Lectures on Supermanifolds and Strings

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Abstract
Lectures presented at the Theoretical Advanced Study Institute, Brown University, June 1988 and at the Spring School on Superstrings, Trieste, April, 1988.

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1. Introduction

These lectures are intended as a fairly extensive introduction to the theory of supermanifolds and in particular super Riemann surfaces. I will assume a fair degree of familiarity with differential geometry and some basic definitions from algebra, as well as a nodding acquaintance with supersymmetry as practiced in the physics literature. I will try to keep the presentation both informal and honest — something of a tightrope. On the one hand some of the nicest material cannot honestly be called informal, and had to be omitted. On the other hand some shortcuts cannot even informally be called honest, and so some sections have become a bit longwinded. My defense is simply that recent events in string theory have shown us that naiveté has its perils in supermanifold theory.

Fermionic string perturbation theory is still not in anything like its final form. What we have been finding this year is that in order to make progress one needs a more geometrical approach to the super Riemann surfaces on which superconformal field theories are formulated. The traditional approach to computing string amplitudes converts the integrals over supermoduli to integrals over ordinary moduli at the expense of introducing ‘picture-changing’ operators [1]. This procedure however is known to suffer from an ambiguity whose form is a total derivative [2]. It has subsequently become clear [3][4] that the origin of this ambiguity is the coordinate-dependence of integration on supermanifolds, described briefly in sect. 3.5 below. While in certain cases the ambiguity seems to vanish [5], to overcome the problem in general will almost certainly require that we adopt a more intrinsic definition of the string integrand. Steps in this direction have been taken in [6][7][8], but the story is not over yet.

Even if one can formulate perturbation theory in components, it seems clear that a manifestly supersymmetric version will be more elegant. It is certainly of great practical utility for certain questions, as I will try to show in chapter 5. Moreover a superspace formulation of the operator formalism for string theory exists [8] and may point the way to a nonperturbative extension.

Finally, the study of supermanifolds is of considerable mathematical interest, especially in complex case. Not everything you know about ordinary manifolds (and in particular ordinary Riemann surfaces) goes through in the super case! There are plenty of issues where somebody needs to have a good idea. I hope that some of you take that as a challenge.

My knowledge about supermanifolds has mainly percolated down from discussions with colleagues. I especially want to thank L. Alvarez-Gaumé, J. Bernstein, J.-B. Bost, R. Bott, P. Deligne, V. Dellapietra, E. D'Hoker, D. Friedan, S. Giddings, C. Gomez, D. Kazhdan, G. Moore, A. Morozov, J. Polchinski, J. Rabin, M. Rothstein, I. Singer, C. Vafa, and H. Verlinde for discussions and collaboration over the years.1

1.1. History

The theory of supermanifolds is becoming a classical subject. There seems to be general agreement that the first seeds were planted with Schwinger’s article describing fermion amplitudes in the classical limit [9]. Slowly people began to understand that such amplitudes could be written as formal functional integrals over a function “space” with anticommuting parameters, and that there was an associated classical dynamics on phase “spaces” of this type. Both points of view are developed in two remarkable papers by J.L. Martin in 1959 [10]. Martin’s work seems to have lain unnoticed for some time. Some of the ideas, and others, were also given by F.A. Berzin [11][12], who focused on the formal similarity between the rules of functional integral and derivative for Bose versus Fermi fields. Starting from this similarity one is tempted to unify the formulas and extend them to other constructions of analysis and geometry, to obtain versions where commuting and anticommuting elements play nearly symmetrical roles. This program was started by Berzin and Kać, who reinvented Lie superalgebras [10][13] and began to classify these [14].

Meanwhile physicists were discovering the useful properties of certain field theories with equal numbers of Bose and Fermi fields, both in their own right

1 Other acknowledgements appear at the end.
and to get a world-sheet interpretation of fermionic string theories [16]. At once it became clear that the nice properties of these models were due to a symmetry mixing bosons and fermions, just as rotations mix various spin components of a single boson or fermion [15][17][18]. Wess and Zumino made the remarkable discovery that not only the $2$T, but also the $4d$ conformal algebra as well as the Lorentz group admit a super extension. They found a renormalizable Lagrangian model, realizing such a symmetry, with a surprising cancellation of field theoretical divergencies [19].

Any theory with bosons and fermions has a field space with even and odd coordinates — a field superspace. Finding appropriate field superspaces and actions to build supersymmetric theories was at first quite an art. Salam and Strathdee, and Ferrara, Wess and Zumino made the decisive step towards sorting out the situation in 1974 [20][21]. They noted that if we discard spacetime itself and replace that by a superspace then there emerges a field superspace. If moreover the new spacetime has a superalgebra of symmetries then we quite naturally obtain corresponding supersymmetry transformations on field space. The situation is entirely analogous to the way that Lorentz symmetry of flat spacetime gives symmetry transformations on fields. As usual, covariant methods then allow us to write down invariant actions for supersymmetric systems.

The success of the above program for flat $R^4$ spacetime, and for the string [22], lent impetus to the study of more general supermanifolds. Actually this program was already well underway. A definition similar to that adopted here was given by Berezin and Leites [23][24] and by Kostant [25].

The original papers released a torrent of pent-up formalism. Here I will mention only an alternate approach to the original notion of supermanifold. This was initiated in various versions by B. DeWitt, A. Rogers, and A.S. Schwars [26]–[28] and elaborated in [29]–[34].

A branch of supermanifold theory of particular interest to string theorists is the study of super Riemann surfaces. Just as the flat superspace $R^{d4}$ was invented as the proper home for field theories with rigid supersymmetry, so SRS are the natural home for two-dimensional theories with a much larger symmetry: superconformal invariance. Early sources for SRS include [35][36]; however, the main ideas were almost present much earlier in the supergravity literature. The papers [37][38] added considerable detail to the picture. Many more references appear below and in [39].

Important early reviews of supermanifold theory include [40][25][41]–[43]. One of many recent reviews is the book of Berezin [44]. Reviews of SRS theory include [36] and [45].

1.2. Outline

The plan of the present notes is as follows. The reader is given fair warning that each chapter assumes a bit more background than the one preceding it.

Chapter 2 introduces the notion of a supermanifold, beginning with Martin’s search for an extension of classical dynamics. From this point of view it is very natural to extend the definition of manifold by fooling around with the set of functions, and so we are led fairly directly to the definition to be used throughout these notes. We extend such notions as vector fields and maps of smooth supermanifolds, and then formally define split, projected, and general supermanifolds. In 2.6 we make the corresponding remarks for the complex analytic case. With the rebirth of string theory this case has come to the fore; as we will see it is more subtle than the smooth case.

Chapter 3 is devoted to the basic structure of supermanifolds and to invariants with no ordinary analog. We also develop more differential geometry and discuss the problem of integration. Chapter 4 introduces super Riemann surfaces and the main structures unique to them. Chapter 5 describes an application of SRS theory to the operator formalism for string theory, and chapter 6 gives a simple example of the compactification of supermoduli space$^2$.

Throughout these notes our approach to SRS theory is analytic or differential-geometric. A promising analog of the algebraic approach to Riemann surfaces has recently emerged [46][47], but I won’t cover it here. Nor will I say much about the supergravity approach, which has recently been reviewed in [7] (see [39][45] for the link to the analytic approach).

$^2$ Chapters 5–6 were originally given as lectures at Trieste in May, 1988.
2. Supermanifolds

2.1. Grassmann algebras

Recall that an algebra is a system with multiplication and addition (i.e. a ring) as well as a scalar multiplication by numbers (a vector space). For example given any real finite-dimensional vector space \( V \) we can build the tensor algebra \( \otimes^* V \) consisting of all the \( n \)-fold tensor products of vectors in \( V \), for all \( n \geq 0 \).\(^3\) If we replace the tensor product by the antisymmetrized tensor, or wedge, product, we get the Grassmann algebra \( \Lambda^* V \). As a real vector space \( \Lambda^* V \) has dimension \( 2^q \) where \( q = \dim V \). As an algebra it is generated by any basis of \( q \) vectors in \( V \), along with the fixed generator \( 1 \) of \( \Lambda^0 V = \mathbb{R} \).

The Grassmann algebra is clearly associative: \((\omega \wedge \eta) \wedge \psi = \omega \wedge (\eta \wedge \psi)\). Furthermore it is graded:

\[
\Lambda^* V = \Lambda^0 V \oplus \Lambda^1 V \oplus \ldots \oplus \Lambda^q V .
\]  
(2.1)

Thus as a vector space it has a basis of vectors each having definite weight \( k \), \( 0 \leq k \leq q \). By an abuse of language we will express (2.1) by saying \( \Lambda^* V \) is graded by the integers \( \mathbb{Z} \). There are no \((q+1)\)-vectors, since the square of anything in \( V = \Lambda^1 V \) is zero. We will generally drop the wedge symbol below.

The multiplication law in the Grassmann algebras is not commutative. However, the product \( \omega \phi \) is related to \( \phi \omega \) in a simple way if \( \omega \) and \( \phi \) are both homogeneous, i.e. if they each belong entirely to one of the vector subspaces \( \Lambda^k V \). In this case one has

\[
\omega \phi = (-)^{|\omega||\phi|} \phi \omega
\]  
(2.2)

where \( |\omega| \) denotes the value of \( k \) above. If \( \omega \) or \( \phi \) is not homogeneous we can uniquely expand it as the sum of homogeneous terms and apply (2.2). Note that all that matters is the value of \( |\omega|, |\phi| \) mod 2! Hence we can adopt a coarser grading of \( \Lambda^* V \):

\[
\Lambda^* V = \Lambda_{ev} V \oplus \Lambda_{od} V .
\]  
(2.3)

We say that \( \Lambda^* V \) is graded by the group \( \mathbb{Z}_2 \), since clearly \(|\phi \omega| = |\phi| + |\omega| \) mod 2. We express (2.2) by saying that the algebra \( \Lambda^* V \) is \( \mathbb{Z}_2 \) graded-commutative.

2.2. Generalized classical dynamics

To apply algebras of this type we will follow Martin’s reasoning [10]. We begin by recalling some facts from Hamiltonian dynamics. The key point is to shift attention from the trajectory of a system through a configuration space to the evolution of the various observables describing the system. In practice this works as follows. We convert Newton’s law from a second order to a first order system of differential equations by the usual trick, replacing configuration space \( N \) by a space \( TN \) of positions and velocities, or equivalently a “phase space” \( \Gamma \equiv T^* N \) of positions and momenta. Since \( \Gamma \) contains all the information about the state of a system, all observables are functions on \( \Gamma \). Two such functions can be added or multiplied by point by point on \( \Gamma \) to get a third; for example the angular momentum is defined by \( \vec{L} = \vec{r} \times \vec{p} \), a function on \( \Gamma \). Thus the set

\[
\mathcal{A} = C^\infty (\Gamma)
\]

of all smooth functions on \( \Gamma \) — all observables — is an algebra.

Hamiltonian dynamics puts a second product structure on \( \mathcal{A} \), the Poisson bracket. Now we can think of the evolution of a system in two ways:

a) “Schrodinger”: given a special observable \( H \in \mathcal{A} \), the hamiltonian, we get a vector field \( X_H \) on \( \Gamma \) via Hamilton’s equations. It tells each state where to go next.

b) “Heisenberg”: given \( H \in \mathcal{A} \) we get a flow on \( \mathcal{A} \) itself. That is, for every observable \( f \) we find the time evolution of the observed value of \( f \) by leaving the state fixed but considering a family \( f_t \). We solve for the family by solving

\[
\frac{d}{dt} f_t = \{H,f_t\}
\]  
(2.4)

Since the flow in (a) has time derivative \( \dot{f} = X_H f \), and since

\[
X_H f = \{H,f\}
\]  
(2.5)
is a definition of the Poisson bracket, the two viewpoints are equivalent.

Eqn. (2.4) makes sense by virtue of two key properties of the poisson bracket. First, since the time derivative of $f_1 f_2$ should obey the product rule, we need

$$\{a b, c\} = a \{b, c\} + \{a, c\} b$$

(2.6)

This property is automatically satisfied by brackets defined as in (2.5). Similarly the time derivative of $\{f_1, f_2\}$ should be $\{f_1, f_2\} + \{f_1, f_3\}$; asking moreover that the hamiltonian be no different from any other function leads to

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad ,$$

(2.7)

the Jacobi identity. Eqn. (2.7) implies that the contact transformations —the flows of the various vector fields $X_f$ corresponding to observables— form a closed Lie algebra.

Starting from version (b) of dynamics, we can now proceed to generalize classical mechanics by discarding phase space! This of course is exactly what quantum mechanics does; the uncertainty principle says that the points of $\Gamma$ have no physical meaning. Instead one takes $\hat{A}$, with its product and bracket structures, and insofar as possible attempts to realize them as products and commutators of operators on some linear space.

We won’t do anything that drastic. We should however keep in mind that we can modify (b) in ways having no literal analog like (a), as long as a good quantization procedure is retained. This was Martin’s insight.

Let’s consider the dynamics of a single pinned spin. In classical dynamics we have a configuration space which is the group $SO(3)$ and generators $\hat{L}$ defined above satisfying

$$\{L_i, L_j\} = \epsilon_{ijk} L_k \quad .$$

When we quantize this system, however, we invariably get integer values for the spin. What system is the classical limit of a pinned spin-1/2 particle?

Consider a generalization of $\mathcal{A}$ to $\hat{\mathcal{A}}$, a $\mathbb{Z}_2$-graded-commutative ring; specifically consider $\hat{\mathcal{A}} = \Lambda^*(\mathbb{R}^3)$. Then $\hat{\mathcal{A}}$ has generators $1, \xi^i$ $i = 1, 2, 3$ satisfying $\xi^i \xi^j = -\xi^j \xi^i (\xi^1)^2 = 0$, etc. We want to give $\hat{\mathcal{A}}$ a Poisson bracket structure satisfying the analogs of (2.6) and (2.7), but corresponding to anticommutators in quantum mechanics. When applied to odd generators we want

$$\{a, b\} = -(-)^{|a| b^{|a|}} \{b, a\} \quad ,$$

(2.8)

to get the Pauli exclusion principle. For consistency we then must require

$$\{a, b\} = a \{b, c\} + (-)^{|a| b^{|a|}} \{a, c\} b$$

(2.9)

$$(-)^{|a| b^{|a|}} \{a, b\} + (-)^{|b| a^{|b|}} \{b, c\} + (-)^{|c| a^{|c|}} \{c, a\} = 0 \quad .$$

(2.10)

We can summarize equations like (2.2), (2.8), (2.9) by saying that whenever something passes through something we get an extra $(-)^{|a| b^{|a|}}$ relative to the usual formula. Eqn. (2.10) is then the consistent modification to (2.7); it is called ‘superjacobi identity’. An algebra satisfying it is called Lie superalgebra. Thus we require a Poisson bracket turning $\hat{\mathcal{A}}$ into a Lie superalgebra. If now $\hat{H}$ is any even element of $\hat{\mathcal{A}}$, then the equation of motion $\dot{\xi} = \{H, \xi\}$ still makes sense thanks to (2.9), (2.10). From the abstract point of view, then, dynamics based on a Lie superalgebra of observables is potentially interesting. The observables commuting with $\hat{H}$ themselves from an algebra of symmetries which could well have odd generators — supersymmetries.

In our example we define the algebra by declaring

$$\{\xi^i, \xi^j\} = -\frac{i}{2} \epsilon^{ij} \quad .$$

(2.11)

If we choose the Hamiltonian to be $\hat{H} = -iB^i \xi^i \xi^k \eta_{jk}$, where $\vec{B}$ is a real vector field, then the equation of motion is $\dot{\xi} = \{H, \xi\} = (\vec{B} \times \vec{\xi})^i$, which indeed describes precession of a spin. Furthermore we can easily realize the algebra (2.11) on a vector space, and hence quantize the system: we can take our Hilbert space to be just $\mathbb{C}^2$, and the operators representing $\xi^i$ to be the Pauli matrices. Thus we recover the fact that upon quantization this system has spin 1/2, something impossible for a system coming from ordinary classical dynamics. To avoid nonsense we can concede that the $\xi^i$ themselves are not observable; only the energy, angular momentum, and other even quantities are true observables.
What has just happened? We abstracted from classical dynamics an algebra \( A \) of observables, a Lie structure on \( A \), and a distinguished observable \( H \). We then modified \( A \) to something not of the form \( C^\infty(\Gamma) \) for any phase space \( \Gamma \), generalized the bracket, and got a "reasonable" description of the classical limit of spin 1/2. In passing, I should mention that the present approach is very natural from the point of view of constrained dynamics, which can now be generalized quite simply to the Grassmannian case.

Still we're not as happy as we could be, however. The bracket structure on \( \hat{A} \) came from nowhere, instead of expressing the geometry of some phase space.

### 2.3. The "space" \( \mathbb{R}^{pl} \)

It would be nice if we could generalize the notion of manifold so that in a sense the new kinds of algebras \( \hat{A} \) above could again be regarded as the "functions" on a "space". In fact it would be more than just nice — we need such a generalization in order to generate examples of \( \hat{A} \) which come from action principles. First note that the pinned spin discussed above generalizes easily to a nonrelativistic spinning point particle. We simply let

\[
\hat{A} = C^\infty(\mathbb{R}^6) \otimes \Lambda^\ast(\mathbb{R}^3) \quad .
\]  

(2.12)

This algebra has generators \( 1, z^i, p^i, \xi^a \), where the \( z^i, p^i \) are even coordinates on \( \mathbb{R}^6 \). (We will make this precise in a moment.) They describe the particle's position and momentum while the \( \xi^a \) describe the spin as before.

That is, any \( f \) in \( \hat{A} \) has a unique expansion

\[
f = f_0(z^i) + f_{1\mu}(z^i)\xi^a + f_{2\mu
u}(z^i)\xi^a\xi^b + \cdots 
\]  

(2.13)

where \( f_0 \) is an ordinary smooth function on \( \mathbb{R}^6 \), \( f_{1\mu} \) are three ordinary functions, and so on\(^4\). We will write \( f \) as \( f(z^i, \xi^a) \), in keeping with our resolve to think of \( z^i, \xi^a \), and the rest of \( \hat{A} \) as all being generalized 'functions' on some 'space'. But what space?

---

\(^4\) Strictly speaking the \( z^i \) aren't generators unless we restrict to real-analytic (not smooth) functions. Eqn. (2.13) is still valid, however.

We 'solve' this problem semantically. The superspace \( \hat{M} = \mathbb{R}^{pl} \) is a fiction; it refers precisely to the ring \( \hat{A} \) in (2.12). There is no underlying set of points whose functions is \( \hat{A} \). We simply agree to treat the two kinds of generator of \( \hat{A} \) symmetrically insofar as possible. We now want to explore the extent to which this is a useful fiction, i.e. the extent to which the constructions of geometry can be phrased in terms of the ring \( \hat{A} \) without specific reference to the (nonexistent) "points" of \( \hat{M} \). We have already seen that this is a very natural generalization from the viewpoint of Hamiltonian dynamics.

The good news is that eventually most of geometry will admit such a formulation. Indeed the beginning of wisdom in algebraic geometry is precisely to focus on the ring of functions, and constructions where this ring contains nilpotents are already well known. The generalisation from commutative to graded commutative rings is often more a notational headache than a problem of principle.

Before we come to the formal definition of a supermanifold, consider the notion of tangent vector. Since \( \hat{M} \) is not a set of points, we don't expect any good notion of a tangent vector at a point. But the notion of an ordinary vector field can be phrased entirely in terms of the ring \( \hat{A} \), and so we expect a simple generalisation to \( \hat{A} \). Namely, we can think of \( X \) as a directional derivative; more precisely, vector field \( X \) is a derivation of \( \hat{A} \), an \( \mathbb{R} \)-linear map of \( \hat{A} \) to itself denoted by \( f \rightarrow Xf \), the derivative of \( f \) along \( X \). It satisfies the product rule, \( X(fg) = (Xf)g + f(Xg) \). Similarly on \( \hat{A} \) we define a (left) vector field as a map \( f \rightarrow Xf \) satisfying

\[
X(fg) = (Xf) \cdot g + (-1)^{\|X\|f} f \cdot (Xg) 
\]  

(2.14)

Clearly the set \( \hat{T} \) of all such derivations is in fact an infinite-dimensional vector space over \( \mathbb{R} \). But in fact we also know how to multiply \( X \) by any super function \( h \) in \( \hat{A} \); just let \( (hX)(f) = h(Xf) \), again a derivation. So \( \hat{T} \) becomes a "vector space" over \( \hat{A} \) itself. Actually we cannot use the words 'vector space' because \( \hat{A} \) is not a field. Instead \( \hat{T} \) is called a free module over the ring \( \hat{A} \). Here 'module' means what was just said, that \( \hat{A} \) acts on \( \hat{T} \) as a scalar product. 'Free' means every vector field can be uniquely expanded in
a basis. This is clear classically: \( T \) is in fact finite-dimensional over \( \mathbb{A} \), since any vector can be expanded in terms of a basis as \( X = X^i \frac{\partial}{\partial x^i} \), where \( X^i \) are ordinary functions and \( x^i \) are coordinates for \( M \). Similarly in the super case we have \( X = X^i \frac{\partial}{\partial x^i} + X^\mu \frac{\partial}{\partial \xi^\mu} \). Here \( X^i, i = 1, \ldots, p, X^\mu, \mu = 1, \ldots, q \) are functions in \( \mathbb{A} \). If \( X \) is homogeneous then the \( X^i \) have the same parity as \( X \) while the \( X^\mu \) have the opposite parity. The elementary derivations \( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^\mu} \) are defined by their actions on the coordinate superfunctions:

\[
\frac{\partial}{\partial x^i}(x^j) = \delta^i_j \quad \frac{\partial}{\partial \xi^\mu}(x^j) = 0
\]

\[
\frac{\partial}{\partial x^i}(\xi^\mu) = 0 \quad \frac{\partial}{\partial \xi^\mu}(\xi^\nu) = \delta^\mu_\nu.
\]  

(2.15)

These actions extend to more complicated functions via the expansion (2.13) and the rule (2.14). For example \( \frac{\partial}{\partial x^i}(\xi^j) = -\xi^j \), and so on. With these rules we find that \( [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0 \) as usual, and \( [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^\mu}] = 0 \), but the anticommutator \( [\frac{\partial}{\partial \xi^\mu}, \frac{\partial}{\partial \xi^\nu}]_{\pm} = 0 \) is what vanishes for odd vectors. We will always just write \([ \cdot, \cdot \] to mean the commutator when either entry is even or the anticommutator when both are odd. With this bracket the vector fields \( \mathfrak{g} \) become a Lie superalgebra; the reader should check (2.8) and (2.10).

Thus the free module \( \mathfrak{g} \) has a basis of \( p \) even and \( q \) odd elements, corresponding to the dimension of \( \mathbb{M} = \mathbb{R}^{p,q} \). We say the rank of \( \mathfrak{g} \) is \( p|q \).

All the definitions so far suffer from the obvious drawback that most manifolds don't admit a single good coordinate chart. In the next section we will develop a global picture.

First however let's consider another example of a concept from geometry which can be expressed in terms of functions, namely a map from \( M = \mathbb{R}^p \) to \( M' = \mathbb{R}^{p'} \). Given \( \phi : M \to M' \), we can take any function \( f \) on \( M' \) and pull it back to a function \( \phi^* (f) \) on \( M \): simply let \( \phi^* f = f \circ \phi \). If \( \phi \) is smooth then \( \phi^* \) takes \( \mathcal{A}_{M'} \) to \( \mathcal{A}_M \). Moreover \( \phi^* \) is clearly a ring homomorphism. The reader should pause to verify that if \( M \xrightarrow{\phi} M' \xrightarrow{\psi} M'' \) then \( (\psi \circ \phi)^* = \phi^* \circ \psi^* \).

To generalize this, consider \( \mathbb{M} = \mathbb{R}^{p|q}, \mathbb{M}' = \mathbb{R}^{p'|q'} \). If a map \( \phi : M \to M' \) is given we can define a map \( \tilde{\phi} : \mathbb{M} \to \mathbb{M}' \) precisely by saying what \( \tilde{\phi}^* \) should do to the functions \( \mathcal{A}_M, \mathcal{A}_M' \). Just as \( \mathbb{M} \) is a fiction, so is \( \tilde{\phi} ; \) we stress that \( \tilde{\phi} \) means no more and no less than the pair \( (\phi, \tilde{\phi}^*) \) consisting of an honest map \( \phi \) and a ring homomorphism \( \tilde{\phi}^* \). There is a compatibility condition between \( \phi, \tilde{\phi}^* \) to be described momentarily. Since \( \tilde{\phi}^* \) is supposed to be a homomorphism, we need only tell what it does to the coordinates \( x^i, \xi^\mu \); the rest will then be determined.

If \( \tilde{\phi}^* \) is to respect the multiplication laws on \( \mathcal{A}_M, \mathcal{A}_M' \), then in particular it must respect the \( \mathbb{Z}_2 \)-grading: even functions should go to even and odd to odd. But recall that \( \Lambda^*(\mathbb{R}^p) \) and hence \( \mathbb{M} = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^p) \), has more structure than just a \( \mathbb{Z}_2 \)-grading: \( \mathbb{M} \) is actually graded by integers, as in the expansion (2.13). So we are faced with a choice: to allow only those \( \tilde{\phi}^* \) which preserve the \( \mathbb{Z}_2 \)-grading, or to allow any \( \tilde{\phi}^* \) which preserve the grading only mod 2. Both cases are interesting. The former are called split maps of \( M \) to \( M' \); they generally look like

\[
\tilde{\phi}^*(x^i) = h_0(x^i)
\]

\[
\tilde{\phi}^*(\xi^\mu) = \xi^\nu h_{\nu \lambda}(x^i)
\]  

(2.16)

Here \( h_0 \) and \( h \) are sets of smooth functions on \( \mathbb{R}^p \). We further demand that \( h_0 \) should reproduce the original ordinary map \( \phi : M \to M' \); this is the compatibility condition mentioned earlier. The form of (2.16) is dictated when we demand that \( x^i \), of weight zero, goes to weight zero and similarly \( \xi^\mu \) goes to weight one.

General maps then admit an expansion

\[
\tilde{\phi}^*(x^i) = h_0(x^i) + \xi^\nu h_{\nu \lambda}(x^i) + \ldots
\]

\[
\tilde{\phi}^*(\xi^\mu) = \xi^\nu h_{\nu \lambda}(x^i) + \xi^\nu \xi^\rho h_{\rho \lambda \nu}(x^i) + \ldots.
\]  

(2.17)

This time weight even \( \to \) even, odd \( \to \) odd. Again \( h_0 \) must agree with the underlying map of points \( \phi \).

---

5 These are not Poisson brackets, so we do not expect an analog of (2.9).

6 Some authors use the word morphism where we use map.
In either case, a diffeomorphism is defined in the usual way: it's a smooth map whose inverse is again a smooth map. The action of \( \tilde{\phi}^* \) on an arbitrary smooth function can be deduced even if that function is not a polynomial in \( z^I \).

Simply expand \( f \) out in a terminating Taylor series in the odd variables:

\[
\tilde{\phi}^*(f) = f_0 \circ \tilde{h}_0 + \xi^\mu \tilde{h}^\mu_\nu (f_\mu \circ \tilde{h}_0) + \xi^\mu \xi^\nu \tilde{h}^\mu_\nu \left. \left( \frac{\partial f_0}{\partial x^\mu} \circ h_0 \right) \right| + (f_\mu \circ \tilde{h}_0) + \cdots
\]

where \( f = f_0 + \xi^\mu f_\mu + \cdots \). Since \( f_0, f_\mu, \ldots \) are differentiable this makes sense; the reader should check that it gives us a ring homomorphism.

For completeness I should list a third kind of map, a middle ground between split and general. These are the projective maps, generally of the form

\[
\tilde{\phi}^*(x^I) = h_0(x) \\
\phi^*(\xi^I) = \xi \cdot h(x) + \xi \xi \cdot k(x) + \cdots
\]

The three classes of maps (2.16), (2.17), (2.19) are important because each closes under composition. In fact one can easily see that the composition split \( \circ \) split is split, split \( \circ \) projective and projective \( \circ \) projective is projective, and of course anything \( \circ \) general is general. The name 'projective' comes as follows. \( \mathbb{R}^{1|0} \) has a simple map to \( \mathbb{R}^{1|0} \), namely \( \tilde{\phi}^*(z^I) = z^I \), where \( z^I \) are coordinates for \( \mathbb{R}^{1|0} \). \( \phi^* \) is into, so we call \( \phi \) a projection. Maps of the form (2.19) are simply the ones which commute with \( p \), in the sense that

\[
\tilde{\phi}^* \tilde{\phi}^* = \tilde{\phi}^* \phi^*
\]

while in general (2.17) does not. We will return to this observation in the next section.

Physicists are often tempted to abbreviate the formula (2.17) to simply

\[
z^I = h_0(x^I) + \xi^\mu h^\mu_\nu (x^I) + \cdots
\]

etc. This leads to sticky questions. "How can the left side, which is 'purely real', equal the right side, which 'takes values in the Grassmann algebra'?" This isn't really the relevant question, though. For one thing, we have agreed not to think of \( \tilde{\mathcal{A}} \) as functions on \( \mathbb{R}^p \) with values in \( \Lambda^n(\mathbb{R}^p) \). Instead we think of \( \tilde{\mathcal{A}} \)

as real functions on a 'space' \( \mathbb{R}^{1|p} \), some of whose coordinates are black boxes. The real question we should ask about (2.20) is, "How can \( z^I \), an element of \( \tilde{\mathcal{A}}_{\mathbb{R}^p} \), be equated to an element of \( \tilde{\mathcal{A}}_{\mathbb{R}^p} \)?" These two rings are not naturally isomorphic. The answer, of course, is that strictly speaking (2.20) wrong; (2.17) is right. Now there is no paradox; the homomorphism \( \tilde{\phi}^* \) can perfectly well include higher-order terms. Once this is understood the reader should have no difficulty dealing with papers where (2.20) is adopted as shorthand for (2.17).

### 2.4. Smooth supermanifolds

An ordinary smooth manifold is a topological space \( M \), together with an atlas, or cover of \( M \) by open sets \( \mathcal{U}_a \) and homeomorphisms \( \phi_a : \mathcal{U}_a \to \mathcal{V}_a ; \mathcal{V}_a \) are open sets in \( \mathbb{R}^p \). (See Fig. 2.1.) Since \( M \) itself has no a priori smooth structure we cannot demand that the \( \phi_a \) be smooth. Instead let \( \mathcal{V}_{ab} = \mathcal{V}_a \cap \mathcal{V}_b \phi_a(\mathcal{U}_a \cap \mathcal{U}_b) \); then we require that \( F_{ab} \equiv \phi_a \circ \phi_b^{-1} \) be a smooth map of \( \mathcal{V}_{ab} \subseteq \mathcal{V}_a \subseteq \mathbb{R}^p \) to \( \mathcal{V}_{ab} \subseteq \mathcal{V}_b \subseteq \mathbb{R}^p \). This does make sense because \( \mathbb{R}^p \) does have a smooth structure. By their definition the \( F_{ab} \) obey the cocycle relations:

\[
(F_{ab})^{-1} = F_{ba} ; \quad F_{ab} \circ F_{bc} \circ F_{ca} = \text{identity}
\]

on the appropriate domains.

---

Fig. 2.1
We can now determine unambiguously whether a function \( f \) on \( M \) is smooth, even if \( f \) doesn't live in just one \( U_\alpha : f \) is smooth if \( (f|_{U_\alpha}) \circ (\phi_\alpha^{-1}) \) is a smooth function on \( \mathbb{R}^p \) for all \( \alpha \). In this way we can associate to any open set \( U \) a ring \( \mathcal{A}_U \) of all smooth function on \( U \). The machine \( U \to \mathcal{A}_U \) will simply be called \( \mathcal{A} \). We say that \( M \) is a topological space and \((M, \mathcal{A})\) is a smooth manifold built on \( M \).

Those who have studied this sort of thing will recognize that \( \mathcal{A} \) is a sheaf. I will try to get by without a systematic introduction to sheaves but I urge the reader to consult the short treatment in [48] or the very short treatment in [49]. The only comments I'll make here are that

a) If \( U' \subseteq U \), then we have a natural ring homomorphism \( \pi_* : \mathcal{A}_U \to \mathcal{A}_{U'} \), namely restriction. We will denote the restriction \( \pi^*(f) \) by \( f|_{U'} \).

b) \( \mathcal{A} \) is nice because it describes the smooth structure of \( M \) without committing us to any particular atlas of patches.

To define a supermanifold we will now play the same game as in the previous section: we augment each \( \mathcal{A}_{V_\alpha} \) and glue them together using maps of the form (2.16), (2.17), or (2.19). The resulting constructions we will call split, general, or projected supermanifolds.

For each \( \alpha \) let \( \mathcal{A}_{V_\alpha} = \mathcal{A}_{V_\alpha} \otimes \Lambda^*(\mathbb{R}^p) \). As usual we invent a fictitious superdomain \( \tilde{V}_\alpha \) whose ring of smooth functions is \( \mathcal{A}_{V_\alpha} \). For each nonempty overlap \( U_\alpha \cap U_\beta \) let \( \mathcal{A}_{V_{\alpha\beta}} = \mathcal{A}_{V_{\alpha\beta}} \otimes \Lambda^*(\mathbb{R}^p) \). We naturally get a restriction map \( \tilde{r} : \tilde{V}_{\alpha\beta} \to \tilde{V}_\alpha \), or \( \tilde{r}^* : \mathcal{A}_{V_{\alpha\beta}} \to \mathcal{A}_{V_\alpha} \). Namely the underlying map \( r \) is the inclusion \( V_{\alpha\beta} \subseteq V_\alpha \), while \( \tilde{r} \) is the usual restriction tensored with the identity.

Suppose we are now given smooth transition maps \( \tilde{F}_{a\beta} : \tilde{V}_{a\beta} \to \tilde{V}_{a\beta} \) whose underlying maps are the \( F_{a\beta} \). If the \( \tilde{F}_{a\beta} \) satisfy

\[
\tilde{F}_{a\beta} = (\tilde{F}_{a\beta})^{-1} ; \quad \tilde{F}_{\gamma \alpha} \circ \tilde{F}_{\alpha \beta} \circ \tilde{F}_{\gamma \beta} = \text{identity}
\]

then we say that they define a supermanifold \( \tilde{M} \). Note that (2.22) implies in particular that \( \tilde{F}_{a\beta} \) are all diffeomorphisms.

Referring back to Fig. 2.1, notice what we have and have not done. At the top of the figure we still have the same topological space \( \tilde{M} \) and the same open sets \( U_\alpha \). Just as we couldn't ask that \( \phi_\alpha \) be smooth maps before, so now we can't ask that they be super maps. We instead supply smooth super maps \( \tilde{F}_{a\beta} \) on regions of \( \mathbb{R}^{2|p} \) which reduce to the \( \phi_\alpha \circ \phi_\beta^{-1} \). Since now the cocycle conditions are not automatically satisfied, we impose them by hand. Note that the order of terms in (2.22) is reversed from (2.21), simply because that's how pullbacks work.

Having defined \( \tilde{M} \) with an atlas we can easily extract a patch-free version. To any open set \( U \) in the underlying manifold \( M \) we associate a ring \( \mathcal{A}_U \) as follows. A super function \( f \in \mathcal{A}_U \) consists of a collection \( f_\alpha \in \mathcal{A}_{V_\alpha} \) of functions, all of which agree:

\[
\tilde{F}_{a\beta} (f_\alpha|_{V_{\alpha\beta}}) = f_\beta|_{V_{\alpha\beta}}.
\]

Because the \( \tilde{F}_{a\beta} \) are all homomorphisms, we can easily define \( f + g, fg \), etc., and \( \mathcal{A}_U \) becomes a \( \mathbb{Z}_2 \)-graded ring. In general, however, the \( \mathbb{Z} \)-gradings on \( V_\alpha, V_\beta \) disagree, so \( \mathcal{A}_U \) has no well-defined \( \mathbb{Z} \)-grading — unless all of the transition function are split. Once again if \( U' \subseteq U \) then we get a natural restriction map \( \tilde{r} : \tilde{U}' \to \tilde{U} \). Thus we can throw away the original atlas and regard a supermanifold \( \tilde{M} = (M, \mathcal{A}) \) as a ringed space, a topological space with a sheaf \( \mathcal{A} \) of \( \mathbb{Z}_2 \)-graded rings. See [44] for a systematic treatment of this elegant viewpoint.

Here is a simple example. Let \( V_1 \) and \( V_2 \) both be \( \mathbb{R}^3 \), and near \( x^2 + y^2 = 1 \) let \( F_{12}(\sigma, \tau) = \left( \frac{x}{x^2 + y^2} , \frac{y}{x^2 + y^2} \right) \). (In our language this means that \( F_{12}(\sigma(1)) = \frac{-x(y)}{x^2 + y^2} \), etc.) Thus \( M = S^3 \), the sphere. Now let \( q = 2 \) and

\[
\begin{pmatrix}
\tilde{F}_{12}(\xi(1)) \\
\tilde{F}_{12}(\eta(1))
\end{pmatrix}
\begin{pmatrix}
1 \\
x^2 + y^2
\end{pmatrix}
\begin{pmatrix}
x(2) \\
y(2)
\end{pmatrix}
\begin{pmatrix}
\xi(2) \\
\eta(2)
\end{pmatrix}.
\]
Since this is the square root of the jacobian of \( F_{12} \), the odd variables \( \xi, \eta \) transform as spinors over \( M \). Lastly we can declare that near \( x^2 + y^2 = 1 \)

\[
F^*_{12}(x(1)) = \frac{-x(2)}{x^2(2) + y^2(2)} + ax(1)\eta(2)
\]

\[
F^*_{12}(y(1)) = \frac{y(2)}{x^2(2) + y^2(2)}
\]

For \( a \neq 0 \) these coordinates do not provide a splitting. But if we let \( \hat{x}(1) = x(1) + \xi(1)\eta(1)\theta(x(1),y(1)) \) then the coordinates \((\hat{x}(1),y(1))\) and \((x(2),y(2))\) do yield a splitting, provided that \( g = \frac{x(2)}{x^2(2) + y^2(2)} \) near \( x^2 + y^2 = 1 \) and \( g \) is regular throughout \( x^2 + y^2 \leq 1 \). In that case the change of variables from \((x(1),y(1))\) to \((\hat{x}(1),y(1))\) is legitimate and we have found a splitting.

### 2.5. Vectors and maps

We can now 'globalize' the two geometric notions introduced in the previous section. A vector field on \( \hat{M} \) is a derivation of \( \hat{\mathcal{A}} \). Since there is no global coordinate system we cannot globally expand vectors into components, just as in ordinary geometry; thus \( \hat{\mathcal{F}}_{\hat{M}} \) is not a free \( \hat{\mathcal{A}}_{\hat{M}} \)-module. Locally however \( \hat{\mathcal{F}}_{\mathcal{U}} \) is free over \( \mathcal{A}_{\mathcal{U}} \), and so we say that \( \hat{\mathcal{F}} \) is locally free over \( \hat{\mathcal{A}} \) of rank \( p|q \).

More generally a bundle over a supermanifold is any locally free sheaf of \( \hat{\mathcal{A}} \)-modules. A little thought shows that this reduces to the usual definition when \( \hat{\mathcal{A}} \) is replaced by \( \mathcal{A} \) since it says that sections of \( \hat{\mathcal{F}} \) are locally just \((r + s)\)-tuples of functions in \( \hat{\mathcal{A}} \) with \( r \) even and \( s \) odd entries.

There is a perennial confusion here which requires a digression. The alert reader will have remarked that at least in the split case the odd variables transform as sections of an ordinary bundle \( E \) over \( M \) (see (2.16)). What then does \( \frac{\partial}{\partial \xi^a} \xi^a = 0 \) mean? Don't we need some sort of covariant derivative here? If so it's bad news — it means we must use a flat connection.

But it's not so. Recall the point of a covariant derivative is to make \( \nabla_\xi \xi^a \) transform "covariantly," i.e. homogeneously. But that's not what we want. Consider for example a manifold with dimension 1|1. Change coordinates to

\[
\tilde{x} = x, \quad \tilde{\xi} = A(x)\xi
\]

Consistent with our vow to treat \( x \) and \( \xi \) symmetrically we compute \( \delta \tilde{\xi}/\delta x = A'\xi \neq 0 \). This is not a homogeneous transformation since \( \delta \bar{x}/\delta x = 0 \). Is it consistent? We should have that

\[
\begin{pmatrix}
\delta/\delta x \\
\delta/\delta \xi
\end{pmatrix} =
\begin{pmatrix}
1 & A' \\
0 & A
\end{pmatrix}
\begin{pmatrix}
\delta/\delta \bar{x} \\
\delta/\delta \bar{\xi}
\end{pmatrix}
\]

\[
\delta/\delta \bar{z} = \delta/\delta x - A^{-1}A'\xi(\delta/\delta \xi)
\]

Hence while \( \delta \tilde{\xi}/\delta x \neq 0 \), we do have \( \delta \tilde{\xi}/\delta \bar{z} = 0 \), so (2.15) is consistent across patch boundaries. So (2.15) is fine as it stands.

On \( \mathcal{U}_\alpha \), \( \mathcal{U}_\beta \) we get two bases for \( \hat{\mathcal{F}}_{\mathcal{U}_\alpha \mathcal{U}_\beta} \), namely \( \frac{\partial}{\partial x^I(\alpha)} \), \( \frac{\partial}{\partial x^I(\beta)} \) and \( \frac{\partial}{\partial \xi^a(\alpha)} \), \( \frac{\partial}{\partial \xi^a(\beta)} \).

Recall what these mean. The former are derivations on \( \hat{\mathcal{A}}_{\mathcal{U}_\alpha} \) and the latter on \( \hat{\mathcal{A}}_{\mathcal{U}_\beta} \), defined by (2.15). We can therefore take the vector fields \( \delta/\delta x^I(\alpha) \) over to derivations on \( \hat{\mathcal{A}}_{\mathcal{U}_\alpha} \) using the transition functions \( \hat{F}_{\alpha \beta} \), then expand them in the basis \( \delta/\delta x^I(\beta) \). Here we introduce the standard convention of generic index, denoted by a capital roman letter; \( I \) stands for either \( i \) or \( \mu \), \( x^I \) for \( x^i \) or \( \xi^\mu \), and so on. We will also write \( |I| \) to denote 0 or 1, respectively. The corresponding matrix of functions on \( \hat{\mathcal{A}}_{\mathcal{U}_\alpha} \) is called the Jacobian matrix of \( \hat{F}_{\alpha \beta} \), just as in ordinary geometry:

\[
(F_{\alpha \beta})_x : \frac{\partial}{\partial x^I(\beta)} \mapsto J^I_J \frac{\partial}{\partial x^I(\alpha)}
\]

\[
"J^I_J = \delta x^I(\beta)/\delta x^J(\alpha)"\text{, etc.}
\]

To be ultra-precise, \( \text{ }^9 \)

\[
J^I_J = F^*_{\alpha \beta} \left[ \frac{\delta (F^*_{\alpha \beta}(x^I(\alpha)))}{\delta x^J(\beta)} \right]
\]

and similarly for the other three blocks. Note that the 'even' blocks \( J^I_j, J^I_j \) consist of even entries, while the 'odd' blocks \( J^I_j, J^I_j \) have odd entries. A matrix with this property is said to be even; it defines an even homomorphism between the vector spaces \( \hat{\mathcal{F}}_{\mathcal{U}_\alpha} \) and \( \hat{\mathcal{F}}_{\mathcal{U}_\beta} \). Thus we can consistently define vector

\(^9\text{ Even this is not totally precise, since it omits the relevant restrictions. We will sacrifice this horrendous degree of precision from now on.}\)
fields as collections of coefficient super functions on the various patches related by the transition matrices \( J_{ij} \). Similarly we require that the transition matrices of any bundle be even matrices, so that the parity of any section, and hence the graded rank \( r \), will be well-defined across patch boundaries.

The second notion defined earlier on \( \mathbb{R}^{p|q} \) is that of a map. This has an easy generalization to arbitrary supermanifolds. We supply an ordinary map \( \phi : M \rightarrow M' \), and for every \( U' \subseteq M' \) a homomorphism \( \hat{\phi}^* : \hat{\mathcal{A}}_{U'} \rightarrow \hat{\mathcal{A}}_{\phi^{-1}(U')} \). Again there is a compatibility condition, which we again defer for a moment.

Suppose \( (M, \hat{\mathcal{A}}) \) is defined starting from a smooth manifold and an atlas of super transition functions as in (2.17). We can define an inclusion \( \hat{\iota} : M \hookrightarrow \hat{M} \) as follows. The underlying map is the identity \( \hat{\iota} : M \hookrightarrow M' \). The map \( \hat{\iota}^* \) takes \( \hat{\mathcal{A}} \) to \( \mathcal{A} \) by setting all \( \xi \)'s to zero, certainly a homomorphism. Thus \( \hat{\iota}^* (\hat{\xi}^i) = \hat{\xi}^i \); \( \hat{\iota}^* (\xi^i) = 0 \). (To avoid confusion we have again distinguished corresponding coordinates for \( M \) and \( \hat{M} \) by a bar.) This is consistent across patches thanks to the compatibility conditions on the patching maps \( \hat{F}_{\alpha \beta} \), which guarantee that \( \hat{\iota}_\alpha \circ F_{\alpha \beta} = \hat{F}_{\alpha \beta} \circ \hat{\iota}_\beta \). Clearly \( \hat{\iota} \) takes \( \hat{\mathcal{A}}_{\alpha} \) to \( \mathcal{A}_\alpha \) with no kernel, so we can say that \( \hat{\iota} \) is an injection.

We can now state the compatibility condition on a general map \( (\phi, \hat{\phi}^*) \): we require that globally

\[
\hat{\phi} \circ \hat{\iota} = \iota' \circ \phi ,
\]

(2.25)

where \( \hat{\iota}, \iota' \) are the inclusions of \( M, M' \) into \( \hat{M}, \hat{M}' \) described above. The reader should verify that this reduces to the compatibility condition given in the previous section when \( \hat{M} = \mathbb{R}^{p|q} \).

A second important map is defined only for projected (and hence in particular split) supermanifolds. Consider again the map \( \hat{\phi} : \mathbb{R}^{p|q} \rightarrow \mathbb{R}^p \) obtained by letting

\[
\hat{\phi}^*(\hat{\xi}^i) = \xi^i .
\]

(2.26)

The reader can easily show that the various \( \hat{\phi}_\alpha \) defined on \( \hat{\mathcal{V}}_\alpha \) will patch together globally only if the transition functions satisfy (2.19). Then \( \hat{\phi} \circ \hat{\iota} \) is the identity on \( M \) and we call \( \hat{\phi} \) a projection (or retraction) of \( \hat{M} \) to its base. Conversely, if a projection \( \hat{\phi} \) is given we can easily find an atlas of coordinate systems joined by transformations of the form (2.19). Simply choose an atlas of coordinate charts \( \hat{z}^i_{\alpha} \) for \( \hat{M} \) and let \( \hat{z}^i_{\alpha} = \hat{\phi}^*(\hat{z}^i_{\alpha}) \) on \( \hat{M} \). Then on each patch choose \( q \) additional odd coordinates \( \xi^i \).

On the other hand for a general supermanifold there is no natural choice of projection! Certainly (2.26) does not work. We have

\[
\hat{\phi}_\alpha \circ \hat{F}_{\alpha \beta}^* (\hat{z}^i_{\alpha}) = h^i_0 (\hat{z}^i_{\beta}) ,
\]

(2.27)

while

\[
\hat{F}_{\alpha \beta}^* \circ \hat{F}_{\alpha \gamma}^* (\hat{z}^i_{\alpha}) = \hat{F}_{\alpha \beta}^* (\hat{z}^i_{\alpha}) = h^i_0 (\hat{z}^i_{\beta}) + \cdots .
\]

(2.28)

(In the future we will drop the bars when the meaning is clear.) When the dots are present, i.e. when \( \hat{M} \) is not projected, then the two expressions disagree.

The reader should ponder this counterintuitive situation for a while.

Naively there seems to be an obvious projection from \( \hat{M} \) to \( M \) obtained by "discarding the Grassmann part" of the coordinates of a point. But \( \hat{M} \) has no points. In no sense is it just the Cartesian product of \( M \) with \( \Lambda^*(\mathbb{R}^q) \). The only way to define \( \hat{\phi} \) is to say what \( \hat{\phi}^* \) does to functions, and this requires additional information not contained in the transition functions of a general supermanifold.

2.6 Complex supermanifolds

It's not hard to define complex functions on a smooth supermanifold. We just replace \( \hat{\mathcal{A}} \) by \( \hat{\mathcal{A}}_c = \hat{\mathcal{A}} \otimes \mathbb{C} \), with the rule \( \hat{f}(x, \xi) \otimes w = f(x, \xi) \otimes \bar{w} \) for \( f \in \hat{\mathcal{A}}_c \) and \( w \in \mathbb{C} \). There is also a complexified tangent space \( \hat{T}_x = \bar{T}_x \otimes \mathbb{C} \) and so on. Here again we have an involution; it satisfies

\[
\bar{x} \hat{f} = \hat{f} \bar{x} .
\]

In particular \( \bar{X} \) still acts from the left.

A complex supermanifold is a very different sort of thing. Recall [48] that a complex manifold is a topological space with an atlas \( \phi_\alpha : U_\alpha \rightarrow V_\alpha \), where now \( V_\alpha \subseteq \mathbb{C}^p \) and \( F_{\alpha \beta} \) are holomorphic functions. This is a lot of information. Some topological spaces, even smooth 2n-manifolds, admit no complex structure at
all, while others admit many inequivalent ones. For example every compact orientable smooth 2-surface admits a complex structure, and indeed every one except the sphere has a continuum of different choices, even though they are all indistinguishable when viewed as smooth surfaces. That is, we can change one set of holomorphic $F_{a\beta}$ into any other set $F'_{a\beta}$ by choosing diffeomorphisms $\psi_a : \mathcal{V}_a \to \mathcal{V}'_a$ and taking (Fig. 2.2)

$$F'_{a\beta} = \psi_a \circ F_{a\beta} \circ \psi_a^{-1},$$

but we cannot in general do so using holomorphic functions $\psi_a$.

Fig. 2.2

This brings out a general point. Two manifolds are said to be the "same" whenever the patching data of one can be brought to those of the other by a transformation like (2.29).10 $F$ and $F'$ are said to differ by a coboundary. Equivalently we can say that $M, M'$ are the 'same' if there exists an invertible map between them preserving all the relevant structures. Any structures on $M, M'$ must be preserved by the maps $\psi$. Thus the more structure $M, M'$ have, the harder it is to find appropriate $\psi$ and the larger the set of inequivalent manifolds in that category.

10 Possibly after passing to a finer open cover.

Given a set of holomorphic transition maps we can say unambiguously whether a function on $M$ is analytic. This gives a ring $\mathcal{O}_M$ of analytic functions on any open set $U$, with the obvious restriction map.

Given a complex manifold $(M, \mathcal{O})$, we now define a complex supermanifold as follows. Let $\mathcal{O}_\nu = \mathcal{O}_\nu \otimes \Lambda^\nu (\mathbb{C}^\nu)$, a $\Lambda^\nu$-graded-commutative ring over $\mathbb{C}$. As usual we build a sheaf $\tilde{\mathcal{O}}$ of such rings on $M$ using transition maps

$$\tilde{F}_{a\beta}^{\nu}(x^{(a)}_{(\alpha)}) = h^{\nu}_{(\alpha)}(x^{(\beta)}_{(\beta)}) + \cdots \quad (2.30)$$

$$\tilde{F}_{a\beta}^{\nu}(x^{(a)}_{(\alpha)}) = \theta^{\nu}_{(\alpha)} h_{(\beta)}^{\nu}(x^{(\beta)}_{(\beta)}) + \cdots \quad (2.31)$$

Here $x^{(a)}_{(\alpha)}$ are coordinates for $\mathcal{V}_{a\alpha}$ while $\theta^{(a)}_{(\alpha)}$ are a basis of $\mathbb{C}^\nu$, regarded as odd super functions; the functions $h^{\nu}_{(\alpha)}, h_{(\beta)}^{\nu}, \ldots$ are now required to be holomorphic. Again we require that $\Lambda^\nu$ come from the ordinary transitions $F_{a\beta}$ and that (2.30), (2.31) be invertible and satisfy the cocycle relation. A complex supermanifold is holomorphically projected or split if complex coordinates can be chosen such that the ellipses are absent in (2.30) or (2.30)-(2.31), respectively.

We can again define derivations of $\tilde{\mathcal{O}}$ to be holomorphic tangent vector fields. Multiplying such a derivation by an analytic superfunction gives another, so $\tilde{T}$ is in this case a locally free sheaf of $\tilde{\mathcal{O}}$-modules with $p$ even and $q$ odd generators. More generally a holomorphic bundle is any such sheaf $\mathcal{F}$; it has a graded rank, or dimension, denoted $rk \mathcal{F} = r|s$.

A map of complex supermanifolds is defined exactly as before, with $\tilde{\mathcal{O}}$ replacing $\mathcal{O}$. Again there is always an inclusion $i : M \hookrightarrow \tilde{M}$, and a holomorphic projection $\tilde{p} : \tilde{M} \to M$ is available only if $M$ has projective transition functions.

Given a manifold with some extra structure we can always 'forget' the extra structure to get a simpler space. For example since holomorphic functions are in particular smooth we can build in a canonical way a smooth supermanifold of real dimension $2p|2q$ starting with a complex one. We just substitute $z^i = z^i + iy^i$, $\theta^\nu = \xi^\nu + iy^\nu$ in (2.30), (2.31) and take the real and imaginary parts. To every holomorphic superfunction is canonically associated a smooth function. Any section of $\tilde{T}$ then gives a section of the corresponding smooth
tangent bundle, \( \mathcal{T}_c = \text{Der} \, \mathcal{A}_c \). These vector fields annihilate every \( f \) where \( f \in O \). They generate a subbundle \( \mathcal{T}^{1,0} \subset \mathcal{T}_c \); one has \( \mathcal{T}_c = \mathcal{T}^{1,0} \oplus \mathcal{T}^{0,1} \).

2.7. Heterotic supermanifolds

This section is optional and will not be used later.

If we are willing to sacrifice the real structure (complex conjugation), an even more bizarre possibility presents itself. Given an ordinary complex manifold we have seen how one can always construct a corresponding smooth manifold: just use the fact that the holomorphic transition functions can also be regarded as smooth. This constructs the ring \( \mathcal{A}_c \) of smooth complex functions. Moreover \( \mathcal{A}_c \) has a natural real structure, ordinary complex conjugation, which is preserved by the transition functions. The same remarks apply to supermanifolds; a complex supermanifold of dimension \( p|q \) becomes a smooth supermanifold of real dimension \( 2p|2q \).

Suppose now that one is given a ringed space where the rings \( \mathcal{A} \) are over the complex numbers, are \( \mathbb{Z}_2 \)-graded, and are locally isomorphic to \( \mathcal{A}_c \otimes \Lambda^* \mathbb{C}^e \), where \( \mathcal{A}_c \) are the smooth complex functions on an ordinary manifold. We do not suppose that \( q \) is even, however, nor that \( \mathcal{A} \) has any involution. All we assume is that the reduced \( \mathcal{A}_c = \mathcal{A}/\mathcal{N} \) have such an involution, as befits the complex functions on an ordinary manifold. The resulting structure is called a chiral, or heterotic supermanifold.

Any ordinary smooth supermanifold provides an example of this construction. For a more interesting choice, suppose we are given two complex supermanifolds \( \mathcal{M}, \mathcal{M}' \) over the same \( M \). Choose atlases such that \( h^q_i(z^j) = h^q_i(z^j) \). The rest of the coefficients in (2.30), (2.31) will in general be totally different, since we have not even assumed \( q = q' \); e.g. \( q' \) could be zero. For each \( \alpha \) build the ring \( \mathcal{A}_{\mathcal{V}_c} = \mathcal{A}_{\mathcal{V}_c} \otimes \Lambda^* \left( \mathbb{C}^e \oplus \mathbb{C}^{e'} \right) \). Complex coordinates for \( \mathcal{A}_{\mathcal{V}_c} \) are then \( z^i, z'^j \); generators for \( \Lambda^* \mathbb{C}^e \) and \( \Lambda^* \mathbb{C}^{e'} \) will be called \( \theta^i, \theta^{i'} \) respectively. Once again, the \( \theta^i \) can be completely absent. We can use \( \tilde{\mathcal{F}}_{\alpha \beta} \) and \( \tilde{\mathcal{F}}_{\alpha \beta} \) to define ring homomorphisms \( \tilde{\mathcal{F}}_{\alpha \beta} : \tilde{\mathcal{A}}_{\mathcal{V}_c} \rightarrow \tilde{\mathcal{A}}_{\mathcal{V}_c} \), again using the Taylor expansion trick in (2.18). The resulting ringed space \( \tilde{\mathcal{M}} = (M, \tilde{\mathcal{A}}) \) is then a chiral supermanifold. Such spaces play a key role in fermionic string theory.

To make sure that \( \tilde{\mathcal{M}} \) is well defined we must verify that it changes by an equivalence whenever \( \tilde{\mathcal{F}}_{\alpha \beta} \) and \( \tilde{\mathcal{F}}_{\alpha \beta} \) do, subject to \( h^q_i = h^q_i \). This is easily seen to be the case. For chiral supermanifolds the two tangent spaces \( \mathcal{T}^{1,0} \) and \( \mathcal{T}^{0,1} \) exist, but their odd parts are unrelated; \( \mathcal{T} = \mathcal{T}^{1,0} \oplus \mathcal{T}^{0,1} \) is not the complexification of any real vector bundle.

3. Structures on supermanifolds

In this chapter we will investigate the structure of supermanifolds. We will see that they possess certain invariants with no analogs on ordinary manifolds. These invariants include some which tell whether a general supermanifold admits any projection, whether a projected supermanifold admits a splitting, and whether two splittings or projections are the same. We will also develop more differential geometry on supermanifolds.

3.1. A simple example

To fix ideas consider a very simple example. Again we will take the base to be a sphere, but this time work in the complex category. Thus there are two patches \( \alpha = 1, 2 \), and

\[ F_{13}^a(z_{(1)}) = z_{(2)}^{-1} \]  

(3.1)

If the odd dimension is 1, then clearly (2.17) reduces to (2.16), which is not so interesting, so let \( \tilde{\mathcal{M}} \) have two odd coordinates \( \theta_{(\alpha)}, \psi_{(\alpha)} \). Let \( \theta \) and \( \psi \) both transform as one-forms:

\[ \tilde{F}_{12}^a(\theta_{(1)}) = -z_{(2)}^{-2}\theta_{(2)} \quad \tilde{F}_{12}^a(\psi_{(1)}) = -z_{(2)}^{-2}\psi_{(2)} \]  

(3.2)

Now take

\[ \tilde{F}_{12}^a(z_{(1)}) = z_{(2)}^{-1} + \alpha\psi_{(2)}\theta_{(2)} \]  

(3.3)

where \( \alpha \) is some complex number and \( k \) is an integer. Eqns. (3.1)-(3.3) are certainly consistent — with only two patches the cocycle condition is vacuous. Also obviously we can set \( \alpha = 0 \) or 1 by rescaling. If \( \alpha = 1 \) then \( \tilde{\mathcal{M}} \) is not obviously projected. Let's try to find a projection anyway.
We can replace the coordinates \( z_{(a)} \) by any functions \( w_{(a)}(z_{(a)}, \theta_{(a)}, \psi_{(a)}) \) well-defined and such that \( w; \theta, \psi \) are invertible functions of \( z, \theta, \psi \). But clearly it won’t do us any good to change the lowest term of \( w_{(a)} \), so we may as well take
\[
\begin{align*}
  w_{(a)} &= z_{(a)} + f_{a}(z_{(a)})\psi_{(a)}
  &= z_{(a)} \left[ 1 + z^{-1}_{(a)}f_{a}(z_{(a)})\psi_{(a)} \right].
\end{align*}
\]

The question is then, can we choose functions \( f_{a} \) such that (3.3) is split when expressed in terms of \( w_{(a)} \)? An easy calculation shows that
\[
\begin{align*}
  \tilde{F}_{12}^{-1}(w_{(1)}) &= w^{-1}_{(2)} + \psi_{(2)}f_{2}(w_{(2)}) + w^{k}_{(2)}f_{1}(w^{-1}_{(2)})
  &= w^{-1}_{(2)} + \psi_{(2)}f_{2}(w_{(2)}) + w^{k}_{(2)}f_{1}(w^{-1}_{(2)}).
\end{align*}
\]

If we now insert the Taylor expansions of \( f_{a} \), we see that for nearly every value of \( k \) the offending term can indeed be removed. Only for \( k = -3 \) are we stuck with it.

We conclude that over the sphere \( \mathbb{P}^1 \) there is precisely one supermanifold whose bundle is \( \omega \otimes \omega \), where \( \omega \) is the canonical (or cotangent) line bundle, and which cannot be holomorphically projected. The reader should verify using the same argument that when the bundle has total degree less than \(-4\) (in the above example we had \( \deg(\omega \otimes \omega) = -4 \)), there is a whole continuum of different non-projectable spaces, while for \( \deg > -4 \) they can all be projected.

In the next section we will dissect what just happened and generalize.

### 3.2. Obstructions to projection

Let \( \bar{M} \) be a smooth supermanifold with two odd dimensions. Finding a projection means finding \( \bar{p} \) such that
\[
\begin{align*}
  \bar{p}^* : \mathcal{A} &\to \bar{\mathcal{A}}.
\end{align*}
\]

On each patch there’s an obvious choice: since \( \bar{\mathcal{A}}_{\psi_{(a)} = \mathcal{A}_{\psi_{(a)}} \otimes \Lambda^0(\mathbb{R}^2)} \), we let \( \bar{p}_{\alpha}^* \) be the identity mapping
\[
\begin{align*}
  \bar{p}_{\alpha}^* : \mathcal{A}_{\psi_{(a)}} &\to \mathcal{A}_{\psi_{(a)}} \otimes \Lambda^0(\mathbb{R}^2),
  \bar{z}^i &\to z^i.
\end{align*}
\]

Recall that the barred coordinates live downstairs; \( \bar{z}^i \equiv \iota^*(z^i) \). As we have seen these \( \bar{p}_{\alpha}^* \) disagree; we do however have from (2.27), (2.28) that on \( \bar{\nabla}_{\alpha} \) the error
\[
\begin{align*}
  \Delta_{\alpha\beta}[\bar{z}_{(a)}] &= \left[ \bar{p}_{\beta}^* \circ \bar{F}_{\alpha\beta} - \bar{F}_{\alpha\beta} \circ \bar{p}_{\alpha}^* \right]\left( \bar{z}_{(a)} \right)
  &= \xi_{(\beta)}^\mu \xi^\nu_{(\mu)} h_{\mu\nu} \left( \bar{z}_{(\beta)} \right)
\end{align*}
\]
is nilpotent. This form is a bit misleading, however, since the left hand side is written in terms of coordinates on \( \mathcal{U}_{\alpha} \) while the right side is in terms of \( \mathcal{U}_{\beta} \) coordinates. Writing it more carefully one has for any \( f(\bar{z}^i) \)
\[
\begin{align*}
  \Delta_{\alpha\beta}[f] &= \xi_{(\beta)}^\mu \xi^\nu_{(\mu)} h_{\mu\nu} \left( \bar{z}_{(\beta)} \right)
  = \xi_{(\beta)}^\mu \xi^\nu_{(\mu)} h_{\mu\nu} \left( \bar{z}_{(\beta)} \right)
  = \xi_{(\beta)}^\mu \xi^\nu_{(\mu)} h_{\mu\nu} \left( \bar{z}_{(\beta)} \right)
  = \xi_{(\beta)}^\mu \xi^\nu_{(\mu)} h_{\mu\nu} \left( \bar{z}_{(\beta)} \right)
\end{align*}
\]
or simply
\[
\begin{align*}
  \Delta_{\alpha\beta} &= \xi_{(\beta)}^\mu h_{\mu\nu} \left( \frac{\partial h_{\beta\alpha}}{\partial \bar{z}^i_{(\beta)}} \right)^{-1} \frac{\partial f}{\partial \bar{z}^i_{(\beta)}}
  &= \xi_{(\beta)}^\mu h_{\mu\nu} \left( \frac{\partial h_{\beta\alpha}}{\partial \bar{z}^i_{(\beta)}} \right)^{-1} \frac{\partial f}{\partial \bar{z}^i_{(\beta)}}
\end{align*}
\]

(3.5)

Let \( \mathcal{E} \) be the two-dimensional ordinary bundle over \( M \) describing the \( \xi^\mu \). Then (3.5) shows that \( \Delta_{\alpha\beta} \) gives a section of \( (\mathcal{E} \wedge \mathcal{E}) \otimes T \) over \( \mathcal{U}_{\alpha\beta} \), where \( T \) is the ordinary tangent of \( M \).

The reader should invert (2.17) and use the transformation rules for sections of \( \mathcal{E} \wedge \mathcal{E} \otimes T \) to verify that
\[
\begin{align*}
  \Delta_{\alpha\beta} &= \Delta_{\beta\alpha} ;
\end{align*}
\]
similary the cocycle condition (2.22) gives that
\[
\begin{align*}
  \Delta_{\alpha\beta} + \Delta_{\beta\gamma} + \Delta_{\gamma\alpha} &= 0
  \quad (3.6)
\end{align*}
\]
on the appropriate overlaps. (We are suppressing some pullbacks from the notation, as in (2.20), (2.23).) Thus \( \Delta \) defines a Čech cocycle on \( M \) with values in \( (\mathcal{E} \wedge \mathcal{E}) \otimes T \) [48].

What can be done? We can play around with new coordinates as in the previous section, but it’s better to leave the coordinates fixed forever and modify the choice (3.4):
\[
\begin{align*}
  \bar{p}_{\alpha}^* : \bar{z}^i_{(a)} &\to z^i_{(a)} + \xi^\mu_{(a)} \theta^\nu_{(a)} \eta^\mu_{(a)} \eta^\nu_{(a)}(z^i_{(a)})
\end{align*}
\]

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for some collection of ordinary functions \( \eta_{\mu \nu}(\alpha) \). We still have that \( \hat{\pi} \cdot \hat{\eta}^* = 1 \).

Then \( \Delta \) becomes

\[
\Delta_{\alpha \beta} = \Delta_{\alpha \beta} + \eta_{\alpha} - \eta_{\beta} .
\]

(3.7)

Clearly \( \Delta' \) still obeys (3.6).

In other words we can modify \( \Delta \) by any coboundary without changing \( \hat{M} \). The question now becomes, can we choose \( \eta \) so as to obtain \( \Delta \equiv 0 \)? The situation is thus almost exactly analogous to the issue of spin structures (see e.g. [50]); the answer is 'yes' provided that the cohomology class

\[
[\Delta] \in H^1(M; (E \wedge E) \otimes T)
\]

(3.8)

vanishes. \( [\Delta] \) just means \( \Delta \) modulo (3.7). The notation \( H^1 \) in (3.8) means that \( \Delta \) is defined on two-fold overlaps, modulo something defined on "1-fold" overlaps, i.e. the patches themselves. One can show that the class \( [\Delta] \) of \( \Delta \) is unchanged if we change all of the coordinate systems by a coboundary (2.29).

In any number of odd dimensions we have a similar situation. We express it succinctly as follows:

\[
0 \to N_{\alpha \nu} \xleftarrow{\varphi_{\alpha \nu}} A \to 0 .
\]

(3.9)

Here \( N_{\alpha \nu} \) are the even nilpotent functions, and the notation just means that they are precisely the even super functions annihilated by \( \hat{\pi} \). Eqn. (3.9) is called an exact sequence. Any map \( \hat{\eta}^* \), inverting \( \hat{\pi}^* \) on the right,

\[
0 \to N_{\alpha \nu} \xleftarrow{\varphi_{\alpha \nu}} A \to 0
\]

(3.10)

is said to split the exact sequence. In any number of dimensions the first obstruction to defining a \( \hat{\eta}^* \) is always a class \( [\Delta] \) as in (3.8). The higher obstructions are more subtle [51][44][52].

At this point we can simply apply the bludgeon. On a smooth manifold \( H^q(M; \mathcal{F}) = 0 \) for any bundle \( \mathcal{F} \) and \( q > 0 \) [48][53]. For example, when \( q = 1 \) one can choose a partition of unity \( \rho_{\alpha} \) subordinate to \( \{U_{\alpha}\} \) and take

\[
\eta_{\alpha} = \sum_{\beta} \rho_{\alpha} \cdot \Delta_{\alpha \beta} .
\]

Then

\[
\eta_{\alpha} - \eta_{\tau} = \sum_{\beta} \rho_{\beta} (\Delta_{\alpha \beta} - \Delta_{\alpha \tau})
\]

\[
= \left( \sum_{\beta} \rho_{\beta} \right) \Delta_{\alpha \tau}
\]

\[
= \Delta_{\alpha \tau} ,
\]

by the cocycle condition (3.6), and so \( [\Delta] = 0 \).

The existence of a partition of unity is just a fancy way of saying that with smooth functions we can always interpolate between one sort of behavior near the boundary of a patch and another on the interior. It was exactly this freedom which let us split the example space in sect. 2.4. So it should be no surprise that for a smooth supermanifold all obstructions vanish; every smooth supermanifold is split [54][55].

For complex supermanifolds the situation is totally different. The behavior of an analytic function anywhere determines it everywhere, leading to a great rigidity in any object built from analytic data. By arguments identical to those above we find \( [\Delta] \in H^1(M; (E \wedge E) \otimes T) \), where now \( E \) and \( T \) are holomorphic bundles and so we permit only \( \Delta \) and \( \eta \) built from holomorphic sections. Let us now reexamine the situation in sect. 3.1. The bundle \( T = \omega^{-1} \) has degree 2, while as mentioned \( E \wedge E \) has degree \( -4 \), for a total of \( -2 \). The Serre duality theorem then asserts that \( H^1(M; (E \wedge E) \otimes T) \simeq \mathbb{C} \), but we can see the same thing directly.

First suppose \( f \) is a function on \( M = \mathbb{P}^1 \), so that \( f = (f_1, f_2) \), and \( f_1(z_{(1)}) = f_2(z_{(1)})^{-1} \). Then

\[
\Delta_{12}[f] = \Delta_{12}[f_1(z_{(2)})] - \Delta_{12}[z_{(2)}] = \psi_{(2)} \theta_{(2)} \zeta_{(2)}
\]

\[
= -\psi_{(2)} \theta_{(2)} \zeta_{(2)} \frac{\partial f_1}{\partial z_{(2)}}
\]

\[
= \psi_{(2)} \theta_{(2)} \zeta_{(2)} \frac{\partial f_2}{\partial z_{(2)}}
\]

\[
\Delta_{12} = \psi_{(2)} \theta_{(2)} \zeta_{(2)} \frac{\partial}{\partial z_{(2)}}
\]

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Since $E = \omega \otimes \omega$ and $T \simeq \omega^{-1}$ we can think of $\Delta_{12}$ as a section of $\omega$, i.e. as the one-form $\Delta_{12} = \Delta^{k+2}_{(2)} dx_{(2)}$. Then for $k = -3$ we get $\Delta_{12} = \Delta^{k+2}_{(2)} dx_{(2)} = -\Delta^{k+2}_{(1)} dx_{(1)}$, and so $\Delta$ cannot be written as $\eta_1 - \eta_2$ with $\eta_a$ analytic throughout $U_a$. For $k \neq -3$, however, $\Delta_{12}$ extends either to $U_1$ or to $U_2$, and so a suitable $\eta$ exists. Thus we reproduce the conclusion of sec. 3.1.

If a supermanifold $\tilde{M}$ with projection is given we can examine bundles on $\tilde{M}$. Some will simply be given by bundles downstairs: $\tilde{\mathcal{F}} \simeq \pi^* \mathcal{F}$ for some ordinary bundle. Again an obstruction cocycle defined by $\tilde{\mathcal{F}}$ determines whether this is the case [52]; again in the smooth case all obstructions vanish and every bundle comes from an ordinary bundle on $M$.

Now suppose a projection $\tilde{\pi}$ is given, i.e. a splitting of (3.10) for the case of $q = 2$ odd dimensions. Any other projection $\tilde{\mathcal{F}}'$ can be written as

$$ (\tilde{\mathcal{F}}')^* = \tilde{\mathcal{F}}^* + \tau. \quad (3.11) $$

Here $\tau : \mathcal{A} \to N_{ev}$, since both $\tilde{\mathcal{F}}^*$, $\tilde{\mathcal{F}}'^*$ have the same restriction to $M$. Requiring that $(\tilde{\mathcal{F}}')^*$ should be a homomorphism, $\tilde{\mathcal{F}}'^*(fg) = \tilde{\mathcal{F}}'(f)\tilde{\mathcal{F}}'(g)$, gives that $\tau$ is a derivation of $\mathcal{A}$ with values in $N_{ev}$. Thus $\tau$ is a global section of $N_{ev} \otimes T$. One writes

$$ \tau \in H^0(M; (E \wedge E) \otimes T) $$

where we used $N_{ev} = E \wedge E$ for $q = 2$. In this way one can show that when $|\Delta| = 0$ then the space of all splittings is a space modeled on the vector space $H^0((E \wedge E) \otimes T)$ [44]. There is however no canonical, or preferred choice of splitting — all we found was the difference of two.\footnote{Note again the analogy to spin structures: They exist when a certain class in $H^2(M; \mathbb{Z}_2)$ vanishes. When they exist, they form a homogeneous space modeled on the group $H^1(M; \mathbb{Z}_2)$. There is in general no canonical, or preferred, spin structure.} This is still useful, for example when we seek a splitting with some extra property (see chap. 6). As before the situation in higher dimensions is more subtle [44].

3.3. An alternate approach

This section is optional and will not be used later.

In the Introduction I mentioned another approach to supermanifolds. If one is unhappy at the thought of spaces without points one can associate to any supermanifold a set of points by the following recipe. We will follow most closely the discussion of [31], but very similar ideas were put forward in [26], [27], [30], and [28].

Given an ordinary manifold $M$ one can recover the points of $M$ in a roundabout way as the set of all maps from $\ast$ to $M$, where $\ast$ is the set consisting of just one point. With our definition of maps, however, the set $\text{Mor}(\ast \to \tilde{M}) \equiv \text{Hom}(\tilde{\mathcal{A}}_M \to \tilde{\mathcal{A}})$ of all maps recovers only the points of $M$, not $\tilde{M}$. This is because any homomorphism must take $\xi^a$ to zero, since $\mathcal{A}_a = \mathbb{R}$ has no odd generators. One can invent a fancier definition of maps between two supermanifolds, for example as sketched in [56], but for the present purposes that's not what we want — it merely reproduces $\tilde{M}$, which still has no points!

Instead, given any Grassmann algebra $\Lambda$ consider the supermanifold $\ast \Lambda = (\ast, \Lambda)$ whose base is just a point, and associate to $\tilde{M}$ the set of points

$$ \tilde{M}_\Lambda \equiv \text{Mor}(\ast \Lambda \to \tilde{M}) \equiv \text{Hom}(\tilde{\mathcal{A}}_M \to \Lambda). $$

If $\Lambda$ has at least $q$ odd generators then $\tilde{M}_\Lambda$ faithfully captures the structure of $\tilde{M}$. For example let $\tilde{M} = \mathbb{R}^{p|q}$. Then to specify a homomorphism from $\tilde{\mathcal{A}}_M$ to $\Lambda$ we need to say what happens to $\xi^a$ and $\xi^a$. Thus a point of $\tilde{M}_\Lambda$ is a $p$-tuple of even elements of $\Lambda$ corresponding to $x^i$, and a $q$-tuple of odd elements corresponding to $\xi^a$:

$$ \mathbb{R}^{p|q}_\Lambda = (\bigoplus_i \Lambda_{ev}) \oplus (\bigoplus_a \Lambda_{od}) \quad (3.12). $$

Note that $\tilde{M}_\Lambda$ is indeed an ordinary set; in fact we can easily turn it into a manifold. In exchange for concreteness, however, we have paid a price: $\tilde{M}_\Lambda$ has a lot of information not intrinsic to $\tilde{M}$. For example, its dimension grows with that of the auxiliary algebra $\Lambda$, which has nothing to do with $\tilde{M}$.

Given any function $f \in \tilde{\mathcal{A}}_M$ and any $P : \tilde{\mathcal{A}}_M \to \Lambda$ we get $P(f) \in \Lambda$. Since $P$ can be regarded as a point of $\tilde{M}_\Lambda$, each super function on $\tilde{M}$ thus yields
an ordinary function \( f_A : \overline{M}_A \to \Lambda \). (Just let \( f_A(P) \equiv P(f) \).) Once again, however, only a tiny subset of all \( \Lambda \)-valued functions on \( \overline{M}_A \) actually arise in this way, roughly speaking those with finite Taylor expansions in the last of directions in (3.12). The rest of the smooth functions are unphysical baggage; for example in classical mechanics they do not correspond to any observables. Again as we increase the size of \( \Lambda \) we get more and more such bogus functions.

We can improve the situation a bit by noticing that given a homomorphism \( \rho : \Lambda \to \Lambda' \) we get a map \( \overline{\rho} : \overline{M}_A \to \overline{M}_{A'} \) and \( f_A \mapsto \overline{\rho}[f_A] = f_{A'} \). Thus instead of thinking of a supermanifold as a ringed space \( (M, \overline{\Lambda}_M) \) we can instead think of it as a machine taking \( \Lambda \) to the space \( \overline{M}_A \), with specified behavior as \( \Lambda \) is extended to include more auxiliary generators. We can then take the direct limit as \( \Lambda \) becomes large and call the whole system a supermanifold. This point of view is adopted in [29], [31], [27]. Each of these papers proves that one or another variant of this definition is exactly equivalent to the one discussed here. (See also [32], [57] for more on the relationships between various definitions.)

The statement is sometimes made that supermanifolds in the sense of this section are inherently more general than those of other sections because their transition functions can contain 'extra' odd parameters. Consider the following example. We build a super-torus \( \tilde{T} \) from the complex plane \( \mathbb{C}^{1|1} \) by the following identifications:

\[
\begin{align*}
\tilde{F}^* (z) = z & \quad \tilde{G}^* (z) = z + t + \theta \zeta \\
\tilde{F}^* (\theta) = \theta & \quad \tilde{G}^* (\theta) = \theta + \zeta
\end{align*}
\]  

(3.13)

Here \( t \) is a complex parameter and \( \zeta \) is an anticommuting complex parameter. \( z \) and \( \theta \) are the actual coordinates on \( \tilde{T} \).

To describe (3.13) in the language of the rest of this paper we introduce the idea of families. Momentarily suppose that the \( \zeta \) terms of (3.13) are set to zero. Then we can regard (3.13) as defining a single \( 1|1 \) supermanifold \( \tilde{T} \) given a complex number \( t \). But we can also think of these equations for all values of \( t \) at once; this gives a supermanifold \( \tilde{C} \) of dimension \( 2|1 \) with coordinates \( z, t; \theta \). Moreover we have an obvious projection from \( \tilde{C} \) to the parameter space, the complex \( t \)-plane: \( \pi : \tilde{C} \to \mathbb{C} \) takes \( t \) back to \( \pi^*(t) = t \). (Don't confuse \( \pi \) with a projection from \( \tilde{C} \) to its base!)

Now we can see how to bundle the full (3.13). It describes a supermanifold \( \tilde{C} \) of dimension \( 2|2 \), a family of super tori with parameter space \( \tilde{S} = \mathbb{C}^{1|1} \), the \( (t, \zeta) \)-plane, and a projection \( \pi : \tilde{C} \to \tilde{S} \) whose fibers are super tori. From this point of view the only real difference between the DeWitt-Rogers-Schwars approach and the Beresin-Leites-Kostant approach seems to be that the latter requires us to account explicitly for all the odd parameters we use, while the former keeps a pool of parameters on hand. This is not really a hardship, since it is often important to keep the parameter space and projection \( \pi \) out in the open, regardless of which approach one uses, because their topology is of interest.

In such a topologically nontrivial case the parameter space \( \tilde{S} \) is itself pieced together from patches, and one must verify that on each patch overlap the given maps \( \pi_m : \tilde{C}_m \to \tilde{S}_m \) agree. An example of this sort of construction is given in chap. 6.

Once again the idea of families of objects is commonplace throughout algebraic geometry. It is also central to covariant string perturbation theory, where the objects are super Riemann surfaces and the parameter space \( \tilde{S} \) will be supermoduli space. We can for example define two kinds of tangents: ordinary tangents to \( \tilde{C} \), and vertical or relative tangent vector fields, those annihilating every function \( \pi^* f \) constant on the fibers. Unless otherwise stated, from now on all supermanifolds will tacitly be families of supermanifolds; all tangents, forms, and so on will be relative objects (see e.g. [58]).

3.4. Differential forms

We have seen how to define a space of tangent vector fields on a supermanifold \( \tilde{M} \). The vector space \( \tilde{T}_U \) of vector fields over \( U \) is \( \mathbb{Z}_2 \)-graded: it is the sum of two vector spaces consisting of even and odd derivations. By the rules of linear algebra on \( \mathbb{Z}_2 \)-graded vector spaces [44][52], one can define the dual space \( \tilde{\Omega}^1 \equiv \tilde{T}^* \) of 1-forms. We will use the convention for dual bases that

\[
\langle E_A, E^B \rangle = \delta^B_A \; ;
\]

12 For a precise definition of fiber as used here see [58].
in particular,
\[ \left( \frac{\partial}{\partial z^M}, dz^N \right) = \delta^N_M \]
defines \( dz^N \), where \( z^M = (z^m, \xi^p) \). Thus if \( E_A = E_A^N \frac{\partial}{\partial z^N} \) and \( E^A = dz^N E_A^N \) we get \( E_A^N E_N^B = \delta^B_A \). We will always adhere to this rule for index summation, which is convenient when the odd variables transform as spinors.

One can build a de Rham complex as follows (see e.g. [25][59]). \( \hat{\Omega}^* \) is generated by the symbols \( dz^m \), \( d\xi^p \) over \( \hat{\hat{\mathcal{M}}} \). It is \textit{bigraded}. Super functions \( f \in \hat{\hat{\mathcal{M}}} \) have grading \( (p, q) = (0, |f|) \), where \( |f| \) is the parity of \( f \). The symbol \( d \) increases the grading \( p \) and does not affect \( q \). \( \hat{\Omega}^* \) is subject to the relations
\[ \omega \wedge \omega' = (-)^{pq} (-)^{q'q'} \omega' \wedge \omega. \]
Thus \( dz^1 dz^2 = -dx^2 dx^1 \), \( d\xi^1 d\xi^2 = +d\xi^2 d\xi^1 \), but \( dz^1 d\xi^1 = -d\xi^1 dz^1 \), in contrast to the definition in [43]. Define \( d : \hat{\Omega}^p \rightarrow \hat{\Omega}^{p+1} \) by
\[ df = \frac{\partial f}{\partial z^N} dz^N , \]
\[ d(z^M w_M) = dz^M \wedge dz^N (\partial_N w_M) , \]
for a function \( f \) and 1-form \( \omega \). Thus \( d \) acts from the \textit{right}. It does not agree with the usual \( d \) when \( \hat{\hat{\mathcal{M}}} \) has no odd coordinates! This unfortunate situation is forced on us by our insistence on the above \( \bigwedge \) summation rule, which says e.g. that one-forms must be written as \( \omega = dz^N w_N \). The reader should verify that with these rules \( d^2 = 0 \). For a two-form as usual we define
\[ \omega = \frac{1}{2} dz^M dz^N w_{NM} = \frac{1}{2} E^B E^A w_{AB} , \]
where \( E^A \) is any frame for \( \hat{\Omega}^1 \). Then \( (d\omega)_{NM} = 2\partial_N w_{MM} \equiv \partial_N w_M - (-)^{NM} \partial_M w_N \).

Starting with these definitions one can develop a great deal of the usual machinery of differential geometry. (See for example [60][61].) For example, the operator \( d \) gives us a sequence of differential forms:
\[ 0 \rightarrow \mathbb{R} \leftarrow \hat{\hat{\mathcal{M}}} \leftarrow \hat{\hat{\Omega}}^1 \leftarrow \hat{\hat{\Omega}}^2 \rightarrow \ldots . \]

The Poincaré lemma says that for ordinary manifolds this sequence is exact, i.e. locally every closed form is \( d \) of something, and it's not hard to go from there to the corresponding fact for supermanifolds.

We can now return to the problem mentioned at the end of sect. 2.2. To obtain a Poisson bracket superalgebra on \( \hat{\hat{\mathcal{M}}} \) we suppose \( \hat{\mathcal{M}} \) to be equipped with a symplectic structure, a closed nondegenerate 2-form \( K \in \hat{\hat{\Omega}}^2 \). The usual formulas then give the bracket from \( K \); as in classical mechanics \( dK = 0 \) guarantees the super-Jacobi identity. For the example of sect. 2.2 one has simply that \( K = \sum d\xi^i \wedge d\xi^i \), and similarly for more complicated examples.

On complex supermanifolds we define \( \hat{\Omega}^p \) similarly, starting now with the holomorphic derivations. As described in sec. 2.6, we can if we like choose to "forget" the complex structure on \( \hat{\hat{\mathcal{M}}} \) and build an associated smooth supermanifold. Every \( \hat{\hat{\mathcal{M}}} \) will then have a distinguished subring consisting of \( \hat{\Theta}_{\mathcal{M}} \), the holomorphic functions. Similarly, the space of all complex tangent vector fields \( \hat{\hat{\mathcal{F}}} \) has a distinguished subspace \( \hat{\hat{\mathcal{F}}}^{1,0} \) consisting of those which annihilate \( \hat{\Theta} \). These are the vector fields whose expansions contain \( \frac{\partial}{\partial z} \) but not \( \frac{\partial}{\partial \bar{z}} \), but whose coefficients are not necessarily holomorphic. As mentioned, \( \hat{\hat{\mathcal{F}}}^{1,0} \) together with the complex conjugate space \( \tilde{\mathcal{T}}^{0,1} \) span all of \( \hat{\mathcal{T}} \).

We can again take the dual space \( \hat{\hat{\mathcal{A}}}^{1,0} \), the one-forms containing \( dz \) but not \( d\bar{z} \). More generally \( \hat{\hat{\mathcal{A}}}^{p,q} \) are the \((p+q)\)-forms with \( p \) \( dz \)'s and \( q \) \( d\bar{z} \)'s. As in ordinary geometry we can split the \( d \) operator into \( d = \partial + \bar{\partial} \). The only functions annihilated by \( \bar{\partial} \) are the holomorphic ones, and so we now get a new exact sequence:
\[ 0 \rightarrow \hat{\Theta} \rightarrow \hat{\hat{\mathcal{A}}} \rightarrow \hat{\hat{\Omega}}^{1,0} \rightarrow \hat{\hat{\Omega}}^{0,2} \rightarrow \ldots \] (3.14)
or more generally
\[ 0 \rightarrow \hat{\hat{\Omega}}^p \rightarrow \hat{\hat{\Omega}}^{p,0} \rightarrow \hat{\hat{\Omega}}^{p,1} \rightarrow \hat{\hat{\Omega}}^{p,2} \rightarrow \ldots \] (3.15)
(Recall that on a complex supermanifold \( \hat{\hat{\Omega}}^p \) denotes the holomorphic \( p \)-forms.)

\[ \text{3.5. Integration} \]
Recall the fundamental change-of-variables formula for integration over an open set $U$ in $\mathbb{R}^n$ [62]: if $g: U \to \mathbb{R}^n$ is a 1-1, $C^1$ map whose derivative is never singular and which preserves orientation, then

$$\int_{g(U)} f = \int_U (f \circ g) \det g' .$$

(3.16)

Recall what that determinant is doing there. If $g$ is close to the identity map, $g(x) = x + V$ where $V$ is an infinitesimal vector field, and

$$f \circ g = f + Vf$$

$$\det g' = 1 + \theta \cdot V$$

$$\int_U (f \circ g) \det g' = \int_U f + \int_U \theta \cdot (fV) .$$

The second term becomes a boundary integral by Stokes' theorem; it's what's needed to give $\int_{g(U)} f - \int_U f$. (see Fig. 3.1).

We see that to get a coordinate-invariant notion of integral we need to consider objects associated to the representation $\det$ of $GL^+$. Fortunately however, all representations of $GL^+$ can be realized as tensor products of the vector representation. So the things we can integrate are some sort of tensors. As is well known these are just the top differential forms. To indicate the dependence of the integral on the coordinates we then write $\int_U f d\omega^1 \wedge \cdots \wedge d\omega^n$ instead of $\int_U f$. Letting $\omega = f d\omega^1 \wedge \cdots \wedge d\omega^n$, (3.16) becomes

$$\int_{g(U)} \omega = \int_U g^* \omega .$$

(3.17)

Eqn. (3.17) means that we can define integration over a general manifold as follows. Given a volume form $\omega$ and a partition of unity $\rho_\alpha$ we integrate the forms $\omega_\alpha = (\rho_\alpha^{-1})^* (\rho_\alpha \cdot \omega)$ over the various sets $V_\alpha \subseteq \mathbb{R}^p$ and then add them all up. Eqn. (3.17) and the composition law then imply that the result is unchanged if we change coordinate systems by (2.29).

We now turn to the super situation. The Berezin integral of a super-function over a domain $\tilde{U} \subseteq \mathbb{R}^{p|q}$ is defined as follows. In the usual coordinates let

$$f(x, \xi) = f_0(x) + \xi^a f_\alpha(x) + \cdots + \xi^1 \cdots \xi^q f_{1\ldots q}(x) .$$

Then

$$\int_{\tilde{U}} f = \int_U f_{1\ldots q} ,$$

(3.18)

where on the left appears an ordinary integral over $U$. We would like to know what sort of object can be invariantly integrated starting from this rule.

Suppose $\tilde{g}: \tilde{U} \to \mathbb{R}^{p|q}$ is a map whose underlying map $g$ is as above and $\tilde{g}$ is close to the identity. Then

$$f \circ \tilde{g} = \tilde{g}^* f = f + V^i \theta_i f ,$$

where $V^i$ are even functions while $V^\mu$ are odd. $\tilde{g}$ has a jacobian matrix (2.24), which we call $\tilde{g}'$. Let

$$\operatorname{ber} \tilde{g}' = 1 + \theta_i V^i - \theta_\mu V^\mu .$$

(3.19)
Once again it is customary to suppress the $\hat{g}^*$ from the notation on the left side.

Let us now return to (3.20). Consider the super domain $g(\mathcal{U})$ over the open set $g(\mathcal{U})$. One has from the definition that

$$\int_{g(\mathcal{U})} f - \int_{\mathcal{U}} f = \int_{\mathcal{U}} \mathcal{A}_i [V^j_i f_{1,...,q}]$$

(3.22)

where $V^i = V^i_0 + \xi^i V^i_1 + \cdots$. Unfortunately the right hand side does not agree with (3.20), since

$$(V^i f)_{1,...,q} = V^i_0 + f_{1,...,q} + \sum_{\mu=1}^q (-)^{q+\mu} V^i_\mu f_{1,...,\mu-1,...,q} + \cdots$$

(3.23)

All the terms after the first in (3.23) are trouble. The analogous error terms for a finite diffeomorphism say that even if one is given a section of ber, still its integral over $\tilde{M}$ is not well-defined across patch boundaries.

There is a humble but instructive example of this phenomenon. Consider the space $\mathbb{R}^{112}$ and the open set $\mathcal{U}$ over the interval $(0,1)$. Let $\tilde{g}: \mathcal{U} \to \mathbb{R}^{112}$ be defined by $g=\text{identity}$, so that $g(\mathcal{U}) = \mathcal{U}$, and by

$$g^*(-y) = z + \xi^1 \xi^2$$

$$\tilde{g}^*(\xi^2) = \xi^\mu$$

Then the jacobian $\text{ber} \tilde{g} \equiv 1$. A typical volume form is now $\omega = y(dz^1, \ldots, dz^{112}; d\xi^1, \ldots, d\xi^{112})$; the above rules say that $\int_{\mathcal{U}} \omega = 0$. Rewriting this, however, we get

$$\int_{\mathcal{U}} (a + \xi^1 \xi^2)(\text{ber} \tilde{g})(dx^1, dz^1, d\xi^1, d\xi^2) = 1 \neq 0$$

Of course the problem in (3.23) disappears if $V^i = V^i_0$; exponentiating, we see that if all patch transition functions are projective, eqn. (2.19), then there is no error. Hence we have a

Definition. On a projected supermanifold $\tilde{M}$, a volume form is a section of ber.

We have seen, however, that while every smooth supermanifold admits a projection, still the choice is not unique. This means that in general one cannot
integrate sections of \( \text{ber} \); further information is needed. This is the notorious integration ambiguity of fermionic string theory [3]. It turns out that no split atlas of coordinate charts for \( \widetilde{\mathcal{M}} \) is known; certainly the traditional choice [63]–[66] does not have split transitions.

There is sometimes a way out. Note that the error terms always amount to the integral over \( \mathcal{M} \) of a total divergence. The reader should verify that globally if two different projections \( \tilde{\rho}, \tilde{\rho}' \) are given which differ by \( \tau \in H^0((\mathcal{E} \wedge \mathcal{E}) \otimes T) \) (see (3.11)), then the integrals \( \int_{\tilde{\mathcal{M}}} \omega \) of a fixed ber form \( \omega \) computed using \( \tilde{\rho}, \tilde{\rho}' \) differ by a globally-defined total derivative:

\[
\int_{\tilde{\mathcal{M}}} \omega - \int_{\mathcal{M}} \omega = \int_{\mathcal{M}} d\lambda
\]

for some ordinary \((p-1)\)-form \( \lambda \), depending on \( \omega, \tilde{\rho}, \tau \). Thus if \( \mathcal{M} \) is compact without boundary, or if \( \omega \) has compact support, then there is no ambiguity. Unfortunately, for string amplitudes we are given a ber form with noncompact support, on a space with (in general) no canonical choice of projection, so there is still a gap in the integration prescription. This was first noticed in [2] and described in this language in [3].

As usual there are various alternate formulations of the Beresin integral. A very clear treatment along roughly the lines given here is that of Rothstein [67]. If we think of supermanifolds in terms of sets of points (sect. 3.3) then it is possible to realize the Beresin integral as a kind of contour integral on \( \widetilde{\mathcal{M}}_\Lambda \), an idea due to DeWitt [26][68]. In this language the above "ambiguity", or dependence on a projection, gets replaced by a dependence on the chosen contour.

4. Super Riemann surfaces

In this chapter I will use a lot of facts and terminology from Riemann surface theory. One review with applications to strings is [50].

A 2-dimensional real manifold \( \Sigma \) does not have enough structure to define a string action. One needs in addition a conformal structure, essentially in order to distinguish right-moving excitations (corresponding to holomorphic functions) from left-moving excitations (corresponding to antiholomorphic functions). Similarly, a supermanifold \( \tilde{\Sigma} \) of real dimension \( 2|2 \) does not have enough structure to define a string action. In this section we will explore what that new structure is.

On two occasions we have seen how some extra structure can be defined on a manifold or supermanifold. One can proceed in two ways. If the manifold is given as patching data, one can require that that data take some special form. Given two such manifolds, we then only consider them to be equivalent if there exists a correspondence of the patches\(^\text{13} \) which everywhere takes the same special form as that required of the transition functions (see sect. 2.6). For example one can require that the transitions be projected, or split, or holomorphic.

Alternately one can introduce some global geometrical object, like a projection or complex structure. Requiring that all coordinate charts respect this object then implies that they are all related by transition functions as above. In the first example mentioned, one can require that all coordinate charts respect the projection in the sense that the even coordinates are all pullbacks \( z^i = \tilde{\sigma}^i(\tilde{z}^i) \) of coordinates on the base; this guarantees that \( z^a(\tilde{z}^i, \tilde{\xi}^a) = \tilde{z}^a(\tilde{z}^i) \) is independent of \( \tilde{\xi}^a \). In the second example, requiring that \( z^a \) and \( z^\alpha \) be holomorphic functions on \( \mathcal{M} \) guarantees that \( z^a(\tilde{z}^i) \) is holomorphic.

A third example will be useful. To define a conformal structure on a 2-dimensional surface one can follow the first approach, regarding each patch as a piece of \( \mathbb{C} \) and requiring that all transitions be holomorphic. Alternately one can introduce a metric on the surface and single out those local coordinate systems \( x, y \) in which \( dx \) is orthogonal to \( dy \). One first shows that such isothermal coordinate systems always exist. Letting \( z = x + iy \), one next finds that any two such coordinate systems differ by a holomorphic transformation, and so a conformal structure in the first sense has been defined.

There is a useful variant of the second approach. Suppose that instead of introducing a metric on \( \Sigma \) we introduce everywhere a frame of vector fields

\(^{13}\) Perhaps after a suitable refinement of the cover.
that vector's flow lines. In this sense $D_{\theta}$ resembles more the derivative $\theta_{a}$ on a one-dimensional space than it does the derivative $\theta_{a}$ on an ordinary two-dimensional space. We will develop the resemblance further below.

We'd like to modify $D_{\theta}$ to get something intrinsic, so that we can use it on arbitrary super Riemann surfaces — whatever they are. Consider for example the sphere $\Sigma^{11}$, where $\theta$ transforms as a spinor:

\[
\begin{align*}
F^{*}(z') &= -z^{-1} \\
F^{*}(\theta') &= z^{-1} \theta \\
F_{*}(D_{\theta}) &= -z'D_{\theta'}.
\end{align*} \tag{4.1}
\]

So while $D_{\theta}$ isn't invariant, it's at least covariant under this particular transformation. Again the situation is reminiscent of one complex dimension: on a Riemann surface $\partial_{x}$ transforms into $\left(\frac{\partial}{\partial z}ight)^{-1} \partial_{x}$. In each case the thing to do is to find a line bundle whose transition functions match the indicated factors, making $Df$ (resp. $\partial f$) a globally well-defined section of that bundle. Of course for Riemann surfaces this bundle is the cotangent $\omega$, and $\theta = dz \otimes \theta_{z}$ is the usual Dolbeault operator, invariant under any holomorphic change of coordinate.

For the super case we first need to find the most general transformation under which $D_{\theta}$ transforms covariantly, i.e. homogeneously. The reader should verify that any new $z', \theta'$ must verify [35][36]

\[
D_{\theta}z' = \theta'D_{\theta}\theta' \quad ; \quad D_{\theta}z' = D_{\theta'}z = 0 \quad , \tag{4.2}
\]

and that under such a transformation

\[
D_{\theta} = (D_{\theta'})D_{\theta'} \quad . \tag{4.3}
\]

Any diffeomorphism of $\Sigma^{11}$ which satisfies (4.2) is called superconformal. It's clear from the definition that the composition of two superconformal maps is again superconformal, but the reader should verify that as well.

---

14 Some authors refer this as $D$, a notation we will reserve for something slightly different.
If a supermanifold $\hat{\Sigma}$ of dimension 1|1 is patched from maps obeying (4.2), we call it a super Riemann surface, or SRS. On such a surface (4.3) defines a line bundle. In fact we've already seen the dual of this bundle. Note that

$$\text{ber} \left( \begin{pmatrix} \partial z' \\ \partial \theta' \end{pmatrix}, \begin{pmatrix} \partial z \\ \partial \theta \end{pmatrix} \right) = \text{ber} \left( \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \partial z' + \theta' \partial \theta' \\ 0 \partial \theta' \end{pmatrix} \begin{pmatrix} \partial z \\ \partial \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \theta' & 1 \end{pmatrix} \right)$$

$$= \frac{1}{D_{\theta} \theta'} \frac{\partial z' + \theta' \partial \theta'}{D_{\theta} \theta'}$$

$$= D_{\theta} \theta' .$$

(4.4)

In particular for a nonsingular superconformal map $D_{\theta} \theta'$ never vanishes. Eqn. (4.4) means that on a SRS

$$D = [dz|d\theta] \otimes D_{\theta} = ds \otimes D_{\theta}$$

is globally defined, since (4.4) are the transition functions of ber. Note the traditional notation $s$ for $(z, \theta)$ and $ds$ for $[dz|d\theta]$. We will refer to ber on a SRS as $\hat{\omega}$; like the cotangent $\omega$ on a Riemann surface it is called the canonical line bundle over the SRS.

4.2. Frame definition

What geometrical object lurks beneath the patching condition (4.2)? How can we describe it in a coordinate-invariant way? The answer will be a reduction of the structure group, somewhat like the third example of sect. 4.

We can restate the previous section by saying that a SRS has, not a distinguished vector field $D_{\theta}$, but at least a distinguished subspace of $\hat{T}$, namely the space $\mathcal{D} \subseteq \hat{T}$ spanned by $D_{\theta}$, and that locally $\hat{T} = \mathcal{D} \oplus \mathcal{C}$, where $\mathcal{C}$ is locally spanned by $\partial_z$. Also we saw that $\mathcal{D} \cong \hat{\omega}$ where $\hat{\omega}$ is the canonical line bundle and the check means dual.

When a supermanifold has a distinguished subbundle of $\hat{T}$ (sometimes called a distribution), we can restrict attention from all frames of $\hat{T}$ to those whose first $k$ elements span $\mathcal{D} \subseteq \hat{T}$. For our case $k=1$; two good frames are thus related by matrix of the form:

$$\begin{pmatrix} E_{+} \\ E'_{+} \end{pmatrix} = \begin{pmatrix} A & \Gamma \\ 0 & B \end{pmatrix} \begin{pmatrix} E_{+} \\ E'_{+} \end{pmatrix} .$$

(4.5)

Here again $v$ is a frame index running over just one value, as is $+$. Note that for $\{E_{+}, E'_{+}\}$ to be a nondegenerate basis we must have $E_{+}$ even and $E'_{+}$ odd. Then $A, B$ are even invertible super functions and $\Gamma$ is odd, so that the transformation (4.5) is overall even. The group of such matrices we will call $G$. Any two frames $E_{A}$ and $E'_{A}$ related by (4.5) will be deemed equivalent, much as in sect. 4.1.

We've seen that we get a collection of frames related by $G$ on patch overlaps given a superconformal structure in the sense of sect. 4.1.1.

Suppose now we take the frame $E'_{+} = D_{\theta}, E_{+} = \partial_z$ and disguise it by applying (4.5). Then we no longer have $\{E'_{+}, E_{+}\} = 2E_{+}, \{E'_{+}, E_{+}\} = 0$, but the reader can verify that in any case one always has

$$\{E'_{+}, E_{+}\} = \alpha E_{+} + \beta E'_{+}, \quad \alpha \text{ invertible} \quad .$$

(4.6)

Thus (4.6) is a necessary condition for a frame to have come from a SRS. Conversely a simple argument shows that given any frame $E_{A}$ of holomorphic tangent vector fields satisfying (4.6), one can find local superconformal coordinates $z, \theta$ in which $E_{+} \propto D_{\theta}$ [58]. For, let $(w, \lambda)$ be any holomorphic coordinate system. Then by a rescaling we can take $E_{+}$ to be of the form $\partial_{\lambda} + \alpha \partial_{w}$ for an odd function $\alpha$. Then the nondegeneracy condition (4.6) implies that $\partial_{w}$ is invertible. Thus $\alpha = \alpha_{0}(w) + \lambda \alpha_{1}(w)$, where $\alpha_{1}$ is even and invertible. Let $z = z(w, \lambda) = z_{0}(w) + \lambda z_{1}(w)$ and $\theta = \lambda$. Then

$$E_{+} = \partial_{\theta} + \left( \frac{\partial z}{\partial w} + \frac{\partial \theta}{\partial \lambda} \right) \partial_{z} ,$$

so we want to choose $z_{0}, z_{1}$ such that $\alpha \frac{\partial z}{\partial w} + \frac{\partial \theta}{\partial \lambda} = \lambda$. We must therefore solve two differential equations locally:

$$z_{1} + \alpha_{0} \frac{\partial z_{0}}{\partial w} = 0 \quad ; \quad \alpha_{1} \frac{\partial z_{0}}{\partial w} + \alpha_{0} \frac{\partial z_{1}}{\partial w} = 1 .$$

Letting $y = -\alpha_{0} \frac{\partial z_{0}}{\partial w}$ we can locally solve for $y$ and then $z_{1}, z_{0}$.

The argument just given is a simple example of an integrability theorem. It says that certain classes of frames admit coordinate systems which put them into canonical form. The existence of isothermal coordinates on a metric 2-surface, mentioned at the beginning of this chapter, is another example. There
the role of the nondegeneracy condition (4.6) is played by the condition that \( e^a \) and its conjugate \( e^\bar{a} \) must everywhere be linearly independent. In general, however, integrability requires more than nondegeneracy conditions. One usually has to demand that some of the Lie brackets of the frame vectors vanish, as happens for example in the case of complex structures [48].

Given a set of superconformal coordinates \( x, \theta \), any other such set \( x', \theta' \) will clearly be a superconformal transformation of \( x, \theta \). Hence given a frame satisfying (4.6) and defined modulo \( G \), we can find a unique superconformal structure in the sense of sect. 4.1.

Not every complex supermanifold \( \hat{\Sigma} \) of dimension 1|1 admits a frame family defined up to \( G \) and satisfying (4.6) everywhere. For one thing \( E_4 \) is in a sense a square root of \( E_8 \). Not surprisingly this boils down to saying that the odd variable \( \theta \) must transform as a spinor on the underlying \( \Sigma \). What is surprising is that when a superconformal structure does exist, it’s unique [39], in contrast to the situation with, say, spin structures. This fact is useful for instance when one wants to describe SRS as algebraic varieties [47].

One can also begin with a smooth 2|2-supermanifold and attempt to impose both a complex and a superconformal structure at once. The resulting Lie bracket conditions (see above) are just the torsion constraints of 2d supergravity. When they are satisfied, an integrability theorem again guarantees the existence of superconformal coordinates [39]. This interpretation of those constraints is discussed at length in [69] and [39], so I won’t pursue it here. Suffice to say that the supergravity approach leads to a practical set of explicit holomorphic coordinates for supermoduli space [66]. One takes a fixed 2|2-surface and finds a family of frames obeying the torsion constraints and parametrized by a region of \( \mathbb{C}^{3n-3|2n-2} \). A similar construction gives holomorphic coordinates for chiral supermoduli space [70].

4.3. Two examples

We have already encountered an example of a SRS, namely the super sphere \( \mathbb{P}^{11} \) with its operators \( D_9 \propto D_9 \); see (4.1). We can readily generalize this. Given any ordinary Riemann surface \( \Sigma \), build the split complex supermanifold \( \hat{\Sigma} = (\Sigma, \Lambda \omega^{1/2}) \), where \( \omega^{1/2} \) is any holomorphic spin bundle. Let \( \theta \) be the nonvanishing local section of \( \omega^{1/2} \) whose square is \( dz \); regarded as a coordinate in \( \hat{\Sigma} \), \( \theta \) obeys \( \theta' = \pm \left( \frac{dz'}{dz} \right)^{1/2} \theta \). Then \( D_9 \) again defines a superconformal structure, being related to \( D_9 \) by the invertible super function \( \pm \left( \frac{dz'}{dz} \right)^{-1/2} \). Hence every Riemann surface with spin structure yields a SRS; conversely every individual SRS is certainly split (having but one odd coordinate) and moreover has its superconformal structure equivalent to \( D_9 \).

To see that this is not all, recall the example (3.13). Beginning with the torus and following the above recipe leaves us unprepared for the \( \zeta \) terms. Of course the point is that we were unwise to consider only individual SRS; these correspond to points in the moduli space of super tori, while we know that points are not the whole story. We must define a family of SRS as a family of supermanifolds together with a family of \( D \) operators, vertical vector fields with values in the relative \( \omega \) bundle. Eqn. (3.13) is just such an example, with \( D = [dz|d\theta] \otimes D_9 \) as usual; the transitions are readily seen to be superconformal. For each value of the modulus \( t \) the total space (of dimension (1|2)) can be seen to admit no splitting, so this family certainly cannot come from the construction of the proceeding paragraph. Such a family, in which the transition functions are not all split, will be called a nonsplit family.\(^{15}\)

Given an arbitrary family of SRS, we can find a family with no odd parameters by restricting to the base of the parameter space. We have seen that every individual SRS, and similarly every family with no odd parameters, is split. The resulting family is accordingly called the split locus of the original family. For (3.13) it consists of those SRS with \( \zeta = 0 \).

Every family of SRS can locally be described as deformation away from a split family. However for certain purposes this description is clumsy. It is better to use constructions which work for any family with any chosen coordinates. An example of this sort of approach will be given in chapter 5.

4.4. Superdifferentials

\(^{15}\) Some authors use the term “nonsplit SRS” to denote such a family.
The holomorphic bundle $\hat{\omega}$ defined in sect. 4.1 was useful in that there was an intrinsic operator $D$ taking values in $\hat{\omega}$. This in turn is important because by definition $\hat{\omega} \equiv \text{ber}$, and ber is related to volume elements. This is another way in which SRS seem to be 'almost' one-dimensional, since an ordinary manifold of complex dimension greater that one certainly has no such operator.

Specifically, we can define the bundle

$$\text{vol} \equiv \hat{\omega} \otimes \hat{\omega}$$

and see that it is just the full volume-form bundle for $\hat{\Sigma}$, just as on ordinary Riemann surfaces. The correspondence is simply

$$[dz|d\theta] \otimes [d\bar{z}|d\bar{\theta}] \rightarrow [dzd\bar{z}|d\theta d\bar{\theta}]$$

which factors through any holomorphic change of coordinates. Thus sections of vol can be integrated over $\hat{\Sigma}$, at least when $\hat{\Sigma}$ is compact. In particular given a function $f$ we can define a functional

$$S[f] = \int_{\hat{\Sigma}} (Df)(\bar{D}f)$$

the action for the fermionic string.

More generally, define a $\frac{1}{2}$-superdifferential to be a meromorphic section of $\hat{\omega}^{\otimes 2}$. Similarly, a $(\frac{1}{2}, \frac{1}{2})$-superdifferential is a smooth section of $\hat{\omega}^{\otimes 2} \otimes \hat{\omega}^{\otimes 2}$ [35][36]. Hence $(\frac{1}{2}, \frac{1}{2})$-superdifferentials can be integrated, much as ordinary $(1, 1)$ differentials can be integrated on an ordinary Riemann surface. Since these are all line bundles, one has simply that the product of a $(\frac{1}{2}, \frac{1}{2})$-superdifferential times one of weight $(\ell', \frac{\ell'}{2})$ is a $(\frac{\ell + \ell'}{2}, \frac{\ell + \ell'}{2})$ superdifferential. The transition functions for superdifferentials of weight $(\frac{\ell}{2}, \frac{\ell}{2})$ under superconformal transformations are just

$$\varphi = (D\theta')^p(D\bar{\theta}')^q \cdot \varphi'$$

where $\varphi$ is the coefficient of $[dz|d\theta]^p \otimes [d\bar{z}|d\bar{\theta}]^q$, and similarly $\varphi'$. The central to the usefulness of superdifferentials is a simple exact sequence. Recall that on an ordinary Riemann surface one has an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{A} \rightarrow \Omega^{0,1} \rightarrow 0$$

(4.7)

Here $\mathcal{O}$ are the local holomorphic functions, $\mathcal{A}$ are the smooth ones, and $\Omega^{0,1}$ are smooth (0,1)-forms. The sequence (4.7) is called a "resolution of $\mathcal{O}$," and it forms the basis of the theory of analytic torsion [71]. Unfortunately nothing like this can be true in the super case. $\Omega^{0,1}$ is just too big; since it's two-dimensional it contains many sections which are not of the form $\bar{\theta} f$. Indeed we instead get the nonterminating sequence (3.14).

To remedy the situation we note that we have at hand another operator with values in a one-dimensional bundle, namely $\bar{D}$, and as noted in sect. 4.1 any function annihilated by $\bar{D}$ is holomorphic, in contrast to the ordinary case. It is also easy to show that every differential in $\tilde{\mathcal{A}}^{0,1}$ is locally $\bar{D}$ of something, and so we do get a short exact sequence

$$0 \rightarrow \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{A}} \rightarrow \tilde{\Omega}^{0,1} \rightarrow 0$$

(4.8)

the fundamental sequence of a SRS. Eqn. (4.8) plays the same role as the Dolbeault sequence (4.7) on ordinary Riemann surfaces.

There is a simple relationship between superdifferentials and differential forms [72][39][73]. For example one has

$$\tilde{\mathcal{A}}^{0,1} \cong \mathcal{Z}^{0,1}$$

(4.9)

where $\mathcal{Z}^{0,1}$ are the $\bar{\partial}$-closed (0,1)-forms. To verify (4.9), choose local superconformal coordinates and identify $\beta|dz|d\theta$ with $(d\bar{z} - d\bar{\theta} \cdot \bar{\theta})d\beta + d\bar{\theta} \beta$, where $\beta$ is any function. The reader should verify that with the rules of sect. 3.4 this differential form is indeed the most general $\bar{\partial}$-closed form, and that this identification factors through a superconformal change of coordinates. Moreover, we can identify $\bar{D}$ with $\bar{\theta}$ under (4.9).
Similarly one can show that $\mathcal{R}^{2\nu}$ corresponds to the $d$-closed $(1,1)$-forms [72][73]. Thus while not every $(1,1)$ differential form can be integrated, at least some can.

In addition to area integrals, one often wants to perform contour integration on an ordinary Riemann surface. Here the objects to integrate are ordinary 1-forms; the conformal structure is not needed at all. The situation is slightly different on SRS, again basically because differential forms are not the right thing to integrate. The integration of holomorphic superdifferentials was discussed in [36] and elsewhere; arbitrary forms appear in [72], [8], and [74]. Here I will consider only the simplest case, that of holomorphic superdifferentials on closed contours. Just as in the case of ordinary Riemann surfaces it turns out that the precise contour chosen is immaterial; all that matters is its homology class, and so subtleties in defining the contour in the odd directions are avoided.

Suppose one has a contour $C$ defined on the underlying Riemann surface and lying entirely in a coordinate patch, with $x, \theta$ superconformal coordinates and $\Phi = \varphi(dx d\theta)$ a holomorphic $\frac{1}{2}$-superdifferential. Then we simply let

$$\int_C \Phi \equiv \int_C dx \varphi_1 ,$$

(4.10)

where $\varphi = \varphi_0 + \theta \varphi_1$ and the right hand side is an ordinary integral over $\Sigma$. Note that any coordinate system gives us locally a projection to the base, and this is what we use in interpreting $\varphi_1$ as a function on $\Sigma$. Eqn. (4.10) certainly depends only on the homology class of $C$, and in particular is zero if $C$ is contractible on a set where $\Phi$ is holomorphic. Moreover, the reader should show that this definition is invariant under superconformal transformations, from which it follows that (4.10) is well-defined for any contour. However (4.10) certainly does depend on the superconformal structure chosen.

Thus given a holomorphic $\frac{1}{2}$-superdifferential $\Phi$, we can define its periods to be the integrals round the $2\gamma$ nontrivial homology generators, where $\gamma$ is the genus of $\Sigma$. Given a meromorphic superdifferential $\Phi$ we can also define in a similar way its residues round each of the poles. In the Laurent expansion of $\Phi$ the residue at the origin is the coefficient of $x^{-1}dx d\theta$; this definition can be

directly shown to be invariant under superconformal transformations preserving the origin.

Thus $\frac{1}{2}$-superdifferentials are the analogs of abelian differentials on a Riemann surface; they are often called super abelian differentials.

4.5. Serre, Dolbeault, and Riemann-Roch theorems

In this section I will just quote some useful results from [75][72][73]. They are all analogs of famous facts on ordinary Riemann surfaces (see e.g. [53]).

In sect. 3.2 we defined a vector space $H^1(\mathcal{F})$ associated to any sheaf $\mathcal{F}$ on $\Sigma$. It consisted of sections $\Delta_{\omega,\beta}$ of $\mathcal{F}$ defined on the 2-fold patch overlaps $U_\alpha \cap U_\beta$ and satisfying the cocycle condition (3.6). Two such $\Delta$'s were called equivalent if they differed by something of the form $\eta_\alpha - \eta_\beta$, where each $\eta_\alpha$ extends to the entire patch $U_\alpha$. We saw that if $\mathcal{F}$ was a differentiable bundle then $H^1(\mathcal{F}) = 0$, but that this was not true in the holomorphic case.

In order to compute $H^1$ in the holomorphic case we need something more powerful than the brute-force analysis used in the example of sect. 3.1. To get there, we first consider the case where $\mathcal{F}$ is the trivial bundle. An important result in the theory of Riemann surfaces says that to compute cohomology it suffices to use just two patches, one of which can be taken to be a small disk about any point [76]. Then $\Delta = \Delta_{12}$ is just a holomorphic super function on the perimeter $C$ of the chosen small disk, defined modulo those functions $\eta_1$, $\eta_2$ which extend holomorphically to either the disk or the rest of the Riemann surface. If now one is given any super abelian differential $\Phi$ on $\Sigma$, then a linear map can be defined:

$$\lambda_\Phi: H^1(\mathcal{O}) \to \mathcal{C}$$

$$\lambda_\Phi[\Delta] \equiv \int_C \Delta \Phi .$$

This certainly is linear, and really is well-defined on $H^1$, since if $\Delta$ extends without poles to the whole disk then the residue of $\Delta \Phi$ is zero, and similarly in the other direction.
For arbitrary $\mathcal{F}$ we just replace $\Phi$ by a section of the bundle $\hat{\omega} \otimes \mathcal{F}$, so that again $\Delta \Phi$ is a $\frac{1}{2}$-superdifferential. The content of the Serre theorem is that the pairing obtained in this way is an isomorphism:

$$H^1(\mathcal{F}) \cong H^0(\hat{\omega} \otimes \mathcal{F})^* .$$  

(4.11)

Another useful way to compute $H^1$ is to relate it to the more traditional, de Rham-type definition in terms of differential forms. This is what the Dolbeault theorem does. The appropriate definition in the complex case is

$$H^p_D \equiv \Gamma(\mathbb{F}^p(1)/D\Gamma(\mathbb{F}^p,0))$$

where $\Gamma(\cdot) = H^0(\cdot)$ is the space of sections of a differentiable bundle. We want to relate this to the Cech definition of $H^1$ used in these notes. The relation is just

$$H^1(\hat{\omega}^p) \cong H^p_D$$

$$H^q(\hat{\omega}^p) = 0 \ , \ q > 1 .$$  

(4.12)

Again one can generalise this for arbitrary bundles. Eqn. (4.12) is useful in the classification of line bundles on SRS and their connections [72][73].

Finally I will quote a result bearing not on the cohomology spaces themselves, but rather on their dimensions. On ordinary Riemann surfaces the dimensions of $H^0$ and $H^1$ are related for any bundle. This is because of the Serre duality theorem and a certain index theorem relating the spaces of sections of $\mathcal{F}$ and $\omega \otimes \mathcal{F}$. In the super case the spaces $H^r$ are all graded vector spaces; any bundle has both even and odd sections, since we took care in the definition of bundle to stipulate that the transition functions preserve the parity. Hence the dimension of $H^r$ is a pair of integers.\footnote{In more precise language, given a family of SRS $H^r$ becomes a family of vector spaces, i.e. a bundle over the parameter space, and the dimension is the rank of that bundle. I am glossing over complications; this rank is not always well-defined. See [72][73][77].} Suppose now that the ordinary Riemann surface associated to $\hat{\Sigma}$ has a nondegenerate even spin structure. Then the super Riemann-Roch theorem again relates $H^0$ to $H^1$, as follows.

$$h^0(\mathcal{O}) \equiv \dim H^0(\hat{\omega}) = (g|0) .$$

Similarly there is one even holomorphic function (the constant) and no odd ones:

$$h^0(\hat{\mathcal{O}}) = (1|0) .$$

These statements illustrate a special case of the general super Riemann-Roch theorem [72]: for a line bundle $\mathcal{F}$ of parity $|\mathcal{F}|$ we have

$$h^0(\mathcal{F}) - h^1(\mathcal{F}) = \Pi|\mathcal{F}|(d + 1 - g|d) .$$  

(4.13)

(Our parity conventions differ slightly from [72].) Here $\Pi|\mathcal{F}|$ exchanges the two numbers following it if $\mathcal{F}$ is odd. The result of Rosly, Schwarz, and Voronov is that (4.13) continues to hold away from the split locus. Since the left side is an index, while all excursions in the odd moduli must be regarded as "infinitesimal", this is to be expected.

In the next section we will use the Serre and Riemann-Roch theorems to find the dimension of the space of moduli of SRS.

4.6. Moduli of SRS

A fundamental result in Riemann surface theory states that the moduli space $\mathcal{M}$ of curves is itself a complex space (see e.g. [78]). We can think of this result in two ways. From a complex analytic point of view we can first define the notion of a holomorphic family of Riemann surfaces, roughly a projection

$$\pi : C \to S$$  

(4.14)
whose fibers are all Riemann surfaces.

Given such a family we can ask whether it contains "every" Riemann surface, and if so whether it contains each one "only once." More precisely, one can first ask a local question: near each point \( Q \) of \( S \) does moving in \( S \) constitute the most general deformation of the Riemann surface \( X_\alpha \equiv \pi^{-1}(Q) \)? To answer the question one defines the "Kodaira-Spencer map" associated to \( \pi \) at \( Q \). This map takes the space \( T_0 S \) of tangents to \( S \) to an abstract space of deformations of \( X_\alpha \). If it is everywhere an isomorphism then locally \( \pi: C \to S \) describes a general deformation with no redundant parameters.

In general no good family (4.14) of deformations of a complex manifold will exist. For the case of Riemann surfaces, however, such a family exists and is moreover universal, in the sense that any other family can be built from it. The parameter space \( S \) of their universal family is called the moduli space of Riemann surfaces, and is denoted by \( \mathcal{M} \). It is by construction a complex space. An analysis of SRS along these lines has been carried out in [58]; again one finds a universal holomorphic family of SRS with parameter space the supermoduli space \( \overline{\mathcal{M}} \), Alternate analyses from the viewpoint of super uniformization theory have been developed in [58] and [37].

A Riemann surface can be thought of as patches of \( \mathbb{C} \) with transition functions \( F_{\alpha\beta} \) respecting the holomorphic structure. To deform the surface we can deform each patching function, replacing it by a new one \( F'_{\alpha\beta} \) again respecting the complex structure of \( \mathbb{C} \). The new and old are said to differ by a local automorphism of the standard complex structure on \( \mathbb{C} \). The generators of such automorphisms are clearly just the holomorphic vector fields, local sections of the tangent \( T \). Requiring that \( F'_{\alpha\beta} \) continue to satisfy the cocycle condition, and that it not be related to \( F_{\alpha\beta} \) by a mere change of local trivializations, leads by a standard argument to the identification\(^{18}\)

\[
\mathcal{T}_\mathcal{M} \cong H^1(T)
\]

On the left we have the tangent to the parameter space \( \mathcal{M} \), while on the right we have the tangent space to the Riemann surface itself. Since \( T \cong \omega^{-1} \) for a Riemann surface, we recover the result that the cotangent space to \( \mathcal{M} \) is the space of quadratic differentials, using Serre duality: \( H^1(\omega^{-1}) = H^0(\omega \otimes \omega) \).

For complex 1|1 manifolds the situation is much the same. Holomorphic vector fields again generate automorphisms of the complex structure on patch overlaps, and so we get that the tangent to the space \( \overline{\mathcal{M}} \) of complex 1|1 supermanifolds is

\[
\mathcal{T}_{\overline{\mathcal{M}}} \cong H^1(\overline{T})
\]

For SRS, however, the automorphisms must preserve the full superconformal structure. The infinitesimal analog of (4.2) says that local automorphisms are generated by vector fields \( V = V^\alpha \frac{\partial}{\partial z^\alpha} + V^\theta \frac{\partial}{\partial \theta} \) satisfying

\[
D_\theta(z + V^z) = (\theta + V^\theta)D_\theta(\theta + V^\theta)
\]

whose most general solution is (verify this!)

\[
V_\alpha \equiv v^\alpha \frac{\partial}{\partial z^\alpha} + \frac{i}{2}(D_\theta v^\alpha)D_\theta
\]  

(4.15)

for some super function \( v^\alpha \). The reader should also show that under a superconformal change of variables \( v^\alpha \) transforms as a \(-1\)-superdifferential. Thus the sheaf of local automorphisms is

\[
\text{aut} \equiv \omega^{-1}
\]  

(4.16)

and the holomorphic tangent to supermoduli space is

\[
\mathcal{T}_{\overline{\mathcal{M}}} \cong H^1(\text{aut})
\]  

(4.17)

We can now use the Serre theorem (4.11) to show that the cotangent is \( H^1(\omega^{-2}) = H^0(\omega^3) \); thus the \( \frac{3}{2} \)-superdifferentials play the role in SRS theory of the quadratic differentials on ordinary Riemann surfaces.

If we know the dimension of the space of \(-1\)-superdifferentials, then the Riemann-Roch theorem will tell us the dimension of (4.17). Just as in ordinary Riemann surface theory, however, one finds that there are no global \(-1\)-superdifferentials if the genus \( g > 1 \) [58], and so

\[
\dim \overline{\mathcal{M}} = (3g - 3|2g - 2), \quad g > 1
\]

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---

\(^{18}\) More precisely, \( \mathcal{T}_\mathcal{M} = R^1\pi_*(\mathcal{T}_G|\mathcal{M}) \).
One can easily check that this is plausible by working it out for a split family of SRS. Here we can expand any superdifferential of weight $\frac{p}{2}$ into ordinary differentials of weight $\frac{p}{2}$ and $\frac{p+1}{2}$ and of opposite parities. For $p = 3$ we find that the cotangent space at $\Sigma$ is $H^0(\omega^2)H^0(\omega^3/2)$; applying the ordinary Riemann-Roch theorem one then recovers the above dimensions. The two parts of $H^0$ are seen to correspond to quadratic differentials and to ordinary $\frac{3}{2}$-differentials. The latter are sometimes called gravitino zero modes, from their origins in supergravity.

One can now construct explicit complex coordinate patches for $\mathcal{M}$, parameterized families of SRS with the numbers of parameters indicated above with holomorphic transitions among themselves, using either the method of super Beltrami differentials [77][79], or the related supergravity formalism [68], or the method in [80]. However, the above abstract characterisation of the cotangent space will prove to be very convenient in the next chapter, since a simple formula exists for the variation of string states under a tangent to $\mathcal{M}$ specified by an element of $H^1(\text{aut})$.

5. Operator formalism and holomorphic factorisation

In this chapter I'd like to describe part of a recent paper on the operator formalism in superstring theory [8]. I will assume more string theory background than in the previous chapters (see also [81][82]). One of the things we did was to use SRS theory to make contact with the operator formalism in a manifestly supersymmetric way. To begin with I should say what the previous sentence means.

When we say that the Polyakov action is manifestly coordinate-invariant we mean that it depends only on geometrical objects, and not on their explicit components. These are 26 scalars and one new object, namely a conformal structure on the world sheet $\Sigma$. Invariance is "manifest" because the action is built from bits which have previously been shown to be individually invariant, namely the Dolbeault operators $\partial, \bar{\partial}$ and integration of $(1,1)$-forms.

In the string case scalars get replaced by scalar superfunctions $X^P$. And we have now seen that a superconformal structure replaces the conformal structure. Again there are primitive operations which one can check to be intrinsically-defined and then assemble into complicated objects, which will automatically be invariant. So far we have the following constructions:

- A canonical bundle $\omega$ of superdifferentials of weight $1/2$.
- D. It takes scalars to sections of the canonical bundle $\omega$.
- Integration. It takes $\left(\frac{1}{2}, \frac{1}{2}\right)$ - superdifferentials to numbers, or more precisely to functions on any parameter space.
- Contour integration. It takes holomorphic $\frac{1}{2}$-superdifferentials (or antiholomorphic ones) to numbers, given a homology cycle in the underlying Riemann surface.

We will add to this list, but recall that the first two are enough to get an intrinsic action. Note that a $\left(\frac{1}{2}, \frac{1}{2}\right)$-superdifferential can never be holomorphic. Also for contour integration to work it's essential that the integrand be holomorphic near the contour, just as in the ordinary case: an arbitrary $\left(\frac{1}{2}, 0\right)$ - superdifferential has an integral depending on the curve, not simply its homology class, because it's not a closed form. We'll later see how to specify a curve, and hence the integral of an arbitrary form.

5.1. Operator formalism

The operator formalism for string theory has a long history, but its extension to multiloop amplitudes is relatively recent. The problem is that globally on a compact Riemann surface there is no vector field we can call Euclidean "time", and hence no consistent Hilbert space formulation of amplitudes. We can however find local patches where a radial quantization exists. The decisive
step comes with the realisation that the rest of the surface, however complicated, can be summarised by a state |\Sigma\rangle in the corresponding Hilbert space:

\[ P \]

\[ D \]

\[ \Sigma \]

Fig. 5.1

Let's make this more precise. To associate a state to the full surface \( \Sigma \) we must choose a circle \( C \) somewhere on \( \Sigma \) and a normal to \( C \). We do this by supposing \( \Sigma \) to be equipped with a chosen point \( P \) and a local coordinate \( z \), such that \( x_P = 0 \). Then \( C = \{ |z| = 1 \} \) and the normal, "time", is given by the conformal structure. Similarly we build a state \( \langle 0 | \) from the interior of \( C \), a disk \( D \), with the standard local coordinate.

To construct \( \langle 0 | \) and \( | \Sigma \rangle \), consider a real scalar field \( y(x, \bar{z}) \), with conformally-invariant action \( S = \int_B \theta y \wedge \bar{\theta} \). Such a field has one oscillator for every Fourier mode of \( y \). The quantum wave functional for \( y \) is thus a function of each Fourier coefficient, or in other words a functional \( \Psi[f] \) of the boundary value \( f \) of \( y \) at time 0, i.e. on \( C \). We can build such a functional:

\[ \Psi[f] = \int_{y|z=0} [d\bar{y}] e^{-S[y]} \]

The corresponding state is either \( |0 \rangle \) or \( |\bar{\Sigma} \rangle \) depending on whether \( y \) and the action are taken to be defined on the disk \( D \) or on \( \Sigma \equiv \Sigma \setminus D \).

Pretty clearly the full partition function is:

\[ Z = \int_\Sigma [d\bar{y}] e^{-S} = \int [df] \bar{\Psi}_B[f] \Psi_D[f] \equiv \langle 0 | \bar{\Sigma} \rangle \]

Similarly if we glue two cylinders together we get the same propagator as on one long cylinder — the "semigroup property" of path integrals.

Why is this interesting? Why not simply do path integrals? Several reasons were sketched in the papers [81], [8]. First of all, the state \( |\Sigma\rangle \) contains much more than just the partition function. We can instead compute \( \langle 0 | \psi(P) \cdots |\Sigma\rangle \) where the \( y \) are evaluated on \( C \); since \( y \) is harmonic this gives correlations everywhere. Secondly the operator formalism is intimately tied into the constructions of string field theory and may help make them more natural. It may also serve as a bridge to nonperturbative string theory via the universal Grassmannian. In the super case the operator formalism may provide a better understanding of GSO projection than just as a sum over spin structures. Finally it may help us to rephrase, and resolve, the integration ambiguity, by connecting it to BRST.

That was a long list of maybe's. My point here is that the operator formalism sometimes holds substantial practical advantages over path integral methods. We will see how it affords us a tremendous simplification in the proof of holomorphic factorisation. Also in [81], [8] other practical advantages are discussed involving modular invariance and the construction of vertex and spin operators.

Recall that in the bosonic string moduli space has \( 3g - 3 \) even coordinates \( \theta^i \). The string measure is a top form on moduli space, that is it eats \( 3g - 3 \) holomorphic tangent vectors and \( 3g - 3 \) antiholomorphic and yields a number. Let \( V_1, \ldots, V_{3g-3} \) be holomorphic tangent vector fields which vary holomorphically, and similarly \( \bar{V}_1, \ldots, \bar{V}_{3g-3} \). Then \( \mu(V_1, \ldots, V_{3g-3}) \) is a function on moduli space. If factorization of left- and right-movers were perfect we would expect \( \mu(\cdots) \) to be of the form \( \mu(\cdots) = f(t)g(t) \) locally. This property is clearly unaffected if we replace \( V_1, \ldots, V_{3g-3} \) by another such basis. The theorem of Belavin-Knizhnik however says that instead we have

\[ \delta \delta \log \mu(V_1, \ldots, V_{3g-3}) = -13\delta \delta \log \det \tau_3 \]  

(5.1)

where \( \tau_3 \) is the imaginary part of the period matrix. Thus \( \mu \) itself is of the form

\[ \mu = (\det \tau_3)^{-13} \psi \wedge \bar{\psi} \]
where $\psi$ is a holomorphic $(3g - 3)$-form. All kinds of goodies come from this formula. (See e.g. [78].) We would like the super analog, and indeed one exists [63][83][84][85][80][7]. In the past however it's turned out to be a bit messy.

5.2. The period matrix

Let's begin with the RHS of (5.1). Let $\omega^i$ be the super Abelian differentials on a SRS $\Sigma$. How many are there? In the usual case we can answer using the Riemann-Roch theorem: there are always $g$ of them. In the super case we first suppose from now on that we work at an even nonsingular spin structure. (Otherwise the partition function vanishes!) Then the only holomorphic function is the constant and so $h^0(\mathcal{O}) = (1|0)$. The super Riemann-Roch theorem (sect. 4.5) now implies that $h^0(\omega) = (g|0)$, since $\omega$ is odd: $|\omega| = 1$. We can therefore normalize the $g$ even super abelian differentials by $\int_{x_a} \omega^j = \delta^j_{ai}$ as usual.

Since this exhausts our freedom we find a matrix [72][85][7]

$$\hat{\tau}^{ij} = \int_{x_b} \omega^j,$$

the super period matrix. It depends on a canonical homology basis as always. Note however that one need not check that $\hat{\tau}$ is otherwise invariant; this is manifest since it is built from invariant objects. Unlike the bosonic case $\hat{\tau}$ does not contain enough information to reconstruct all the moduli $\omega^i$ and supermoduli $\zeta^a$ — after all, every entry of $\hat{\tau}$ is commuting. Nevertheless it is what we want.

We are supposed to find the variation of $\hat{\tau}$ as the SRS is changed. We could attempt to write an explicit coordinate system for moduli space, for example as in [66] or [80], then compute $\delta\hat{\tau}/\delta\epsilon^i$, $\delta\hat{\tau}/\delta\zeta^a$, but this is quite messy. Instead we will use a very nice collection of tangents to super moduli space provided by [85]. These tangents aren't integrable — but for the computation of $\delta\hat{\tau}$ it doesn't matter, so long as we put the same thing on both sides of (5.1).

Recall that we deal not with a Riemann surface $\Sigma$ but a triple $(\Sigma, F, z) \equiv \Sigma$. We will refer to the moduli space of such triples as $P$ in the bosonic case. (In the super case $\tilde{P}$ consists of $(\Sigma, \tilde{P}, z, \theta)$ where $\tilde{P}$ is a marked point and $z, \theta$ are superconformal coordinates centered at $\tilde{P}$.) How can we deform $\Sigma$? Referring to Fig. 5.1, suppose one has a vector field $v^i(z)$ defined and holomorphic on the unit disk, except possibly for poles at the center. We can take the clutching function which glues $D$ onto $\Sigma_1$ and modify it by composing with the infinitesimal diffeomorphism $1 + v$. Using this new clutching function we glue $D$ back onto $\Sigma_1$ to obtain a new surface $\Sigma'$, close to $\Sigma$ (Fig. 5.2). The marked point still corresponds to the center of the disk; the local coordinate still corresponds to the standard $z$ on $D$. Thus $v$ deforms $\tilde{\Sigma}$ to $\tilde{\Sigma}'$ and hence defines a tangent $\delta_\epsilon$ to every point of $P$. These tangents do not commute; instead the Lie brackets are

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']},$$

where on the left appears the Lie bracket of vector fields on $P$; on the right is a bracket of vector fields on the Riemann surface itself.

Fig. 5.2

For every generator $v \in \text{Vect} S^1$, the Virasoro algebra, we get $\delta_\epsilon$. What is the kernel of the map $\delta_\epsilon$? Suppose first that $v$ extends holomorphically to the entire disk $D$. Then we have changed clutching functions by a coboundary so $\Sigma' = \Sigma$. However, $\Sigma' \neq \Sigma$, because the local parameter $z$ gets replaced by $z + v^i$. Such $v$ therefore modify the coordinate and (if $v(0) \neq 0$) the location of the marked point $P$. On the other hand if $v^i$ extends not to $D$ but to $\Sigma_1$ then again $\Sigma' = \Sigma$. Since no extra information on $\Sigma_1$ enters $\Sigma$, such $v$ give trivial deformations, $\delta_\epsilon = 0$. The subalgebra of such $v$ is called Borel $(\Sigma)$, in analogy to Lie group theory.

For SRS we first must find the analog of $v$. Not every holomorphic diffeomorphism is permitted now, since some spoil the superconformal structure.
Indeed to be permissible a vector field $V$ on $\mathbb{S}$ must be of the form (4.15). The vector fields $V_{\gamma}$ preserving $D$ form a closed subalgebra by the super Jacobi identity. The reader should check that in this way we get a bracket on the $(-1)$-superdifferentials [36]:

$$[v, v'] = (v^2 \partial_{v^2} - v^2 \partial_{v^2} + \frac{1}{2} D_0 v^2 D_0 v^2 - (D_0)^2)$$

and that this satisfies (2.10). Since $v = v_\gamma + \partial_{v^2} \eta$ is a holomorphic superfield, it has enough degrees of freedom to correspond to the Neveu-Schwarz algebra $\text{Vect} S^{11}$. In exactly the same way as before we get a map $\delta$ from $\text{Vect} S^{11}$ to tangent vector fields $\delta v$ on $\mathbb{S}$. Again there is a "Borel" subalgebra at each point of $\mathbb{S}$, Borel $(\mathbb{S})$, containing those $v$ extending to all of $\mathbb{S}$. Again these form the kernel of $\delta$. Again the Riemann-Roch theorem can be used to show that $\delta$ is onto: every tangent to $\mathbb{S}$ arises this way. Of course not every tangent vector field arises this way unless we allow $v$ to depend on the chosen $\mathbb{S}$. In this case we find

$$[\delta v, \delta v'] = \delta (v^{i'} \delta_{v^{i'}} - v^{i'} \delta_{v^{i'}} + \frac{1}{2} D_0 v^{i'} D_0 v^{i'}) .$$

Thus e.g. if $v_{A}$ are chosen to correspond to coordinate differentials $\partial/\partial t_{A}$ we must have

$$\partial v_{A}/\partial t_{B} - (-)^{A B} \partial v_{B}/\partial t_{A} = [v_{A}, v_{B}] .$$

The set of tangents obtained in this way is very pleasant; unlike the case with super Beltrami differentials everything is holomorphic. (Other subtleties are eliminated as well.) For example it is obvious what happens to the various holomorphic differentials on $\mathbb{S}$. If $s$ is such a differential (e.g. a super abelian differential), use the given local coordinate to get a differential on the standard disk. Then the new differential $D$ given by

$$s' = (1 - L_{\gamma}) s$$

clearly extends holomorphically to $\mathbb{S}$. Here $L_{\gamma}$ is the Lie derivative along $V_{\gamma}$ given in (4.15). From this one can easily prove (5.2). Instead we will now use it to find $\delta v^{i' j'}$.

Consider a BC system of weight $\frac{1}{2}$. We will shortly see that its correlations obey

$$\delta_{\gamma} (B(\mathbf{F}) \cdots B(\mathbf{G}) C(Q)) = (B(\mathbf{F}) \cdots T_{\gamma})$$

where $T_{\gamma} = \oint T_{\gamma}(s)s^i(s) ds$, and $T$ is the stress tensor. The correlation function $B(\mathbf{F}) \cdots C(Q)$ is a super abelian differential on each of $\mathbb{S}$. Writing the infinitesimal variation of $\mathbb{S}$ as $v$ folded with some kernel $\eta$,

$$\delta_{\gamma} \mathbb{S} = \left[ \oint ds \eta^{i} \mathbb{S}(s) \eta_{i} \right] ds$$

we see from (5.3) that $\eta$ is a $\frac{3}{2}$-superdifferential at $s$, and a $\frac{3}{2}$-superdifferential at $s$ with pole structure dictated by the operator product expansion of $T$ with $B$. For $B$ of weight $\frac{1}{2}$:

$$T(\mathbf{F}) B(s) = \frac{1}{2} \theta_{1.2} B(s) + \frac{1}{2} \theta_{1.2} D_{ss} B(s) + \frac{1}{2} \theta_{1.2} D_{ss} B(s)$$

Now the super Riemann-Roch theorem (4.13) implies that for fixed $s$ there are just $g + 1$ even $\frac{1}{2}$-superdifferentials with a given second-order pole (take $\mathcal{F} = \mathcal{O}(2\mathbf{S})$, degree $d = 2$). But $\eta$ must satisfy $g$ conditions: the a-periods of $\delta\mathbb{S}$ must all vanish. Furthermore the normalization of $\eta$ is fixed by the pole structure above — so $\eta$ is unique.

Now cut $\mathbb{S}$ into a polygon; as usual we multiply by 1 and add zero:

$$\oint ds \eta^{i} (s, s') = \oint \oint \eta^{i} - \oint \oint \eta^{i}$$

using the known periods of $\mathbb{S}$ and $\eta$. Now let $\mathbb{S} = Df^{i}$ for functions $f$ defined on the cut surface. As usual our expression becomes

$$\oint f^{i} \eta^{i} = \oint f^{i} \eta^{i}$$

where $C'$ is a small contour surrounding the pole.

The poles in $\eta$ give, by the super Cauchy formula, terms with $f^{i} (s)$ and $Df^{i} (s)$. The former cancel while the latter is $\mathbb{S}^{i}$; one obtains

$$\delta_{\gamma} \mathbb{S}^{i} = -i \pi \oint \mathbb{S} (D\mathbb{S}^{j} \cdot \mathbb{S}^{k} + \mathbb{S}^{j} D\mathbb{S}^{k})$$
is Weyl-invariant. The super-period matrix enters as the normalization matrix of the zero modes of $\dd^1$, i.e. $(\omega^i, \omega^j) = \frac{1}{2} \delta^i_{\theta^j} = \phi_{\theta^j}^\dagger \phi^i_{\theta^j} - \phi^i_{\theta^j} \phi^j_{\theta^j} - \phi_{\theta^j}^\dagger \phi_{\theta^j}$ by the super Riemann bilinear relations [72].

We can now vary the matter sector state and compare the answer to (5.4). (In the Weyl-invariant regularization we are using, the ghosts will not contribute to the holomorphic anomaly; this is obvious since they have completely separate Fock spaces.) To do this we again vary the state (see below):

$$\delta^i_\delta^j \log(\cdots) = \langle \cdots | T_{\delta^i_{\theta^j}} | T_{\delta^j_{\theta^i}} \rangle$$ .

(5.6)

Note that $\delta^i_\delta^j$ commutes with $\delta_{\theta^j}$. Using the known stress tensor $T = \frac{1}{2} : (DX)(D^2X) :$, we can compute the contractions which contribute to (5.6). From the non-factorizing second term of (5.5) these are easily seen to reproduce the structure of terms in (5.5), the variation of $\omega_{i\theta^j}$. Doing it carefully one finds

$$\delta \delta \log(\cdots) = -\frac{d}{2} \delta \delta \log \det \omega_{i\theta^j}$$

or for $d = 10$,

$$\langle \cdots \rangle = (\det \omega_{i\theta^j})^{-\frac{d}{2}} |F|^\frac{d}{2}$$

where $F$ is holomorphic on supermoduli space. This is the desired result — the super Belavin-Kaishnik theorem.

We have twice used the variation of a state with respect to moduli. Let me sketch briefly where this formula comes from. Parallel to the bosonic case let

$$S[X] = -i \int_{\Sigma} (DX)(D^2X)$$

$$\Psi[F] = \int_{X|_{\partial = F}} [dX] e^{-S[X]}$$

At the split locus $\Psi$ splits into $\Psi_{\text{Bose}} \cdot \Psi_{\text{Fermi}}$. Here $C = S^{1|1}$, a real super circle embedded in $\Sigma$. $C$ is to be regarded as a “real axis”:

$$x = e^{i\alpha} \quad \bar{x} = e^{-i\alpha} \quad \theta = \bar{\theta} = \chi$$.
$F$ is a function on $\tilde{\mathcal{C}} : F = f_0(\alpha) + \chi f_1(\alpha)$. This is the correct amount of boundary data for one scalar and one Weyl fermion. $\tilde{\mathcal{C}}$ sits in $\Sigma$ in a way dictated by the given superconformal coordinates.

We now wish to characterize the state $\Psi$, which we will again write as $[\Sigma]$. It turns out that

$$Q(H) = \oint_{\tilde{\mathcal{C}}} \left( H(\partial - \bar{\partial})X - X(\partial - \bar{\partial})H \right)$$

(5.7)

all annihilate the state, for any real super function $H$ on the circle $\tilde{\mathcal{C}}$. The contour integral used in (5.7) is quite different from (4.10), since the integrand is not holomorphic. To define it we restrict the integrand to $\tilde{\mathcal{C}}$ by the previous embedding and then integrate over $\alpha$ and $\chi$. The answer does not depend on the given circle only through its homology class, since again the integrand is not holomorphic.

At the split locus these charges include the usual bosonic charges, $Q_{\text{Bose}}(h_0) = \oint_{\tilde{\mathcal{C}}} [h_0(\theta - \bar{\theta})\psi - \psi(\theta - \bar{\theta})h_0]$. In this formula and (5.7) the normal derivatives are defined by taking $H$ (resp. $h_0$) and extending it harmonically: $\partial \bar{\partial} H = 0$ (resp. $\partial \bar{\partial} h_0 = 0$). This can always be done in the ordinary case via the existence theorem for harmonic functions. Alternately we can display enough harmonic functions (one for each fourier mode of $h_0$), as a consequence of the Weierstrass gap theorem. Similarly a super analog of this theorem, proved using the super Riemann-Roch theorem (4.13), gives the existence of $H$.

We therefore have one "conserved" charge (a charge annihilating the state with one boundary) for each super function on $S^{11}$. These suffice to determine $[\Sigma]$ up to a constant. This fact is very useful because now we can investigate the variation of $[\Sigma]$ by studying that of $Q(H)$. If we can show

$$Q((1 + V_s)H) = Q(H) + [T_s, Q(H)]$$

(5.8)

where $V_s$ is as in (4.15) then

$$Q((1 + V_s)H)[\Sigma] = 0$$

will imply that $[\Sigma] = (1 + T_s)[\Sigma]$ up to a constant normalization. But (5.8) is not hard to verify explicitly; using the operator product expansion $T_s$ differentiates the fields in $Q$, and by parts integration this is the same as differentiating $H$.

We can rephrase this situation in a geometrical way. If we let $\nabla_s = \delta_s + T_s$ then it defines a covariant derivative, for which

$$\nabla_s[\Sigma] = 0$$

(5.9)

This makes sense, since from the operator product expansion of $T$ with itself we find that $\nabla$ is flat, i.e. $[\nabla_s, \nabla_s'] = \nabla_{s,s'}$ for constant $s, s'$. The formula (5.9) expresses the variations needed in (5.3) and (5.6).

In conclusion I again want to stress that the operator formalism has the potential to solve various outstanding problems in string theory. But even at the practical level a number of constructions become quite simple in this framework, and that seems reason enough to study it for now.

9. Compactification of supermoduli space

This chapter outlines some recent work by Giddings, Rothstein, Vafa and myself on the geometry of supermoduli space. Specifically we were looking for concrete examples which were so simple that one can describe completely the global geometry of the space, including for example the issue of splitness, after compactification. The example to be described below is the moduli superspace $\mathcal{M}_{g,0}$ of spheres with four Neveu-Schwarz punctures, and also briefly $\tilde{\mathcal{M}}_{1,2,0}$. The former space was chosen for scrutiny because its dimension is 1/2 and its base is an ordinary sphere. Thus it has the potential to be isomorphic to the simplest nonsplit space, discussed in sect. 3.1. In fact we will see that this space is split — but not in a "nice" way to be defined below. Indeed no "nice"

19 Actually there is a central term, which vanishes when we include the ghosts.

20 $\tilde{\mathcal{M}}_{g,p,q}$ denotes the moduli superspace of super Riemann surfaces (SRS) of genus $g$ with $p$ super and $q$ spin punctures.
splitting exists. The implications of this fact for string theory are not clear; it is best to regard the whole discussion as an exercise for learning about the compactification of $\mathcal{M}$.

6.1. Degeneration of Riemann surfaces

First let's recall the bosonic situation. The moduli space $\mathcal{M}_0$ is of course trivial by the uniformisation theorem: the sphere is conformally rigid. If we introduce punctures, however — ultimately the locations of vertex operators — then not all configurations are related by conformal automorphisms. In our case any configuration can be brought to one where in the z-plane punctures $P_1, P_2, P_3$ are located at $z = -1, 1, \infty$ while $P_4$ is at some arbitrary point $z = w$. To be somewhat pedantic we can say that we have a trivial family of Riemann surfaces of the form $C = \mathbb{P}^1 \times \mathbb{P}^1$ over a parameter space $\mathbb{P}^1$; coordinates for the two spheres are $z$ and $w$. $C$ is called the universal curve; the parameter space is the moduli space. In addition we have four sections $s_i : \mathbb{P}^1 \to C$, of which three are at $z(w) =$constant while the last is the diagonal, $z(w) = w$.

Conformal field theory tells us to be careful when two punctures collide. To avoid double poles we should perform another conformal map, from a family where $P_4 \to P_4'$ to one where the $P'$s stay fixed but a pinch develops:

![Diagram](image)

**Fig. 6.1a,b**

We can say this precisely. Within the parameter space $\mathbb{P}^1$ we define an open set $\tilde{U} = \mathbb{P}^1 \setminus \{\pm 1, \infty\}$. Over $\tilde{U}$ we can put the product family and mark points as in Fig. 6.1a. Next we define little open sets $U_-, U_+, U_\infty$, each of them a small disk with coordinate $q_-, q_+, q_\infty$ centered at $w = \pm 1, \infty$ respectively. Over $U_+$, for example, we now build a family of pinching curves as in Fig. 6.2 (see e.g. [78]). Each curve will be built from a fixed disk on the left with coordinate $z$ and two fixed punctures $z = -1, \infty$ and similarly a disk on the right with coordinate $y$ and fixed punctures $y = 1, \infty$, joined by a central tube. We take the latter to be the standard pinching family, or “plumbing fixture”:

**Fig. 6.2**

Fig. 6.2 depicts one fiber of the family $C_+ \to U_+$; for each $q_+ \in U_+$ the shaded region is excluded while the annular regions are glued via $y = z^{-1}q_+$. Having defined a family $C_+$ over $U_+$ we must now proceed to glue it to $\tilde{C} = \tilde{U} \times \mathbb{P}^1$ over $\tilde{U}$. The gluing map must satisfy very restrictive conditions:

a) It glues fibers isomorphically to fibers. Thus $q_+ (w; z)$ is independent of $z$, and $s(w; z)$ or $y(w; z)$ give a conformal automorphism of fig 6.1a to fig 6.2 for each fixed $w$ near 1, $w \neq 1$.

b) It sends the four punctures $P_i$ onto the $P'_i$ in order.

It's easy to see that these requirements fix the gluing map uniquely:

$$z = \frac{1}{2}(z - 1)$$

$$q_+ = \frac{1}{2}(w - 1)$$

Since $q_+$ is a regular function of $w$ all the way down to $w = 1$, we see that we may as well forget about $q_+$ and continue to use $w$ as our coordinate throughout $\tilde{U} \cup U_+$. Similarly $w$ is a good coordinate near $w = -1$. In the neighborhood of $w = \infty$, however we find that the good pinching coordinate $q_\infty$ is related to $w$ by

$$q_\infty = w^{-1}$$
Thus $\mathcal{M}_{0,4}$ is covered by two patches with transition functions which make it into a sphere — as we guessed all along.

6.2. Punctures on SRS

Before proceeding with the construction I should make more precise the notion of punctures. Even on an ordinary Riemann surface we do not mean that the "punctures" are literally deleted points; e.g. we do not admit functions with essential singularities at those points. Instead we view them simply as chosen, or marked points $P_t$ and admit poles by tensoring in $\mathcal{O}(nP_t)$, the sheaf of local meromorphic functions holomorphic away from $P_t$ and with at most an $n^{th}$ order pole at $P_t$.

On a SRS there are two natural notions of puncture, and both are needed in string theory. Similarly to the previous paragraph one can let a super puncture (or NS puncture) be just a marked super point, or more precisely a map $\tilde{\sigma}_t$ from any parameter space $\tilde{\mathcal{S}}$ into a family $\overset{\sim}{\mathcal{C}} \rightarrow \tilde{\mathcal{S}}$ of SRS satisfying $\pi \sigma_t =$ identity. It can be given as functions $(z_i(t,\zeta), \theta_i(t,\zeta))$: the even functions $(z - z_i)$ then define divisors $\Delta_i$ on the SRS just as in the bosonic case and we can talk about poles at the puncture. This is the only kind of puncture I will use in the example in the next section.

The other case, a spin puncture (or R puncture) is more subtle [8].\textsuperscript{21} Unlike the super case something is actually singular at the spin puncture, but not the curve itself. Instead we define a smooth SRS with $n$ spin punctures to be a smooth complex 1|1-supermanifold with a nonsingular divisor $\Delta = \Delta_1 \cdots \Delta_n$ and a derivation $D$. $D$ maps the super functions $\tilde{\mathcal{S}}$ to sections of $\tilde{\omega} \equiv [\nabla \mathcal{S}(\Delta)]$. Away from $\Delta$ there must always exist good local coordinates $z, \theta$ such that $D = [dz|d\theta](\theta_\theta + \eta \theta_z)$ as usual. Near $\Delta_i$, however, we require that there be good complex coordinates $z, \eta$ such that $\Delta_z = \{z = 0\}$ and

$$D = [dz|d\eta](z^{-1}\theta_\eta + \eta \theta_z).$$  \hspace{1cm} (6.1)

Thus $D$ takes us to sections of the modified canonical bundle $\tilde{\omega}$ above. There is nothing inherently ramified about the SRS at the punctures.

Of course one can replace the good, non-superconformal coordinates $z, \eta$ by multivalued "coordinates" $z, \theta = z^{1/2} \eta$, whereupon $D$ regains its usual form. But the system of cuts usually taken to join pairs of spin punctures is not a part of the SRS proper.

As an example of a punctured SRS consider the smooth complex supermanifold with base the sphere and coordinate maps $z_1 = z_1^{-1}$, $\eta_2 = \eta_1$. Since $\eta$ is not a spinor this is an example of a complex 1|1-supermanifold for which no nonsingular superconformal structure exists. Now however let $\Delta_i$ be defined by $z_i$ and $D$ as (6.1); Then $D$ is globally defined. To put this into a more traditional form we can now let

$$u_1^2 = -z_1, \quad u_2^2 = z_2, \quad \theta_i = \pm u_i \eta_i.$$

Then we recover the usual form of the super sphere with two spin punctures.

6.3. Degeneration of SRS

The construction of the moduli space for the four-punctured sphere was not very challenging in the bosonic case, but matters become somewhat more interesting in superspace. Consider the fixed SRS $\mathbb{P}^{11}$ built from $\mathbb{P}^1$ with its spin bundle, and the moduli space $\mathcal{M}_{0,4,0}$ with four super punctures. Now we find that automorphisms can fix the bosonic coordinates of 3 punctures, but the fermionic coordinates of just 2. We will depict a choice of coordinates by a picture:
Again this says $P_1'$ is at $(z, \xi) = (-1, \tau_+)$ etc. Now however the gluing map is:

$$xy = -t_+^2 \ ; \ z\psi = -t_+\xi \ ; \ u\xi = t_+\psi \ ; \ \xi\psi = 0 \ .$$

(6.3)

Note that as in the bosonic case there is a normalization map from a SRS with super pinch to a nonsingular one with two super punctures, and that in particular normalisation commutes with the $D$ operator. One simply pulls apart the SRS into two parts, with superconformal coordinates $(z, \xi)$ and $(y, \psi)$ respectively, and makes the evident projection from that disconnected space to (6.3) at $t_+ = 0$.

The surprise is that $t_+^2$, not $q_+$, parametrises the pinch. This is necessary in order that $\psi$ and $\xi$, which are spinors, should be single-valued with respect to each other as we walk around the “boundary” $t_+ = 0$ of supermoduli space, or equivalently that spinors on one side should have a good meromorphic extension to the whole family. Thus $\tilde{M}$ must ramify over its 3 points at infinity! [87] Let $r$ be a coordinate for this covering, with $w = \frac{1}{2}(r^2 + r^{-2})$. Then the $r$-plane covers the $w$-plane four times, except at the points at infinity. The latter are located now at $w = \pm1, \pm i, 0, \infty$.

Again we can glue the pinching family $\tilde{C}_{+1}$ to the big set $\tilde{C}$, requiring the map to satisfy:

a) It is a superconformal automorphism fiber by fiber on the overlap.

b) It sends the super punctures to their counterparts.

This map is again unique. To describe it we first redefine $\delta, \eta$ to $\tilde{\delta} = (r + r^{-1})^{-1} \delta$, $\tilde{\eta} = (r - r^{-1})^{-1} \eta$. This is permissible since on $\tilde{U}$ the extra factors just introduced do not blow up. These factors rescale $\delta, \eta$ into coordinates regular at some of the points at infinity. Indeed, solving the above

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[22] There is also a good normalization map taking a spin-pinched curve to a SRS with spin punctures and commuting with $D$; the two branches of the punctured curve are now glued along the $0|1$-subvariety $\Delta_1$ [87].
conditions one finds

\[
\begin{align*}
\tau &= 2z + 1 \\
\theta &= \sqrt{2} \xi \\
\bar{t}_+ &= \frac{1}{2} (r - r^{-1}) \\
\bar{\zeta}_+ &= -i\sqrt{2} \bar{\eta} \\
\gamma_+ &= \frac{1}{\sqrt{2}} (r + r^{-1}) \delta
\end{align*}
\]  \hspace{1em} (6.4)

Note that again \( t_+; \bar{t}_+, \gamma_+ \) are regular functions of \( r; \bar{\eta}, \delta \) as \( r \to \pm 1 \), so again on \( \tilde{\mathcal{U}}_{\pm 1} \) we can simply use the same coordinates as on the big patch \( \tilde{\mathcal{U}} \). A similar argument works on \( \tilde{\mathcal{U}}_{\pm \epsilon} \). We note from (6.4) that the new coordinates \( r, \bar{\eta}, \delta \) are related to the natural ones \( t, \gamma, \zeta \) by a split transformation.

As before the last kind of pinch (or “channel”) causes troubles. (There is of course nothing significant in the fact that one channel is singled out, since we have all along been treating the pinches asymmetrically.) The problem is that the coordinates depicted by

![Diagram](image)

become singular as \( r \to 0, \infty \); unlike the previous cases the two free coordinates are on the same side, which is impossible at the pinch. Thus the unique gluing map for \( r \) close but not equal to 0 must include a super-Moebius transform taking one free odd coordinate to the other side. This yields

\[
\tau - \tau^{-1} = \frac{2i}{t} (1 - \frac{1}{2} \gamma \zeta_0 (t_0 - t_0)) \\
\bar{\eta} = \frac{1}{\sqrt{2}} \xi_0 \\
\bar{\delta} = \frac{1}{\sqrt{2}} \frac{r + r^{-1}}{r - r^{-1} \xi_0} + i \frac{\sqrt{2}}{r + r^{-1} \gamma_0}
\]

In particular \( r \) is a nonsplit function of the natural coordinates \( t_0; \gamma_0, \xi_0 \), and similarly at \( r \to \infty \).

6.4. Splittings and good splittings

This does not mean that \( \tilde{\mathcal{M}}_{0,4,0} \) cannot be split! Indeed we see that we can once again use \( \bar{r} \) itself as the even coordinate on \( U_0 \), and \( \bar{r} = r^{-1} \) on \( U_\infty \). Since \( \tau \) is a regular function of \( t_0; \gamma_0, \xi_0 \) throughout \( \tilde{\mathcal{U}}_0 \), we see that \( \tau; \gamma_0, \xi_0 \) are good coordinates for \( \tilde{\mathcal{U}}_0 \), and similarly near \( \infty \); this suffices to show that \( \tilde{\mathcal{M}}_{0,4,0} \) can be split. But something important has been lost: this time the new coordinates, while regular, are related to the natural ones by nonsplit transformations. In this sense the splitting given is not “good”. A completely analogous situation obtains for \( \tilde{\mathcal{M}}_{1,2,0} \) at least when the ordinary modulus is suppressed.

We can say this is an intrinsic way. A splitting, or just a projection, provides a map \( \tilde{\mathcal{M}}_{0,4,0} \to \tilde{\mathcal{M}}_{0,4} \). The fibers of this map are divisors, subvarieties of codimension 1/0 in \( \tilde{\mathcal{M}} \). But \( \tilde{\mathcal{M}} \) has more structure than just a supermanifold: it also comes equipped with a special divisor, the locus \( \tilde{\Delta} \) of pinched SRS. Similarly \( \tilde{\mathcal{M}} \) has a divisor \( \Delta \) of pinched Riemann surfaces. The projection \( \tilde{\mathcal{M}} \) defined by taking \( \tau, \bar{r} \) to be the good even coordinates conflict with this extra structure in the sense that \( \tilde{\Delta} \neq \tilde{\pi}^{-1}(\Delta) \). For example at \( r \to 0 \) the divisor is given by \( \{ t_0 = 0 \} \), while the fiber is given by \( \{ r = 0 \} \). Since for small \( r \) we have \( r \sim \frac{1}{2} (t_0 + \frac{1}{2} \gamma \xi_0) \), these two divisors differ.

I should briefly discuss another approach to this problem. As on the ordinary sphere we can define a cross ratio

\[
\mathcal{P} = \frac{x_1 x_2 x_3}{x_4 x_2 x_3}; \quad z_{ij} = z_i - z_j - \theta_i \theta_j
\]
The quantity $p$ is invariant under super-Mobius transformations. Naively it seems that we can take $p$ and $\bar{p} = p^{-1}$ to be good even coordinates on all of $\bar{M}_{0,4,0}$, and clearly $\bar{p}$ is a split function of $p$ and the odd coordinates — i.e. a function only of $p$. Moreover at each of the points at infinity as $t \to 0$ we have that $p$ or $\bar{p}$ approach 1 or 0, independently of the odd parameters. Doesn’t this mean that the even functions $p, \bar{p}$ define a splitting compatible with the divisor at infinity? Alas, no. Both $p$ and $\bar{p}$ are bad coordinates at infinity. For example, as $t_+ \to 0$ we have $\bar{p} = \frac{2+1+\gamma p+4}{1+3+q}$, so $1-\bar{p} \sim t_+^2 + \text{nilpotent}$. A good coordinate would be $(1-\bar{p})^{1/2}$, but at $t_+ = 0$ this is not longer zero; instead $(1-\bar{p})^{1/2} \to -\gamma_+ \zeta_+ / 2\sqrt{3}$. Again the “natural” choice of splitting doesn’t match the one we want at the boundary.

There is a great deal of rigidity in an analytic object such as a splitting. Many supermanifolds admit no splitting at all; while $M_{0,4,0}$ is not of this type, still it seems clear that a splitting compatible with its extra structure cannot be found. A real proof of this assertion requires that we find the space of all possible splittings and check them all. In section 3.2 we saw that the splittings form an affine space modeled on $H^0(T \otimes \Lambda^2 \epsilon)$, where $T$ is the tangent space to $\bar{M}$ and $\epsilon$ is the bundle where the odd variables live. In the present case this is a vector space of dimension five. But an element taking our bad projection to a good one must satisfy one condition at each of the six boundary points, and so no good choice exists.

Why does this matter? As stated earlier, I don’t really know. However, to compute 4-point string amplitudes we must integrate a density over $\bar{M}$. Even after the GSO projection this density will in general blow up at infinity, and so in general one needs to pick up a residue there. But in superspace the definition of residue is not so obvious; unlike a SRS $\bar{M}$ has no superconformal structure to help us out. We can always expand in powers of the pinch coordinate $t$ — but only if the splitting used to integrate on the rest of $\bar{M}$ is compatible with the natural $t$. If this cannot be arranged, as happens even in this simple case, then corrections must be added to the integral, similar to the ones discovered by Green and Seiberg [99].

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References


