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# The $C$ -infinity jet of non-concave manifolds and lens rigidity of surfaces

Xiaochen Zhou  
zx@sas.upenn.edu

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# The C-infinity jet of non-concave manifolds and lens rigidity of surfaces

## Abstract

In this thesis we work on the boundary rigidity problem, an inverse problem on a manifold with boundary, which studies the unique determination of, and algorithms towards total recovery of, the metric tensor, based on the information of distances between boundary points. There are three main results in this thesis. The first result is an algorithm to recover the Taylor series of the metric tensor (C-infinity jet) at the boundary. The data we use are the distances between pairs of points on the boundary which are close enough to each other, i.e. the "localized" distance function. The restriction we impose on the shape of the manifold near the boundary is the minimal possible, i.e., the localized distance function does not completely coincide with the localized in-boundary distance function at any point. Here "in-boundary" distance means the length of the shortest path lying entirely on the boundary. Such a boundary we call "non-concave". A different algorithm has already been published in [26], but our result in this thesis is much more elementary. Our second result is a counter-example to the statement "Lens data uniquely determine the C-infinity jets at boundary points". It is the first known pair of manifolds with identical lens data but different C-infinity jets. Our first example is easy to construct, but the jets of the metrics only differ in the second order. With a careful modification to preserve smoothness, we can construct a pair with different C1 jets, meaning different second fundamental forms of the boundaries. The results above are already published in the author's paper [28]. Our third result is, if two surfaces with the same boundary are conformal, have the same lens data, and have no trapped geodesic or conjugate points, then they are isometric. The proof applies techniques in integral geometry, and uses results in [3] and [14]. If we combine this with not yet published results in [10] of Croke, Pestov, and Uhlmann, then we can drop the conformal assumption.

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Christopher Croke

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Xiaochen Zhou

A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial  
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Christopher Croke, Professor of Mathematics  
Supervisor of Dissertation

---

Jonathan Block, Professor of Mathematics  
Graduate Group Chairperson

Dissertation committee:

Tony Pantev, Professor of Mathematics

Ryan Blair, Lecturer of Mathematics

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## ABSTRACT

The  $C^\infty$  Jet of Non-Concave Manifolds and Lens Rigidity of Surfaces

Xiaochen Zhou

Christopher Croke

In this thesis we work on the boundary rigidity problem, an inverse problem on a manifold with boundary, which studies the unique determination of, and algorithms towards total recovery of, the metric tensor, based on the information of distances between boundary points.

There are three main results in this thesis. The first result is an algorithm to recover the Taylor series of the metric tensor ( $C^\infty$  jet) at the boundary. The data we use are the distances between pairs of points on the boundary which are close enough to each other, *i.e.* the “localized” distance function. The restriction we impose on the shape of the manifold near the boundary is the minimal possible, *i.e.*, the localized distance function does not completely coincide with the localized in-boundary distance function at any point. Here “in-boundary” distance means the length of the shortest path lying entirely on the boundary. Such a boundary we call “non-concave”. A different algorithm has already been published in [26], but our result in this thesis is much more elementary.

Our second result is a counter-example to the statement “Lens data uniquely determine the  $C^\infty$  jets at boundary points”. It is the first known pair of manifolds

with identical lens data but different  $C^\infty$  jets. Our first example is easy to construct, but the jets of the metrics only differ in the second order. With a careful modification to preserve smoothness, we can construct a pair with different  $C^1$  jets, meaning different second fundamental forms of the boundaries.

The results above are already published in the author's paper [28].

Our third result is, if two surfaces with the same boundary are conformal, have the same lens data, and have no trapped geodesic or conjugate points, then they are isometric. The proof applies techniques in integral geometry, and uses results in [3] and [14]. If we combine this with not yet published results in [10] of Croke, Pestov, and Uhlmann, then we can drop the conformal assumption.

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# Chapter 1

## Introduction

### 1.1 Boundary Rigidity Problem

Let  $(M, g)$  be a compact Riemannian manifold with smooth boundary, and let

$$\tau : M \times M \rightarrow \mathbb{R}$$

be the distance function given by  $g$ . The boundary rigidity problem asks whether we can recover  $g$  from  $\tau|_{\partial M \times \partial M}$ . That is, whether we can uniquely determine the Riemannian metric of  $M$ , knowing the distances from boundary points to boundary points. Obviously if we pullback the metric  $g$  by a diffeomorphism  $f : M \rightarrow M$  that fixes every boundary point, the resulting metric  $f^*g$  gives the same boundary distance function as before, but it is different from the original metric  $g$ . So the natural question is, whether this is the only obstruction to unique determination. If the answer is positive for  $(M, g)$ , then it is called *boundary rigid*.

One would like to know whether a given manifold with boundary is boundary rigid. If in some cases we have affirmative answers, we further want to have a procedure to recover the interior metric structure from the information of boundary (“chordal”) distances. For a survey of boundary rigidity problem, see [6].

There are certainly many manifolds with boundary that are not boundary rigid. For example, consider the  $n$ -dimensional hemisphere with the standard metric. Its boundary is the equator, and the distances between boundary points can all be realized inside the boundary. Therefore, if we arbitrarily dilate (increase the Riemann tensor) the hemisphere somewhere far away from the boundary, it will not change the boundary distance  $\tau$  function at all.

From the example above, we can imagine that if we want some manifold with boundary to be boundary rigid, then for each interior point there must be a geodesic between two boundary points passing through it, and the geodesic should minimize distance all along. Otherwise the interior point will not be even detected from the boundary distance function  $\tau$ . However, this condition is necessary but far from sufficient. For interesting examples, see [9] section 6, and [4] section 2.

People naturally wonder: what condition can we pose on a manifold with boundary, to guarantee the boundary rigidity? We have the condition “being a subspace of a constant curvature manifold”, but if the curvature is positive, then the domain is restricted to be contained inside an open hemisphere.

**Theorem 1.1.1.** *If  $(M, \partial M, g)$  is a compact subdomain with smooth boundary  $\partial M$*

of Euclidean space  $\mathbb{R}^n$ , hyperbolic space  $\mathbb{H}^n$ , or the open hemisphere of  $S^n$ , then  $(M, \partial M, g)$  is boundary rigid.

The proof of the hemisphere case can be found in [18], for Euclidean case see [12] or [4] section 6, and for hyperbolic case see [1].

It is also proved that manifolds with metrics which are sufficiently close to the flat metric are boundary rigid. See [2] for the following theorem.

**Theorem 1.1.2.** *Let  $M$  be a compact domain in  $\mathbb{R}^n$  with a smooth boundary. There exists a  $C^2$ -neighborhood  $U$  of the Euclidean metric  $g_E$  such that, every  $g \in U$  is a minimal orientable filling and is boundary rigid.*

Here “minimal orientable filling” leads us to an important technique to study boundary rigidity problem, and for the study of minimal periodic geodesic on manifolds without boundary, see [12] and [2].

Another rigidity result for a more general category of metrics was discovered independently by Croke [5] and Otal [21]. The main theorem is the following:

**Theorem 1.1.3.** *If  $(M, \partial M, g)$  is a compact, non-positively curved, SGM surface with boundary, then it is boundary rigid.*

In the hypothesis above, SGM manifold is defined in [4]. Roughly speaking, to say a manifold with boundary is SGM (Strong Geodesic Minimizing), means that all maximal non-constant geodesic hit the boundary on both ends, and all geodesics minimize the distance between any pair of its points. When a geodesic

has infinite length (but may intersect the boundary tangentially), then we call it a “trapped” geodesic, and call the manifold “trapping”. The SGM property can be defined completely using the boundary distance function  $\tau$ , see [4] Definition 1.1. It is conjectured that all SGM manifolds are boundary rigid.

A notion narrower than SGM is “simple”. We say a manifold with boundary is simple, if the boundary is strictly convex, and there is a unique geodesics between any pair of boundary points. By looking at geodesics leaving a fixed point, we can show that the manifold is topologically a ball. R. Michel conjectured that simple manifolds are boundary rigid, see [18]. The best result about simple manifolds is in dimension 2, with the following theorem from [22].

**Theorem 1.1.4.** *Two dimensional simple compact Riemannian manifolds with boundary are boundary rigid.*

This proof uses the canonical (with an orientation) rotation by an angle of  $\frac{\pi}{2}$ , together with some integral geometry techniques, which translates the distance information into information about Dirichlet-to-Neumann map. In dimension 2, the Dirichlet-to-Neumann map will determine the conformal class of the Riemannian manifold, see [17]. Then the conformal class and the boundary distance function will determine the metric, see [20] or [6]. The last step can be proved using Santaló’s formula and Hölder inequality. For more about integral geometry, see [23].

## 1.2 The $C^\infty$ Jet

The  $C^\infty$  *jet* at a point of a Riemannian manifold is, roughly speaking, the Taylor series of the metric tensor at the point. Therefore to recover the  $C^\infty$  jet at boundary points is the first step of the recovery of the entire interior metric structure.

In arguments about the boundary rigidity problem, often one needs to extend  $(M, g)$  beyond its boundary, and here people care about the smoothness of the extension. An extension of  $g$  is smooth if and only if the  $C^\infty$  jets computed from both sides of the boundary agree, or strictly speaking, if  $(M, \partial M, g_M)$  and  $(N, \partial M, g_N)$  are Riemannian manifolds with the same (isometric) boundary  $\partial M$ , then the glued manifold without boundary  $(M \cup N)/\partial M$  has a smooth metric tensor which agrees with  $g_M$  on  $(M - \partial M)$  and  $g_N$  on  $(N - \partial M)$ , if and only if the  $C^\infty$  jets of  $\partial M$  in  $M$  and  $N$  are exactly the same in even orders and the same but with opposite signs in odd orders, computed under some common boundary normal coordinates. The definitions are provided in section 2.1. This will guarantee that, given  $M_1$  and  $M_2$  with isometric boundaries and the same  $C^\infty$  jets along the boundaries, if we can smoothly fit  $M_1$  into a larger outer manifold, then we can smoothly fit  $M_2$  into that same outer manifold as well.

There are results on the boundary determination of  $C^\infty$  jets. Michel [18] proved that boundary distances uniquely determine the Taylor series of  $g$  up to order 2, and in [19] he proved the same result without order limitations but with  $\dim(M) = 2$ , both with convex boundaries. Here convexity roughly means that the distance of

two sufficiently close boundary points should be realized by a geodesic whose interior does not intersect the boundary. In [16] there is an elementary proof that the  $C^\infty$  jet is uniquely determined by the boundary distance function if the boundary is convex. However the results above are not constructive. In [26], Uhlmann and Wang applied the result of [24] and used a suitably chosen reference metric, and gave a recovery procedure of  $C^\infty$  jet on the boundary from localized boundary distance function. Here “localized” means we do not need to know  $\tau$  for all pairs of points in  $\partial M$ , but we only should know  $\tau$  restricted to an open neighborhood of the diagonal of  $\partial M \times \partial M$ , that is, the distance between close enough pairs. It should be noted that the arguments in [16] and [26] also apply to non-concave boundary (see Definition 2.1.1) without much modification.

Up to now, the only result for possibly concave boundary is [25], Theorem 1. The statement is, if a geodesic segment  $\gamma$  is tangent to the boundary at one end  $p$ , and the other end  $q$  lies on the boundary, then under some generic no conjugate points condition, we can recover the  $C^\infty$  jet at  $p$  based on the lengths of geodesic segments in a neighborhood of  $\gamma$ . The proof is constructive because they gave an algorithm to find the derivatives of all orders. Our recovery procedure in this thesis is similar to [25].

In the next chapter of this thesis, we give the same results as in [26], that is, a procedure to recover the  $C^\infty$  jet at boundary points, but our argument is relatively elementary. We also directly adopt the weaker assumption that the boundary is

non-concave, as opposed to “convex” as in previous results.

In chapter 3, we give the first known example that shows the lens data do not always determine the boundary  $C^\infty$  jet. Here “lens data” include the information of  $\tau|_{\partial M \times \partial M}$  and the lengths of all maximal geodesics, together with the locations and the directions whenever they hit the boundary (see Definition 3.1.1, or [25] section 1 for detail). So the results in [16], [26], and the first part of this thesis show that lens data uniquely determine  $C^\infty$  jet near non-concave points. Meanwhile, the results in [25] should imply: We can uniquely recover  $C^\infty$  jet near “generic” concave points, from the lens data of geodesics with bounded length, which are “almost” tangent to the boundary. In the example in section 3.1, the boundary is totally geodesic, and nearby geodesics have unbounded length, although each of them hits the boundary in finite time. In the example in section 3.2, the boundary is strictly concave, but every complete geodesic tangent to the boundary has infinite length. Therefore, the examples in this thesis fall in the gap between non-concave results (Theorem 2.2.8 of this thesis, [16], and [26]) and the concave result [25].

### 1.3 Lens Rigidity of Surfaces

If the manifold with boundary is not simple or SGM, then certainly lens data carry essentially more information than the boundary distance function. If the manifold has no trapped geodesics (*i.e.* the manifold is non-trapping), then lens data seem to have gathered information in any direction at any point. However, if the manifold



has a trapped geodesic, then there are examples where the lens data do not uniquely determine the metric, see [9].

There are few results about lens rigidity outside of the simple or SGM cases. Notice that the lens rigidity problem is equivalent to the boundary rigidity problem, if the manifold is *SGM* or simple.

In [25], Stefanov and Uhlmann generalized their local result for simple metrics to obtain a local lens rigidity result. In [27], Vargo proved a lens rigidity result in the category of analytic metrics. He used the result of [25] to determine the  $C^\infty$  jet, and used the lens data to recover the whole metric under the assumption that the metric and boundary are both analytic.

Almost all results until now cannot avoid the assumption that the manifold has no trapped geodesics, *i.e.* all directions are contained in the image of the geodesic flow of vectors originating from the boundary. The only result for a manifold with trapped geodesics is about  $D^n \times S^1$ , with a generalization, see [7]. Also see [11] for interesting examples.

In the last part of this thesis, we prove that if two metrics on the same surface with boundary are conformal, they have the same lens data, and neither of them has conjugate points or trapped geodesics, then the conformal factor is 1 everywhere. In this proof we use the techniques of integral geometry, *i.e.* Santaló's formula and Hölder inequality, see [8] and [4] for applications.

A crucial theorem we will use in this thesis can be found in [3], with the following

statement,

**Theorem 1.3.1.** *If  $(M, \partial M, g)$  is lens rigid and has no trapped geodesics, and it admits a free action by a finite group  $\Gamma$  of isometries, then the quotient  $(M/\Gamma, \partial M/\Gamma, g)$  is also non-trapping and lens rigid.*

In the same paper C. Croke also gave an example which shows that we cannot change the two appearances of “lens rigid” into “boundary rigid” in the theorem above.

With the theorem above, We can lift the metric to a finite normal covering, where the lengths of geodesics are far smaller than the systole (see Definition 4.2.1, or [15]). Finally, we apply the results from the not yet published paper [10], to drop the conformal assumption.

# Chapter 2

## Recovery of the $C^\infty$ Jet near Non-Concave Points

### 2.1 Preliminaries

Throughout this paper we let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with smooth boundary  $\partial M$ . Let  $\tau$  be the distance function, and let  $\rho = \tau^2$ . We further introduce the notation  $\tau_x(\cdot) = \tau(\cdot, x)$ , and  $\rho_x(\cdot) = \rho(\cdot, x)$ . Notice that the distance might not be realized by a geodesic, and a curve realizing it can have non-geodesic parts in the boundary.

Let

$$\mu : \partial M \times \partial M \rightarrow \mathbb{R} \tag{2.1.1}$$

be the distance function on the Riemannian manifold  $(\partial M, g|_{\partial M})$ . Note that  $\mu$

is not  $\tau|_{\partial M}$  ( $\mu \geq \tau$  in general) although they may agree on some subset. Near “non-concave” points,  $\tau|_{\partial M}$  contains more information than  $\mu$ .

**Definition 2.1.1** (Concave and Non-Concave points). Let  $x \in \partial M$ . We say  $\partial M$  is *concave* at  $x$  if the second fundamental form of  $\partial M$  is negative semi-definite at  $x$ , with respect to  $\nu_x$  the inward-pointing unit normal vector. We call  $\partial M$  *non-concave* at  $x$  if it is not concave at  $x$ , that is, the second fundamental form has at least one positive eigenvalue.

In order to detect non-concave points from  $\tau|_{\partial M}$ , we state the following elementary proposition without proof.

**Proposition 2.1.2.** *Let  $x \in \partial M$ . If  $\partial M$  is concave in an open neighborhood of  $x$ , then there exists  $\varepsilon > 0$  such that whenever  $p, q \in \partial M$  satisfy  $\mu(x, p) < \varepsilon$  and  $\mu(x, q) < \varepsilon$ , we have  $\tau(p, q) = \mu(p, q)$ . That means, for a pair of points close enough to  $x$ , the shortest path between them is along the boundary.*

Since  $(M, g)$  is extendable, we fix a compact  $n$ -dimensional Riemannian manifold without boundary  $(\tilde{M}, \tilde{g})$ , such that  $\partial M$  is an  $(n-1)$ -dimensional submanifold, the interior of  $M$  is a connected component of  $\tilde{M} - \partial M$ , and  $g$  is the restriction of  $\tilde{g}$ .

Next we define *boundary normal coordinates* of  $M$  near  $\partial M$ . Let  $(x_1, \dots, x_{n-1})$  be a coordinate chart on the manifold  $\partial M$ . For  $p \in M$  close enough to  $\partial M$ , there is a unique point  $y = (y_1, \dots, y_{n-1}) \in \partial M$  that is closest to  $p$ . We then give  $p$  the coordinates  $(y_1, \dots, y_{n-1}, y_n)$  where  $y_n$  is defined to be the distance from  $p$  to  $y$ .

In such coordinates,  $g_{in} = \delta_{in}$ , and the curves  $c(t) = (y_1, \dots, y_{n-1}, t)$  are geodesics perpendicular to  $\partial M$  at  $t = 0$ .

Knowing the  $C^\infty$  jet of  $g$  on  $\partial M$  is equivalent to, with respect to a given boundary normal coordinates, knowing the derivatives

$$\left. \frac{\partial^k}{\partial x_n^k} g_{ij} \right|_{\partial M}$$

for all  $k \geq 0$  and indices  $i, j$ , where  $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ . Clearly if we know the jet with respect to one choice of boundary normal coordinates, we are able to find the jet with respect to every choice of boundary normal coordinates, knowing the coordinate change on the boundary. For each integer  $l \geq 0$ , knowledge of  $C^l$  jet means knowledge of all the  $\frac{\partial^k}{\partial x_n^k} g_{ij}$  with  $k \leq l$ . In this paper we find the jet only under boundary normal coordinates, and see [16] Theorem 2.1 for the precise statement for general coordinates.

The key identity in the jet recovery procedure is the *Eikonal equation*,

$$|\nabla \tau_p| = 1, \quad p \in M,$$

wherever the function  $\tau_p$  is smooth. In coordinate charts the Eikonal equation is

$$g^{ij} \frac{\partial \tau_p}{\partial x_i} \frac{\partial \tau_p}{\partial x_j} = 1.$$

Here we adopt Einstein summation convention, where  $i, j$  ranges from 1 to  $n$ , and matrix  $(g^{ij})$  is the inverse of  $(g_{ij})$ . In boundary normal coordinates, this becomes

$$g^{\alpha\beta} \frac{\partial \tau_p}{\partial x_\alpha} \frac{\partial \tau_p}{\partial x_\beta} + \left( \frac{\partial \tau_p}{\partial x_n} \right)^2 = 1, \quad (2.1.2)$$

where  $\alpha$  and  $\beta$  range from 1 to  $(n - 1)$ .

We will use the convention that  $i, j$  range from 1 to  $n$ , and  $\alpha, \beta$  range from 1 to  $(n - 1)$ . We assume we are always in boundary normal coordinates near  $\partial M$ . We will write  $\partial_{x_i}$  for  $\frac{\partial}{\partial x_i}$ , and  $\partial_{x_i x_j}$  for  $\frac{\partial^2}{\partial x_i \partial x_j}$ , and so on. The reader should view the function  $\tau$  as  $\tau(x, y)$  and

$$\tau(x_1, \dots, x_n, y_1, \dots, y_n),$$

so that formulas like  $\partial_{x_i} \tau(p, q)$  and  $\partial_{y_i} \tau(p, q)$  will make sense. We treat  $\rho = \tau^2$  similarly.

## 2.2 Recovery Algorithm

We have the following lemmas.

**Lemma 2.2.1.** *Let  $V_\varepsilon \subset M \times M$  be the set of pairs  $(x, y)$  satisfying the following properties:  $\tau(x, y)$  is realized by a geodesic in  $M$ , and  $\tau(x, y) \leq \varepsilon$ .*

*Then there exists an  $\varepsilon > 0$  such that  $\rho$  is a smooth function on  $V_\varepsilon$ .*

The lemma is easy to prove if  $M$  has no boundary. But when  $M$  has a boundary, we may first prove the property for the extension  $(\tilde{M}, \tilde{g})$  with its corresponding  $\tilde{\rho}$ , and use the fact that  $\rho|_{V_\varepsilon}$  is the same as  $\tilde{\rho}|_{V_\varepsilon}$ . Recall that a function being smooth in (a subset of) a manifold with boundary (and possibly corner) means the manifold together with the function can be extended into a bigger one without boundary such that the function is still smooth.

Notice that we cannot replace  $\rho$  in the last lemma with  $\tau$ , because  $\tau$  is not smooth where  $x = y$ . Smoothness is the reason why we use distance squared rather than distance itself.

**Lemma 2.2.2.** *Let  $c : (-\varepsilon, +\varepsilon) \rightarrow M$  be a smooth curve in  $M$ , which may intersect  $\partial M$ . If for each  $t$  the distance between  $c(t)$  and  $c(0)$  is realized by a minimizing geodesic of  $M$ , then we have*

$$2|c'(0)|^2 = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \rho(c(t), c(0)). \quad (2.2.1)$$

*Proof.* If  $c$  is a geodesic the statement is clearly true. If  $c'(0) = 0$  the statement is also easy to prove.

Otherwise, we may think of  $c'(t)$  as coming from a vector field  $X$  in a neighborhood of  $c(0) \in M$ . This will give rise to a vector field  $\tilde{X} = (X, 0)$  in  $M \times M$ . Then we look at the right side,

$$\begin{aligned} \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \rho(c(t), c(0)) &= \tilde{X}(\tilde{X}\rho) \\ &= \text{Hess}\rho(\tilde{X}, \tilde{X}) + \left( \nabla_{\tilde{X}} \tilde{X} \right) \rho \\ &= \text{Hess}\rho(\tilde{X}, \tilde{X}), \end{aligned}$$

where all expressions are evaluated at  $(c(0), c(0))$ , a critical point of  $\rho$ . However,  $\text{Hess}\rho(\tilde{X}, \tilde{X})$  only depends on  $\tilde{X}$  at the point, which is  $(c'(0), 0)$ , so the right side of equation (2.2.1) only depends on  $c'(0)$ . This means we might as well assume  $c$  is a geodesic. □

Now we are ready to recover the jet from the localized boundary distance function. We present the recovery procedure in four steps, *i.e.* Proposition 2.2.3, 2.2.4, 2.2.6, and 2.2.7.

**Proposition 2.2.3.** *We can recover the  $C^0$  jet from the localized boundary distance function.*

This is easy because  $C^0$  jet is simply  $g_{ij}|_{\partial M}$  the Riemannian metric tensor. From the localized boundary distance function, we are able to compute the length of any smooth curve in  $\partial M$ . The curve lengths will tell us the metric tensor.

We start the recovery procedure for higher order jets. The idea underlying the proofs of the following propositions (Proposition 2.2.4, 2.2.6, and 2.2.7) is derived from [25], section 3.

**Proposition 2.2.4.** *If  $\partial M$  is non-concave at  $y$ , then we can recover the  $C^1$  jet near  $y \in \partial M$  from the localized boundary distance function, with respect to a given boundary normal coordinates.*

We need a definition for the proof of this proposition.

**Definition 2.2.5** (Convex direction). Let  $\xi$  be a vector tangent to  $\partial M$ . We can find a geodesic  $\gamma : (-\varepsilon, +\varepsilon) \rightarrow \partial M$  with  $\gamma'(0) = \xi$ . (Here  $\gamma$  may not be a geodesic in  $M$ .) Let  $\nabla$  be the covariant derivative in  $M$ , and  $\nu$  the inward-pointing unit normal at appropriate points in  $\partial M$ . We call  $\xi$  a *convex direction* if  $\langle \nabla_{\gamma'(0)} \gamma', \nu \rangle > 0$ .



Certainly the set of convex directions compose an open subset of  $T(\partial M)$ . By definition,  $\partial M$  is non-concave at  $y$  if and only if there is at least one, and hence a nonempty open set of convex directions based at  $y$ .

*Proof of Proposition 2.2.4.* After possibly changing coordinates, we assume  $\partial_{x_1}$  is a convex direction at  $y$  (and in a neighborhood too). Let  $c(t)$  be a curve in  $M$  such that  $c'(t) = \partial_{x_1}$ , which means its coordinates representation is  $(x_1 + t, x_2, \dots, x_n)$ . Applying lemma 2.2.2 we know

$$2g_{11}(p) = \partial_{x_1 x_1} \rho(p, p), \quad (2.2.2)$$

where  $p$  is not assumed to be on the boundary. Clearly both sides of the equation are smooth functions of  $p$ . We now let the point  $p$  move in the direction  $\partial_{x_n}$  and take the derivative of equation (2.2.2),

$$\begin{aligned} 2\partial_{x_n} g_{11} &= \partial_{x_n} \partial_{x_1 x_1} \rho + \partial_{y_n} \partial_{x_1 x_1} \rho \\ &= \partial_{x_1 x_1} (\partial_{x_n} \rho + \partial_{y_n} \rho). \end{aligned} \quad (2.2.3)$$

We let  $c : (-\varepsilon, +\varepsilon) \rightarrow \partial M$  be the curve in  $\partial M$  with  $c(0) = y$  and  $c' \equiv \partial_{x_1}$ . Since  $\partial_{x_1}$  is a convex direction, we may assume for any point  $x$  on  $c$  which is not the same point as  $y$ , the distance between  $y$  and  $x$  is realized by a geodesic segment whose interior does not intersect  $\partial M$ , and the geodesic is transversal to  $\partial M$  at both endpoints. So we know the value of  $(\partial_{x_n} \tau)(x, y)$  from first variation of arc length,

and similarly  $(\partial_{y_n}\tau)(x, y)$ . The values  $(\partial_{x_n}\rho)(x, y)$  and  $(\partial_{y_n}\rho)(x, y)$  are then easily recovered from localized  $\tau|_{\partial M}$ .

Since  $\partial_{x_1x_1}(\partial_{x_n}\rho + \partial_{y_n}\rho)|_{(y,y)}$  only depends on the value of  $(\partial_{x_n}\rho + \partial_{y_n}\rho)(x, y)$  where  $x$  is along the curve  $c$ , from equation (2.2.3) we find  $\partial_{x_n}g_{11}|_y$ .

Now we use the fact that a symmetric  $n \times n$  tensor  $(f_{ij})$  can be recovered by knowledge of  $f_{ij}v_k^i v_k^j$  for  $N = n(n+1)/2$  “generic” vectors  $v_k$ ,  $k = 1, \dots, N$ , and we can find such  $N$  vectors in any open set on the unit sphere.

We may choose appropriate  $N$  perturbations of  $\partial_{x_1}$ , say  $v_k$ , which are all convex directions at  $y$ . Letting  $(\partial_{x_n}g_{ij})$  be the tensor described above, We find the values of  $\partial_{x_n}g_{ij}v_k^i v_k^j$  using the same method as above (change  $\partial_{x_1}$  into  $v_k$ ). They will tell us the values of  $\partial_{x_n}g_{ij}|_y$ . □

Next we give the recovery procedure of  $C^2$  jet, which applies Eikonal equation. The cases of higher order jets are essentially the same as  $C^2$  jet.

**Proposition 2.2.6.** *If  $\partial M$  is non-concave at  $y$ , then we can recover the  $C^2$  jet near  $y \in \partial M$  from the localized boundary distance function, with respect to a given boundary normal coordinates.*

*Proof.* Clearly being non-concave is an open property for points in  $\partial M$ , so we have already recovered  $C^1$  jet near  $y$  by Proposition 2.2.4. Again we assume without loss of generality that  $\partial_{x_1}$  is a convex direction at  $y$ .

Now look at equation (2.2.2) again, and let  $p$  move in the direction  $\partial_{x_n}$ , but this

time we look at the second derivative:

$$\begin{aligned} 2\partial_{x_n x_n} g_{11} &= (\partial_{x_n} + \partial_{y_n})^2 (\partial_{x_1 x_1} \rho) \\ &= \partial_{x_1 x_1} (\partial_{x_n x_n} \rho + 2\partial_{x_n y_n} \rho + \partial_{y_n y_n} \rho). \end{aligned} \quad (2.2.4)$$

Again we let  $c$  be a short enough curve in  $\partial M$  with  $c(0) = y$  and  $c' \equiv \partial_{x_1}$ . For the same reason as in the proof of Theorem 2.2.4, to compute the value of right side of equation (2.2.4) we only need to know  $(\partial_{x_n x_n} \rho + 2\partial_{x_n y_n} \rho + \partial_{y_n y_n} \rho)$  at  $(x, y)$  where  $x$  lies on  $c$ .

If  $x = y$ , it is easy to see the value of  $(\partial_{x_n x_n} \rho + 2\partial_{x_n y_n} \rho + \partial_{y_n y_n} \rho)$  at  $(x, y)$  is 0.

If  $x \neq y$ , then we look at the Eikonal equation as in (2.1.2), in the following form,

$$g^{\alpha\beta}(x)(\partial_{x_\alpha} \tau_y(x))(\partial_{x_\beta} \tau_y(x)) + (\partial_{x_n} \tau_y(x))^2 = 1. \quad (2.2.5)$$

Taking  $\partial_{x_n}$  we get (with all terms evaluated at  $x$ )

$$\partial_{x_n} g^{\alpha\beta} (\partial_{x_\alpha} \tau_y) (\partial_{x_\beta} \tau_y) + 2g^{\alpha\beta} (\partial_{x_\alpha x_n} \tau_y) (\partial_{x_\beta} \tau_y) + 2(\partial_{x_n} \tau_y) (\partial_{x_n x_n} \tau_y) = 0. \quad (2.2.6)$$

In equation (2.2.6), the term  $\partial_{x_n} g^{\alpha\beta}$  we already know because  $(g^{\alpha\beta})$  is the inverse of  $(g_{\alpha\beta})$  and we know  $g_{\alpha\beta}$  and  $\partial_{x_n} g_{\alpha\beta}$ . Also we know  $\partial_{x_\alpha} \tau_{q_2}$  from the localized boundary distance function. We know  $\partial_{x_\alpha x_n} \tau_y$  because from the first variation formula we know  $\partial_{x_n} \tau_y$  in a neighborhood of  $x$  along the boundary.

Therefore, so far the only term in equation (2.2.6) we do not know is  $\partial_{x_n x_n} \tau_y(x) = \partial_{x_n x_n} \tau(x, y)$ , whose coefficient is  $2\partial_{x_n} \tau_y(x)$ , a nonzero number because of the transversality of the segment between  $x$  and  $y$  to  $\partial M$ . We can now immediately find value

of  $\partial_{x_n x_n} \tau(x, y)$  from the other terms. Then, we can find  $\partial_{x_n x_n} \rho(x, y)$ .

If we interchange the roles of  $x$  and  $y$ , we can find  $\partial_{y_n y_n} \rho(x, y)$ . As for  $\partial_{x_n y_n} \rho(x, y)$ , we simply take derivative of equation (2.2.5) with respect to  $y_n$ , that is, let  $y$  move away from the boundary, and get (assuming all are taken at  $(x, y)$ )

$$2(\partial_{x_\alpha y_n} \tau)(\partial_{x_\beta} \tau) + 2\partial_{x_n y_n} \tau \partial_{x_n} \tau = 0,$$

where we know all but  $\partial_{x_n y_n} \tau(x, y)$ . So we can find the value of  $\partial_{x_n y_n} \tau(x, y)$  and hence  $\partial_{x_n y_n} \rho(x, y)$ .

Up to now, we have computed  $(\partial_{x_n x_n} \rho + 2\partial_{x_n y_n} \rho + \partial_{y_n y_n} \rho)$  at  $(x, y)$  with  $x \in c$ , so by equation (2.2.4), we can find  $\partial_{x_n x_n} g_{11}|_y$ .

Once again, we perturb  $\partial_{x_1}$  a little to get sufficiently many vectors  $v_k$  with convex directions. Carry out the procedure for every  $v_k$  to know  $(\partial_{x_n x_n} g_{ij})v_k^i v_k^j$ , and combine the values all together to find out all the  $\partial_{x_n x_n} g_{ij}|_y$ .  $\square$

We may now proceed by induction.

**Proposition 2.2.7.** *Let  $k \geq 3$ . If we have recovered the  $C^{k-1}$  jet in an open neighborhood of  $y \in \partial M$ , then with respect to a given boundary normal coordinates, we can recover the  $C^k$  jet of the same neighborhood from localized boundary distance function.*

*Proof.* We let  $p$  in equation (2.2.2) move towards the  $n$ th direction and take the

$k$ th derivative, and get the following equation,

$$2\partial_{x_n}^k g_{11}|_p = (\partial_{x_n} + \partial_{y_n})^k (\partial_{x_1 x_1} \rho)|_{(p,p)} \quad (2.2.7)$$

$$= \partial_{x_1 x_1} \left( \sum_{i=0}^k \binom{k}{i} \partial_{x_n}^i \partial_{y_n}^{k-i} \rho \right)_{(p,p)}. \quad (2.2.8)$$

Here we borrow notation from Theorem 2.2.6. The right side of equation (2.2.8) evaluated at  $(y, y)$  only depends on

$$\partial_{x_n}^i \partial_{y_n}^{k-i} \rho(x, y), \quad (2.2.9)$$

where  $x$  lies on the curve  $c$ , and  $i = 0, 1, \dots, k$ .

If  $x = y$  all are simple to compute, and the values do not even depend on the manifold.

If  $x \neq y$ , We first solve the problem when  $i = k$ . We apply the operator  $\partial_{x_n}^{k-1}$  to equation (2.2.5), recalling the Eikonal equation holds wherever the gradient is smooth. The resulting equation has terms involving  $g^{\alpha\beta}$ ,  $\partial_\alpha \tau_y$ ,  $\partial_\beta \tau_y$ , and  $\tau_{q_2}$ , and each of them may carry the operator  $\partial_{x_n}$  at most  $(k-1)$  times, except the last term

$$2(\partial_{x_n} \tau_y)(\partial_{x_n}^k \tau_y),$$

where  $\partial_{x_n} \tau_y$  is nonzero at  $x$  by transversality. Since  $x \neq y$  (which means  $\tau \neq 0$ ), knowing the derivatives of  $\rho$  up to order  $(k-1)$  is equivalent to knowing the derivatives of  $\tau$  up to order  $(k-1)$ . It is also okay to move  $x$  along the boundary (*i.e.* take  $\partial_{x_\alpha}$ ) because all procedures work in some open neighborhoods, with a change of coordinates if necessary. So by the inductive hypothesis, we know  $g^{\alpha\beta}$ ,

$\partial_{x_\alpha}\tau$ ,  $\partial_{x_\beta}\tau$ , and  $\tau$ , along with their derivatives involving  $\partial_{x_n}$  up to  $(k-1)$  times. Therefore, we can compute  $\partial_{x_n}^k\tau$ , and hence  $\partial_{x_n}^k\rho$ . Now we have finished the case  $i = k$ .

If  $i = 0$ , we do the same procedure after interchanging  $x$  and  $y$ .

Finally, if  $0 < i < k$ , we have at least one  $\partial_{x_n}$  and one  $\partial_{y_n}$  applied to  $\rho$  in formula (2.2.9). To proceed, we can apply  $\partial_{x_n}^{i-1}\partial_{y_n}^{k-i}$  to Eikonal equation (2.2.5). We then use the same method as in the case  $i = k$ . Note that  $\partial_{y_n}g^{\alpha\beta}(x) \equiv 0$ , because  $\partial_{y_n}$  does not move point  $x$ .

So far we have found  $\partial_{x_n}^k g_{11}|_y$ .

We perturb  $\partial_{x_1}$  a little to get sufficiently many vectors  $v$  with convex directions. Carry out the procedure for every such  $v$  to know  $(\partial_{x_n}^k g_{ij})v^i v^j$ , and put the results all together to determine all the  $\partial_{x_n}^k g_{ij}|_y$ .  $\square$

If we combine the results of Proposition 2.2.4, Proposition 2.2.6, and Proposition 2.2.7, we have the following

**Theorem 2.2.8.** *Suppose  $\partial M$  is non-concave at  $y$ , and  $D \subset \partial M \times \partial M$  is an open neighborhood of  $(y, y)$ . Then we can recover the  $C^\infty$  jet of  $g$  at  $y$  based on the information of  $\tau|_D$ .*

If we want to weaken the assumption in the theorem, we can try to detect non-concave points of  $\partial M$  by information about  $\tau|_{\partial M}$  only. The contrapositive statement of Proposition 2.1.2 is, if in any open neighborhood of  $y$  in  $\partial M$ , we can find  $x_1, x_2$  with  $\tau(x_1, x_2) < \mu(x_1, x_2)$ , then  $y$  is not in the interior of the (closed) set

of concave points, *i.e.*  $y$  is in the closure of non-concave points. But we can recover  $C^\infty$  jets near non-concave points, and jets are continuous (because  $g$  is extendable), so we know the jet at  $y$ .

**Theorem 2.2.9.** *Suppose  $y \in \partial M$ . If for every neighborhood  $D$  of  $(y, y) \in \partial M \times \partial M$ , we have  $\tau|_D$  and  $\mu|_D$  do not entirely agree, then we can recover  $C^\infty$  jet of  $g$  at  $y$ .*

This can help us know the interior metric structure if we a priori assume the manifold, metric, and boundary are analytic. Observe that the set of non-concave points is open, and we have the following

**Theorem 2.2.10.** *Suppose  $(M, \partial M, g)$  is analytic. If for any connected component of  $\partial M$ , we have a point  $y$  satisfying the hypothesis of Theorem 2.2.8, then we can recover the  $C^\infty$  jet of  $g$  at all points of  $\partial M$ .*

This can lead to lens rigidity results in the category of analytic metrics, with some assumptions such as “every unit speed geodesic hits the boundary in finite time”, see [27].

In Theorem 2.2.9 and 2.2.10, the hypothesis is simply “the localized chordal distance function at the boundary does not agree with the localized in-boundary distance function”. One is tempted to remove the words “localized”, which means we now have the question: for an analytic Riemannian manifold with boundary, if  $\tau$  does not entirely agree with  $\mu$ , can we compute the  $C^\infty$  jet? The answer is negative, because of the examples described in chapter 3.

Before we move on to the examples, we will implement this recovering procedure on the standard disk as a submanifold  $D^n$  of  $\mathbb{R}^n$ , with the Euclidean metric. The fact that  $D^n$  has the highest degree of symmetry makes our calculation easy enough with bare hands.

## 2.3 Algorithm Validation on the Standard Disk up to Order 3

In this section we use the algorithm we just designed, to recover the  $C^\infty$  jet of a standard ball  $D^n \in \mathbb{R}^n$ . We should forget about any structure we know about the ball, but only use the distance function between boundary points. After we find out the jet, we compare it with the real jet which we find using the standard ball structure, and they should match. Since the computation blows up quickly, we only compute up to the  $C^3$  jet.

First let's see what results we should reach. We fix a geodesic normal coordinates near a point on the boundary  $S^{n-1}$ . Since all directions are equivalent, we only need to compute one. Let  $\theta$  be the (unit-speed) parameter along the first coordinate. Then  $\theta$  is also the angle between the points involved, because the ball has unit radius, and from now on we view  $\theta$  in this way. Let the  $n$  stand for the direction perpendicular to  $S^n$ , where increasing the  $n$ -th coordinate means going towards the center. We expand the coordinate system to boundary normal coordinates described



before.

$$g_{\alpha\beta} = (1 - x_n)^2 \cdot \delta_{\alpha\beta},$$

so at the boundary, where  $x_n = 0$ ,

$$\partial_{x_n} g_{11} = -2, \tag{2.3.1}$$

$$\partial_{x_n}^2 g_{11} = 2, \tag{2.3.2}$$

$$\partial_{x_n}^3 g_{11} = 0, \tag{2.3.3}$$

and all higher derivatives are all zero.

From now on, we forget the equations (2.3.1), (2.3.1), and (2.3.1). Instead, we will recover them from the fact that the chordal distance between two points which are  $\theta$  apart along the boundary, is indeed  $2 \sin \frac{\theta}{2}$ . We have

$$2\partial_n g_{11} = \partial_\theta^2 (\partial_{x_n} \rho + \partial_{y_n} \rho). \tag{2.3.4}$$

Because  $\tau = 2 \sin \frac{\theta}{2}$ , we have  $\rho = 4 \sin^2 \frac{\theta}{2}$ . Therefore

$$\begin{aligned} \partial_\theta \tau &= \cos \frac{\theta}{2}, \\ \partial_{x_n} \tau &= -\sin \frac{\theta}{2}, \end{aligned} \tag{2.3.5}$$

so

$$\begin{aligned} \partial_{x_n} \rho &= 2\tau \cdot \partial_{x_n} \tau = -4 \sin^2 \frac{\theta}{2}, \\ \partial_{y_n} \rho &= -4 \sin^2 \frac{\theta}{2}, \end{aligned}$$

and from (2.3.4) we get

$$\begin{aligned}
\partial_{x_n} g_{11} &= \frac{1}{2} \cdot \partial_\theta^2 \left( -4 \sin^2 \frac{\theta}{2} - 4 \sin^2 \frac{\theta}{2} \right) \\
&= \frac{1}{2} \cdot \left( -4 \cos^2 \frac{\theta}{2} + 4 \sin^2 \frac{\theta}{2} \right) \\
&= -2 \cos^2 \frac{\theta}{2} + 2 \sin^2 \frac{\theta}{2} = -2 \cos \theta.
\end{aligned} \tag{2.3.6}$$

When  $\theta = 0$ , the equation above has the form

$$\partial_{x_n} g_{11} = -2.$$

This coincides with the equation (2.3.1). We now have finished recovering the  $C^1$  jet.

For  $C^2$  jet, we have

$$2\partial_{x_n}^2 g_{11} = \partial_\theta^2 (\partial_{x_n}^2 \rho + \partial_{y_n}^2 \rho + 2\partial_{x_n y_n} \rho). \tag{2.3.7}$$

We have

$$\begin{aligned}
\partial_{x_n}^2 \rho &= \partial_{x_n} (\partial_{x_n} \rho) \\
&= \partial_{x_n} (2\tau \cdot \partial_{x_n} \tau) \\
&= 2\tau \cdot \partial_{x_n}^2 \tau + 2(\partial_{x_n} \tau)^2
\end{aligned} \tag{2.3.8}$$

In the last expression, we know everything except  $\partial_{x_n}^2 \tau$ . So we now find it:

$$g^{\alpha\beta} \partial_\alpha \tau \cdot \partial_\beta \tau + (\partial_{x_n} \tau)^2 = 1. \tag{2.3.9}$$

Applying  $\partial_{x_n}$  to the above equation, we have

$$\partial_{x_n} g^{\alpha\beta} \cdot \partial_\alpha \tau \cdot \partial_\beta \tau + 2g^{\alpha\beta} \partial_\alpha (\partial_n \tau) + 2\partial_n \tau \cdot \partial_{x_n}^2 \tau = 0 \tag{2.3.10}$$

Because the two points involved are along the first coordinate axis and the coordinate system is normal, the derivative  $\partial_\alpha \tau$  ( $\partial_\beta \tau$ ) is zero unless  $\alpha = 1$  ( $\beta = 1$ ). So for  $g^{\alpha\beta}$ , we only need to consider  $g^{11}$ . Since  $g_{\alpha\beta} = \delta_{\alpha\beta}$ , and  $g_{11} = -2$ , the entry of the inverse matrix  $g^{11} = 2$ . The equation now translates into

$$2 \cdot \cos^2 \frac{\theta}{2} + 2 \left( -\frac{1}{2} \cos \frac{\theta}{2} \right) \cdot \cos \frac{\theta}{2} + \left( -2 \sin \frac{\theta}{2} \right) \cdot \partial_{x_n}^2 \tau = 0.$$

Solving this equation for  $\partial_{x_n}^2 \tau$ , we have

$$\partial_{x_n}^2 \tau = \frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}. \quad (2.3.11)$$

Plug this equation into equation (2.3.8), and then we have

$$\partial_{x_n}^2 \rho = 2 \cos^2 \frac{\theta}{2} + 2 \sin^2 \frac{\theta}{2} = 2. \quad (2.3.12)$$

and  $\partial_{y_n}^2 \rho = 2$ .

Now we find  $\partial_{x_n y_n} \rho$ . We have

$$\begin{aligned} \partial_{x_n y_n} \rho &= \partial_{y_n} (2\tau \cdot \partial_{x_n} \tau) \\ &= 2\tau \cdot \partial_{x_n y_n} \tau + 2 \cdot \partial_{x_n} \tau \cdot \partial_{y_n} \tau. \end{aligned} \quad (2.3.13)$$

Apply  $\partial_{y_n}$  to the Eikonal equation (2.3.9), we have

$$2g^{\alpha\beta} \cdot \partial_{x_\alpha y_n} \tau \cdot \partial_{x_\beta} \tau + 2\partial_{x_n} \tau \cdot \partial_{x_n y_n} \tau = 0.$$

This now translates into

$$2 \left( -\frac{1}{2} \cos \frac{\theta}{2} \right) \cos \frac{\theta}{2} + \left( -2 \sin \frac{\theta}{2} \right) \cdot \partial_{x_n y_n} \tau = 0.$$

so

$$\partial_{x_n y_n} \tau = -\frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}.$$

and

$$\begin{aligned} \partial_{x_n y_n} \rho &= 4 \sin \frac{\theta}{2} \left( -\frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \right) + 2 \sin^2 \frac{\theta}{2} \\ &= -2 \cos^2 \frac{\theta}{2} + 2 \sin^2 \frac{\theta}{2} \\ &= -2 \cos \theta. \end{aligned}$$

Now equation (2.3.7) becomes

$$\begin{aligned} 2\partial_{x_n}^2 g_{11} &= \partial_{\theta}^2 (2 + 2 - 4 \cos \theta) \\ &= 4 \cos \theta. \end{aligned}$$

If we let  $\theta = 0$ , we get  $\partial_{x_n}^2 g_{11} = 2$ . This result agrees with equation (2.3.2). Now we finished recovering the  $C^2$  jet.

For  $C^3$  jet, the base equation is

$$2\partial_{x_n}^3 g_{11} = \partial_{\theta}^2 (\partial_{x_n}^3 + \partial_{y_n}^3 + 3\partial_{x_n}^2 \partial_{y_n} + 3\partial_{y_n}^2 \partial_{x_n}) \rho. \quad (2.3.14)$$

The relation between the derivatives of  $\rho$  and of  $\tau$  is

$$\begin{aligned} \partial_{x_n}^3 \rho &= \partial_{x_n} (2\tau \cdot \partial_{x_n}^2 \tau + 2(\partial_{x_n} \tau)^2) \\ &= 2\tau \cdot \partial_{x_n}^3 \tau + 6 \cdot \partial_{x_n} \tau \cdot \partial_{x_n}^2 \tau. \end{aligned} \quad (2.3.15)$$

We will find  $\partial_{x_n}^3 \tau$ . We apply  $\partial_{x_n}$  to equation (2.3.10), and get

$$\begin{aligned} & \partial_{x_n}^2 g^{\alpha\beta} \cdot \partial_\alpha \tau \cdot \partial_\beta \tau + 4\partial_{x_n} g^{\alpha\beta} \cdot \partial_{x_\alpha x_n} \tau \cdot \partial_{x_\beta} \tau \\ & \quad + 2g^{\alpha\beta} \cdot \partial_{x_\alpha} \partial_{x_n}^2 \tau \cdot \partial_{x_\beta} \tau \\ & + 2g^{\alpha\beta} \cdot \partial_{x_\alpha x_n} \tau \cdot \partial_{x_\beta x_n} \tau + 2(\partial_{x_n}^2 \tau)^2 + 2\partial_{x_n} \tau \cdot \partial_{x_n}^3 \tau = 0. \end{aligned} \quad (2.3.16)$$

We again can restrict our sights to  $\alpha = 1$  and  $\beta = 1$ , because otherwise the corresponding terms will be zero. So the equation above has the following form.

$$\begin{aligned} & \partial_{x_n}^2 g^{11} \cdot (\partial_\theta \tau)^2 + 4\partial_{x_n} g^{11} \cdot \partial_\theta \partial_{x_n} \tau \cdot \partial_\theta \tau \\ & \quad + 2g^{11} \cdot \partial_\theta \partial_{x_n}^2 \tau \cdot \partial_\theta \tau \\ & + 2g^{11} \cdot (\partial_\theta \partial_{x_n} \tau)^2 + 2(\partial_{x_n}^2 \tau)^2 + 2\partial_{x_n} \tau \cdot \partial_{x_n}^3 \tau = 0. \end{aligned} \quad (2.3.17)$$

We already know that  $g_{\alpha\beta}$ ,  $\partial_{x_n} g_{\alpha\beta}$ , and  $\partial_{x_n}^2 g_{\alpha\beta}$  all are multiples of the identity matrix, so an easy computation will give us the equation  $\partial_{x_n}^2 g^{11} = 6$ . Then we plug in all known expressions into equation (2.3.17), and have the following.

$$\begin{aligned} & 6 \cdot \cos^2 \frac{\theta}{2} + 4 \cdot 2 \cdot \left( -\frac{1}{2} \cos \frac{\theta}{2} \right) \cdot \cos \frac{\theta}{2} \\ & \quad + 2 \cdot \left( -\frac{1}{2} \cos^2 \frac{\theta}{2} - \frac{1 \cos^4 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} \right) \\ & + 2 \cdot \frac{1}{4} \cos^2 \frac{\theta}{2} + 2 \cdot \frac{1 \cos^4 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} + 2 \left( -\sin \frac{\theta}{2} \right) \cdot \partial_{x_n}^3 \tau = 0. \end{aligned}$$

Solve for  $\partial_{x_n}^3 \tau$ , and we have

$$\partial_{x_n}^3 \tau = \frac{1}{2 \sin \frac{\theta}{2}} \left( \frac{3}{2} \cos^2 \frac{\theta}{2} \right) = \frac{3 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2}},$$

and  $\partial_{y_n}^3 \tau = \frac{3 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2}}$ . Therefore

$$\begin{aligned} \partial_{x_n}^3 \rho &= 4 \sin \frac{\theta}{2} \cdot \frac{3 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2}} + 6 \left( -\sin \frac{\theta}{2} \right) \frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\ &= 0, \end{aligned}$$

and  $\partial_{y_n}^3 \rho = 0$ , too.

To find  $\partial_{x_n}^2 \partial_{y_n} \tau$ , we apply  $\partial_{y_n}$  to equation (2.3.10), and get

$$\begin{aligned} 2\partial_{x_n} g^{\alpha\beta} \cdot \partial_{x_\alpha y_n} \tau \cdot \partial_{x_\beta} \tau + 2g^{\alpha\beta} \cdot \partial_{x_\alpha x_n y_n} \tau \cdot \partial_{x_\beta} \tau \\ + 2g^{\alpha\beta} \cdot \partial_{x_\alpha x_n} \tau \cdot \partial_{x_\beta y_n} \tau \\ + 2\partial_{x_n y_n} \tau \cdot \partial_{x_n}^2 \tau + 2\partial_{x_n} \tau \cdot \partial_{x_n}^2 \partial_{y_n} \tau = 0. \end{aligned} \quad (2.3.18)$$

Again, take  $\alpha$  and  $\beta$  to be 1, and plug in all known expressions. We have

$$\begin{aligned} 2 \cdot 2 \left( -\frac{1}{2} \cos \frac{\theta}{2} \right) \cdot \cos \frac{\theta}{2} + 2 \cdot \left( \frac{1}{2} \cos^2 \frac{\theta}{2} + \frac{1 \cos^4 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} \right) \\ + 2 \cdot \frac{1}{4} \cos^2 \frac{\theta}{2} \\ + 2 \cdot \left( -\frac{1 \cos^4 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} \right) - 2 \sin \frac{\theta}{2} \cdot \partial_{x_n}^2 \partial_{y_n} \tau = 0. \end{aligned} \quad (2.3.19)$$

Solve for  $\partial_{x_n}^2 \partial_{y_n} \tau$ , and we get

$$\partial_{x_n}^2 \partial_{y_n} \tau = \frac{1}{2 \sin \frac{\theta}{2}} \left( -\frac{1}{2} \cos^2 \frac{\theta}{2} \right) = -\frac{1 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2}}.$$

The corresponding derivative of  $\rho$  is

$$\begin{aligned}
\partial_{x_n}^2 \partial_{y_n} \rho &= \partial_{y_n} (2\tau \cdot \partial_{x_n}^2 \tau + 2(\partial_{x_n} \tau)^2) \\
&= 2\partial_{y_n} \tau \cdot \partial_{x_n}^2 \tau + 2\tau \cdot \partial_{x_n}^2 \partial_{y_n} \tau + 4\partial_{x_n} \tau \cdot \partial_{x_n y_n} \tau \\
&= 2 \left( -\sin \frac{\theta}{2} \right) \frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} + 4 \sin \frac{\theta}{2} \left( -\frac{1}{4} \frac{\cos^2 \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right) \\
&\quad + 4 \left( -\sin \frac{\theta}{2} \right) \left( -\frac{\cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \right) \\
&= 0.
\end{aligned} \tag{2.3.20}$$

By symmetry, we also have

$$\partial_{y_n}^2 \partial_{x_n} \rho = 0$$

We put everything back into equation (2.3.14), which is

$$2\partial_{x_n}^3 g_{11} = \partial_\theta^2 (\partial_{x_n}^3 \rho + \partial_{y_n}^3 \rho + 3\partial_{x_n}^2 \partial_{y_n} \rho + 3\partial_{y_n}^2 \partial_{x_n} \rho) = 0.$$

This agrees with the equation (2.3.3), and now we finished the recovery of  $C^3$  jet.

# Chapter 3

## Counter-Examples of Lens

### Rigidity

#### 3.1 Examples of Different $C^2$ Jets

In this section we are going to give an example of two manifolds which have the same boundary and the same lens data but different  $C^\infty$  jets. The idea of the example is borrowed from [4] section 2, and [9] section 6. The idea in [4] and [9] is, if we have a surface of revolution with two circles as boundary, then in some sense, the lens data only depends on the measures of the sublevel sets of radius function along a meridian. We can find distinct smooth functions  $f_1, f_2$  both with domain  $[a, b]$ , such that they have the same measure for every sublevel set.

Before giving the example, we give the definition of lens data and lens equiva-



lence.

**Definition 3.1.1.** Let  $(M, \partial M, g)$  be a Riemannian manifold with boundary, and let  $\partial(SM)$  be the set of unit vectors with base point at boundary. Define set  $\Omega \subset \partial(SM) \times \partial(SM) \times \mathbb{R}^+$  to be the set of 3-tuples  $(\gamma'(0), \gamma'(T), T)$  that satisfies: (1)  $\gamma$  is a unit speed geodesic, (2)  $\langle \gamma'(0), \nu \rangle > 0$  *i.e.*  $\gamma'(0)$  points inwards, and (3)  $T$  is the first moment at which  $\gamma$  hits  $\partial M$  again. The description above depends on the interior structure, so we orthogonally project  $\partial(SM)$  to  $\overline{B(\partial M)}$  the closed ball bundle on  $\partial M$ . This projection maps  $\Omega$  to  $\Omega' \subset B(\partial M) \times \overline{B(\partial M)} \times \mathbb{R}^+$ .

We define *lens data* to be the information of  $\Omega'$  and  $\tau|_{\partial M}$ . We say two Riemannian manifolds with boundary are *lens equivalent* if they have the same boundary and the same lens data *i.e.* the same  $\Omega'$  and  $\tau|_{\partial M}$ . We say a manifold is lens rigid, if all lens equivalence between this manifold and another one can be represented as an isomorphism which fixes every boundary point.

Consider the strip  $S$  defined as  $\mathbb{R} \times [0, L]$ , with standard coordinates  $(x, y)$  where  $0 \leq y \leq L$ . Obviously  $S$  has a natural structure of manifold with boundary. Define a Riemannian metric  $g$  on  $S$  as

$$g_{yy} = 1, \quad g_{xy} = 0, \quad g_{xx} = f(y).$$

Here  $f : [0, L] \rightarrow \mathbb{R}$  is a smooth function, such that  $f(0) = f(L) = 1$  and  $f(y) \geq 1$  for all  $y \in (0, L)$ . Under certain circumstances, this manifold can be viewed as the universal cover of a surface of revolution in  $\mathbb{R}^3$ .

Obviously the curves  $\gamma_0(t) = (x_0, t)$  and  $\gamma_L(t) = (x_0, L - t)$  are unit speed geodesics, for any  $x_0 \in \mathbb{R}$ . So the normal (*i.e.* “ $n$ -th” in previous sections) direction is simply the  $y$  direction, and the  $C^\infty$  jet of  $S$  is determined by

$$\frac{\partial^k f}{\partial y^k}, \quad k = 1, 2, \dots$$

If we let index 1 stand for  $x$  and 2 for  $y$ , a straightforward computation of Christoffel symbols shows

$$\Gamma_{22}^2 = \Gamma_{12}^2 = \Gamma_{22}^1 = \Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 = -\frac{1}{2}f'(y), \quad \Gamma_{12}^1 = \frac{1}{2}\frac{f'(y)}{f(y)}.$$

Let  $(x(t), y(t))$  be a geodesic of  $S$  parameterized by arc length. Then it will satisfy the second order system

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt}, \quad k = 1, 2,$$

where  $i, j$  ranges over 1, 2, and  $x_1 = x$ ,  $x_2 = y$ .

**Lemma 3.1.2.** *Along a geodesic,  $\frac{dx}{dt} \cdot f(y)$  is constant.*

*Proof.*

$$\begin{aligned} \frac{d}{dt} \left( \frac{dx}{dt} \cdot f(y) \right) &= \frac{dx}{dt} \cdot \frac{d}{dt} f(y) + \frac{d^2 x}{dt^2} \cdot f(y) \\ &= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \cdot \sum_{i,j} \Gamma_{ij}^1 \frac{dx_i}{dt} \frac{dx_j}{dt} \\ &= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \cdot 2\Gamma_{12}^1 \frac{dx}{dt} \frac{dy}{dt} \\ &= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \cdot \frac{f'(y)}{f(y)} \frac{dx}{dt} \frac{dy}{dt} \\ &= 0. \end{aligned}$$

□

The Lemma above is Clairaut's relation when  $S$  is a surface of revolution. The Lemma does not require that the geodesic is unit speed, but from now on we assume all geodesics in discussion are of unit speed.

Since  $f(y)$  is never 0, we know either  $\frac{dx}{dt}$  is constant zero or never changes sign. Since  $g_{ij} = \delta_{ij}$  at the boundary, we know each geodesic leaves  $S$  at the same angle as when it enters  $S$ . Also, in each geodesic,  $|\frac{dx}{dt}|$  assumes its maximum on the boundary because  $f(y)$  is minimal there. Therefore, since

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 g_{xx} = 1,$$

we know  $\frac{dy}{dt}$  never changes sign in the interior. This means each entering geodesic transversal to the boundary goes all the way to the other component of the boundary, and hits the boundary with the "same" direction as it entered.

Suppose  $(x(t), y(t))$ ,  $t \in [0, T]$  is such a maximal geodesic, and without loss of generality we assume  $\frac{dy}{dt} > 0$  which is equivalent to the geodesic entering  $S$  at  $y = 0$  and leaving  $S$  at  $y = L$ . Since  $\frac{dy}{dt}$  is positive and smooth, we have

$$\begin{aligned} T &= \int_0^L \frac{dt}{dy} dy \\ &= \int_0^L \left(\frac{dy}{dt}\right)^{-1} dy \\ &= \int_0^L (1 - x'(t)^2 \cdot f(y))^{-\frac{1}{2}} dy \\ &= \int_0^L \left(1 - \frac{x'(0)^2}{f(y)}\right)^{-\frac{1}{2}} dy, \end{aligned}$$

and

$$\begin{aligned}
x(T) - x(0) &= \int_0^L \frac{dx}{dy} dy \\
&= \int_0^L \frac{dx}{dt} \left( \frac{dy}{dt} \right)^{-1} dy \\
&= \int_0^L \frac{x'(0)}{f(y)} \left( 1 - \frac{x'(0)^2}{f(y)} \right)^{-\frac{1}{2}} dy,
\end{aligned}$$

Now let's consider two different strips of this kind,  $S_1, S_2$  with  $L = 2\pi$  and

$$f_1(y) = 2 - \cos(y),$$

$$f_2(y) = 2 - \cos(2y).$$

Consider a geodesic in  $S_1$  and one in  $S_2$  entering them at the same location and same direction, *i.e.*  $x_1(0) = x_2(0)$  and  $x'_1(0) = x'_2(0)$ . Then obviously  $T_1 = T_2$  and  $x_1(T_1) = x_2(T_2)$ , because for each real number  $r$ , the sublevel sets  $\{f_1 \leq r\}$  and  $\{f_2 \leq r\}$  have the same measure.

If we take quotients of  $S_1$  and  $S_2$ , both by  $x$ -axis slides of multiples of  $100$ , we have two cylinders with identical lens data but different  $C^\infty$  jets. Furthermore, both are compact and analytic. If we want the boundary to be connected, we can take the quotients of the cylinders by an orientation-reversing involution, which gives us two Möbius bands.

**Theorem 3.1.3.** *There is an example of two analytic Riemannian manifolds with isometric boundaries and identical lens data, but different  $C^\infty$  jets at the boundaries.*

The examples  $S_1$  and  $S_2$  have same lens data, same  $C^1$  jet, but different  $C^2$  jet. If one wants a pair of examples of different  $C^1$  jets, then the idea still works, but to

construct an example we need to care about the smoothness at the peaks and the smooth extendability at boundary, as in the following section.

### 3.2 Examples of Different $C^1$ Jets

In this section we give an example of two manifolds which have the same boundary and the same lens data but different  $C^1$  jets. Knowledge of the  $C^1$  jet is equivalent to knowledge of the second fundamental form of the boundary as a submanifold, so different  $C^1$  jet means different “shape” of the embedding.

The setup is the same as in the previous section. The only modifications are the functions  $f_1$  and  $f_2$ . Let  $L = 14$ .

Let  $f_1 : [0, 14] \rightarrow \mathbb{R}$  be a smooth function that satisfies the following properties:

$$f_1(x) = x + 1, \text{ if } 0 \leq x \leq 1;$$

$$f_1'(x) > 0, \text{ if } x \in [1, 3), \text{ and } f_1'(3) = 0;$$

$$f_1(3 + t) = f_1(3 - t) \text{ if } t \in [0, 1];$$

$$f_1'(x) < 0, \text{ if } x \in (3, 6);$$

$$f_1(x) = 1, \text{ if } 6 \leq x \leq 7;$$

$$f_1(7 + t) = f_1(7 - t) \text{ if } t \in [0, 7].$$

We have some freedom of choice here, but it is crucial and possible to make  $f_1$  smooth. Intuitively,  $f_1$  starts at 1 and increases linearly in the first time period, near the peak  $f_1$  is symmetric, then it smoothly decreases to the constant function

1, and later it copies its own mirror image.

In order to define  $f_2$ , we define the non-increasing function  $H : [1, +\infty) \rightarrow [0, 6]$ ,

$$H(y) = m(\{x \in [0, 6] \mid f_1(x) \geq y\}),$$

where  $m(\cdot)$  is the Lebesgue measure.

We think of  $f_2$  as the “horizontal central lineup” of  $f_1$ . That is,  $f_2 : [0, 14] \rightarrow L$  should satisfy the following:

$$\text{if } x \in [0, 3), \text{ then } f_2(x) = y \text{ if and only if } x = 3 - \frac{H(y)}{2};$$

$$f_2(3) = f_1(3);$$

$$f_2(3+t) = f_2(3-t), \text{ if } t \in [0, 3];$$

$$f_2(x) = 1, \text{ if } x \in [6, 7];$$

$$f_2(7+t) = f_2(7-t) \text{ if } t \in [0, 7].$$

Obviously  $f_2$  is uniquely determined by  $f_1$ , and they have the same measure for every sublevel set. The only possibilities of non-smoothness of  $f_2$  are at 0, 3, 6, 8, 11, 14. We have  $f_2 = f_1$  near 3 and 11 from the symmetry of  $f_1$  near the peaks. It is not hard to see  $f_2$  is smooth at 6, because the graph of  $f_2$  near (6, 1) is a linear transformation of that of  $f_1$  near the same point. The smoothness near 8 is guaranteed for the same reason. Finally, from the symmetry of  $f_2$ , the smooth extendability of  $f_2$  at 0 and 14 immediately follows.

Observe that  $f_1'(0) = 1 = -f_1'(14)$  but  $f_2'(0) = f_2'(14) = 0$ , *i.e.* the boundary is concave in  $S_1$  but totally geodesic in  $S_2$ . We now use the same argument as in last

section. The strips  $S_1, S_2$  defined by  $f_1, f_2$  are lens equivalent, but have different  $C^1$  jets. If we want, we can take quotients to make the strips compact, and to make the boundaries connected, as in the last section.

**Theorem 3.2.1.** *There is an example of two Riemannian manifolds with isometric boundaries and identical lens data, but different  $C^1$  jets at the boundaries.*

# Chapter 4

## Lens Rigidity of Surfaces

### 4.1 Preliminaries

Throughout this chapter, we assume that the following statement is true.

**Statement 4.1.1.** *For a compact surface with boundary, the lens data determines the metric up to a conformal factor and isometry. That is, if  $(M, \partial M)$  is a surface, and two metrics  $g_1$  and  $g_2$  of  $M$  give rise to the same lens data, then there exist a smooth positive function  $\rho$  on  $M$  which equals 1 on  $\partial M$ , and a diffeomorphism  $f : M \rightarrow M$  which is identity on  $\partial M$ , such that  $g_1 = \rho^2 \cdot f^*(g_2)$ .*

For the definitions of lens data, lens equivalence, and lens rigidity, see Definition 3.1.1. It is reasonable to believe Statement 4.1.1, because the authors plan to prove it in the not yet published paper [10]. We want to take advantage of this result, and prove the following theorem.



**Theorem 4.1.2.** *Let  $(M, \partial M, g)$  be a compact surface with boundary, which has no trapped geodesics or conjugate points. Then  $M$  is lens rigid.*

We need the following statements.

**Proposition 4.1.3.** *If compact surfaces  $(M_1, \partial M, g_1)$  and  $(M_2, \partial M, g_2)$  are lens equivalent, and if neither of them has trapped geodesics, then there exists a diffeomorphism  $f : M_1 \rightarrow M_2$  which fixes every boundary point.*

*Proof.* See [3], section 3. We illustrate the idea here. Since none of the manifolds has trapped geodesics, there is a homeomorphism between  $UM_1$  and  $UM_2$  the unit vector bundles, which (is defined as the only map that) fixes all boundary unit vectors and preserve the geodesic flow. This homeomorphism gives a group isomorphism between  $\pi_1(UM_1)$  and  $\pi_1(UM_2)$ , and this isomorphism descends to an isomorphism between  $\pi_1(M_1)$  and  $\pi_1(M_2)$  because the respective kernels ( $\pi_1(S^1)$ ) can be matched along the boundary  $\partial M$ . Since 2-dimensional smooth manifolds with boundary can be characterized by their fundamental groups, there is a diffeomorphism between  $M_1$  and  $M_2$ . It is not hard to see that if this diffeomorphism respect the previous isomorphism between the fundamental groups, then we can make the diffeomorphism preserve all the boundary points.  $\square$

**Proposition 4.1.4.** *If compact surfaces  $(M_1, \partial M, g_1)$  and  $(M_2, \partial M, g_2)$  are lens equivalent, and suppose  $M_1$  has no trapped geodesics or conjugate points, then  $M_2$  has no trapped geodesics or conjugate points either.*

*Proof.* Obviously a trapped geodesic can be detected from lens data, because the presence of a trapped geodesic is equivalent to the length of geodesics starting at boundary points having no upper bounds. For the absence of conjugate points, see [10], Proposition 16. □

The following theorem can be found as Proposition 17 of [10].

**Theorem 4.1.5.** *Let  $M$  be a compact surface with or without boundary. Suppose  $M$  has no conjugate points. Let  $\gamma$  be a geodesic joining  $x$  and  $y$  in  $M$ . Then, for any curve  $\tau$  joining  $x$  and  $y$  that is path-homotopic to  $\gamma$ , we have  $L(\gamma) \leq L(\tau)$ . Here  $L$  stands for the arc length.*

The idea of the proof of Theorem 4.1.5 in [10] is a standard minimax argument. Assume  $\gamma$  is not minimizing, and then we find the path homotopy between  $\gamma$  and the minimizing path, such that the maximal energy of the intermediary paths is minimal among all path homotopies. Such a minimax path should be a path which is not locally the shortest, which contradicts the hypothesis that no conjugate points exist.

## 4.2 The Systole Approach

In this section we prove Theorem 4.1.2.

We first introduce the notion of *systole* and then prove a lemma about it.

**Definition 4.2.1** (Systole). For a manifold with or without boundary, if its fundamental group is non-trivial, then its systole is defined as the greatest lower bound of the lengths of non-contractible loops. Write the systole of  $M$  as  $\text{Sys}(M)$ . We define the systole of a simply connected manifold to be  $+\infty$ .

We have the following lemma about systole. For the definition and properties of normal covering spaces, see [13] section 1.3.

**Lemma 4.2.2.** *Let  $M$  be a compact surface with or without boundary. Then for any positive number  $C$ , we can find a compact normal covering space  $\widetilde{M}$  of  $M$ , such that  $\text{Sys}(\widetilde{M}) > C$ .*

*Proof.* Let  $\widehat{h} : \widehat{M} \rightarrow M$  be the universal covering of  $M$ , and let  $\widehat{p}$  be a fixed point of  $\widehat{M}$ . Let  $p = \widehat{h}(\widehat{p})$ . Then every covering space of  $M$  corresponds to a subgroup of  $\pi_1(M, p)$ .

Let  $D$  be the diameter of  $M$ . We define the finite set  $\widehat{P}$  to be

$$\{\widehat{q} \in \widehat{M} \mid \widehat{h}(\widehat{q}) = p \text{ and } \widehat{d}(\widehat{q}, \widehat{p}) \leq C + 2D\},$$

where  $\widehat{d}(\cdot, \cdot)$  is the distance function on  $\widehat{M}$ . The set  $\widehat{P}$  corresponds to a finite subset of  $\pi_1(M, p)$ , which we call  $\Gamma$ . So  $\Gamma$  is given by

$$\Gamma = \{[\gamma] \in \pi_1(M, p) \mid \gamma \text{ is a loop with base point } p \text{ and length } \leq C + 2D\}.$$

By the result of [14], we know the fundamental groups of surfaces are residually finite, which means for each non-identity element in the group, there is a normal

subgroup of finite index not containing that element. For each element  $\alpha \in \Gamma$ , we can find a  $\Phi_\alpha$  which is a finite-index normal subgroup of  $\pi_1(M, p)$  not containing  $\alpha$ . Now let

$$\Phi = \bigcap_{\alpha \in \Gamma} \Phi_\alpha.$$

As a finite intersection of finite-index normal subgroups,  $\Phi$  is again a finite-index normal subgroup of  $\pi_1(M, p)$ . Now  $\Phi$  corresponds to a normal covering of  $(M, p)$ , say  $(\widetilde{M}, \widetilde{p})$ . The claim is  $\text{Sys}(\widetilde{M}) > C$ .

Let  $\gamma : [0, 1] \rightarrow \widetilde{M}$  be a loop with length less than or equal to  $C$ . Let  $\widetilde{d}(\cdot, \cdot)$  be the distance function on  $\widetilde{M}$ . Since  $D$  is the diameter of  $M$ , we can find a lift of  $p$ , say  $\widetilde{q} \in \widetilde{M}$ , such that  $\widetilde{d}(\widetilde{q}, \gamma(0)) \leq D$ . This means we can find a path  $\tau$  from  $\widetilde{q}$  to  $\gamma(0)$ , with arc-length no more than  $D$ . Since the covering  $\widetilde{\pi} : \widetilde{M} \rightarrow M$  is normal, we can find a deck transformation  $F : \widetilde{M} \rightarrow \widetilde{M}$  such that  $F(\widetilde{q}) = \widetilde{p}$ . We now look at the loop  $\sigma = F \circ (\tau \cdot \gamma \cdot \tau^{-1})$  based at  $\widetilde{p}$ , where “ $\cdot$ ” means path concatenation. The length of  $\sigma$  is less than  $C + 2D$ , but by the construction of  $(\widetilde{M}, \widetilde{p})$  we know that any loop based at  $\widetilde{p}$  must be contractible if its length is no more than  $C + 2D$ . So  $\sigma$  is contractible in  $\widetilde{M}$ , and hence  $\gamma$  is contractible, too. This means  $\text{Sys}(\widetilde{M}) > C$ .  $\square$

Now we can proceed to prove Theorem 4.1.2 of this thesis. The idea of the proof comes from [4]. Remember that the following proof needs Statement 4.1.1, which we have not used by now.

*Proof of Theorem 4.1.2.* Since  $M$  has no trapped geodesics, we can find an upper

bound  $L$  for the lengths of geodesics (because otherwise we can find an length-increasing sequence of unit-speed geodesics, where we can take a limit direction by compactness, and find a trapped geodesic). From Lemma 4.2.2, we can find a normal covering  $\tilde{\pi} : \tilde{M} \rightarrow M$ , such that  $\text{Sys}(\tilde{M}) > 2L$ . Recall the main theorem of [3] states that if a manifold is lens rigid, then its finite quotients are lens rigid. So it suffices to show that  $(\tilde{M}, \partial\tilde{M}, \tilde{g})$  is lens rigid, because a normal covering map is a quotient map.

Suppose we have another surface  $(\tilde{M}_1, \partial\tilde{M}, \tilde{g}_1)$  with the same boundary, which is lens equivalent with  $(\tilde{M}, \partial\tilde{M}, \tilde{g})$ , then from Proposition 4.1.3, we may assume that  $\tilde{M}_1 = \tilde{M}$  as differential manifolds. Notice that we no longer need to deal with  $(M, \partial M, g)$ , so for clarity, from now on we write  $\tilde{g}$  as  $g$ , and  $\tilde{g}_1$  as  $g_1$ . Therefore, we have a surface  $\tilde{M}$  with boundary  $\partial\tilde{M}$ , and two metric tensors  $g$  and  $g_1$  which give the same lens data.

From Statement 4.1.1, there is a diffeomorphism  $f : \tilde{M} \rightarrow \tilde{M}$  fixing every boundary point, such that  $g = \rho^2 \cdot f^*(g_1)$  for some smooth function  $\rho : \tilde{M} \rightarrow \mathbb{R}^+$  that has value 1 at boundary. We let  $g_2 = f^*(g_1)$ , so  $g = \rho^2 g_2$ . By hypothesis,  $M$  has no conjugate points or trapped geodesics, so  $(\tilde{M}, g)$  and  $(\tilde{M}, g_2)$  have the same properties, too.

We need some notations for further reasoning. Let  $\mu_2$  be the measure on  $\tilde{M}$  given by metric  $g_2$ , and let  $U_2\tilde{M}$  be the  $g_2$ -unit vector bundle of  $\tilde{M}$ . Let  $N$  be the unit inward-pointing normal vector at appropriate boundary points. Let  $U^+(\partial\tilde{M})$  be

the set of inward-pointing ( $\langle \cdot, N \rangle > 0$ ) unit vectors on the boundary (no ambiguity concerning  $g$  or  $g_2$  because  $\rho = 1$  at boundary). Let  $\gamma_v$  be the  $g_2$ -geodesic with initial vector  $v$ , and  $l_2(v)$  is its  $g_2$ -length. Then by Santaló formula,

$$\begin{aligned} 2\pi \int_{\widetilde{M}} \rho(x) d\mu_2(x) &= \int_{U_2\widetilde{M}} |u|_g du \\ &= \int_{U^+(\partial\widetilde{M})} \int_0^{l_2(v)} |\gamma'_v(t)|_g \cdot \langle v, N \rangle dt dv. \end{aligned}$$

We now claim

$$\int_0^{l_2(v)} |\gamma'_v(t)|_g dt \geq \int_0^{l_2(v)} |\gamma'_v(t)|_{g_2} dt = l_2(v) \quad (4.2.1)$$

for every  $v \in U^+(\partial\widetilde{M})$ . Define  $\tau_v$  to be the  $g$ -geodesic with initial vector  $v$ . Since  $(\widetilde{M}, g)$  and  $(\widetilde{M}, g_2)$  are lens equivalent,  $\tau_v$  and  $\gamma_v$  have the same starting point and ending point. If  $\tau_v$  is path homotopic to  $\gamma_v$ , then by Theorem 4.1.5, the  $g$ -length of  $\gamma_v$  (the left side of inequality (4.2.1)) is greater than or equal to the  $g$ -length of  $\tau_v$ , which is the same as the  $g_2$ -length of  $\gamma_v$  (the right side of inequality (4.2.1)) from lens equivalence. If  $\tau_v$  is not path homotopic to  $\gamma_v$ , then we have a non-contractible loop  $\gamma_v \cdot \tau_v^{-1}$ . The  $g$ -length of the loop is at least  $2L$  because  $\text{Sys}(\widetilde{M}) > 2L$ , and the  $g$ -length of  $\tau_v$  is at most  $L$ . Therefore the  $g$ -length of  $\gamma_v$  (the left side of inequality (4.2.1)) is at least  $L$ , which is greater than or equal to the  $g_2$ -length of  $\gamma_v$  (the right side of inequality (4.2.1)). We have finished the proof of the claim  $\int_0^{l_2(v)} |\gamma'_v(t)|_g dt \geq \int_0^{l_2(v)} |\gamma'_v(t)|_{g_2} dt$ . We now have

$$\begin{aligned}
2\pi \int_{\widetilde{M}} \rho(x) d\mu_2(x) &= \int_{U^+(\partial\widetilde{M})} \int_0^{l_2(v)} |\gamma'_v(t)|_g \cdot \langle v, N \rangle dt dv \\
&\geq \int_{U^+(\partial\widetilde{M})} \int_0^{l_2(v)} |\gamma'_v(t)|_{g_2} \cdot \langle v, N \rangle dt dv \\
&= \int_{U^+(\partial\widetilde{M})} \int_0^{l_2(v)} 1 \cdot \langle v, N \rangle dt dv \\
&= \int_{U_2\widetilde{M}} 1 du \\
&= 2\pi \text{Vol}(\widetilde{M}, g_2).
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
\text{Vol}(\widetilde{M}, g) \cdot \text{Vol}(\widetilde{M}, g_2) &= \left( \int_{\widetilde{M}} \rho^2 d\mu_2 \right) \cdot \left( \int_{\widetilde{M}} 1 d\mu_2 \right) \\
&\geq \left( \int_{\widetilde{M}} \rho d\mu_2 \right)^2 \tag{4.2.2}
\end{aligned}$$

$$\geq \left( \text{Vol}(\widetilde{M}, g_2) \right)^2. \tag{4.2.3}$$

But from lens equivalence, we know  $\text{Vol}(\widetilde{M}, g) = \text{Vol}(\widetilde{M}, g_2)$ , which is a simple application of Santaló formula, see [6]. So the inequalities in (4.2.2) and (4.2.3) are in fact equalities. The equality of Cauchy-Schwarz (4.2.2) means that  $\rho$  is constant almost everywhere. Since  $\rho = 1$  at boundary and  $\rho$  is smooth, we have  $\rho \equiv 1$ . This means  $g_2 = g$  everywhere, and  $g = f^*(g_1)$ . So the lens data determine  $g$  up to a diffeomorphism fixing every boundary point, which is the definition of lens rigidity of  $(\widetilde{M}, g)$ .

Therefore, since lens rigidity is preserved after taking quotients,  $(M, g)$  is lens rigid. □

We have also reached the following conclusion.

**Proposition 4.2.3.** *Let  $(M, \partial M, g)$  be a compact surface with boundary, which has no trapped geodesics or conjugate points. Then  $M$  has a compact normal covering space  $\widetilde{M}$ , such that all geodesics in  $\widetilde{M}$  are length-minimizing paths.*



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