.58$, or roughly $360$ percent larger than average income in the economy. In contrast, the maximum amount that an individual would wish to borrow in the current garnishment regime (with no discharge) is $-0.93$, or roughly $210$ percent larger than average income in the economy.\(^{34}\)

This logic makes clear that the optimality of the “zero tolerance” regime hinges on the size of the lowest efficiency level.\(^ {35}\) If this level is very low – “a disaster state” – the optimal garnishment economy will not be the “zero tolerance” one. This point is verified by augmenting the income process with an efficiency level that corresponds to a super-poor state, which happens with a very small probability and is very transitory.\(^ {36}\) Addition of this state does not change model statistics in the baseline economy because the probability of the super-poor state is very low and the individual debtor always has the option to declare bankruptcy.\(^ {37}\) However, once bankruptcy protection is eliminated, the event looms large in the utility calculation of individuals. Basically, the presence of this state raises the welfare gain from elimination of discharge for low punishment regimes and lowers it for high punishment regimes. For instance, the welfare gain for the current regime rises from $0.841$ percent to $0.917$ percent and that for the “zero tolerance” regime declines from $3.710$ percent to $2.076$ percent. Furthermore, the “zero tolerance” regime is no longer optimal; the optimal regime now has $\gamma = 0.50$ and $c_{\text{min}} = 0$. These results are indicative of how the welfare results would change if wealth/liability shocks had been included in the model. As noted in Chatterjee et al. (2007) and in Livshits et al. (2007), wealth shocks stemming from

\(^{34}\)The most that a lender would wish to borrow is the debt level at which $q(a', s) a'$ is maximized. Although the individual can borrow more than this, he would not want to because the borrower gets less in terms of current consumption and is saddled with more debt in the future.

\(^{35}\)In the discretized income process of the baseline economy, the minimum efficiency level is a positive number although the true distribution allows for efficiency levels arbitrarily close to zero with vanishingly small probability.

\(^{36}\)The efficiency level is 5 standard deviations below the unconditional mean of the log-efficiency process. It is iid and occurs with a probability of 1 in 3.5 million (which is the mass 5 standard deviations or more below the mean of a normal distribution).

\(^{37}\)For example, both the population in debt and the population filing for bankruptcy are unchanged. The capital-output ratio declines slightly to 3.03 (from 3.07).
uninsured medical expenses are an important trigger for bankruptcy. Including such shocks will likely reduce the welfare gain from elimination of discharge. In this sense, our welfare estimates should be viewed as an upper bound.

2.5 Conclusion

It is useful to conclude by putting the findings of this project into perspective and drawing some lessons regarding important future research lines in the area of consumer default.

A question one might ask is whether the findings reported in this paper firmly support the notion that the US economy would be better off without bankruptcy protection. Although a 1 percent increase in welfare is large by the standards of macroeconomic analysis, this gain must be set against costs ignored in this study. One cost that has been ignored is the cost of enforcing garnishment laws. In the baseline economy, less than 2 percent of the population suffer garnishment each period; in the counterfactual, close to 16 percent do. To process such a large volume of garnishments, the administrative and legal resources devoted to this task must be increased very substantially. Second, a society with as extreme a wealth distribution as the garnishment-only economy would presumably suffer from political and social costs. These missing costs suggest that the findings reported in this paper do not provide a convincing case for elimination of bankruptcy protection, at least given current garnishment laws.

This, then, raises the question of whether default policy should move in the direction of eliminating bankruptcy protection and toughening up garnishment laws. A strong commitment to honor one’s debt works well if it is combined with forward-looking behavior that steers individuals away from truly bad financial situations. But the train of events of the last several years shows that individuals may not possess the requisite foresight to pull this off. After decades of low aggregate volatility, the Great Contraction caught people and policymakers by surprise. The design of legal institutions that can facilitate the optimal societal response to unexpected situations remains an open and important task. In this quest, historical experience can serve as a guide: In the pre-discharge era, state legislatures granted delinquent debtors one-time debt forgiveness when macroeconomic conditions
were particularly bad. Estimating the net benefits of tough garnishment laws coupled with state-contingent bankruptcy protection would seem to be a useful line of research.

Another question one might ask is what lessons do quantitative-theoretic models offer for conventional empirical research on consumer default? Empirical studies (well-known examples are Fay, Hurst and White (2002) and Gross and Souleles (2002)) focus on testing the sign restrictions on regression coefficients implied by simple models of default. Quantitative-theoretic models go beyond simple default models in making predictions regarding the magnitude of the various effects as well. This added quantitative information can be useful in interpreting empirical findings, in terms of the importance of the various causal mechanisms at work. A case in point is the muted effect of variation in garnishment restrictions (all else remaining the same) on bankruptcy filing rates in the model. As noted earlier, this difference may arise from the presence of real world features missing in the model such as informal bankruptcy and the effects of state variations in Chapter 7 homestead exemptions. Furthermore, the fact that quantitative-theoretic models are fully articulated artificial economies composed of heterogeneous agents operating through time can be leveraged to generate artificial data of the type available to empirical researchers and which can then be approached in the same way that researchers approach real data. This procedure can illuminate the strengths and weaknesses of empirical specifications in uncovering causal links that researchers believe exist in reality – and which exist for sure in the model-generated data – but can get distorted, or masked, by data limitations (Chen, 2010).

Aside from enriching the interplay between theory and empirics, quantitative-theoretic models may alert empirical researchers to regularities that should exist in the data, if the underlying theory is correct. One example of this is the prediction that the costs of garnishment ought to be lower than the costs of bankruptcy and that debts collected in garnishment should be smaller, on average, than debts written off in bankruptcy. The feedback can also go the other way: Fay, Hurst and White’s finding that many individuals forgo bankruptcy even when it is financially beneficial has motivated quantitative theorists to include the heterogeneous non-pecuniary costs of bankruptcy filings (Athreya, Tam and Young, 2009b).
Chapter 3

Computing Dynamic Heterogeneous-Agent Economies: Tracking the Distribution

Grey Gordon

Summary

Theoretical formulations of dynamic heterogeneous-agent economies typically include a distribution as an aggregate state variable. This paper introduces a method for computing equilibrium of these models by including a distribution directly as a state variable if it is finite-dimensional or a fine approximation of it if infinite-dimensional. The method accurately computes equilibrium in an extreme calibration of Huffman’s (1987) overlapping-generations economy where quasi-aggregation, the accurate forecasting of prices using a small state space, fails to obtain. The method also accurately solves for equilibrium in a version of Krusell and Smith’s (1998) economy wherein quasi-aggregation obtains but households face occasionally binding constraints. The method is demonstrated to be not only accurate but also feasible with equilibria for both economies being computed in under ten minutes in Matlab. Feasibility is achieved by using Smolyak’s (1963) sparse-grid inter-
polation algorithm to limit the necessary number of gridpoints by many orders of magnitude relative to linear interpolation. Accuracy is achieved by using Smolyak’s algorithm, which relies on smoothness, only for representing the distribution and not for other state variables such as individual asset holdings.

3.1 Introduction

The evolution of prices in dynamic heterogeneous-agent economies typically depends on the state of every agent thereby requiring that a distribution be a state variable. The contribution of this paper is to introduce a method for computing equilibrium in these models by including an entire distribution, if finite-dimensional, or a fine approximation of it, if infinite-dimensional, as a state variable. The insight of Krusell and Smith (1997, 1998) is that this approach is not necessary if a model features quasi-aggregation, the condition where prices can be accurately forecasted using just a few state variables. However, not all economies feature quasi-aggregation and I show that the method presented in this paper is capable of accurately computing equilibrium in at least one of these: Huffman’s (1987) overlapping-generations (OLG) economy paired with an extreme calibration used in Krueger and Kubler (2004). Even when quasi-aggregation obtains, including a distribution as a state variable may be desirable from a conceptual or purely pragmatic perspective. I show that the method accurately computes equilibrium in an economy of this type also: a version of Krusell and Smith’s (1998) (KS) economy where households face occasionally-binding constraints. The method is feasible for these two economies with equilibrium for both computed in just a few minutes in Matlab.¹ As discussed momentarily, Smolyak’s (1963) sparse-grid interpolation algorithm introduced to economics by Krueger and Kubler (2004) makes this possible.

Smolyak’s algorithm is a projection method that uses collocation on a very sparse grid.² The algorithm approximates a function by interpolating its value at a set of predefined

¹Carroll’s (2006) endogenous gridpoints method is used to solve the household problem. Value function iteration is also feasible, just slower and less accurate than Carroll’s Euler-equation based method.
²For an excellent introduction to projection methods, including projection methods that use collocation, the reader is referred to Judd (1998). For an accessible exposition of the Smolyak algorithm, the reader is referred to Krueger, Kubler, and Malin (2011). Section 2 of this paper also discusses the algorithm.
gridpoints (collocation points) using weighted sums of polynomials. The fineness of the approximation is controlled by using different “levels of approximation.” For the lowest level of approximation, which is the only one used in this paper, the number of gridpoints grows only linearly in dimension. More specifically, given a function of dimension $d$, Smolyak’s algorithm gives $2d + 1$ points that the function must be evaluated at in order to approximate it. In contrast, linear interpolation or any tensor-product interpolation method would require at least $2^d$ points. To see the difference this makes, consider that the distributions (and hence state spaces) used in this paper have up to 200 elements: to approximate a function of this dimension using linear interpolation would require more than $10^{48}$ trillion function evaluations compared to only 401 for Smolyak interpolation. Not only is the Smolyak algorithm computationally efficient, but Barthelmann, Novak, and Ritter (2000) prove the approximation has nearly optimal error bounds for smooth functions.\(^3\) The disadvantage of Smolyak’s algorithm is that approximations to non-smooth functions may be quite poor.

The application of Smolyak’s algorithm presented in this paper leverages the strengths of the Smolyak algorithm, computational efficiency and accuracy for smooth functions, while avoiding its main weakness, poor approximation of non-smooth functions. While many heterogeneous-agent models feature policy functions that are kinked in individual wealth or income and hence are not smooth, as long as they are smooth in the aggregate state, they can be approximated well by the Smolyak algorithm. This is accomplished through indexing policy functions by individual states and constructing a Smolyak approximation to each indexed policy function. For example, given a capital policy function $k'(k, \mu)$ where $k$ is a household’s current capital holdings and $\mu$ is a distribution of holdings across households, the Smolyak approximation to $k'(k, \mu)$ would likely be poor if $k'$ were kinked in $k$. However, if $k'$ is fairly smooth in $\mu$ for fixed $k$, then the indexed policy function $k'_k(\mu)$ could be accurately approximated using Smolyak interpolation.\(^4\) By indexing policies in this way, the resulting Smolyak approximations may be accurate even if the policies are “not smooth.” I refer to this approach as the Smolyak method.

\(^3\)The error bounds depend not only on the number of times a function is continuously differentiable but also on how little curvature a function has. The term “smooth” is used atypically here to cover both of these properties.

\(^4\)Of course approximating each function requires that the number of values $k$ takes on is finite. Section 3 discusses how to apply the Smolyak algorithm even if $k$ takes on an infinite number of values.
Recognizing the computational challenge posed by solving a model where the distribution was part of the state space, Krusell and Smith (1997, 1998) found a clever way of circumventing it. By replacing the distribution with a few aggregate statistics and assuming that households perceive prices to be functions of only these statistics (and the aggregate shocks), a law of motion for them enables households to predict current and future prices and hence optimize. Given the optimal household policies, it is then possible to check the accuracy of the perceived prices and law of motion through simulation. If a small set of statistics can be found that results in an accurate law of motion and accurate price forecasts, then quasi-aggregation is said to obtain, in which case it is hoped that the computed bounded-rationality equilibrium is close to the equilibrium of the full-rationality model. Equilibrium has typically been computed by guessing on a law of motion, solving the household problem, simulating the economy, and updating the law of motion using data from the simulation. I refer to this approach as the KS method.

The Smolyak method has three advantages over the KS method. First, the method does not rely on quasi-aggregation, an equilibrium property which is not known a priori. Second, there is no need to simulate the economy in order to compute the solution. Not only can this result in substantial time-savings, but it also means the computed solution is not a random variable. Third, for certain classes of models, namely those where the distribution is finite-dimensional, the solution can be regarded as a full rational-expectations equilibrium.\footnote{Even if the distribution has infinite dimension but is represented by a finite number of elements, as is the case when using the method of Ríos-Rull (1997), the solution may be regarded as an approximate full rational-expectations equilibrium.}

While the Smolyak method has several advantages over the KS method, this paper is in no way a critique of it. When the KS method works, that is when quasi-aggregation obtains, it is extremely powerful. Indeed, whereas the Smolyak method has gridpoints growing linearly in the dimension of the underlying state space, the KS method’s gridpoints need not grow at all! Moreover, quasi-aggregation has obtained in a wide variety of models. The KS method is robust, conceptually simple, and easy to program, and so is a powerful tool.

Yet there are cases where the KS method does not work well. As already mentioned, I present one such OLG economy that has a known solution due to Huffman (1987) and
calibration due to Krueger and Kubler (2004) (KK). In the most extreme case where there are only three generations, a linear forecasting rule for the aggregate capital stock results in an $R^2$ statistic of .676 and a maximum percent error of 3.17. In contrast the Smolyak method’s forecast performs very well resulting in an $R^2$ statistic of .999 and a maximum percent error of 0.07. While the performance of the KS method could be improved by adding more moments, here that would mean the distribution could be completely summarized as only the oldest two generations have positive capital holdings. Moreover, the Smolyak method is already faster in this case than the KS method. The Euler-equation errors, a measure of household optimization error, are similar across the two models with the KS method better in terms of maximum errors and the Smolyak method better in terms of average errors.

Even when the KS method does work well, the Smolyak method may achieve a similar level of accuracy and possibly be even faster to run. As for accuracy, in the modified Krusell and Smith (1998) economy studied, I find the computed equilibria are virtually identical across methods both in terms of optimization errors and simulated aggregate moments: Euler-equation errors for the Smolyak method (-2.22 maximum and -4.97 average) are slightly smaller than those of the KS method (-1.89 maximum and -4.79 average) and the simulated capital series are at most .28% apart. As for speed, the KS method will typically be faster. However, this depends on the cost of household optimization relative to the cost of simulation. In the KS economy, the former cost dominates and the Smolyak method takes 6.4 minutes compared to 3.3 minutes for the KS method. However in the OLG economy, the latter cost dominates when the number of generations is less than 50 making the Smolyak method faster.

The Smolyak method may also be more intuitive than the KS method for certain classes of models. For instance, dynamic models of voting do not typically have a natural “suffi-

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6 A more realistic example where quasi-aggregation does not obtain comes from the equity premium literature. Chien, Cole, and Lustig (2009) find that including even five moments of the distribution results in an $R^2$ of only .50 to .75 when forecasting the pricing kernel (cf. Table 2 of their paper).

7 I report the Euler errors in log10 scale.

8 Parallelization of the household problem, which was not used, could shift this balance substantially in favor of the Smolyak method. Value function iteration in particular is known to parallelize well (see Aldrich, Fernández-Villaverde, Gallant, and Rubio-Ramírez 2011).
cient statistic” representation. Instead, researchers have typically used coarse histograms to summarize the distribution, as is done in Krusell and Rios-Rull (1999). While there is usually some way of approximating the aggregate state space with just a few statistics, the Smolyak method provides an alternative that may be both feasible and accurate, as it is in two non-trivial economies.

Since the seminal papers of Krusell and Smith (1997, 1998), many methods have been developed to solve dynamic heterogeneous-agent models. For a thorough review of current methods the reader is referred to the January 2010 special issue of the Journal of Economic Dynamics and Control “Computational Suite of Models with Heterogeneous Agents: Incomplete Markets and Aggregate Uncertainty.” I highlight a few of these that are most closely related to the Smolyak method. The first approach is the “backward induction” method of Reiter (2010). As in the KS method, the aggregate state space is a small set of statistics. A distinguishing aspect of his approach is that these statistics map into a specific “proxy” distribution which agents use to make forecasts. A qualitatively similar approach is due to Algan, Allais, and Den Haan (2010): like Reiter (2010) they link a few moments to a specific distribution but do so in a different way. A third approach is due to Den Haan and Rendahl (2010). Roughly speaking, they construct an approximation to the true policy function that results in exact aggregation. While these methods have many merits, they place special structure on either the distribution (in the case of Reiter 2010 and Algan et al. 2010) or on the policy functions (in the case of Den Haan and Rendahl 2010) to construct a law of motion. The Smolyak method does neither of these.

In the same OLG economy studied here Krueger and Kubler (2004) use Smolyak interpolation to solve for a full rational-expectations equilibrium with a large state space. Their approach differs from the one I present because they make no distinction between individual and aggregate states. This means that the same approximation must be used for both

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9An exception to this is Azzimonti, de Francisco, and Krusell (2006) where the mean and median of wealth are proven to be sufficient statistics.

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11For KK, the state space is a distribution of capital holdings \(k\) and a particular generation’s capital holdings is just “read off” this distribution. However, by expanding the state space to \((k, k)\) where \(k\) is a particular generation’s capital holdings, it is possible to use a fine approximation for the individual state \(k\) and a coarse approximation for the aggregate state \(k\). Essentially this separates the role of prices, which are determined from \(k\), from the role of individual wealth, which is proportional to \(k\). While in equilibrium \(k\) must be consistent with \(k\), this only matters when simulating the economy and is trivial to enforce.
and so a higher-order Smolyak approximation (one that grows quadratically or cubically in
dimension) is required to achieve sufficient accuracy. Consequently, their method can only
handle economies with relatively small state-space dimensions.\footnote{KK report that the Euler-equation errors and computation times increase rapidly in the number of
generations. They conclude their algorithm can only be applied if the number of generations is less than 30
(p. 19). The Smolyak method presented in this paper can easily handle 100 generations and the maximum
errors appear to asymptotically approach $-2.46$ (roughly a 1 dollar mistake for every 300 dollars spent). See
Table 3.4 of this paper.}

Smolyak interpolation is not the only method that could be used to include the dis-
tribution as a state variable. In particular, the recently developed cluster-grid projection
method of Judd, Maliar, and Maliar (2010) is capable of handling problems of very large
dimension. Relative to Smolyak interpolation, their method provides greater flexibility in
terms of where gridpoints are placed and which basis functions are used. This comes at
the cost of using weakly more gridpoints else equal.\footnote{Judd et al. (2010) find the method works best (both in terms of accuracy and numerical stability) when
the number of gridpoints is larger than the number of basis functions. The authors argue that a 20% increase
in the number of points (relative to collocation which has the fewest possible number) has a “sizable effect
on accuracy” (p. 30).} Whether the cluster-grid projection
method provides a feasible and accurate alternative to Smolyak interpolation in this context
is left as a question for future research.

This paper is organized as follows. Section 2 describes the Smolyak algorithm. Section
3 discusses the Smolyak method, i.e. the application of the Smolyak algorithm used to
approximate equilibrium. Section 4 presents the OLG and KS models and their calibrations.
Section 5 discusses implementation details specific to the models. Section 6 analyzes the
performance of the Smolyak and KS methods. Section 7 concludes. The appendix examines
alternative implementations of the Smolyak method.

### 3.2 The Smolyak Algorithm

This section describes how functions can be approximated using the Smolyak algorithm. To
distinguish the algorithm from its application to approximating equilibrium, the latter is
referred to as the Smolyak method.

Let $f$ be an arbitrary function mapping $\mathbb{R}^d$ to $\mathbb{R}$ with typical element $x$. The Smolyak
algorithm is best described in three steps which I present as a “black box.” See Krueger
et al. (2011) (KKM) for a careful exposition of all the necessary steps. The code provided is organized similarly to the description given here.\textsuperscript{14} Attention is restricted to the lowest level of approximation.\textsuperscript{15}

**Step 1 – Setup**

Fix bounds $\underline{x}$ and $\bar{x}$ in $\mathbb{R}^d$ on the state space such that $\underline{x} < \bar{x}$. These bounds define a hypercube. The Smolyak algorithm then provides $n := 2d + 1$ collocation points $\{x^i\}_{i=1}^n$ within this hypercube. The advantage of the Smolyak algorithm lies in the construction of these points whose number grows only linearly in dimension.

**Step 2 – Polynomial Construction**

Evaluate $f$ at each of the $n$ collocation points. The Smolyak algorithm then provides polynomial coefficients $\theta$. The coefficients $\theta$ implicitly define an approximating polynomial $\hat{f}$.

**Step 3 – Polynomial Evaluation**

Given coefficients $\theta$, the Smolyak algorithm provides a way to evaluate $\hat{f}$ at arbitrary $x$ (inside or outside of the hypercube).

The collocation points and interpolating polynomial are constructed in such a way that the following conditions are guaranteed:

1. $\hat{f}$ agrees with $f$ at each collocation point, i.e. $\hat{f}(x^i) = f(x^i)$ for all $i \in \{1, \ldots, n\}$.

2. If $f$ is a linear combination of the polynomials $x_j^2$, $x_j$, and 1 for $j \in \{1, \ldots, d\}$, then

\textsuperscript{14}The code is available at https://sites.google.com/site/greygordon. There are several alternatives to my code. In particular “spinterp” is a free Matlab sparse-grid interpolation toolbox available at http://www.ians.uni-stuttgart.de/spinterp/. This toolbox has more features than what I provide. Additionally, KKM provide Fortran routines.

\textsuperscript{15}This is the only one that’s feasible for very large distributions. However for small to medium-sized distributions a higher level of approximation may be feasible. It is easy to try a higher level of approximation when using the provided code.
\( \hat{f} \) agrees with \( f \) everywhere in the hypercube.\(^{16}\)

3. If \( f \) is not perfectly reproduced but is at least continuous then the polynomial \( \hat{f} \) is an almost optimal approximation in a certain sense.\(^{17}\) In general, the less curvature \( f \) has, the better \( \hat{f} \) will be as an approximation.

For additional details on the algorithm and its properties, the interested reader is referred to KKM and Barthelmann et al. (2000).

### 3.3 The Smolyak Method

This section describes the Smolyak method, that is the application of the Smolyak algorithm to approximating equilibrium. First, a typical definition of equilibrium is redefined as a set of functions of only the aggregate state. Second, the algorithm is used to approximate these functions.

**Redefining Equilibrium**

Let \( f \) represent a typical policy function, value function, price function, or law of motion. Without loss of generality assume that \( f \) is a function of some “individual” state \( x \in X \) and an “aggregate” state \( \omega \in \Omega \) that is common across functions.\(^{18}\) For notational convenience also assume \( X \) is shared by all functions and is non-empty. Indexing the functions \( f \) by \( i \in I \) with \( I \) possibly uncountable, a typical definition of equilibrium is then a collection of functions

\[
\{ f_i(x; \omega) \}_{i \in I} \tag{3.1}
\]

that satisfy conditions which are not explicitly stated such as optimality, budget balance, market clearing, and consistency of a law of motion. Consider a new definition of equilibrium

\(^{16}\)Unfortunately there are no cross terms for this level of approximation. However, this does not prevent obtaining an accurate solution for the two non-trivial economies considered in this paper.

\(^{17}\)Barthelmann et al. (2000) show it is not the best (in the sense of minimizing the sup norm) interpolating polynomial, but it is close to it in that it’s error bounds are the same up to a logarithmic factor in the number of collocation points. See Theorem 2 and Remark 4 in their paper.

\(^{18}\)Note that correspondences can be treated as a possibly uncountable collection of functions. Also note that if a function does not depend on the aggregate state, it can just be regarded as a trivial function of the aggregate state.
comprised of indexed functions

\[ \{ f_i^x(\omega) \mid f_i(x; \omega) \forall \omega \in \Omega \}_{i \in I, x \in X} \] (3.2)

that satisfy implicitly the same conditions as before. Now the original equilibrium has been represented as a (large) collection of functions of only the aggregate state.

**Applying the Smolyak Algorithm**

With equilibrium redefined, it is now straightforward to approximate it using the Smolyak algorithm. First consider the easiest case where \( \Omega \) is a subset of \( \mathbb{R}^d \) for some \( d < \infty \) and \( X \) is a finite set.

1. Fix bounds \( \underline{\omega} \) and \( \bar{\omega} \) on the aggregate state space such that \( \underline{\omega} < \bar{\omega} \).

2. Use the Smolyak algorithm to generate collocation points \( \Omega^c := \{ \omega^j \}_{j=1}^n \) where \( n = 2d + 1 \).

3. Make a guess on \( f_i^x(\omega) \) for each \( \omega \in \Omega^c, x \in X, \) and \( i \in I \). Alternatively, make a guess on \( f_i^x(\omega) \) for each \( \omega \in \Omega^c, x \in X, \) and \( i \in \tilde{I} \), where \( \tilde{I} \subset I \) is a subset from which the remaining functions can be recovered through equilibrium conditions.\(^{19}\)

4. Use the Smolyak algorithm to construct approximations \( \hat{f}_i^x(\omega) \) for each \( \omega \in \Omega, x \in X, i \in I \). If the guesses were made for a subset \( \tilde{I} \), then approximations will only be explicitly constructed for this subset with the other functions approximated implicitly.

5. Determine whether the approximated functions nearly satisfy all the equilibrium conditions. If they do, stop. If they do not, proceed from step (3) with new guesses. Alternatively change the bounds and proceed from (1), explicitly approximate other functions in (3) and (4), or pursue different definitions of equilibrium functions or the aggregate state space (e.g. using logs instead of levels).\(^{20}\)

---

\(^{19}\)For instance, one could explicitly approximate consumption and price functions with the savings function being given implicitly through the budget constraint.

\(^{20}\)One could also check whether a higher level of approximation is feasible.
While in abstract this is complicated, the process is simple. Basically, guess on function values at the collocation points, construct Smolyak approximations, and check whether the approximated functions constitute an approximate equilibrium.

The preceding algorithm assumed that $X$ was a finite set and that $\Omega$ had finite dimension. If $X$ is not a finite set, then $f_i(x; \omega)$ must be approximated by its values in a finite set $\tilde{X}$ for each $i \in I, \omega \in \Omega$. This set will typically just be the nodes used for an interpolation, projection, or quadrature method. If $\omega \in \Omega$ has infinite dimension, then it must be approximated using a vector $\tilde{\omega}$ in a subset $\tilde{\Omega}$ of $\mathbb{R}^d$ for some $d < \infty$. If $\omega$ is a distribution, then a natural way to accomplish this is with the method of either Rios-Rull (1997) or Young (2010). If $\omega$ is not a distribution, then some other method must be used which will depend on the application. Using $\tilde{X}$ and $\tilde{\Omega}$ in place of $X$ and $\Omega$, the algorithm above can then be applied.

### 3.4 Models and Calibrations

This section describes the OLG and KS models and calibrations. In the case of the OLG economy, an analytic solution is also given. The OLG model is setup to be qualitatively similar to the KS model so that both feature capital, inelastic labor supply, production, total factor productivity shocks, and log utility. The model calibrations are similar in several respects but differ drastically with respect to time discounting and depreciation.

#### OLG economy

The OLG economy is very similar to Krueger and Kubler (2004) and based on Huffman (1987). The model is setup in sequential rather than recursive form to simplify notation.

A neoclassical production firm operates a production technology $z_t K_t^\alpha N_t^{1-\alpha}$ with $\alpha \in (0, 1)$ that uses as inputs capital $K_t$ rented at rate $r_t$ and labor $N_t$ hired at wage $w_t$ and is subject to a productivity shock $z_t$ that evolves according to a Markov chain. Capital depreciates at a stochastic rate $\delta_t$ that also evolves according to a Markov chain. The firm

---

21These methods handle distributions of the type $\mu(a, s)$ where $a \in [\underline{a}, \bar{a}]$ and $s \in S$ where $S$ is a finite set. If $S$ is infinite, it must be discretized.
takes prices as given and so the equilibrium rental and wage rates are \( r_t = z_t \alpha (K_t/N_t)^{\alpha-1} \) and \( w_t = z_t (1-\alpha)(K_t/N_t)^{\alpha} \) respectively.

Households consist of generations 1 through \( T < \infty \) with no intra-generational heterogeneity. The measure of households is constant across generations with the total measure of households normalized to 1. It is assumed, and this is key for tractability, that households have log utility, a strictly positive labor endowment in their first period of life, and no labor endowment for the rest of their life. The time \( t \) labor endowment of the youngest generation denoted \( l_1^t \) and normalized to \( T \) is supplied inelastically resulting in total labor supply \( N_t = 1(= l_1^t / T) \). The time \( t \) labor endowment for generation \( i \) in \( 2, \ldots, T \) is denoted \( l_i^t \) and is equal to zero. At time \( t = 0 \), households are endowed with capital holdings denoted by a vector \( k_0 = (k_1^0, k_2^0, \ldots, k_T^0) \) where \( k_i^t \) denotes the capital holdings of generation \( i \) at time \( t \). The resulting time 0 aggregate capital endowment is \( K_0 = \sum k_0 \). Assume that newborn households have zero capital holdings.

Households maximize expected discounted lifetime utility subject to a budget constraint, nonnegative consumption, and a natural borrowing limit (equal to zero). The budget constraint at time \( t \) is given by

\[
c_i^t + k_i^{t+1} = (1 + r_t - \delta_t)k_i^t + w_t l_i^t
\]

(3.3)

for generations \( i \in \{1, \ldots, T-1\} \) and

\[
c_T^t = (1 + r_T - \delta_t)k_T^t + w_T l_T^t
\]

(3.4)

for generation \( i = T \). Utility of a household beginning life in period \( t \) is given by

\[
\mathbb{E}_t \sum_{j=1}^T \beta^{j-1} \log(c_{t+j-1}^j)
\]

(3.5)

where \( \beta \in (0,1) \) is the time discount factor.

The necessary and sufficient first-order condition of an age \( i < T \) household at time \( t \) is given by

\[
1/c_i^t = \beta \mathbb{E}_t (1 + r_{t+1} - \delta_{t+1})/c_{t+1}^{i+1}
\]

(3.6)
Using backward induction, the solution to the household problem is shown to be

\[ k_{t+1}^i = \gamma^i(w_t l_t^i + (1 + r_t - \delta_t)k_t^i) \]

\[ \gamma^i = \frac{\beta \sum_{j=0}^{T-1-i} \beta^j}{\sum_{j=0}^{T-1} \beta^j} \] (3.7)

for all \( i \in \{1, \ldots, T-1\} \) and for all \( t \). Note that \( \gamma^i \) is the marginal propensity of generation \( i \) to save, and it is constant over time.

With this solution to the household problem, it is straightforward to calculate the law of motion. Let the time \( t \) distribution of capital holdings across generations be given by the vector \( k_t = (0, k_{t}^2, \ldots, k_{t}^T) \). Then the time \( t \) capital stock is \( K_t = \sum k_t / T \), and since total labor supply equals one, the marginal product of capital is \( r_t = z_t \alpha K_t^{\alpha-1} \) and the marginal product of labor is \( w_t = z_t (1 - \alpha) K_t^{\alpha} \). Using (3.7), the law of motion \( \Gamma \) for capital holdings is shown to be

\[ k_{t+1} = \Gamma(\delta_t, z_t, k_t) = (0, \gamma^1 w_t l_t^1, \gamma^2 k_t^2(1 + r_t - \delta_t), \ldots, \gamma^{T-1} k_t^{T-1}(1 + r_t - \delta_t)) \] (3.8)

which is a correspondence of only the time \( t \) aggregate shocks \((\delta_t, z_t)\) and distribution \( k_t \). This law of motion will be used to check the forecast accuracy of both the KS and Smolyak methods. Equilibrium is given by the capital policies in (3.7), the law of motion in (3.8), and competitive factor prices. Goods market clearing is ensured by Walras’ law.

**KS economy**

The KS economy used is a slightly modified version of the original Krusell and Smith (1998) model and is laid out in Den Haan, Judd, and Juillard (2010). The only difference between the two is Den Haan et al. (2010) add unemployment insurance so that the zero-borrowing constraint is sometimes binding. The model is setup in recursive form to save on notation.

A neoclassical production firm operates a production technology \( z K^\alpha N^{1-\alpha} \) with \( \alpha \in (0,1) \) that uses as input capital \( K \) rented at rate \( r \) and labor \( N \) hired at wage \( w \) and is subject to a productivity shock \( z \). The productivity shock \( z \) takes on one of two values \( z \in \{g, b\} \) and evolves according to a Markov chain \( \Pi_{zz'} \). Capital depreciates at a constant rate \( \delta \). Perfect competition ensures \( r = z\alpha(K/N)^{\alpha-1} \) and \( w = z(1 - \alpha)(K/N)^\alpha \).
Households have stochastic employment status \( s \) taking on one of two values \( s \in \{1, 0\} \) with \( s = 1 \) representing employment and \( s = 0 \) representing unemployment. Employed workers receive a labor endowment \( \bar{e} \) that they supply inelastically to the firm for labor income \( w\bar{e} \). Unemployed workers receive unemployment insurance equal to \( w\bar{u} \) from the government. Unemployment insurance is funded by the government which levies labor income tax \( \tau \) on employed workers and runs a balanced budget.

Employment status evolves with the productivity shock according to a Markov chain \( \Pi_{ss',zz'} \). The (exogenous) stock of unemployed workers \( U \) is assumed to be a function of only the current shock and so is denoted \( U_z \). The employment process implies total labor supply is known as a function of \( z \) with \( N_z = (1 - U_z)\bar{e} \). For the government budget to balance, \( \tau \) must be a function of \( z \) with \( \tau_z = \frac{\bar{u}}{\bar{e}} \frac{U_z}{1 - U_z} \). Households seek to maximize the expected discounted lifetime log-utility of consumption discounted at rate \( \beta \).

The problem of the household is

\[
V(k, s; z, \mu) = \max_{c, k', \mu'} \log(c) + \beta \sum_{s' z'} \Pi_{ss',zz'} V(k', s'; z', \mu')
\]

subject to

\[
c + k' = (1 + r - \delta)k + sw\bar{e}(1 - \tau_z) + (1 - s)w\bar{u}
\]

\[
c \geq 0
\]

\[
k' \in [0, \bar{k}]
\]

\[
\mu' = \Gamma_{zz'}(\mu)
\]

where \( r = r(z, \mu) \) and \( w = w(z, \mu) \), \( \mu \) is a joint distribution of capital holdings and employment status across households (giving \( K \) and \( N \)), and \( \bar{k} \) is an exogenous upper bound on possible capital choices (chosen large enough so as to not be binding in equilibrium). Equilibrium is a collection of policy, value, and price functions \( c, k', V, r, w \), together with a law of motion \( \Gamma_{zz'} \) (for each \( z, z' \)) such that \( V, c \) and \( k' \) solve the household problem taking \( r, w \) and \( \Gamma_{zz'} \) as given, factor prices \( r \) and \( w \) are competitive, and the law of motion \( \Gamma_{zz'} \) is consistent with individual policies and exogenous transition probabilities. Goods mar-

\[\text{22For ease of exposition I setup the model as being “initialized” from a long-run distribution. In general U and as well as N and } \tau \text{ must be determined from the distribution.}\]
ket clearing is ensured by Walras’ law. Unfortunately, there is no known solution for this model.\textsuperscript{23} 

\section*{Calibration}

For the OLG economy, I focus on the extreme calibration presented by KK in which quasi-aggregation fails for small $T$. Depreciation takes on one of two values $\delta \in \{0.9, 0.5\}$ and the productivity shock takes on one of two values $z \in \{1.05, 0.95\}$. Both of these are iid and the four combinations of $\delta$ and $z$ occur with equal probability $\Pi_{\delta z} = 1/4$. The discount factor $\beta$ is taken to be .7.\textsuperscript{24} The parameters are summarized in Table 3.1.

\begin{tabular}{|c|c|}
\hline
Parameter & Value \\
\hline
$\beta$ & .70 \\
$\alpha$ & .36 \\
$\delta$ & [.9,.5,.9,.5] \\
$z$ & [1.05,1.05,.95,.95] \\
$\Pi_{\delta z}$ & 1/4 \\
\hline
\end{tabular}

Table 3.1: OLG Calibration

For the KS economy, the calibration is the same as in Den Haan et al. (2010) which is only a slight modification of the original KS calibration. The calibration matches select business-cycle statistics at a quarterly frequency. Relative to the OLG calibration, households are much more patient with a discount factor of .99, capital depreciates much more slowly at .025, the productivity shocks are somewhat smaller at 1.01 and .99, and the productivity shock is not iid but has persistence $\Pi_{gg} = \Pi_{bb} = 7/8$. All the parameters including the employment process parameters are listed in Table 3.2.

\section*{3.5 Implementation}

This section discusses how the Smolyak method developed in Section 3 is applied to solve the OLG and KS models, as well as how the KS method is applied. In abstract, the procedure

\textsuperscript{23}Interestingly though, there is a solution for a “nearby” economy. If capital is the only input to production, i.e. the production technology is $zK^\alpha$, then the equilibrium capital policy function is $k' = \beta(1+r-\delta)k$. This is the limiting case of the Huffman (1987) example.

\textsuperscript{24}A constant $\beta$ was chosen for simplicity, but a future version of this paper will examine the case of $\beta = .95^{60/T}$ which is used in KK.
for computing equilibrium is the same across both economies and both methods. Fixing a law of motion, backward induction, along with Carroll’s (2006) endogenous gridpoints method and linear interpolation in the capital dimension, is used to solve the household problem. The household capital policies are then used to update the law of motion. This procedure is repeated until the change in the consumption policy and law of motion is less than $10^{-7}$ in levels. The rest of this section discusses the solution procedures in more detail.

**Table 3.2: KS Calibration**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>.99</td>
<td>$\Pi_{00,gg}$</td>
<td>$\frac{2}{3}\Pi_{gg}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>.36</td>
<td>$\Pi_{00,bb}$</td>
<td>$\frac{2}{3}\Pi_{bb}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>.025</td>
<td>$\Pi_{00,gb}$</td>
<td>$\frac{5}{4}\Pi_{gb}$</td>
</tr>
<tr>
<td>$\bar{e}, \bar{u}$</td>
<td>10/9, .15</td>
<td>$\Pi_{00,gg}$</td>
<td>$\frac{3}{4}\Pi_{gg}$</td>
</tr>
<tr>
<td>$U_g, U_b$</td>
<td>.04, .1</td>
<td>$\Pi_{10,zz'}$</td>
<td>$(U_{z'} - U_{zz'})/(1 - U_z)\Pi_{zz'}$</td>
</tr>
<tr>
<td>$g, b$</td>
<td>1.01, .99</td>
<td>$\Pi_{01,zz'}$</td>
<td>$\Pi_{zz'} - \Pi_{00,zz'}$</td>
</tr>
<tr>
<td>$\Pi_{gg}, \Pi_{bb}$</td>
<td>7/8, 7/8</td>
<td>$\Pi_{11,zz'}$</td>
<td>$\Pi_{zz'} - \Pi_{10,zz'}$</td>
</tr>
</tbody>
</table>

**Specific Implementation for the OLG Economy**

There are several implementation choices to be made when using the Smolyak method to compute the OLG economy. One pertains to representing the distribution of capital holdings. In particular, the distribution can be represented as in the theoretical model by using the levels of capital holdings across agents, $k = (0, k^2, \ldots, k^T)$. However, one can instead use $(K, s)$ where $s$ represents the capital holding shares $s = k/\sum k$. Of course it is possible to switch between the state spaces using $k = sKT$, but the two state spaces will result in different numerical solutions. It was found that using $(K, s)$ produced less error in both the forecasted capital stock and in the Euler equations, and so this is adopted as the benchmark method. The appendix presents accuracy numbers for the other state space representation. The state space bounds were taken to be $\pm 20\%$ of the (non-stochastic) steady-state capital stock and $\pm 40\%$ of the steady-state share distribution. In general, it is a good idea to place state space bounds as $\pm X\%$ of the steady-state values as this will cluster the collocation points around the steady-state values.

Another implementation choice applies only if using shares in the state space and regards
handling of the restriction $\sum s = 1$. The Smolyak algorithm is not designed to handle this case because it gives collocation points $\{(K,s)\} \subset \mathbb{R}^{T+1}$ that in general will not satisfy this restriction. The method I adopt is to use a mapping from the hypercube $[0,1]^T$ into the unit-simplex $\Delta(T-1) \subset [0,1]^T$. In particular, given a collocation point $(K,\tilde{s})$ with $\sum \tilde{s} \neq 1$, the mapping $s = \tilde{s} / \sum \tilde{s}$ is used to recover $(K,s)$ with $\sum s = 1$. For the reverse mapping, $\tilde{s} = s$ is used. The appendix explores a different mapping that is more uniform in a probabilistic sense but produces a worse approximation.

A final implementation choice regards simulating the economy. One can construct an approximation to the law of motion and use it to find the distribution of capital next period. Alternatively, one can construct approximations to the capital policy functions and use these to find the distribution. In solving the model, this is a non-issue because the two agree at the collocation points. It was found that approximating the capital policies produced less error, and so this is adopted in the benchmark. The appendix presents accuracy numbers for the other method.

To solve for equilibrium using the KS method, the law was updated by non-stochastically simulating the economy for 5000 periods, discarding the first 1000 periods, and using least squares regression to obtain a new law of motion (no relaxation was used).\footnote{No relaxation was used for the Smolyak method either.} The grid for aggregate capital was set to cover $\pm 60\%$ of the steady-state capital stock with 11 evenly-spaced points. A linear rather than log-linear functional form for the law of motion was assumed, but the two result in nearly identical approximations.\footnote{The Smolyak method’s law of motion was also represented using levels so this makes for a straightforward comparison. In section 6 I argue no functional form will result in quasi-aggregation for small $T$.}

Specific Implementation for the KS Economy

To solve for equilibrium in the KS economy using the Smolyak method, the following implementation was used. After choosing a set $\mathcal{K}$ of capital gridpoints, the infinite-dimensional distribution $\mu(k,s)$ over $k \in [0,\bar{k}], s \in \{0,1\}$ was approximated by a discrete distribution $\tilde{\mu}(k,s)$ over $k \in \mathcal{K}, s \in \{0,1\}$ using the method of Young (2010). The capital grid was constructed using 100 gridpoints resulting in a distribution of dimension 200. Typically the population would be normalized to unity implying the restriction $\sum \tilde{\mu}(k,s) = 1$ in which...
case the collocation points would not all satisfy this restriction. However, all that matters
for prices is the capital-labor ratio, and so I did not impose this.\textsuperscript{27} The bounds on the state
space were taken to be ±100% of the steady state-distribution.\textsuperscript{28} Instead of iterating to
convergence on the household problem every time before updating the law, it was found
that iterating only ten times converged to arbitrary precision, did not require relaxation,
and was fast (this process was repeated until both the law of motion and policies fully
converged).

To solve for equilibrium using the KS method, the law was updated by non-stochastically
simulating the economy for 5000 periods, discarding the first 1000 periods, and obtaining
new coefficients through least squares regression. The law was only updated after iterating
to convergence on the household problem. When updating the law, a relaxation parameter
of .5 was used as a looser value of .25 did not converge. The grid for aggregate capital was
set to cover ±30% of the steady-state capital stock using 11 evenly-spaced points. A linear
functional form for the law of motion was assumed.

3.6 Performance

This section analyzes the performance of the Smolyak and KS methods in computing the
OLG and KS economies.

OLG economy

To evaluate the accuracy of the Smolyak and KS solution methods for the OLG economy,
I focus on capital-stock forecast errors and Euler-equation errors along a long simulated
path. The path is simulated using the \textit{true} law of motion. The simulation length is set to
15000 periods and the first 1000 periods are discarded.

To assess the accuracy of the approximate law of motion, the one-step ahead capital-
stock forecasts are compared with the realized values in several ways. One measure of the

\textsuperscript{27}Originally the collocation points \{\hat{\mu}\} were mapped into the simplex using the transformation \hat{\mu} =
\hat{\mu}/\sum \hat{\mu}. However, it was found that not using this mapping resulted in smaller Euler-equation errors and a
more stable solution.

\textsuperscript{28}This isn’t entirely true as a small constant $10^{-6}$ was added to the upper bound to ensure the hypercube
had positive volume. Also, in general a lower bound of zero could cause problems, but this won’t be the case
when the level of approximation is one as it is here.
accuracy is given by the largest forecast error $|\hat{K}' - K'|/K'$ observed during the simulation where $\hat{K}'$ is the forecasted value and $K'$ the actual. Another measure is the $R^2$ statistic which indicates how much of the variation in $K'$ is explained by $\hat{K}'$. As there is a separate approximate law of motion for each $(\delta, z)$ pair, there are four $R^2$ statistics and the worst of these is referred to as “minimal $R^2$.” For the KS method, an upper bound on the minimal $R^2$ value is found by running a linear regression ex post on the simulated data. As a robustness check, a log-linear regression is also run to calculate the best minimal $R^2$ were a log-linear law of motion to be assumed for the KS method. The maximum error, minimal $R^2$, and best-possible minimal $R^2$ values are reported in Table 3.3.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Max % Error</th>
<th>Minimal $R^2$</th>
<th>Best Minimal $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KS Smolyak</td>
<td>KS Smolyak</td>
<td>Linear Log-Lin</td>
</tr>
<tr>
<td>3</td>
<td>3.17 0.07</td>
<td>.67568 .99940</td>
<td>.67587 .67635</td>
</tr>
<tr>
<td>6</td>
<td>1.03 0.27</td>
<td>.99505 .99987</td>
<td>.99520 .99766</td>
</tr>
<tr>
<td>10</td>
<td>1.05 0.65</td>
<td>.99784 .99982</td>
<td>.99792 .99982</td>
</tr>
<tr>
<td>25</td>
<td>1.04 0.85</td>
<td>.99807 .99981</td>
<td>.99816 .99984</td>
</tr>
<tr>
<td>50</td>
<td>1.04 0.85</td>
<td>.99807 .99980</td>
<td>.99816 .99984</td>
</tr>
<tr>
<td>100</td>
<td>1.04 0.85</td>
<td>.99807 .99980</td>
<td>.99816 .99984</td>
</tr>
</tbody>
</table>

Table 3.3: Error in the Law of Motion

When the number of generations is small, the Smolyak method performs much better than the KS method. The KS method produces maximum errors as large as 3.17% and an $R^2$ value as low as .676. In contrast, the Smolyak method’s maximum error is only .07% and its minimal $R^2$ statistic is .999. That the Smolyak method performs better in this case is confirmed visually in Figure 3.1 which plots the capital-stock forecasts made by both the Smolyak and KS methods with the true values. While the Smolyak forecasts and true values are virtually indistinguishable, the KS forecasts deviate noticeably.

The reason quasi-aggregation fails to obtain for small $T$ is clear. When there are only three generations, marginal propensities to save (which are roughly .54, .41 and 0 for generations 1, 2, and 3 respectively) differ substantially. Moreover a generation’s share of total capital and labor income fluctuates greatly because of large depreciation shocks that only affect capital-rich generations. When the youngest generation holds most of the income, the

---

29Formally this is computed as $R^2 = 1 - \frac{\sum(\hat{K}' - K')^2}{\sum(K' - \bar{K}')^2}$ where the summation is over a particular sample and $\bar{K}'$ is the sample mean of next period’s capital stock.
aggregate propensity to save is roughly .54. If instead the middle-aged or oldest generation holds most of the income, the aggregate propensity to save is closer to .41 or 0 respectively. Hence what matters here is not just aggregate income (given by the capital stock), but also the share of income held by each generation which varies substantially with the history of aggregate shocks. As argued in KK, if either the marginal propensities to save were similar or the distribution did not vary much, quasi-aggregation would obtain.

Furthermore, this failure of quasi-aggregation for small $T$ is not due to the chosen functional form of the law of motion. This is made clear in Figure 3.2 where a scatter plot of today vs tomorrow’s capital stock is contrasted against the best linear rules (one for each pair of shocks) one could have. While a linear rule does not work well, this figure also demonstrates that any forecast rule that is a function of today’s shocks and capital stock will fail to produce a good fit because the capital stock “clouds” are stacked one on another.

For a larger number of generations, the KS and Smolyak method result in similar performance. For example, when there are 100 generations, the maximum observed error is 1.04% for the KS method and 0.85% for the Smolyak method with minimal $R^2$ values of .9981 for the KS method and .9998 for the Smolyak method. The KS method’s performance
noticeably improves as $T$ is increased, while the Smolyak method’s performance improves by one measure and worsens by another.

The reason for the KS method’s improved performance is clear. In the limiting economy as $T$ goes large, $\gamma^i$, the marginal propensity to save of generation $i$, converges to $\beta$ for any fixed $i$. Hence nearly all households have nearly the same marginal propensity to save resulting in quasi-aggregation. Since quasi-aggregation obtains, the KS method performs well.

While quasi-aggregation obtains for large $T$, the Smolyak method’s performance does not noticeably improve, in fact worsening by one measure. However, the advantage of the Smolyak method is that, by keeping track of the entire distribution, its performance is not tied to quasi-aggregation. Rather, the method’s performance hinges on the polynomial structure of the law of motion which does not fundamentally change as $T$ increases.

To test the accuracy of the household policy functions, both maximum and average Euler-equation errors are computed along the simulated path. The errors are computed
following Judd (1992) as

\[
EEE_i^t(\omega_t) = \log_{10} \left| 1 - \frac{u^{-1} \left( \beta E u'(c_{t+1}^{i+1}(k_{t+1}^{i+1}; \omega_t); \hat{\omega}_{t+1}) \right)}{c_t^i(k_t^i; \omega_t)} \right |
\]

where \( R(\hat{\omega}_{t+1}) = 1 + z_{t+1} \alpha \hat{K}_{t+1}^{\alpha-1} - \delta_{t+1} \) and \( u(\cdot) = \log(\cdot) \). For the Smolyak method, \( \omega_t = (z_t, \delta_t, K_t, s_t) \), \( \hat{\omega}_{t+1} = (z_{t+1}, \delta_{t+1}, \hat{K}_{t+1}, \hat{s}_{t+1}) \), and \((\hat{K}_{t+1}, \hat{s}_{t+1})\) is the aggregate state next period according to the perceived law of motion. For the KS method, \( s_t \) and \( \hat{s}_{t+1} \) are simply dropped from the definition of \( \omega_t \) and \( \hat{\omega}_{t+1} \). The interpretation of these errors, derived from Judd and Guu (1997), is that a one-dollar mistake in optimization is made for every \( 10^{-EEE_i^t} \) dollars spent. For example, if \( EEE_i^t \) is \(-3\), then a one-dollar mistake is made for every 1000 dollars spent. Note that as has typically been done in the literature the Euler errors are measured with respect to the perceived state next period. In this sense they isolate household optimization error conditional on a law of motion from error in the law of motion.

Table 3.4 reports the maximum and average errors (across both generations and time). For the most part, the optimization errors of the two methods are comparable. Whereas the KS method results in smaller maximum errors, the Smolyak methods results in smaller average errors. For large \( T \), the maximum percent errors for the Smolyak method are noticeably larger than those for the KS method and result in a one-dollar mistake for every 290 dollars spent compared to 830 for the KS method. The performance of both methods tends to decrease as \( T \) increases but appears to level off for \( T \geq 25 \).

<table>
<thead>
<tr>
<th>T</th>
<th>Max Euler Errors</th>
<th>Avg Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KS</td>
<td>Smolyak</td>
</tr>
<tr>
<td>3</td>
<td>-3.15</td>
<td>-3.28</td>
</tr>
<tr>
<td>6</td>
<td>-3.08</td>
<td>-2.99</td>
</tr>
<tr>
<td>10</td>
<td>-2.91</td>
<td>-2.65</td>
</tr>
<tr>
<td>25</td>
<td>-2.92</td>
<td>-2.46</td>
</tr>
<tr>
<td>50</td>
<td>-2.92</td>
<td>-2.46</td>
</tr>
<tr>
<td>100</td>
<td>-2.92</td>
<td>-2.46</td>
</tr>
</tbody>
</table>

Table 3.4: Euler-Equation Errors in the OLG Economy

Despite using a fairly coarse approximation for the aggregate state, the Smolyak method produces both optimization errors and errors in the law of motion that are quite small. Having an analytic solution makes it possible to see why this is the case. The chosen imple-
mentation of the Smolyak method effectively constructs an approximation of the function

\[ k_{i+1,i,z,\delta}(K, s) = \gamma^i (1 + z\alpha K^{\alpha-1} - \delta)k^i \]  

(3.12)

for each \( i > 1 \), \( k^i \) in a grid, and \((z, \delta)\) combination (\( i = 1 \) is similar). Because \( \alpha \) is in \((0, 1)\), this function is not a polynomial. Hence, away from the collocation points the approximation is not perfect. However, note that this is a function of only one variable, \( K \), and that the polynomial basis used has terms \( K \) and \( K^2 \). Because of this, the approximation is quite good for any generation and any level of capital holdings.

It is also possible to see what indexing the policy functions and separating the individual from the aggregate state accomplishes. If the policy function were not indexed, then the Smolyak approximation would be applied to

\[ k_{i+1,i,z,\delta}(k^i, K, s) = \gamma^i (1 + z\alpha K^{\alpha-1} - \delta)k^i \]  

(3.13)

which has a term \( k^i \) and a cross term \( k^i K^{\alpha-1} \). To capture the impact of this cross term, one would need a finer Smolyak approximation. If in addition the aggregate and individual states were combined, the Smolyak approximation would be applied to

\[ k_{i+1,i,z,\delta}(K, s) = \gamma^i (1 + z\alpha K^{\alpha-1} - \delta)s_i KT \]  

(3.14)

which has cross terms \( s_i K \) and \( s_i K^{\alpha} \). This also would require a higher level of approximation. Indexing policy functions and separating the individual and aggregate states makes a fairly coarse Smolyak approximation accurate.

Now the running times of the two methods are briefly considered. It is important to remember that while the Smolyak method’s aggregate state space grows linearly in \( T \), the KS method’s aggregate state space does not grow at all. Hence, for large \( T \) it is guaranteed that the KS method will be faster. However, for small \( T \), the Smolyak method may be faster because it doesn’t need to simulate the economy in order to update the law of motion. Table 3.5 reports the running times. For \( T < 50 \), the Smolyak method is faster than the KS method, for \( T = 50 \) the times are roughly even, and for \( T > 50 \) the KS method is faster. Note that while computation time for the KS method grows linearly (roughly) in the number of generations, it grows quadratically for the Smolyak method.
Table 3.5: Running Times in Minutes for the OLG Economy

<table>
<thead>
<tr>
<th>$T$</th>
<th>KS</th>
<th>Smolyak</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.12</td>
<td>0.004</td>
</tr>
<tr>
<td>6</td>
<td>0.20</td>
<td>0.02</td>
</tr>
<tr>
<td>10</td>
<td>0.25</td>
<td>0.05</td>
</tr>
<tr>
<td>25</td>
<td>0.59</td>
<td>0.29</td>
</tr>
<tr>
<td>50</td>
<td>1.21</td>
<td>1.18</td>
</tr>
<tr>
<td>100</td>
<td>2.29</td>
<td>4.97</td>
</tr>
</tbody>
</table>

KS economy

Because the KS economy does not have an analytic solution, it is difficult to assess the accuracy of the KS and Smolyak methods. This is especially true for the Smolyak method. To test the law of motion, researchers have typically compared simulated series generated using only household policies with series generated using an approximate law of motion. This is not really applicable to the Smolyak method: if the law of motion is not explicitly approximated but rather given implicitly by the policy functions, then there is no disagreement between the series. In other words, the Smolyak method has an $R^2$ of 1 in this case. However, that does not mean there is no error in the law of motion because interpolating the policies is not typically an error-free process.

In light of this the accuracy of the Smolyak method is assessed in three ways. The first is to compute Euler-equation errors with respect to the realized aggregate state next period rather than the perceived aggregate state. This measure gives an idea of how much error in the law of motion translates into error in household optimization. The second is to compare the capital-stock series from the Smolyak and KS simulations. In addition to the convincing argument made by KS that their computed equilibrium must be close to the true equilibrium, many different solution methods have computed nearly the same equilibrium as the KS method (cf. Den Haan 2010). Hence the KS method’s solution can be used as an accuracy check. The third and final test is the comparison of a simulated series generated by explicitly approximating the capital policies with a simulated series generated by explicitly approximating the law of motion. Because it is possible to simulate the economy with either approximation, this may be helpful in assessing how well the Smolyak method is working.\footnote{This is not sure to be helpful however because the capital policies and law of motion have different...}

\footnote{This is not sure to be helpful however because the capital policies and law of motion have different...}
These three tests are conducted using a simulation length of 5000 periods with the first 1000 periods discarded. The accuracy of the KS method is assessed with the first test and also the typical comparison of the forecasted and realized capital-stock sequences.

First, Euler-equation errors are computed along the simulated path and both the maximum and average errors reported. The Euler errors are calculated analogously to (3.11) except that the rental rate and consumption next period are found using the realized next-period moment or distribution and the errors are only counted if the capital choice is strictly positive. The errors for the Smolyak method are slightly smaller in both maximum and average terms but in this measure the KS and Smolyak methods are roughly equivalent.

Second, the capital sequence generated by the KS method is compared with that of the Smolyak method. Table 3.7 reports the maximum and average differences between the Smolyak and KS aggregate capital series and Figure 3.3 plots them. Visually the series are almost indistinguishable although at times the Smolyak series lies slightly below the KS one. Over the entire simulation the series exhibit a maximum difference of only 0.28% and an average difference of 0.14%. However, while this difference is small, it is systematic with the average non-absolute difference also being 0.14% (measured using the Smolyak series subtracted from the KS series) confirming what was noticed visually. As the KS method is likely very close to the truth, the proximity of these two series confirms the accuracy of the Smolyak method for this economy.

Third and finally, the maximum and average capital-stock “forecast errors” at 1, 25, and 100-steps ahead are examined. As already mentioned, the errors for the Smolyak method are not forecast errors in the typical sense, but rather the discrepancy in capital-stock series properties. For instance, typically the law of motion will vary with the distribution even if the capital policy does not. In this case it would be better to approximate the capital policies and compute the law of motion indirectly.

Table 3.6: Euler-Equation Errors in the KS Economy

<table>
<thead>
<tr>
<th>Max Euler Errors</th>
<th>Avg Euler Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>KS       Smolyak</td>
<td>KS       Smolyak</td>
</tr>
<tr>
<td>-1.89     -2.22</td>
<td>-4.79     -4.97</td>
</tr>
</tbody>
</table>

31 When the capital choice equals zero, the no-borrowing constraint is almost certainly binding in which case the Euler-equation error is not useful.
constructed by two simulation methods. For the KS method, I proceed as usual comparing the forecasts made by the perceived and actual law of motion. Table 3.8 reports the errors, which are clearly small for both methods: even 100-steps ahead the capital stock forecast is off by at most 0.18% for the Smolyak and 0.15% for the KS method. The KS method outperforms the Smolyak method in all measures except the $R^2$, but its performance tends to deteriorate faster as the forecast length increases. Of note is the systematic bias in the
Smolyak’s forecasts as seen in the mean absolute errors being the same as the mean errors. While the KS method is guaranteed to be correct on average (because it makes unbiased forecasts by construction), the Smolyak method is guaranteed to be correct only as the level of approximation goes large. It appears however that the lowest level of approximation produces small errors in the KS economy just as it does in the OLG economy.

Since the Smolyak method places no restrictions on the functional form of the law of motion and appears to be accurate, it is interesting to ask whether the Smolyak-approximated economy exhibits quasi-aggregation. I test this by running linear and log-linear regressions \textit{ex post} on the simulated capital stock series from the Smolyak method. As should have been expected, the fit is extremely good with all the $R^2$ values exceeding .999999. This verifies directly, inasmuch as the Smolyak solution approximates the truth, that KS’s argument for quasi-aggregation was indeed correct: only the mean matters for this model.

The running times of the two methods are now briefly considered. While the KS method (paired with Carroll’s (2006) endogenous gridpoints method) is very fast at 3.25 minutes, the Smolyak method is also quite fast at 6.37 minutes. The Smolyak method performs quite well in this regard as it has 401 collocation points compared to the KS method’s 11 moment gridpoints. Its comparative advantage lies in avoiding the simulation step where much of the computation time is spent for the KS method.

3.7 Conclusion

The Smolyak method is a promising technique for computing equilibrium in dynamic heterogeneous-agent economies. While including a distribution as a state variable massively increases the dimensionality of the state space, the Smolyak sparse-grid interpolation algorithm makes this increase manageable. This technique developed in Smolyak (1963) and Barthelmann et al. (2000) shows great promise for economic applications as Krueger and Kubler (2004) first illustrated. The application of the Smolyak algorithm here results not only in tractability, but also in very good accuracy. In the KS economy the Smolyak method produces errors similar to those of the KS method. In the OLG economy, the Smolyak method performs much better than the KS method when the number of genera-
tions is small because it does not rely on quasi-aggregation which fails in this case. Moreover, for models where the distribution is finite dimensional, like in the OLG model, the method can be regarded as solving for a full rational-expectations equilibrium. In models where the distribution is infinite-dimensional, like in the KS model, the method comes very close to a full rational-expectations equilibrium.

Keeping track of the distribution does come at a cost: the Smolyak method is not as easy to implement nor as fast as the KS method in most cases. With regards to implementation, the code provided with this paper is meant to reduce the programming costs as much as possible. With this code in hand, the Smolyak algorithm is not much more complicated than any other interpolation scheme: values at a few predefined points define an interpolating function. With regards to speed, this paper has shown the Smolyak method need not be much slower than the KS method. For models with larger aggregate state spaces or where Carroll’s (2006) efficient solution method cannot be used, parallelization (which has not been used in this paper) could prove very helpful as the work by Aldrich et al. (2011) has shown.

While the application of the Smolyak algorithm here has been to approximating full-rationality equilibrium as closely as possible, the Smolyak algorithm could also be useful in solving for bounded-rationality equilibrium quickly and accurately. While many methods could benefit from the Smolyak algorithm, of particular promise is the explicit aggregation technique of Den Haan and Rendahl (2010). This method avoids the simulation step of the KS method by explicitly aggregating “auxiliary” policy functions. In its most basic implementation, the auxiliary policy functions are constructed to be linear in asset holdings but as close as possible to the original policies. Because the auxiliary policies are linear, they aggregate perfectly and the minimal aggregate state space is average capital holdings for each type of agent. Having more types of agents or more curvature in the auxiliary policy functions requires having more moments, but Smolyak interpolation is well adapted to handling this increase in aggregate moments. This pairing of Den Haan and Rendahl’s (2010) method with the Smolyak algorithm could achieve some of the benefits of the Smolyak method (no simulation, high accuracy) while being extremely efficient. The exploration of this idea is left for future research.
Appendix A

Evaluating Default Policy: The Business Cycle Matters

A.1 Extended Model Description

This appendix contains extended model descriptions for both the steady state and business cycle models. The objective is to arrive at the simplified definitions of equilibrium presented in the main text.

Steady State Model

The Intermediary’s Problem

The financial intermediary’s problem is to maximize the net present value of financial wealth using contract holdings, capital, and a bond. This present value is discounted using the bond’s return. It is useful to think of a contract as a pair \((a', s)\) and the intermediary’s portfolio of contract holdings as a mapping \(l : A \times S \to \mathbb{R}\). To aggregate the intermediary’s contract holdings, I introduce two functions

\[
C(m) := \sum_{a, s} m(a, s) \rho_{s-1} p(a, s-1) a
\]

(A.1)

and

\[
C'(m) := \frac{1}{q_B} \sum_{a', s} m(a', s) q(a', s) a'.
\]

(A.2)
In this notation, $C(l)$ gives the aggregate yield of portfolio $l$ while $C'(l)$ gives the aggregate price of the portfolio $l$ deflated by the bond’s price (ensuring $C(l) = C'(l)$ in equilibrium).

With this notation, the intermediary’s problem can be written

$$P(K, B, l) = \max_{K' \geq 0, B', l'} D + \bar{q}_B P(K', B', l')$$

$$D + \bar{q}_B C'(l') + K' + \bar{q}_B B' = C(l) + K(1 + r - \delta) + B$$

(A.3)

Let the policy functions for the intermediary be denoted $K'(K, B, l)$, $B'(K, B, l)$, and $l'(K, B, l)$.

**Equilibrium**

A steady state equilibrium is a collection of prices $r, w, q, \bar{q}_B$, repayment rates $p$, policy functions $c, a', d, K', B', l'$, value functions $V, P$, aggregates $K, N, B, l$, and a distribution of households $\mu$ such that all of the following hold:

1. Household policies $c, a', d$ solve their problem.
2. Financial intermediary policies $K', B', l'$ solve their problem.
3. Capital holdings are positive: $K > 0$.
4. Factor prices are competitive (ensuring the production firm optimizes): $w = (1 - \alpha)(K/N)^{\alpha}$ and $r = \alpha(K/N)^{\alpha - 1}$.
5. The intermediary has zero net financial income: $C(l) + K(1 + r - \delta) + B = 0$.
6. The labor market clears: $N = \int ed\mu$.
7. The bond market clears: $B' = 0$.
8. Each contract market clears:

$$-l'(K, B, l)(a', s) = \int [a' = a'(a, e, s, h)] \mu(da, de, s, dh)$$

for all $a', s$.
9. The goods market clears.
10. Repayment probabilities are consistent:

\[ p(a', s) = \mathbb{E}_s(1 - d(a', e', s', 0) + d(a', e', s', 0)\chi we' / (-a')). \]

11. The model is in steady state: \( K' = K, B' = B, l' = l \), and \( \mu \) is consistent with household policies and stochastic transitions.

**Equilibrium Characterization**

I now characterize any equilibrium.\(^1\) First consider the first order conditions from the firm’s problem:

\[ 1 = \bar{q}_B(1 + r - \delta) + \theta \]  \hspace{1cm} (A.4)

\[ \bar{q}_B = \bar{q}_B \]  \hspace{1cm} (A.5)

\[ q(a', s)a' = \bar{q}_B\rho_s p(a', s)a' \]  \hspace{1cm} (A.6)

where \( \theta \geq 0 \) is the Lagrange multiplier on the constraint \( K' \geq 0 \). For \( a' \neq 0 \), this gives the price schedule from the main text

\[ q(a', s) = \bar{q}_B\rho_s p(a', s) \]  \hspace{1cm} (A.7)

in the case of \( K' > 0 \) which is required in equilibrium. This price schedule can also be used for \( a' = 0 \) as a loan to repay nothing with a zero price satisfies the intermediary’s first order condition.

**Proposition 1.** Aggregate capital can found from the distribution as the unique \( K \) solving

\[ (1 + r - \delta)K = \int (a + d(a, e, s, h)(-a - \chi we))d\mu \]

where \( r = \alpha(K/N)^{\alpha-1} \) and \( w = (1 - \alpha)(K/N)\alpha \) (with \( N = \int e d\mu \)).

**Proof.** Define

\[ \tilde{p}(a', s') := \int_{e'} (1 - d(a', e', s', 0) + d(a', e', s', 0)\chi we' / (-a')) f(e'|s')de' \]

so that \( p(a', s) = \sum_{s'} F(s'|s)\tilde{p}(a', s') \).

\(^1\)Chatterjee et al. (2007) prove that an equilibrium exists for the environment with \( \chi = 0 \) and \( \rho_s \) age-invariant when the production function satisfies certain conditions.
The zero-profit condition of the firm requires $C(l) + (1 + r - \delta)K + B = 0$ and bond market clearing requires $B = 0$. Consequently

$$(1 + r - \delta)K = \sum_{a',s} -l(a', s)\rho_s p(a', s)$$

(A.8)

$$= \sum_{a',s} (\sum_{a,h} \int 1_{a'=a'(a,e,s,h)}\mu(a, de, s, h))\rho_s p(a', s)$$

(A.9)

$$= \sum_{a'} \sum_{a,s,h} p(a', s)\rho_s \int 1_{a'=a'(a,e,s,h)}\mu(a, de, s, h)$$

(A.10)

$$= \sum_{a',s'} \sum_{a,s,h} F(s'|s)\tilde{p}(a', s')\rho_s \int 1_{a'=a'(a,e,s,h)}\mu(a, de, s, h)$$

(A.11)

$$= \sum_{a',s'} \tilde{p}(a', s') \sum_{a,s,h} F(s'|s)\rho_s \int 1_{a'=a'(a,e,s,h)}\mu(a, de, s, h)$$

(A.12)

$$= \sum_{a',s'} \tilde{p}(a', s') \int \rho_s F(s'|s) 1_{a'=a'(a,e,s,h)} d\mu(a, e, s, h)$$

(A.13)

where (A.8) uses the definition of $C(\cdot)$, (A.9) uses contract market clearing, (A.10) just rearranges, (A.11) uses the definition of $p$ and $\tilde{p}$, and (A.12)-(A.13) rearrange again. Now because households are born with zero assets, $a'\mu(a', s') = a' \int \rho_s F(s'|s) 1_{a'=a'(a,e,s,h)} d\mu$ for any $a', s'$. Using this fact,

$$(1 + r - \delta)K$$

(A.14)

$$= \sum_{a',s'} \tilde{p}(a', s') a' \mu(a', s')$$

(A.15)

$$= \sum_{a',s'} \tilde{p}(a', s') a' \sum_{h'} \mu(a', s', h')$$

(A.16)

$$= \sum_{a',s',h'} \tilde{p}(a', s') a' \mu(a', s', h')$$

(A.17)

$$= \sum_{a',s',h'} \left( \int e' (1 - d(a', e', s', 0) + d(a', e', s', 0)\chi we')/(\chi we') f(e'|s')de' a' \mu(a', s', h') \right)$$

(A.18)

$$= \sum_{a',s',h'} \left( \int e' ((1 - d(a', e', s', 0)a' - d(a', e', s', 0)\chi we') f(e'|s')\mu(a', s', h')de' \right)$$

(A.19)

$$= \int ((1 - d(a', e', s', 0)a' - d(a', e', s', 0)\chi we')d\mu(a', e', s', h')$$

(A.20)

where (A.15) substitutes, (A.16) uses a probability measure property, (A.17) rearranges, (A.18) uses the definition of $\tilde{p}$, (A.19) rearranges, and (A.20) uses the property $\mu(a, e, s, h) = f(e|s)\mu(a, s, h)$ (by a law of large numbers).
I also claim $K$ is unique. This is clear once one notices $w$ and $rK(=\alpha Y)$ are both increasing in $K$.

Using these characterizations, the simplified definition of equilibrium in the main text is readily obtained.

**Business Cycle Model**

**The Intermediary’s Problem**

The financial intermediary’s problem is to maximize the net present value of financial wealth using contracts, bonds, capital, and an aggregate-complete set of Arrow securities. He discounts future wealth using the state-contingent prices $\bar{q}_{z'}(S)$ of the Arrow securities. It is useful to think of a contract as a tuple $(a', s, z')$ specifying an amount $a'$, the counterparty to the contract as summarized in $s$, and the contingency $z'$ in which the debt (or savings) will be delivered. The intermediary’s contract holdings can be thought of as a pair of portfolios $l_{z'}: A \times S \to \mathbb{R}$, one for each $z'$. It is useful to introduce two functions that aggregate the prices and yields of these portfolios. For any given portfolio of contract holdings $m: A \times S \to \mathbb{R}$, define

$$C(m; S) := \sum_{a,s-1} m(a,s-1)\rho_s p(a,s-1; S) a$$

and

$$C'_{z'}(m; S) := \frac{1}{\bar{q}_{z'}(S)} \sum_{a', s} m(a', s)q_{z'}(a', s; S)a'.$$

$C(m; S)$ is the yield of a portfolio of contracts $m$ and $C'_{z'}(m; S)$ is the price of those contracts normalized by $\bar{q}_{z'}(S)$. This normalization ensures $C'_{z'}(\cdot; S) = C(\cdot; S_{z'})$ in equilibrium.

Using these definitions, the intermediary’s problem can be written

$$P(A, B, K, l; S) = \max_{A_{z'}, B', K' \geq 0, l'_{z'}} D + \sum_{z'} \bar{q}_{z'}(S)P(A'_{z'}, B', K', l'_{z'}; S_{z'})$$

$$D + \sum_{z'} \bar{q}_{z'}(S)(C'_{z'}(l'_{z'}; S) + A'_{z'}) + \bar{q}_B(S)B' + K' = C(l; S) + A + B + (1 + r(S) - \delta)K.$$

Let the policy functions for the intermediary be denoted $A'_{z'}(A, B, K, l; S)$, $B'(A, B, K, l; S)$, $K'(A, B, K, l; S)$, and $l'_{z'}(A, B, K, l; S)$. 

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Equilibrium

For now, let $S$ be defined as $z$ together with a distribution $\mu(a, e, s, h)$ of households and $l, A, B, K, N$ satisfying $B = 0, K > 0, N > 0, A = 0$, and $C(l; S) + A + B + (1 + \alpha(K/N)^{\alpha-1} - \delta)K = 0$. A recursive competitive equilibrium is a collection of price functions $r, w, \bar{q}_B, \bar{q}_G, \bar{q}_B, q_B$, repayment rates $p$, policy functions $c, a'_g, a'_b, d$, value functions $V, P$, and a law of motion $\Gamma$ such that all of the following hold:

1. Household policies solve their problem.
2. The financial intermediary’s policies solve his problem.
3. Factor prices are competitive (ensuring the production firm optimizes):

   \[
   r(S) = \alpha(K/N)^{\alpha-1} \tag{A.23}
   \]
   \[
   w(S) = (1 - \alpha)(K/N)^{\alpha} \tag{A.24}
   \]

4. Asset and factor markets clear:

   \[
   N = \int ed\mu \tag{A.25}
   \]
   \[
   K'(A, B, K, l; S) > 0 \tag{A.26}
   \]
   \[
   B'(A, B, K, l; S) = 0 \tag{A.27}
   \]
   \[
   A'_{z'}(A, B, K, l; S) = 0 \tag{A.28}
   \]

where $A, B, K, l$ and $N$ are read off $S$.

5. Each contract market clears:

   \[
   -l'_{z'}(A, B, K, l; S)(a', s) = \int 1_{(a' = a'_{z'}(a, e, s, h; S)\}} \mu(da, de, s, dh) \forall a', s, z' \tag{A.29}
   \]

6. The goods market clears.

7. Repayment probabilities are consistent:

   \[
   p(a, s-1; S) = \sum_s \int \left( 1 - d(a, e, s, 0; S) + d(a, e, s, 0; S)\chi w(S)e/(-a) \right) f(e|s, z)deF(s|s-1, z) \tag{A.30}
   \]

for all $a', s$, and $S$ (recall $S$ includes $z$).
8. The intermediary makes zero profits:

\[ C(l'_{z'}; S'_{z'}) + A'_{z'} + B' + (1 + r(S'_{z'}) - \delta)K' = 0 \] for each \( z' \) \hspace{1cm} (A.31)

\[ \sum_{z'} \bar{q}_{z'}(S)C'_{z'}(l'_{z'}; S) + \sum_{z'} \bar{q}_{z'}(S)A'_{z'} + \bar{q}_{B}(S)B' + K' = 0 \] \hspace{1cm} (A.32)

9. The law of motion \( S'_{z'} = \Gamma(z', S) \) is consistent: \( \mu' \) is generated by household policies and stochastic transitions, \( N' \) is given by stochastic transitions, and \( l'_{z'}, A'_{z'}, B', \) and \( K' \) are given by their policy functions.

**Equilibrium Characterization**

In this section I give conditions on the allocations of households that greatly simplify the definition of equilibrium. In particular, if these conditions are met then there exist prices and optimal policies of the financial intermediary that satisfy market clearing and consistency conditions.

First, I discuss equilibrium pricing.

**Lemma 1.** The intermediary is indifferent over all feasible policies if

\[ q_{z'}(a', s; S)a' = \bar{q}_{z'}(S)\rho_{a}(a', s; S'_{z'})a' \]

\[ \bar{q}_{B}(S) = \bar{q}_{g}(S) + \bar{q}_{b}(S) \]

\[ 1 = \bar{q}_{g}(S)(1 + r(\mathcal{G}) - \delta) + \bar{q}_{b}(S)(1 + r(\mathcal{B}) - \delta) \]

**Proof.** If these hold, then the first order conditions for contract holdings, bonds, and capital are satisfied. The first order condition for Arrow securities is trivially satisfied: \( \bar{q}_{z'}(S) = \bar{q}_{z'}(S) \). Because the problem is linear, the second order conditions are satisfied. \qed

I restrict attention to the prices satisfying the above.

**Lemma 2.** Equilibrium prices imply \( C'_{z'}(l'_{z'}; S) = C(l'_{z'}; S'_{z'}) \) for any portfolio \( l'_{z'} \). Further, \( C'_{z'}(-l'_{z'}; S) = -C'_{z'}(l'_{z'}; S) \).
Proof. The first part is proved by

\[
\begin{align*}
C^\prime_{z^\prime}(l^\prime_{z^\prime}; S) &:= \frac{1}{\bar{q}^\prime_{z^\prime}(S)} \sum_{a^\prime, s} l^\prime_{z^\prime}(a^\prime, s; S)q^\prime_{z^\prime}(a^\prime, s; S)a' \\
&= \frac{1}{\bar{q}^\prime_{z^\prime}(S)} \sum_{a^\prime, s} l^\prime_{z^\prime}(a^\prime, s; S)\bar{q}^\prime_{z^\prime}(S)p(a^\prime, s; S)_{z^\prime}a' \\
&= \sum_{a^\prime, s} l^\prime_{z^\prime}(a^\prime, s; S)_{z^\prime}p(a^\prime, s; S)_{z^\prime}a' \\
&=: C(l^\prime_{z^\prime}; S)_{z^\prime}.
\end{align*}
\]

The second part is obvious.

Lemma 3. Zero profits for the intermediary obtain if

\[
\begin{align*}
K' &= (-C^\prime_g(l^\prime_g; S) + C^\prime_b(l^\prime_b; S))/(r(G) - r(B)), \\
B' &= (C^\prime_g(l^\prime_g; S)(1 + r(G) - \delta) - C^\prime_b(l^\prime_b; S)(1 + r(B) - \delta))/(r(G) - r(B)),
\end{align*}
\]

and \(A^\prime_g = A^\prime_b = 0\). This portfolio is feasible (and hence by Lemma 1 optimal) if \(-C^\prime_g(l^\prime_g; S) \geq -C^\prime_b(l^\prime_b; S)\).

Proof. For \(A^\prime_g = A^\prime_b = 0\), zero profits obtain if

\[
\begin{align*}
0 &= \sum_{z^\prime} \bar{q}^\prime_{z^\prime}(S)C^\prime_{z^\prime}(l^\prime_{z^\prime}; S) + \bar{q}^\prime_B(S)B' + K' \\
&= C(l^\prime_{z^\prime}; S)_{z^\prime} + B' + (1 + r(S^\prime_{z^\prime}) - \delta)K'.
\end{align*}
\]

In matrix notation, (A.33) is equivalent to

\[
\begin{bmatrix}
C^\prime_g(l^\prime_g; S) & C^\prime_b(l^\prime_b; S)
\end{bmatrix}
\begin{bmatrix}
\bar{q}^\prime_g(S) \\
\bar{q}^\prime_b(S)
\end{bmatrix}
+ \begin{bmatrix}
K' \\
B'
\end{bmatrix}
\begin{bmatrix}
1 \\
\bar{q}^\prime_B(S)
\end{bmatrix}
= 0.
\]

(A.35)

Because prices satisfy

\[
\begin{bmatrix}
\bar{q}^\prime_g(S) \\
\bar{q}^\prime_b(S)
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
1 + r(G) - \delta & 1 + r(B) - \delta
\end{bmatrix}^{-1}
\begin{bmatrix}
\bar{q}^\prime_B(S) \\
1
\end{bmatrix},
\]

(A.36)

if

\[
\begin{bmatrix}
C^\prime_g(l^\prime_g; S) & C^\prime_b(l^\prime_b; S)
\end{bmatrix}
= -\begin{bmatrix}
B' & K'
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 + r(G) - \delta & 1 + r(B) - \delta
\end{bmatrix}
\]

(A.37)
then (A.33) becomes
\[
- \begin{bmatrix} B' & K' \end{bmatrix} \begin{bmatrix} \bar{q}_B(S) \\ 1 \end{bmatrix} + \begin{bmatrix} K' & B' \end{bmatrix} \begin{bmatrix} 1 \\ \bar{q}_B(S) \end{bmatrix} = 0.
\] (A.38)
which holds.

Now if \((l'_g, l'_b, K', B')\) satisfying (A.37) imply (A.34), then zero-profits obtain. Note that (A.37) can equivalently be written
\[
C'_g(l'_g; S) + B' + K'(1 + r(G) − \delta) = 0 \quad \text{(A.39)}
\]
\[
C'_b(l'_b; S) + B' + K'(1 + r(B) − \delta) = 0. \quad \text{(A.40)}
\]
By Lemma 2, \(C'_z(l'_z; S) = C(l'_z; S'_z)\). Therefore (A.34) holds.

Finally, this is a feasible allocation so long as \(K' \geq 0\). Inverting the relationship in (A.37) gives
\[
K' = \frac{-C'_g(l'_g; S) + C'_b(l'_b; S)}{(r(G) - r(B))} \quad \text{(A.41)}
\]
\[
B' = \frac{(C'_g(l'_g; S)(1 + r(B) - \delta) - C'_b(l'_b; S)(1 + r(G) - \delta))/(r(G) - r(B))}. \quad \text{(A.42)}
\]
Consequently, \(K' \geq 0\) is equivalent to \(-C'_g(l'_g; S) \geq -C'_b(l'_b; S)\).

\[\square\]

**Proposition 2.** Define \(\mu'_z\) by
\[
\mu'_z(a', s; S) := \int_{a,e,h} \mathbf{1}_{\{a' = a'_z(a,e,s,h;S)\}} \mu(da, de, s, dh)
\]
so that \(\mu'_z(a', s; S)\) is the measure of households choosing contract \((a', s, z')\). Also, define
\[
K'(\mu'_g, \mu'_b; S) := \frac{1}{r(G) - r(B)}(C'_g(\mu'_g; S) - C'_b(\mu'_b; S)) \quad \text{and}
\]
\[
B'(\mu'_g, \mu'_b; S) := \frac{1}{r(G) - r(B)}((1 + r(G) - \delta)C'_b(\mu'_b; S) - (1 + r(B) - \delta)C'_g(\mu'_g; S)).
\]
If allocations satisfy

1. \(K'(\mu'_g, \mu'_b; S) > 0\) and
2. \(B'(\mu'_g, \mu'_b; S) = 0\),

and if prices satisfy
1. \( q_{z'}(a', s; S) = \tilde{q}_{z'}(S)p_s p(a', s; S_z') \),

2. \( \bar{q}_B(S) = (1 - \bar{q}_B(S)(1 + r(B) - \delta))/(r(G) - r(B)) \), and

3. \( \tilde{q}_B(S) = (\bar{q}_B(S)(1 + r(G) - \delta) - 1)/(r(G) - r(B)) \),

then there exist optimal policies of the financial intermediary that satisfy zero profit conditions and clear Arrow security, capital, bond, and contract markets.

Proof. First note these prices satisfy the conditions in Lemma 1. Hence the intermediary is indifferent over all feasible allocations. Second, specifying \( l'_z = -\mu'_z \) for the intermediary clears contract markets. Further, by Lemma 3, policies that specify \( K' = (-C'_{\bar{q}}(l'_{\bar{q}}; S) + C'_{\bar{q}}(l'_{\bar{q}}; S))/(r(G) - r(B)) \) and \( B' = (C'_{\bar{q}}(l'_{\bar{q}}; S)(1 + r(B) - \delta) - C'_{\bar{q}}(l'_{\bar{q}}; S)(1 + r(G) - \delta))/(r(G) - r(B)) \) and \( A'_z = 0 \) result in zero profits (and clear Arrow security markets). By the second part of Lemma 2, the \( K', B' \) policies are then equivalent to \( K' = (C'_{\bar{q}}(\mu'_g; S) - C'_{\bar{q}}(\mu'_h; S))/(r(G) - r(B)) \) and \( B' = (-C'_{\bar{q}}(\mu'_g; S)(1 + r(B) - \delta) + C'_{\bar{q}}(\mu'_h; S)(1 + r(G) - \delta))/(r(G) - r(B)) \). This policy is feasible if \( K'(\mu'_g, \mu'_h; S) > 0 \). Further, the bond market clears if \( B'(\mu'_g, \mu'_h; S) = 0 \). Hence the specified policies are feasible, optimal, and clear all asset markets.

Proposition 3. Today’s capital stock \( K \) can be found in equilibrium from the joint distribution \( \mu \) as the unique \( K \) solving

\[
(1 + r(S) - \delta)K = \int (a + d(a, e, s, h; S)(-a - \chi w(S)e))d\mu
\]

where \( r(S) = z\alpha(K/N)^{\alpha-1} \) and \( w(S) = z(1 - \alpha)(K/N)^{\alpha} \) (and \( N = \int ed\mu \)).

Proof. The proof is along the same lines as Proposition 1. First, define

\[
\tilde{p}(a, s; S) := \int (1 - d(a, e, s, 0; S) + d(a, e, s, 0; S)\chi w e/(-a))f(e|s, z)de
\]

so that \( p(a', s; S_z') = \sum_{s'} F(s'|s, z')\tilde{p}(a', s'; S_z') \). Now, note that zero-profits, bond market clearing, and Arrow security market clearing imply \( (1 + r(S_z') - \delta)K' = -C(l'_z; S_z') \). From
this,

\[(1 + r(S'_{z'}) - \delta)K' \]

\[= \sum_{a',s} -l'_{z'}(a', s) \rho_s p(a', s; S'_{z'}) a' \]  

(A.43)

\[= \sum_{a',s} \sum_{a,h} 1_{a'=a'(a,e,s,h;S)} \mu(a, de, s, h)) \rho_s p(a', s; S'_{z'}) d' \]  

(A.44)

\[= \sum_{a',s} \sum_{a,s,h} p(a', s; S'_{z'}) a' \int \rho_s 1_{a'=a'(a,e,s,h;S)} \mu(a, de, s, h) \]  

(A.45)

\[= \sum_{a',s} \sum_{a,s,h} \left( \sum_{s'} F(s'|s, z') \tilde{p}(a', s'; S'_{z'}) \right) a' \int \rho_s 1_{a'=a'(a,e,s,h;S)} \mu(a, de, s, h) \]  

(A.46)

\[= \sum_{a',s'} \tilde{p}(a', s'; S'_{z'}) \sum_{a,s,h} F(s'|s, z') a' \int \rho_s 1_{a'=a'(a,e,s,h;S)} \mu(a, de, s, h) \]  

(A.47)

\[= \sum_{a',s'} \tilde{p}(a', s'; S'_{z'}) a' \int \rho_s F(s'|s, z') 1_{a'=a'(a,e,s,h;S)} d \mu(a, e, s, h) \]  

(A.48)

where (A.43) uses the definition of \(C(\cdot)\), (A.44) uses contract market clearing, (A.45) just rearranges, (A.46) uses the definition of \(p\) and \(\tilde{p}\), and (A.47)-(A.48) rearrange again. Now because households are born with zero assets, \(a' \mu_{z'}(a', s') = a' \int \rho_s F(s'|s, z') 1_{a'=a'(a,e,s,h;S)} d \mu\) for any \(a', s', z'\) where I’ve used \(\mu_{z'}\) to denote the distribution implicit in \(S'_{z'}\). Using this fact,

\[(1 + r(S'_{z'}) - \delta)K' \]

(A.49)

\[= \sum_{a',s'} \tilde{p}(a', s'; S'_{z'}) a' \mu_{z'}(a', s') \]  

(A.50)

\[= \sum_{a',s'} \tilde{p}(a', s'; S'_{z'}) a' \sum_{h'} \mu_{z'}(a', s', h') \]  

(A.51)

\[= \sum_{a',s',h'} \tilde{p}(a', s'; S'_{z'}) a' \mu_{z'}(a', s', h') \]  

(A.52)

\[= \int \left((1 - d(a', e', s', 0; S'_{z'})) a' - d(a', e', s', 0; S'_{z'}) \chi w(S'_{z'}) e') d \mu_{z'}(a', e', s', h') \]  

(A.53)

where (A.50) substitutes, (A.51) uses a probability measure property, (A.52) rearranges, and (A.53) uses the definition of \(\tilde{p}\) and the property \(\mu_{z'}(a', e', s', h') = f(e'|s', z') \mu(a', s', h')\) (by a law of large numbers).
A.2 Data and Calibration

Data

The model was constructed to capture the salient features of Chapter 7 bankruptcy in the US. Figure A.1 shows the annual percent of Chapter 7 filings per household from the period 1960 to 2010.\(^2\) The number of households taking advantage of this bankruptcy provision has drastically increased since the 1984 and in 2005 experienced a sharp increase, then decrease, and subsequent recovery. The sharp increase is presumably due to anticipation of the 2005 reform (which mostly applied to filings made on or after October 17, 2005).\(^3\) Clearly, the sample period will drastically affect not only the level of bankruptcies but potentially their cyclicality and volatility. Because of this, I report statistics for each of the subperiods 1960-

\[\text{Figure A.1: Chapter 7 Filings Per Household (1960-2010)}\]

\(^2\)The data that go back to 1960 are only available on a fiscal year basis ending in June (the series ending in December 1990). These are available at the Department of Justice website [http://www.uscourts.gov/Statistics/BankruptcyStatistics.aspx#calendar](http://www.uscourts.gov/Statistics/BankruptcyStatistics.aspx#calendar) in various pdf files. The data for 1960-1982 are at the end of the pdf file labeled “1983-2003 Bankruptcy Filings.” To recover the annual figure for year \(y\) prior to 1990, I use the average of \(y\) and \(y + 1\). To test how well this works, I compare this method’s values with the known values for the period 1990-2004. This produces a good fit with an \(R^2\) of .985. If 2005 is included however, this drops to .827 because of the very large rise at the end of 2005 presumably in anticipation of BAPCPA provisions about to begin (most provisions applied to cases filed on or after October 17, 2005).

1984, 1984-2004, and 1960-2004. In addition to 1984 and 2005 being breakpoints visually, these were years of substantial bankruptcy reform.  

Figure A.2 plots log filings per household and log real GDP detrended using the HP filter with parameter 100 for the sample 1960-2004. Visually bankruptcy filings are much more volatile than output and appear to be countercyclical, but not strongly. This is borne out by the statistics reported in Table A.1. Bankruptcy filings are between 2.8 and 7.0 times more volatile than output and are only mildly countercyclical with a correlation between -.02 and -.45 depending on the sample period.

The lack of strong countercyclicality in filings is at first counterintuitive: one would expect an increase in unemployment and hence a reduced ability to refinance debt to translate

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There were essentially two rounds of substantial bankruptcy legislation. The first round began with the *Marquette* decision in 1979 which was subsequently ammended by the Bankruptcy Amendment Act of 1984. Among other things, these made it easier for credit card companies to charge higher interest. The second round began with the “Bankruptcy Reform Act of 1999” which was passed by Congress but not signed into law. Subsequent revisions of this legislation resulted in the “Bankruptcy Abuse Prevention and Consumer Protection Act” (BAPCPA) which was signed into law in 2005. Several possible explanations for the rise in filings have recently been evaluated by Livshits et al. (2010). They find the rise was most likely due to changes in the credit market environment coming from decreased transaction costs of borrowing and the cost of filing for bankruptcy, not from legislative reform. However, they suggest the increase in information technology may have played a key role in driving down these costs which would also suggest a break in the early 1980s.
into a filings increase. But consider the case of an indebted household who has lost their job in a recession. The household has two choices. They can file for bankruptcy, lose access to their credit cards, and survive off unemployment insurance and other transfers. Alternatively, they can not file, keep their credit cards, and use these to supplement their income until the economy improves and they find a job.

In addition to a fraction of households filing, the model also features output, consumption, investment, and an aggregate labor supply, so I report the cyclical properties of these US data in Table A.2. Consumption, investment, and hours worked are all strongly pro-cyclical. Investment and the fraction of households filing are both much more volatile than output while consumption and hours worked are somewhat less so.

<table>
<thead>
<tr>
<th>Variable (x)</th>
<th>Stddev x in (%)</th>
<th>Stddev x/Stddev y</th>
<th>Correlation of lagged x with y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output (y)</td>
<td>2.44</td>
<td>1.00</td>
<td>0.01 0.58 1.00 0.57 -0.01</td>
</tr>
<tr>
<td>Consumption</td>
<td>1.69</td>
<td>0.69</td>
<td>0.02 0.59 0.92 0.59 0.10</td>
</tr>
<tr>
<td>Investment</td>
<td>7.19</td>
<td>2.94</td>
<td>0.04 0.53 0.89 0.38 -0.26</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>1.60</td>
<td>0.66</td>
<td>-0.23 0.26 0.82 0.69 0.09</td>
</tr>
<tr>
<td>Defaulting Pop</td>
<td>10.25</td>
<td>4.20</td>
<td>0.17 0.00 -0.03 0.26 0.56</td>
</tr>
</tbody>
</table>


I follow Ohanian and Raffo (2011) and take annual hours worked from the Conference Board’s Total Economy Database (TED) available at [http://www.conference-board.org/data/economydatabase/files](http://www.conference-board.org/data/economydatabase/files). Output, consumption, and investment are taken from NIPA as real gross domestic product, real personal consumption expenditures, and real gross private domestic investment. The number of households (used to compute the fraction of households defaulting) is taken from the Census Bureau’s historical tables available at [http://www.census.gov/population/www/socdemo/hh-fam.html#ht](http://www.census.gov/population/www/socdemo/hh-fam.html#ht) (Table HH-1). The number of filings is taken from the Department of Justice website as already described. All variables have been logged and detrended using the HP filter with parameter 100.
Calibration

Table A.3 gives the profiles of all variables omitted in the main text except (for space considerations) $\rho_s$.

### A.3 Computation

The steady state model is computed using a collection of standard techniques. The efficiency process is discretized using the method of Tauchen (1986). The persistent shock is discretized with 11 points with a coverage of $\pm 5$ “average unconditional standard deviations in recessions,” $\bar{\sigma}_{\eta,b}/\sqrt{1-\bar{\rho}^2}$. The bar denotes the numerical average across ages. The transitory shock is discretized with 3 points with a coverage of $\pm 3$ “average” standard deviation, $\bar{\sigma}_\varepsilon$. The “super-rich” efficiency process is discretized with only 3 points linearly spaced. The mass on each point is computed using, again, the method of Tauchen (1986) (e.g. the mass on the low point is the cdf evaluated at the midpoint of the lowest two points). The total number of efficiency states for each age is 36. The asset grid is comprised of 450 points, 70 strictly negative. These are very unevenly spaced between $-5$ and 4000 and concentrated close to zero. The household problem is solved using backward induction with basic grid search assuming the asset policy conditional on repaying or being in bad standing is nearly monotonic (specifically I check 40 grid points below the previous asset choice in addition to all above). Cash-in-hand is not used, and there is no interpolation.

The business cycle model is computed using grid search, backward induction, and the method of Krusell and Smith (1998) (the KS method). The income process is discretized in exactly the same way and the asset grid is the same. The “moments” chosen for the KS method are aggregate wealth $A = (1+r-\delta)K$, aggregate labor $N$, and an equity premium $W$. The equity premium $W$ is defined on the domain $[0,1]$ and controls the risk-free bond price via

$$ \frac{1}{q_B(S)} = \frac{WF(b|z)}{WF(b|z) + (1-W)F(g|z)\hat{R}(S)} + \frac{(1-W)F(g|z)}{WF(b|z) + (1-W)F(g|z)\hat{R}(G)} $$

6Specifically, letting \texttt{linspace}(a,b) denote linear spacing between $a$ and $b$, the $[-5,0]$ has 71 points spaced according to $-\exp(\texttt{linspace}(\log(\kappa),\log(5+\kappa)))-\kappa$ with $\kappa = .3$; the $[0,100]$ range has 381 points spaced according to $\exp(\texttt{linspace}(\log(\kappa),\log(100+\kappa)))-\kappa$ with $\kappa = .5$; and the $[100,4000]$ range has 151 points spaced according to $\exp(\texttt{linspace}(\log(100+\kappa),\log(4000+\kappa)))-\kappa$ with $\kappa = -50$. 

119
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<th>$\theta_s$</th>
<th>$\phi_h$</th>
<th>$\gamma_h$</th>
<th>$\sigma^2_{\eta,h}$</th>
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<td>.990</td>
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<td>.072</td>
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<td>.801</td>
<td>1.050</td>
<td>.011</td>
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<td>1.44</td>
<td>.801</td>
<td>1.000</td>
<td>.017</td>
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</tr>
<tr>
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<td>.927</td>
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<td>.788</td>
<td>1.048</td>
<td>.027</td>
<td>.080</td>
</tr>
</tbody>
</table>

$^*$ρ_s: the conditional probability of survival.

θ_s: adult-equivalent household size.

φ_h: deterministic earnings profile.

γ_h: income shock persistence.

σ^2_{η,h,1}: persistent shock variance.

σ^2_{ε,h}: transitory shock variance.

Table A.3: Parameter Values for Profiles
where $\hat{R}$ is the forecasted gross return to capital (i.e. $1 + r - \delta$). The probabilities ensure the equilibrium value of $W$ is always close to, but slightly above, 0.5. The laws of motion take the following form:

$$
\begin{bmatrix}
K' \\
N' \\
W'
\end{bmatrix} =
\begin{bmatrix}
\Gamma_{11,z} & \Gamma_{12,z} & \Gamma_{13,z} & 0 \\
\Gamma_{21,z'} & \Gamma_{22,z'} & 0 & 0 \\
\Gamma_{31,zz'} & \Gamma_{32,zz'} & \Gamma_{33,zz'} & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
N' \\
A \\
W
\end{bmatrix}
$$

(A.55)

with $A'$ calculated off $K', N'$ and $z'$ using $A' = K'(1 + z'\alpha(K'/N')^{\alpha-1} - \delta)$. Here a subscript of $z$ means the coefficients are $z$-dependent and similarly for $zz'$. Importantly, $W$ does not influence the one-step ahead forecast. This allows the risk-free bond price to be varied holding fixed all other prices.

Because the computational burden is extremely heavy, especially in terms of memory usage, the number of points used for the aggregate moments is kept to a minimum: 3 in the $A$ direction linearly-spaced within $\pm 10\%$ of the steady-state value, 2 in the $N$ direction placed at 0.963 and 1.048, and 2 in the $W$ direction placed at 0.505 and 0.54. The coverage for $N$ is the minimal possible without resorting to extrapolation. Because the natural borrowing limit and welfare depend on the worst possible scenario that can occur, and on how quickly it can be reached, the bounds and the number of gridpoints used influence the results. Somewhat surprisingly, this effect appears to be limited (see Table A.5). At any rate, the bounds I’ve chosen I believe are reasonable especially given past US history where the capital-output ratio declined by some 20% in the Great Depression and even since 1960 hours worked per household has seen declines of 4% (relative to trend).\footnote{Author’s calculations. Hours worked per household are calculated as discussed in Appendix A.2. The capital-output ratio is constructed from NIPA data using the “current cost net stock of fixed assets” divided by nominal GDP.}

The model is simulated non-stochastically as in Young (2010). The economy is simulated for 600 periods and the first 200 periods are discarded. At each point in the simulation, the bond market must be cleared. Because the equity premium is included in the aggregate state space, this is a simple matter of linearly interpolating the asset policies to find a $W$ such that (1.30) holds.
The approximate laws of motion are accurate. This is seen in Table A.4 which records the maximum percent errors and the $R^2$ from 1-step and 50-step forecasts. 50-steps out, the errors do tend to accumulate with the maximum error getting close to .4% for some variables, but overall the fit is good.

<table>
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<tr>
<th>Stat \ Variable</th>
<th>Default $A'$</th>
<th>$N'$</th>
<th>$W'$</th>
<th>No Default $A'$</th>
<th>$N'$</th>
<th>$W'$</th>
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</thead>
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<td>1-step $R^2$</td>
<td>.9996</td>
<td>.9995</td>
<td>.9973</td>
<td>.9997</td>
<td>.9995</td>
<td>.9974</td>
</tr>
<tr>
<td>Max % Error</td>
<td>.098</td>
<td>.127</td>
<td>.026</td>
<td>.097</td>
<td>.127</td>
<td>.023</td>
</tr>
<tr>
<td>50-step $R^2$</td>
<td>.9971</td>
<td>.9951</td>
<td>.9965</td>
<td>.9988</td>
<td>.9951</td>
<td>.9967</td>
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<tr>
<td>Max % Error</td>
<td>.352</td>
<td>.370</td>
<td>.028</td>
<td>.164</td>
<td>.370</td>
<td>.031</td>
</tr>
</tbody>
</table>

Table A.4: Law of Motion Forecasting Accuracy

### A.4 Robustness

This section conducts robustness tests. The focus is on the consequences of eliminating default rather than restricting it.

**Components of the Business Cycle**

One of the key results is that business cycles reduce the welfare gain of eliminating default from a large amount 1.82% to a smaller amount .49%. However, business cycles in the model have three components: a TFP shock $z$, an aggregate labor supply shifter (LSS) $\psi_z$, and counter-cyclical earnings variance (CEV) of the persistent earnings shock $\sigma_{\eta,h,z}$. I now examine how each of these contribute to the total reduction in the welfare gain of eliminating default.

I proceed by starting with all three components and eliminating them successively until the model is equivalent to the steady state one. I try to hold fixed the effects of aggregate moment grids on the results by eliminating these in separate steps. The aggregate moments are $A, W$, and $N$ where $A$ is financial wealth, $W$ is an equity premium, and $N$ is aggregate labor supply; see Appendix A.3 for more on these. I eliminate elements in the following
order: CEV, LSS, the \( N \) grid, TFP, the \( W \) grid, and lastly the \( A \) grid. The last step results in the steady state model.\(^8\)

Table A.5 reports the welfare gains of eliminating default after each of these steps. The most important business cycle components are CEV and the TFP shock. Each of these reduces the gain of eliminating default by around .55%. CEV likely matters because default is the only means of smoothing consumption intratemporally. The TFP shock likely matters because it drives capital accumulation or decumulation and consequently the risk due to fluctuating prices.

<table>
<thead>
<tr>
<th>Eliminating cumulatively from left to right</th>
<th>Welfare Gain (%)</th>
<th>Population in Favor (%)</th>
</tr>
</thead>
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<td>Nothing</td>
<td>.49</td>
<td>57.0</td>
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<tr>
<td>CEV</td>
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<td>58.5</td>
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<tr>
<td>LSS</td>
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<td>58.4</td>
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<td>( N ) Grid</td>
<td>1.03</td>
<td>58.4</td>
</tr>
<tr>
<td>TFP</td>
<td>1.70</td>
<td>57.8</td>
</tr>
<tr>
<td>( W ) Grid</td>
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<td>57.8</td>
</tr>
<tr>
<td>( A ) Grid</td>
<td>1.82</td>
<td>56.8</td>
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</table>

Table A.5: Contribution of Business Cycle Components to Welfare Results

It is worth noting what doesn’t affect the welfare results much. The LSS matters very little and actually makes eliminating default look better by .04%. This is likely because the small earnings swings (±2.5%) have the same persistence as the business cycle and so are very easy to insure through borrowing and saving. What is really surprising is how little the grids matter. Apparently it is not just a positive probability of a deep recession occurring (which is what linear interpolation on a coarse grid for \( A \) that matters but rather how likely it is.\(^9\) This result is similar to when default is not eliminated but just made very costly. There the natural borrowing limit is not in effect but aggregate risk reduces the benefit of moving to a costly-default regime.

\(^8\)So that the model can be computed at every step, the grid coverage cannot be totally eliminated. In the third step the \( N \) grid is changed from \([.963, 1.048]\) to \([1 - 10^{-8}, 1 + 10^{-8}]\). In the fourth step the \( z \) values are changed from \([1 \pm .0224]\) to \([1 \pm 10^{-10}]\) and the \( W \) grid is expanded from \([.505, .54]\) to \([.497, .54]\). The \( W \) grid is expanded because the equilibrium values of \( W \) are now below .505. In the fifth step the \( W \) grid is shrunk to \([.497, .503]\). For the last step I just report results from the steady state model (where \( z \) is always 1, \( N \) is always 1, and \( W \) is always .5). As measured by \( R^2 \), the forecasts become successively worse (because the variation in the actual series goes to zero). Because of this I iterate until the coefficients don’t change much and then stop the computation regardless of \( R^2 \) values.

\(^9\)The \( A \) grid is three linearly-spaced points \([.9A_{SS}, A_{SS}, 1.1A_{SS}]\) where \( A_{SS} \) is the steady state value. This grid with linear interpolation means that when \( A \) is slightly below its steady state value, a weight of almost 1 is placed on \( A_{SS} \) but a small weight is placed on \(.9A_{SS} \). This can be interpreted as a small chance of moving to a deep recession. Apparently it is not so much whether this weight is positive but rather how positive it is.
Guaranteed Earnings Prior to Retirement

It was argued that using a low minimum value for earnings during working life is reasonable. I now explore the robustness of the results to guaranteed earnings prior to retirement.

To provide guaranteed earnings, I do the following. Given the efficiency distribution of the benchmark economy, I replace any values less than a threshold $\tau$ with $\tau$ and re-normalize so that $N = 1$ in steady state. I consider 4 different thresholds $\tau \in \{.008, .058, .127, .233\}$ (the benchmark economy is any $\tau \leq .0043$). These represent a lower bound of roughly $500, $3500, $7,600, and $14,000 taking average household labor income to be $60,000 and are the values considered in Athreya (2008).

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<th>0.058</th>
<th>0.127</th>
<th>0.233</th>
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<td>43.8</td>
<td>48.4</td>
<td>41.9</td>
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</table>

**CD**

- K/Y SS | 3.08 | 3.08 | 3.08 | 3.06 |
- K/Y BC | 3.08 | 3.08 | 3.08 | 3.06 |
- Debt/Y SS | 0.69 | 0.70 | 0.73 | 0.88 |
- Debt/Y BC | 0.53 | 0.53 | 0.53 | 0.70 |
- Population in Debt SS | 10.47 | 10.48 | 12.61 | 13.40 |
- Population in Debt BC | 9.01 | 9.04 | 9.27 | 12.28 |
- Population Filing SS | 0.18 | 0.18 | 0.18 | 0.19 |
- Population Filing BC | 0.26 | 0.25 | 0.30 | 0.27 |
- Worst $R^2$ 1-step Ahead | .9973 | .9973 | .9966 | .9971 |
- Worst $R^2$ 50-step Ahead | .9951 | .9951 | .9910 | .9844 |

**ND**

- K/Y SS | 3.00 | 2.86 | 2.84 | 2.69 |
- K/Y BC | 3.03 | 2.95 | 2.89 | 2.78 |
- Debt/Y SS | 4.75 | 16.27 | 18.77 | 33.99 |
- Debt/Y BC | 2.89 | 9.86 | 13.80 | 25.09 |
- Population in Debt SS | 21.40 | 33.53 | 35.11 | 41.28 |
- Population in Debt BC | 17.85 | 27.38 | 31.23 | 37.53 |
- Worst $R^2$ 1-step Ahead | .9974 | .9962 | .9921 | .9956 |
- Worst $R^2$ 50-step Ahead | .9951 | .2121 | .7478 | .2639 |

Table A.6: Robustness to Guaranteed Income

Table A.6 reports the results. In addition to the usual statistics, I report the worst $R^2$ values from the forecasts. The most glaring observation is that the gains from eliminating
default are very large for $\tau \geq .058$. While the gains are high, in each case considered aggregate risk again reduces the benefit of eliminating default. This is especially true for $\tau = .058$ where the gain goes from 8.2% to 5.4%. While one might expect the results to be monotonic, they are not. In particular, for $\tau = .127$ the drop is only .7%. However, the forecast errors in the ND economies are high for $\tau \geq .058$ and so the results for high values of $\tau$ are likely inaccurate. Overall, these results suggest that, to the extent earnings are guaranteed in working life, it would be substantially welfare improving to eliminate default. However, the gain would be overstated if aggregate risk was not accounted for.

**Retirement Schemes**

In the benchmark calibration, labor income in retirement is comprised of a guaranteed fraction $\kappa_G = .15$ of average earnings and a fraction $\kappa_F = .35$ of earnings from the last period of working life. The average replacement rate is roughly 50% because $\kappa_G + \kappa_F = .5$. However, as already discussed, it is not the replacement rate that really matters but how much of it is guaranteed. I now examine the robustness of the results to alternative replacement schemes $(\kappa_G, \kappa_F)$ subject to keeping $\kappa_G + \kappa_F = .5$.

Table A.7 records the results. What is immediately striking is that when no earnings are guaranteed, eliminating default is a substantial loss, -2.5%, that slightly lessens after including aggregate risk. For the other schemes, the typical pattern of aggregate risk reducing the welfare benefit of eliminating default is observed.

**Profiles**

Many of the parameters in the model are age-dependent. In particular, there are profiles for survival probabilities $\rho_s$, adult-equivalent household size $\theta_s$, deterministic earnings $\phi_h$, income persistence $\gamma_h$, persistent shock variance $\sigma_{\eta,h,z}^2$, and transitory shock variance $\sigma_{\xi,h}^2$. While these bring the model closer in line with the data, they also make interpretation of the results more difficult. I examine the consequences of these profiles by separately setting $\rho_s, \theta_s, \phi_h$ to 1 and also by setting $\gamma_h, \sigma_{\eta,h,z}^2, \sigma_{\xi,h}^2$ to the age-independent estimates.
Table A.7: Robustness of Results to Alternative Retirement Schemes

in Storesletten et al. (2004) (STY).\textsuperscript{10}

Table A.8 reports the results. In each of these tests, aggregate risk reduces the benefit of eliminating default. For the STY estimates the gain is higher in steady state and drops slightly more than in the benchmark. The $\phi_h = 1$ case is worth pointing out because it is indicative of what results from an infinite-horizon economy might look like: because households don’t have life-cycle reasons to borrow, credit and default are less important. Overall, the results look about the same both in terms of welfare and allocations.

\textsuperscript{10}For completeness, these are $\gamma_h = .952, \sigma_{\eta,h,g} = .125, \sigma_{\eta,h,b} = .211, \sigma_{\epsilon,h} = .255$ for all $h$. As in the main text, I set $\sigma_{\eta,h,1} = .5\sigma_{\eta,h,g} + .5\sigma_{\eta,h,b} = .168$. 

Flexible Portfolios

In the calibrated model, most households are restricted to use portfolios that just replicate a risk-free bond, $a'_g = a'_b$. I now relax this assumption, and, in particular, allow every household to choose any portfolio, i.e. to choose $a'_g$ independently of $a'_b$.

Table A.9 reports some results. Under flexible portfolios, the welfare gain of eliminating default falls from 1.82% in steady state to .88% in the business cycle. This shows the result that aggregate risk reduces the welfare gain of eliminating default is robust to the assumption on portfolio restrictions. However, how aggregate risk affects each economy is different. One way this is seen is in how aggregates change. In particular, the capital-output ratio now falls for the CD economy and stays the same for the ND economy. Additionally, while there is a contraction in debt use, the contraction is smaller than under the restricted portfolio assumption.
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<th>Flexible</th>
<th>Benchmark</th>
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<td>1.82</td>
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<tr>
<td>Welfare Gain of BC (%)</td>
<td>0.88</td>
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<td>Population in Favor SS (%)</td>
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<td>56.8</td>
</tr>
<tr>
<td>Population in Favor BC (%)</td>
<td>58.9</td>
<td>57.0</td>
</tr>
<tr>
<td><strong>CD</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K/Y SS</td>
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<td>3.08</td>
</tr>
<tr>
<td>K/Y BC</td>
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<td>3.08</td>
</tr>
<tr>
<td>Debt/Y SS</td>
<td>0.69</td>
<td>0.69</td>
</tr>
<tr>
<td>Debt/Y BC</td>
<td>0.58</td>
<td>0.53</td>
</tr>
<tr>
<td>Population in Debt SS</td>
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<td>10.47</td>
</tr>
<tr>
<td>Population in Debt BC</td>
<td>9.97</td>
<td>9.01</td>
</tr>
<tr>
<td>Population Filing SS</td>
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<tr>
<td>Population Filing BC</td>
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<td>0.26</td>
</tr>
<tr>
<td>Worst $R^2$ 1-step Ahead</td>
<td>.9970</td>
<td>.9973</td>
</tr>
<tr>
<td>Worst $R^2$ 50-step Ahead</td>
<td>.9890</td>
<td>.9951</td>
</tr>
<tr>
<td><strong>ND</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K/Y SS</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>K/Y BC</td>
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<td>Debt/Y SS</td>
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<td>Debt/Y BC</td>
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<td>Population in Debt SS</td>
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<td>Population in Debt BC</td>
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<td>16.53</td>
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<td>.9974</td>
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<tr>
<td>Worst $R^2$ 50-step Ahead</td>
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Table A.9: Robustness of Results to Flexible Portfolios
Appendix B

Dealing with Consumer Default: Bankruptcy vs Garnishment

B.1 Extended Model Description

Let \( A \subset R \) denote the set of asset positions. In the theory as well as in the computation, \( A \) is taken to be a finite set, which includes 0 and positive and negative elements. \( A^- \) is the set of strictly negative elements and \( A^+ \) the set of non-negative elements. Denote the sequence of current and future factor prices by \( p = \{w_{t+j}, r_{t+j}\}_{j=0}^{\infty} \). Let \( w(p) \) and \( r(p) \) denote the current wage and rental rates, namely, \( w_t \) and \( r_t \). Let \( B \) denote the shift operator \( B(p) = \{w'_{t+j}, r'_{t+j}\}_{j=0}^{\infty} \), where \( w'_{t+j} = w_{t+1+j}, r'_{j} = r_{t+1+j}, j = 0, 1, 2, \ldots, \infty. \)

In addition to the individual states \( a, e, \) and \( s \), there is a state variable \( h \in \{0, 1, 2\} \) that takes on the value 1 if the person has a record of a past bankruptcy filing; the value 2 if the person is currently under garnishment (i.e., has some unpaid debt obligation) or if the person has satisfied the garnishment but the record of the garnishment has not yet been removed from the person’s credit history; the value 0 if there is no record of any filing or garnishment in the person’s credit history. We will denote the optimal lifetime utility of a person in state \( (a, e, s, h) \) by \( V(a, e, s, h; p) \). We index the value functions (and decision rules) by \( p \) to emphasize the fact that the value of these objects depends on the current and future trajectory of factor prices.
Decision Problem of People

A person can be in one of five situations:

1. $a < 0$ and $h = 0$. In this case, the person has three options: he can repay his debt; he can default on his debt and subject himself to garnishment; or he can file for bankruptcy. The value from repayment, which we denote as $V^R(a, e, s, h = 0; p)$, is given by the following dynamic program:

$$V^R(a < 0, e, s, h = 0; p) = \max_{a' \in A, n, c \geq 0, \gamma \in [0,1]} u(c, n, e) + \beta \rho E(s', e') V(a', e', s', h = 0; p')$$

s.t.

$$c = w(p)en + a - I\{a' < 0\}q(a', s; p)a' - I\{a' \geq 0\} \bar{q}(p)a'$$

$$p' = B(p),$$

where $I\{\cdot\}$ denotes an indicator function that takes on the value 1 if the expression in $\{\cdot\}$ is true and 0 otherwise.

The value from default and garnishment, which we denote by $V^G(a, e, s, h; p)$, is given by the following dynamic program

$$V^G(a < 0, e, s, h = 0; p) = \max_{a' \in A, n, c \geq 0, \gamma \in [0,1]} u(c, n, e) + \beta \rho E(s', e') [I\{a' < 0\} V(a', e', s', h = 2; p') + I\{a' \geq 0\} V(a', e', s', h = 2; p')]$$

s.t.

$$a' - a \geq \min\{\max\{0, \gamma(w(p)en - c_{\min}\}\}, -a\}$$

$$c = w(p)en - [a' - a]I\{a' < 0\} - I\{a' \geq 0\}[\bar{q}(p)a' - a]$$

$$p' = B(p).$$

The value from bankruptcy, which we denote by $V^B(a, e, s, h; p)$, is given by the
following dynamic program:

\[ V^B(a < 0, e, s, h = 0; p) = \max_{a' \in A, n \in [0,1], c \geq 0} u(c, n, e) + \beta \rho E(s', e'|s)V(0, e', s', h = 1; p') \]

s.t.

\[ c = w(p)en \]
\[ p' = B(p). \]

The person chooses the best of these options. Therefore, in this case,

\[ V(a < 0, e, s, h = 0; p) = \max\{V^R(a < 0, e, s, h = 0; p), V^G(a < 0, e, s, h = 0; p), V^B(a < 0, e, s, h = 0; p)\} \]

2. \( a < 0 \) and \( h = 2 \). This is the case where the person defaulted in some previous period, hasn’t paid off his obligations and is under garnishment. This person can exit garnishment by filing for bankruptcy or he can continue on in garnishment. The value from doing the former is given by

\[ V^B(a < 0, e, s, h = 2; p) = \max_{a' \in A, n \in [0,1], c \geq 0} u(c, n, e) + \beta \rho E(s', e'|s)V(0, e', s', h = 1; p') \]

s.t.

\[ c = (1 - \chi_g)w(p)en \]
\[ p' = B(p). \]

The value of doing the latter is given by:

\[ V^G(a < 0, e, s, h = 2; p) = \max_{a' \in A, n \in [0,1], c \geq 0} u(c, n, e) + \beta \rho E(s', e'|s)[I_{\{a' < 0\}}V^G(a', e', s', h = 2; p') + I_{\{a' \geq 0\}}V(a', e', s', h = 2; p')] \]

s.t.

\[ a' - a \geq \min\{\max\{0, \gamma(w(p)en - c_{\text{min}})\}, -a\} \]
\[ c = [1 - \chi_g]w(p)en - [a' - a]I_{\{a' < 0\}} - I_{\{a' \geq 0\}}[\bar{\rho}(p)a' - a] \]
\[ p' = B(p). \]

The person chooses the best option, so, in this case

\[ V(a < 0, e, s, h = 2; p) = \max\{V^G(a < 0, e, s, h = 2; p), V^B(a < 0, e, s, h = 2; p)\}. \]
3. \(a \geq 0\) and \(h = 2\). This is the case where the person was in garnishment in the past but the garnishment flag has not yet been removed.

\[
V(a \geq 0, e, s, h = 2; p) = \\
\max_{a' \in A^+, n \in [0,1], c \geq 0} u(c, n, e) + \beta p E(s', e') a'[\lambda_g(1 - \lambda_g) V(a', e', s', h = 2; p') + \lambda_g V(a', e', s', h = 0; p')] \\
\text{s.t.} \\
c = [1 - \chi_g] w(p) e n - \bar{q}(p) a' + a \\
p' = B(p).
\]

4. \(a \geq 0\) and \(h = 1\). This is the case where the person filed for bankruptcy in the past and the bankruptcy flag has not yet been removed.

\[
V(a \geq 0, e, s, h = 1; p) = \\
\max_{a' \in A^+, n \in [0,1], c \geq 0} u(c, n, e) + \beta p E(s', e') a'[\lambda_b(1 - \lambda_b) V(a', e', s', h = 1; p') + \lambda_b V(a', e', s', h = 0; p')] \\
\text{s.t.} \\
c = [1 - \chi_b] w(p) e n - \bar{q}(p) a' + a \\
p' = B(p).
\]

5. \(a \geq 0\), \(h = 0\). In this case, the person has no outstanding obligations and no record of past default.

\[
V(a \geq 0, e, s, h = 0; p) = \\
\max_{a' \in A, n \in [0,1], c \geq 0} u(c, n, e) + \beta p E(s', e') a' V(a', e', s', h = 0; p') \\
\text{s.t.} \\
c = w(p) e n + a - I_{\{a' < 0\}} q(a', s; p) a' - I_{\{a' \geq 0\}} \bar{q}(p) a' \\
p' = B(p).
\]

The solution to this individual problem implies policies \(a'(a, e, s, h; p)\), \(n(a, e, s, h; p)\), and \(c(a, e, s, h; p)\). In addition, for \(a < 0\) and \(h \in \{0, 2\}\) the solution also implies a bankruptcy decision rule \(b(a, e, s, h; p)\) which takes the value of 1 if the person files for
bankruptcy and zero otherwise, and a garnishment decision rule \( g(a, e, s, h; p) \) that takes the value 1 if the person defaults (or continues in default) and subjects himself to garnishment, and 0 otherwise.

**Decision Problem of the Representative Firm**

The representative firm’s optimization problem is static: it rents \( K \) and hires \( N \) each period to maximize current period profits. This decision problem is the same regardless of the legal regime in place.

\[
\max_{K \geq 0, N \geq 0} F(K, N) - w(p)N - r(p)K
\]

**Decision Problem of the Intermediary Sector**

As noted earlier, intermediaries are the counterparty to every borrowing and lending contract entered into by consumers and are also the owners of capital stock in this economy. We will assume that there is one representative intermediary who takes prices as given and chooses (i) how much capital to purchase in the current period (for use in the following period), (ii) how many new borrowing or lending contracts of different types to enter into with consumers, and (iii) if there is a market for defaulted debt, how much of each different types of defaulted debt to purchase.

The net return from the first activity (purchase of capital) is:

\[
-K' + \frac{(1 - \delta) + r(B(p))}{1 + i(p)} K'
\]  

(B.1)

Let \( m'(y, s, p) \) be the measure of newly issued contracts of type \((y, s, p)\) held by the intermediary sector at the end of the current period. Let \( \theta(y, s, p) \) be the fraction of borrowers who default (i.e., either file for bankruptcy or enter into garnishment) conditional on taking out a (new) loan of size \( y \) and having current persistent efficiency level \( s \). And, conditional on there being a default, let \( q^D(y, s, p) \) be the expected (per unit) value of the defaulted debt. We will refer to this as the *expected recovery rate*. Then, the net return from the
second activity (purchase of newly issued loans or deposits) is:

\[
\sum_{y \in A^+, e} m'(y, s, p) y \left[ \tilde{q}(p) - \frac{\rho}{1 + i(p)} \right] + \sum_{y \in A^-, e} m'(y, s, p) y \times \left( -q(y, s, p) + \frac{\rho}{1 + i(p)} [(1 - \theta(y, s; p)) + \theta(y, s; p) q^D(y, s, p)] \right). \\
\text{(B.2)}
\]

Let \( z'(y, s, p) \) be the measure of defaulted debt of type \( y < 0, s, p \) held by the intermediary sector. Let \( \eta(\tilde{a}, y, s'; p) \) be the fraction of delinquent debtors who do not file for bankruptcy and choose \( y \leq \tilde{a} \) next period, given an unpaid debt of \( y \), a persistent efficiency level of \( s' \) and next period’s aggregate state \( B(p) \). Then, the net return from the third activity is:

\[
\sum_{y \in A^-} z'(y, s, p) y \times \left( -x(y, s, p) + \frac{\rho}{1 + i(p)} \sum_{s'} \sum_{\tilde{a} \in A} \eta(\tilde{a}, y, s'; B(p)) \left[ (\min\{(\tilde{a} - y), -y\})/(-y) + I_{\tilde{a} < 0} x(\tilde{a}, s', B(p)) \right] \pi_{s'|s} \right). \\
\text{(B.3)}
\]

Let \( M' \) denote the distribution of newly issued contracts held by the intermediary sector and \( Z' \) the distribution of defaulted debt held by the intermediary sector.

The decision problem is to choose \( K', M', Z' \) to maximize the sum of (B.1), (B.2) and (B.3).

**Equilibrium**

We focus on perfect foresight equilibrium. Let \( \mu(a, e, s, h) \) denote the distribution of people over the individual states. Let \( \Gamma(\mu, p) \) describe the law of motion of this distribution. That is, \( \mu'(a, e, s, h) = \Gamma(\mu(a, e, s, h), p) \) is the distribution of people of individual states next period, given the current distribution \( \mu \) and current and future sequence of factor prices \( p \). The time evolution of the distribution is then given by the recursion \( \mu^{t+1}(a, e, s, h) = \Gamma(\mu^t, B^t(p)) \), where it is understood that \( \mu^0 = \mu \), \( B^0(p) = p \) and \( B^t(p) \) is defined by the recursion \( B^t(p) = B(B^{t-1}(p)) \).

A perfect foresight competitive equilibrium for an initial distribution of people over individual states \( \mu(a, e, s, h) \) and an initial aggregate capital \( K \) is (i) a sequence of current and future factor prices \( p \), (ii) a set of credit market prices \( q(a < 0, s, p), \tilde{q}(p) \), \( x(y < 0, s, p) \) and \( i(p) \), (iii) a set of individual decision rules \( a'(a, e, s, h; p), n(a, e, s, h; p) \),
c(a, e, s, h; p), b(a, e, s, h; p) and g(a, e, s, h; p), (iv) a set of production sector decision rules \( K(p) \) and \( N(p) \), (v) a set of intermediary sector decision rules \( K'(p) \), \( m'(y,s,p) \) and \( z'(y,s,p) \) and (vi) a law of motion \( \Gamma(\mu,p) \) such that:

1. The decision rules solve the individual dynamic optimization problem, given \( p \) and credit market prices \( q(y,s,p) \) and \( \bar{q}(p) \).

2. \( K(p) \) and \( N(p) \) solve the production sector static optimization problem.

3. \( K'(p) \), \( m'(y,s,p) \) and \( z'(y,s,p) \) solve the intermediary optimization problem.

4. The goods market clears:

\[
F(K(p), N(p)) = K'(p) - (1 - \delta)K + \sum_{a,e,s,h} [c(a, e, s, h; p) + (I\{h=1\} + I\{h=2\})w(p)en(a, e, s, h; p)] \mu(a, e, s, h).
\]

(B.4)

5. The labor market clears:

\[
N(p) = \sum_{a,e,s,h} en(a, e, s, h; p)\mu(a, e, s, h).
\]

6. The market for physical capital clears:

\[
K(p) = K.
\]

7. The credit market for newly issued debt clears:

\[
\forall (y,s) \in (A \times S) \quad m'(y,s,p) = \sum_{a,e} I\{a'(a,e,s,h=0;p)=y\}\mu(a,e,s,h=0).
\]

(B.5)

8. The market for defaulted debt clears:

\[
\forall (y,s) \in (A^{-} \times S) \quad z'(y,s,p) = \sum_{h \in \{0,2\}} \sum_{a,e} I\{g(a,e,s,h;p)=1 \text{ and } a'(a,e,s,h;p)=y\}\mu(a,e,s,h).
\]

(B.6)
9. The probabilities \( \theta(y, s, p) \) and \( \eta(a', a, e, s; p) \) and the price \( q^D(y, s, p) \) that appear in the intermediary sector's decision problem must be consistent with individual decision rules and market prices. This requires:

\[
\theta(y, s, p) = \sum_{s', e'} [b(y, e', s', h = 0; B(p)) + g(y, e', s', h = 0; B(p))] \phi_{s'}(e') \pi_{s'|s}, \tag{B.7}
\]

\[
\eta(\tilde{a}, y, s'; B(p)) = \sum_{e'} I_{\{b(y, e', s', h = 0; B(p)) = 0 \text{ and } a'(y, e', s', h = 0; B(p)) = \tilde{a}\}} \phi_{s'}(e'), \tag{B.8}
\]

\[
q^D(y, s, p) = E_{(e', s')} \left[ \left\{ \frac{g(y, e', s', h = 0; B(p))}{\theta(y, s, p)} \right\} \times \left[ \min\{a'(y, e', s', h = 0; B(p)) - y, -y\} + x(a'(y, e', s', h = 0; B(p)), s', B(p)) \right] \right] \tag{B.9}
\]

In the expression for \( q^D \), if for some \( y \) \( g(y, e', s', h = 0; B(p)) = 0 \) and \( b(y, e', s', h = 0; B(p)) = 0 \) for all \((e', s')\), the expectation term on the right-hand side becomes indeterminate (i.e., evaluates to \(0/0\)). In this case, the recovery rate in default is irrelevant and we set the expectation to 1.

10. The law of motion for \( \mu \) is consistent with individual decision rules. This requirement is easiest to describe in three parts.

\[
\Gamma(\mu, s)(\tilde{a}, \tilde{e}, \tilde{s}, \tilde{h} = 0) = (1 - \rho)G(\tilde{a}, \tilde{e}, \tilde{s})
\]

\[
+ \rho \sum_{a < 0, e, s} I_{\{a'(a, e, s, h = 0; p) = \tilde{a}, b(a, e, s, h = 0; p) = 0, g(a, e, s, h = 0; p) = 0\}} \mu(a, e, s, h = 0) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e})
\]

\[
+ \rho \sum_{a < 0, e, s} I_{\{a'(a, e, s, h = 1; p) = \tilde{a}\}} \lambda_b \mu(a, e, s, h = 1) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e})
\]

\[
+ \rho \sum_{a > 0, e, s} I_{\{a'(a, e, s, h = 2; p) = \tilde{a}\}} \lambda_g \mu(a, e, s, h = 2) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e}), \tag{B.10}
\]

\[
\Gamma(\mu, s)(\tilde{a}, \tilde{e}, \tilde{s}, \tilde{h} = 1) =
\]

\[
\rho \sum_{a < 0, e, s} I_{\{a'(a, e, s, h = 1; p) = \tilde{a} \text{ and } b(a, e, s, h = 1; p) = 1\}} \mu(a, e, s, h = 0) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e})
\]

\[
+ \rho \sum_{a < 0, e, s} I_{\{a'(a, e, s, h = 2; p) = \tilde{a} \text{ and } b(a, e, s, h = 0; p) = 1\}} \mu(a, e, s, h = 2) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e})
\]

\[
+ \rho \sum_{a > 0, e, s} I_{\{a'(a, e, s, h = 1; p) = \tilde{a}\}} (1 - \lambda_b) \mu(a, e, s, h = 1) \pi_{\tilde{s}|s} \phi_{\tilde{s}}(\tilde{e}), \tag{B.11}
\]

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\Gamma(\mu, s)(\tilde{a}, \tilde{e}, \tilde{s}, \tilde{h} = 2) = \\
\rho \sum_{a<0,e,s} I\{a'(a,e,s,h=1;p)=\tilde{a} \text{ and } g(a,e,s,h=1;p)=1\} \mu(a, e, s, h = 0) \pi_{\tilde{a}|s} \phi(\tilde{e}) \\
+ \rho \sum_{a\geq0,e,s} I\{a'(a,e,s,h=1;p)=\tilde{a}\} (1 - \lambda_g) \mu(a, e, s, h = 2) \pi_{\tilde{a}|s} \phi(\tilde{e}), \quad (B.12)

where \(G\) is the distribution over \((a, e, s)\) from which newborns are drawn (all newborns start with \(h = 0\)).

11. There is perfect foresight. That is, the sequence \(p\) is implied by the evolution of the economy starting from \(K\) and \(\mu(a, e, s, h)\). Formally, this requires that for any \(t \geq 1\) conditions (1)-(9) are satisfied for \(K = K^{t-1}(p)\) and \(\mu(a, e, s, h) = \mu^t(a, e, s, h)\), where \(\mu^t\) satisfies the recursion \(\mu^t(a, e, s, h) = \Gamma(\mu^{t-1}, B^{t-1}(p))\), with \(\mu^0 = \mu(a, e, s, h)\) and \(B^0(p) = p\).

The factor market clearing conditions and the two credit market clearing conditions impose restrictions on factor prices and on the price of loans and deposits which are used in computing the equilibrium of the model. If there is positive production in each period, then

\(r(p) = F_K(K(p), N(p))\) and \(w(p) = F_N(K(p), N(p))\).

If the intermediary sector holds a positive quantity of capital each period, profit maximization requires that the net return from holding capital is exactly zero. From (B.1), this implies:

\(i(p) = r(B(p)) - \delta\).

If the intermediary sector holds positive amounts of deposits or loans of a given type, then the net return on the deposit or the loan must be zero. From (B.2) this implies that

\(q(p) = \frac{\rho}{1 + i(p)} \quad \text{and} \quad (B.13)\)

\(q(y, s, p) = \frac{\rho}{1 + i(p)} \left[ (1 - \theta(y, s; p) + \theta(y, s; p)q^D(y, s, p) \right]. \quad (B.14)\)

Finally, in the garnishment regime, if the intermediary sector holds positive amounts of defaulted debt then the net return on these debts must be zero as well. From (B.3), this
implies:

\[
x(y, s, p) = \frac{\rho}{1 + i(p)} \times \\
\sum_{s', \tilde{a} \in A} \eta(\tilde{a}, y, s'; B(p)) \left[ (\min\{\tilde{a} - y, -y\})/(-y) + I_{\{\tilde{a} < 0\}}x(\tilde{a}, s', B(p)) \right] \pi_{s'|s}.
\]  

(B.15)

### B.2 Computation

The model is solved using a collection of mostly standard techniques. The persistent and permanent components of the efficiency process are discretized using the recent Rouwenhorst method introduced by Kopecky and Suen (2010). We use 3 permanent states and 5 persistent states. Conditional on having a permanent and persistent state, the probability of having a particular level of efficiency is calculated using Tauchen’s (1986) method. The levels of efficiency (but not their probabilities) are chosen by discretizing the unconditional distribution using the Rouwenhorst method. We use 5 levels of efficiency (for a total of 75 states). The additional super-rich and/or super-poor state is added as discussed in the main text. The household problem is solved using a grid search accelerated with policy function iteration.

In making welfare comparisons, the presence of endogenous labor complicates obtaining the consumption equivalent measure. In particular, there is no analytic formula for it. Instead, we proceed in two steps. First, we obtain the welfare from having \( \gamma \) more consumption in every state of the world (holding fixed the other policies) using policy iteration (denote this utility \( V(a, e, h; \gamma) \)). Second, we compute the value of \( \gamma \) that equates \( V(a, e, h; \gamma) \) with some comparison level of utility \( W \). This \( \gamma \) is the consumption equivalent measure. Given policies \( c(a, e, h), n(a, e, h), \) and \( d(a, e, h) \) corresponding to a value function \( V(a, e, h) \), we construct a new consumption policy \( \tilde{c}(a, e, h) = (1 + \gamma)c(a, e, h) \) and compute the value of using \( \tilde{c}, n, \) and \( d \) forever by using policy iteration. We compute this for 30 values of \( \gamma \) between -.9 and 2. Last, to find the \( \gamma \) for which \( \tilde{V}(a, e, h; \gamma) \) equals some level of utility \( W \), we use a nonlinear equation solver (interpolating \( \tilde{V} \) in the \( \gamma \) dimension).
Appendix C

Computing Dynamic Heterogeneous-Agent Economies: Tracking the Distribution

C.1 Alternative Implementations

This appendix explores alternative implementations of the Smolyak method in the context of the OLG economy (which has a known solution). Four different implementations are considered and their descriptions are given below.

*Sm1 – Benchmark Implementation*

The state space uses shares \( s \); the mapping from \( \tilde{s} \) in the cube to \( s \) in the simplex is \( s = \tilde{s} / \sum \tilde{s} \) and the reverse mapping is \( \tilde{s} = s \); and the distribution is forecasted using an approximated capital policy function (not an approximate law of motion). Because of the loss in dimension in mapping to the simplex, there are many reverse mappings into the cube.\(^1\)

\(^1\)It’s unclear which of these reverse mappings is optimal, but the identity mapping is a natural choice.
Sm2 – Alternative Mapping

The state space uses shares $s$; the mapping from $\tilde{s}$ in the cube to $s$ in the simplex is $s = -\log(\tilde{s}) / \sum -\log(\tilde{s})$ and the reverse mapping is $\tilde{s} = e^{10 \log(0.5)s}$; and the distribution is forecasted using approximated policy functions. The mapping is motivated by a method of drawing uniformly from a unit simplex (which is accomplished by drawing from the Dirichlet distribution with concentration parameter equal to 1). Several reverse mappings were tried, but the one used worked best.\footnote{In some sense, the reverse mapping $\tilde{s} = e^{T \log(0.5)s}$ should be optimal because most of the time, $\sum - \log(\tilde{s})$ is equal to $-T \log(0.5)$. Numerically however, raising values to such a large power creates instability. I found $\tilde{s} = e^{10 \log(0.5)s}$ works fairly well.}

Sm3 – Simpler State Space

The state space uses levels of capital stock for each generation $k$; there is no mapping; and the distribution is forecasted using an approximated capital policy function.

Sm4 – Simpler State Space, Alternative Simulation Method

The state space uses levels of capital stock for each generation $k$; there is no mapping; and the distribution is forecasted using an approximated law of motion.

The accuracy numbers for the laws of motion are displayed in Table C.1. For space, the $R^2$ values are rounded to three decimal places. Alternative implementations of the Smolyak method display similar characteristics to the benchmark implementation: each outperforms the KS method for small $T$ (which is when quasi-aggregation breaks down) and the performance of each implementation decreases as the number of periods increases. Sm1 and Sm2 display similar errors which are less than the errors of Sm3 and Sm4. This suggests that the precise mapping of shares may not matter as much as the choice of whether or not to use shares. That Sm1 and Sm2 perform better than Sm3 and Sm4 is likely due to the ability of Smolyak interpolation to achieve high accuracy in one dimension relative to
accuracy in several dimension as discussed in the main text. Because Sm1 and Sm2 include $K$ in the state space directly rather than indirectly through $K = \sum k/T$, Sm1 and Sm2 exploit this feature of the Smolyak algorithm and so capture more of the general equilibrium effects.

<table>
<thead>
<tr>
<th>T</th>
<th>Max % Error</th>
<th>Minimal $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KS</td>
<td>Sm1</td>
</tr>
<tr>
<td>3</td>
<td>3.17</td>
<td>0.07</td>
</tr>
<tr>
<td>6</td>
<td>1.03</td>
<td>0.27</td>
</tr>
<tr>
<td>10</td>
<td>1.05</td>
<td>0.65</td>
</tr>
<tr>
<td>25</td>
<td>1.04</td>
<td>0.85</td>
</tr>
<tr>
<td>50</td>
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<td>0.85</td>
</tr>
<tr>
<td>100</td>
<td>1.04</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table C.1: Accuracy of the Law of Motion

The Euler equation errors are reported in Table C.2. Again, the alternative implementations display similar patterns to the benchmark one: the errors are smaller than the KS method errors for $T = 3$ but increase in the number of generations. Again, Sm1 and Sm2 fair better than Sm3 and Sm4, with Sm1 outperforming Sm2 most of the time. While the KS method tends to produce less error, the errors of the Smolyak implementations would not typically be considered large, at least in terms of average errors. Note that Sm3 and Sm4 have the exact same solution but different simulations (and so result in different maximum and average errors along the simulation path).

<table>
<thead>
<tr>
<th>T</th>
<th>Max Euler Errors</th>
<th>Avg Euler Errors</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>KS</td>
<td>Sm1</td>
</tr>
<tr>
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<td>-3.08</td>
<td>-2.99</td>
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<tr>
<td>10</td>
<td>-2.91</td>
<td>-2.65</td>
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<tr>
<td>25</td>
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<td>-2.46</td>
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<tr>
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<td>-2.46</td>
</tr>
<tr>
<td>100</td>
<td>-2.92</td>
<td>-2.46</td>
</tr>
</tbody>
</table>

Table C.2: Accuracy of the Policy Functions

The running times for the various implementations are reported in Table C.3. As running times are virtually the same for Sm1 as Sm2 and Sm3 as Sm4, I only report joint Sm1/Sm2 and Sm3/Sm4 times. The Sm3 and Sm4 implementations take slightly longer to run than
the Sm1 and Sm2 implementations. This is not because the actual interpolation takes any longer but because extra iterations required for the law of motion to converge, which is itself due to larger errors in the law of motion.

<table>
<thead>
<tr>
<th>T</th>
<th>KS</th>
<th>Sm1/Sm2</th>
<th>Sm3/Sm4</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>0.00</td>
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<tr>
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<td>0.02</td>
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<tr>
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<td>1.18</td>
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<tr>
<td>100</td>
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<td>4.97</td>
<td>6.00</td>
</tr>
</tbody>
</table>

Table C.3: Running Times in Minutes
Bibliography


