Time-Triggered Implementations of Dynamic Controllers

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Abstract
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Disciplines
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Categories and Subject Descriptors
C.3 [Computer Systems Organization]: Special-purpose and Application-based Systems—Real-time and embedded systems; C.4 [Computer Systems Organization]: Performance of Systems—Modeling techniques, Performance attributes; J.7 [Computer Applications]: Computers in other Systems—Real time, Command and control

1. INTRODUCTION

Bridging the gap between high-level modeling or programming abstractions, and implementation platforms is one of the key challenges for embedded software research [15, 14]. The goal of our research, initiated in a recent paper [19], is to address this challenge in the context of implementing feedback control loops by software [7].

Consider a physical plant interacting with a controller that measures some plant signals and generates appropriate control signals in order to influence the behavior of the plant. The models of both the plant and the controller have well-defined timed semantics that can be used for simulation and analysis. Once the controller design is complete, the designed controller model is typically expressed as a set of control (usually MATLAB) blocks. Each control block is compiled into an executable code in a host language such as C, and the control designer specifies a period for the corresponding task. To implement the resulting periodic tasks on a specific platform, one needs to determine the worst-case-execution-time for each block, and check whether the task set is schedulable (c.f. [5, 12]).

While the real-time scheduling based implementation offers a separation of concerns using the abstraction of real-time tasks with periods and deadlines, it introduces several sources of unpredictability. In particular, there are no guarantees regarding when a control block actually reads its inputs and when its outputs become available to its environment, and the order in which the various blocks execute. As a result, quantifying the error between the timed semantics of the control blocks and the possible executions of scheduled tasks, and understanding its impact on the application-level quality-of-service, remains difficult.
The recent emergence of time-triggered architecture as an implementation platform for embedded systems offers opportunities for a more predictable mapping of control models [13, 12]. In a time-triggered implementation, instead of mapping control blocks to periodic tasks, the compiler can allocate well-defined time slots to control blocks. Given a mapping of all the control blocks to the time slots, one can precisely define the trajectories of the implementation and quantify the error with respect to the model-level semantics.

In our formalization, given a dynamic controller model as a set of interacting control blocks, we define the controller implementation on a time-triggered platform using a dispatch sequence that gives the order in which the blocks are repeatedly executed, and a timing function that gives the number of time slots needed to execute each block. For a given model of the plant, we can precisely define the semantics of the implementation, and measure its quality by metrics, such as the $L_2$-norm, of the discrepancy between the trajectories of the model and the implementation. Given linear control plants, proportional-integral-derivative (PID) controllers, a dispatch sequence, and a timing function, we model the controller implementation naturally as a periodic linear time-varying system (PLTV). Compared to our previous work [19], in this paper we consider more general dynamic controllers (PID). This in turn requires us to consider the error dynamics of the numerical integration and differentiation algorithms in our error computation. Using the lifting technique for transforming a PLTV to a linear time-invariant linear system, we can compute the error with respect to the model semantics measured by $L_2$-norms. Since different implementations correspond to different PLTV’s, this gives us a way of comparing implementations quantitatively, as illustrated via examples in Section 4.

Related Work: Programming abstractions for embedded real-time controllers include synchronous reactive programming (languages such as Esterel and Lustre [9, 8]), and the related Fixed Logical Execution Time (FLET) assumption used in the Giotto project [10]. Research on time-triggered architecture has focused on achieving clock synchronization, fault tolerance, real-time communication, and automotive applications (c.f. [12, 13]). The goal of this research has been ensuring predictable communication between components. In the context of this paper, time-triggered platform offers pre-defined time slots for scheduling, and we study how this can be exploited for predictable execution of control blocks.

Recently, the problem of generating code from timed and hybrid automata has been considered in [2, 11, 18], but the focus has been on choosing the sampling period so as to avoid errors due to switching and communication. The work on mapping Simulink blocks to Lustre focuses on signal dependencies [6]. In [1], relative scheduling as a way of generating a dispatch sequence for a control model for soft real-time applications has been explored but it did not have a framework for quantifying the errors. Many variations of basic scheduling model have been explored, but the emphasis is not on quantifying the errors introduced during mapping control model to the task model. Perhaps the most related of these efforts is control-aware scheduling [17] and control-scheduling co-design [3].

There is a rich literature on sampled control systems [4] along two main approaches. The first approach discretizes a continuous plant given implementation dependent sampling times, and a controller is designed for the discretized plant. The second approach starts with a designed continuous controller and focuses on discretizing the controller on some implementation platform [4]. Even though this is the spirit of our approach, the resulting error analysis has historically focused on quantifying the errors introduced due to sampling without paying attention to more detailed models of implementation platforms that must sequentially execute multiple control blocks.

2. MODELING

In this section, we model embedded control systems and provide two different semantics: model-level semantics used for control design, and implementation-level semantics used for execution on a time-triggered platform. The goal of this paper will be to compute a distance between these two semantics, thus providing a measure for the quality of the implementation.

2.1 Feedback Control Model

Consider a finite set $X = \{x_1, \ldots, x_n\}$ of plant variables, a set $Y = \{y_1, \ldots, y_p\}$ of output variables, a set $U = \{u_1, \ldots, u_m\}$ of control variables, and a set $Z = \{z_1, \ldots, z_q\}$ of internal variables. All variables take values in $\mathbb{R}$. A state over a set of variables is a mapping from the set of variables to corresponding values. The set of all possible plant states is thus $\mathbb{R}^n$, and we obtain similar sets of states for all other variables.

A feedback control model is a tuple $\mathcal{M} = (\mathcal{M}_P, \mathcal{M}_C)$ consisting of a plant model $\mathcal{M}_P$ and a controller model $\mathcal{M}_C$. A plant model $\mathcal{M}_P$ consists of

- A function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ that defines the dynamics of variables $x_i$ in terms of the current plant state and control inputs.

- A function $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ that expresses the observable output of the plant given the current plant state.

A controller model $\mathcal{M}_C = (B_1, B_1, \ldots, B_m)$, consists of

- A control block $B_1$ that describes the function $g : \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ that expresses the dynamics of the internal variables $z_i$ in terms of the current internal variables and plant outputs.

- A finite set of control blocks $(B_1, \ldots, B_m)$, one control block $B_j$ for every control variable $u_j \in U$. Every control block $B_j$ describes a function $k_j : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q-1} \rightarrow \mathbb{R}$ expressing the feedback control law for the control variable $u_j$ as a function of observable plant outputs and their derivatives, internal variables, and other control variables.

We assume that the dependence among the control variables is acyclic so that the feedback law for the control variable $u_j$ is a function of the plant outputs and their derivatives, the internal states, and the control variables $u_{1, \ldots, u_{j-1}}$.

Note that the class of controllers considered by this model is dynamic due to the existence of the internal variables $z_i$ that have their own dynamics. In addition, the derivatives of the plant outputs are included in the computation of control blocks $B_j$. This model captures the widely used
proportional-integral-derivative (PID) controllers as well as more general observer-based controllers. This is a significant
extension of the class of static controllers considered in our
previous work [19].

2.2 Model Level Semantics

Given a feedback control model \( M = \langle M_P, M_C \rangle \) with
variables \( X, Y, U, Z \), a trajectory for \( M \) is a function from
the time domain \( \mathbb{R} \geq 0 \) to the set of states over all variables.
Let \( x(t) = (x_1(t), \ldots, x_n(t)) \) denote plant trajectories in
vector notation, and, similarly, \( y(t) = (y_1(t), \ldots, y_m(t)) \), \( u(t) =
(u_1(t), \ldots, u_n(t)) \), and \( z(t) = (z_1(t), \ldots, z_m(t)) \). Given feed-
back control model \( M \), we denote the continuous-time seman-
tics of the feedback control model by \([M]\) and define \([M]\)
as the collection of all trajectories \((x(t), y(t), u(t), z(t))\)
that for all \( t \geq 0 \) satisfy the following differential and al-
gebraic constraints modeling the feedback interconnection
between the plant and the controller dynamics:

\[
\begin{align*}
M_P : & \quad \dot{x}(t) = f(x(t), u(t)) \\
M_C : & \quad \dot{z}(t) = g(z(t), y(t)) \\
        & \quad u_j(t) = k_j(y_j(t), y(t), z(t)) \\
        & \quad \dot{u}_j(t) = k_j(y_j(t), y(t), z(t), u_1(t), \ldots, u_{j-1}(t)) \\
        & \quad \dot{z}(0) = 0
\end{align*}
\]

(1)

In this paper, we assume that the feedback composition to
be well-posed, meaning that for any initial plant state \( x(0) \)
the above equations have unique solutions. Given \( x(0) \),
we denote the unique solution for the continuous-semantics as

\[
(x(t), y(t), u(t), z(t)) = [M](x(0))
\]

(2)

The continuous-time semantics is implementation indepen-
dent semantics that is used for the mathematical analysis
and design of controllers that achieve desired performance
specifications of the output trajectories \( y(t) \), using a variety
of techniques from control theory. Our goal in this paper is
to quantify the deviation from this ideal semantics when
the controller \( M_C \) is implemented on a given time-triggered
platform.

2.3 Implementation Level Semantics

The ideal model-level semantics assumes that all control
blocks of controller \( M_C = \{B_1, B_2, \ldots, B_m\} \) are “computed”
instantaneously and simultaneously. Of course, any soft-
ware implementation of the controller \( M_C \) will violate both
assumptions. In addition, the continuous-time semantics
assumes that the internal variables, described by the dif-
ferential equation \( \dot{z}(t) = g(z(t), y(t)) \) of control block \( B_i \),
are ideally integrated, with no error. Clearly, any numerical
method for approximately solving this differential equation
will introduce errors depending on the integration step size.
Moreover, ideal differentiation computation assumed by the
model-level semantics cannot be satisfied by the implement-
ation, where an approximation algorithm must be used.

As discussed in the introduction, mapping control blocks to
periodic tasks does not allow a mathematically rigorous execu-
tion semantics. Instead, we assume that the imple-
mentation is on a time-triggered platform in which time
can be allotted in fixed-size slots. To model the order in
which the control blocks are executed we consider a dis-
patch sequence \( \rho \), which is an infinite string over the set
\( \{B_0, B_1, B_2, \ldots, B_m\} \). Here, \( B_0 \) is used to model idling from
the viewpoint of the controller (e.g., idling, or allocation of a
time slot to activities other than the computation of control
outputs). Typically, \( \rho \) will be periodic, and will be specified
by a finite string that repeats. Each control block is to be
executed without pre-emption, and when one control block
completes its execution, the next block can start imme-
diately. For example, given a controller \( M_C = \{B_1, B_2, B_3\}\) with
one internal and three control blocks, possible dispatch
sequences are the uniform sequence \( \{B_1, B_2, B_3\}^\infty \) and
the nonuniform sequence \( \{B_1, B_2, B_1, B_2, B_3, B_1, B_2, B_3\}^\infty \) that
also includes idling. Note that a dispatch sequence contains only
ordering information, and is thus independent of the pro-
cessing speed of the platform.

A time-triggered platform provides an atomic time slot of
length \( \delta \), and each block is assigned to a fixed number of
such slots. The computation of each control block \( B_i \)
consists of reading the relevant plant output variables
using sensors, updating the control variable \( u_i \) (or internal
variables \( z_i \)), and finally writing the computed control value
to the actuators at the end of its allotted time. The com-
putation time of each control block is captured by a timing
function \( \tau : \{B_1, B_2, \ldots, B_m\} \rightarrow \mathbb{N}^+ \) which associates to each
control block the number of time slots needed to execute it.
Without loss of generality, we assume \( \tau(B_0) = 1 \).

Informally, given a feedback control model \( M = \langle M_P, M_C \rangle \),
a dispatch sequence \( \rho \), a timing function \( \tau \) and a time slot
length \( \delta \), we can define the implementation semantics as-
soiated with \( M \), denoted as \([M]_{(\rho, \tau, \delta)} \), to be the set of
trajectories obtained by executing the control blocks of con-
troller \( M_C \) according to the dispatch sequence \( \rho \), where the
number of slots of length \( \delta \) for each control block are chosen
according to the timing function \( \tau \).

Formally, to define the implementation semantics, we note
that the dispatch sequence \( \rho \), timing function \( \tau \) and time
slot length \( \delta \) result in the following sequence of timing inst-
stants \( t_i : t_0 = 0 \) and \( t_i = \sum_{k=0}^{i-1} \tau(\rho(k)) \delta \) for \( i \geq 1 \).
Except for \( t_0 \), these are the precise timing instants when a
control block completes its computation and its outputs are
updated. In addition, we recursively define the time instants
\( \Delta_I(i) \), where \( \Delta_I(0) = 0 \) and

\[
\Delta_I(i+1) = \begin{cases} 
\Delta_I(i) + \tau(\rho(i))\delta & \text{if } \rho(i) \neq B_i \\
\tau(B_i)\delta & \text{if } \rho(i) = B_i 
\end{cases}
\]

(4)
in order to model the time elapsed since the last execution
of the integration block \( B_i \). Similarly, we define the sequence of
time instants \( \Delta_D(i) \), where \( \Delta_D(0) = 0 \) and \( j = 1, 2, \ldots, m \),

\[
\Delta_D(i+1) = \begin{cases} 
\Delta_D(i) + \tau(\rho(i))\delta & \text{if } \rho(i) \in \{B_0, B_1\} \\
\tau(B_j)\delta & \text{if } \rho(i) \in \{B_0, B_1\}
\end{cases}
\]

(5)
to model the time elapsed between executions of blocks \( B_j \),
\( 1 \leq j \leq m \), which are essential to the differentiation com-
putation. The derivatives of plant outputs \( y \) are numerically
computed in the controller and are held in internal variables
\( w \). The implementation semantics \([M]_{(\rho, \tau, \delta)} \) can now be
defined as the collection of trajectories \((x(t), y(t), u(t), z(t))\)
that for all \( t \geq 0 \) satisfy the continuous-time plant dynamics
defined by Eq. (1), and the following controller implementa-
tion constraints for every \( 1 \leq j \leq m \) and \( i \geq 0 \),
Initialization
\[ z(0) = 0, \quad w(0) = 0, \quad u_j(0) = 0 \] (6)

Inter-sample
\[ z(t) = z(t_i), \quad w(t) = w(t_i), \quad u_j(t) = u_j(t_i) \]
for \( t_i < t < t_{i+1} \) (7)

Controller Updates
\[ u_j(t_{i+1}) = \begin{cases} u_j(t_i) & \text{if } \rho(i) \notin B_j \\ k_j(y(t_i), w(t_{i+1}), z(t_i), \\ u_1(t_i) \ldots u_{j-1}(t_i)) & \text{if } \rho(i) = B_j \end{cases} \] (8)

Numerical Differentiation
\[ w(t_{i+1}) = \begin{cases} w(t_i) & \text{if } \rho(i) \in \{B_0, B_1\} \\ D(y(t_i), y(t_i - \Delta_D(i)), \\ \Delta_D(i), w(t_i)) & \text{if } \rho(i) \notin \{B_0, B_1\} \end{cases} \] (9)

Numerical Integration
\[ z(t_{i+1}) = \begin{cases} z(t_i) & \text{if } \rho(i) \notin B_I \\ G(z(t_i), y(t_i), z(t_i - \Delta_I(i)), \\ y(t_i - \Delta_I(i)), \Delta_I(i)) & \text{if } \rho(i) = B_I \end{cases} \] (10)

Implementation constraints (6) capture initialization, constraints (7) express the fact that during the execution of any control block all computed values remain constant, equation (8) describes the equations for updating the control variables, (9) represents the numerical computation of the derivatives of plant outputs, and (10) captures the numerical scheme for integrating the internal variables when control block \( B_I \) is scheduled. The implementation constraints clearly show that \( u_j(t), z(t), \) and \( w(t) \) are piecewise-constant signals.

Note that the function \( G(z(t_i), y(t_i), z(t_i - \Delta_I(i)), y(t_i - \Delta_I(i)), \Delta_I(i)) \) can be used to model a variety of fixed-step numerical integration algorithms. Different choices for this function can model different choices for well known numerical algorithms as shown in Table 1. Euler is a one-step method, whereas Trapezoid and Adams-Bashforth are two-step methods that utilize information in two different time instants in integrating the dynamics of the internal variables. Higher order methods could easily be modeled at the expense of more variables, one for each step\(^1\). Note that every time the integration block \( B_I \) is executed, the numerical methods need to integrate the internal dynamics starting from the last time block \( B_I \) was executed. Therefore the dispatch sequence will have a direct effect on the size of the integration step \( \Delta_I(i) \) and therefore the quality of the approximation.

Similarly, the function \( D(y(t_i), y(t_i - \Delta_D(i)), \Delta_D(i), w(t_i)) \) can be used to model the numerical computation of the derivatives of \( y \). Some commonly used algorithms for approximating the derivatives are

\(^1\)Note that Runge-Kutta methods, even though popular for simulation, are problematic for code generation, as they require evaluations such as \( g(z(t_i + \Delta_I(i)), y(t_i + \Delta_I(i))) \), which in turns requires predicting the sensed input \( y(t_i + \Delta_I(i)) \) in the future.

Backward Difference
\[ w(t_{i+1}) = \frac{1}{\Delta_D(i)} (y(t_i) - y(t_i - \Delta_D(i))) \]

Tustin’s Approximation
\[ w(t_{i+1}) = \frac{2}{\Delta_D(i)} (y(t_i) - y(t_i - \Delta_D(i))) - w(t_i) \]

Given \( x(0) \), we denote the solutions for the implementation semantics as
\[ (x(t), y(t), u(t), z(t)) = [M]_{(\rho, \tau, \delta)}(x(0)) \] (11)

The main goal of this paper is to quantify the quality of the controller implementation for a particular dispatch sequence \( \rho \), timing function \( \tau \) and time slot length \( \delta \). Having defined both the ideal platform-independent semantics, and the platform-dependent semantics, we can directly define the error of the implementation as a function of the initial plant state \( x(0) \) simply as
\[ (x(t), y(t), u(t), z(t)) = [M]_{(\rho, \tau, \delta)}(x(0)) \]

\[ (\tilde{x}(t), \tilde{y}(t), \tilde{u}(t), \tilde{z}(t)) = [M]_{(\rho, \tau, \delta)}(x(0)) \]

\[ e_{M}(\rho, \tau, \delta, x(0)) = \int_{0}^{t_{\infty}} \| y(t) - \tilde{y}(t) \|^2 dt \] (12)

We are therefore measuring the implementation error in the \( L_2 \) sense. Note that we are measuring the implementation error on the output variables of the overall closed loop system, rather than the error on the controller variables. We are therefore directly measuring the effect of controller implementation on the performance of the overall feedback interconnection.

Given a feedback control model \( M \) and a set of initial plant states \( X_0 \), we will say that the implementation \( (\rho_1, \tau_1, \delta_1) \) is more accurate than the implementation \( (\rho_2, \tau_2, \delta_2) \) (noted \( (\rho_1, \tau_1, \delta_1) \preceq_M (\rho_2, \tau_2, \delta_2) \)) if the implementation error of \( (\rho_1, \tau_1, \delta_1) \) is smaller than the one of \( (\rho_2, \tau_2, \delta_2) \) for all initial states:
\[ \forall x(0) \in X_0 \quad e_{M}(\rho_1, \tau_1, \delta_1, x(0)) \leq e_{M}(\rho_2, \tau_2, \delta_2, x(0)). \] (13)

Note that the relation \( \preceq_M \) is a preorder on the set of implementations. The challenge is now to compute the \( L_2 \) norm of the implementation error as a function of \( x(0) \), given implementation specifics \( (\rho, \tau, \delta) \).

3. ERROR ANALYSIS
In this section we provide a method for the computation of the implementation error \( e_{M}(\rho, \tau, \delta, x(0)) \) for an important class of plants and embedded controllers. We assume that the plant model is a linear, time-invariant (LTI) system:
\[ M_P : \begin{cases} \dot{z}(t) = A_p x(t) + B_p u(t) \\ y(t) = C_p x(t) \\ x(0) \in \mathbb{R}^n \end{cases} \]

\[ \text{where } A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m} \quad \text{and } C_p \in \mathbb{R}^{p \times n}. \]

We will also assume that the feedback controller model \( M_C \) has the
where $Q$ consists of one control block class are the so-called proportional-integral-derivative (PID) controllers. We can thus readily see that the controllers captured in this sequence with an infinite sum. The computation of matrix $Q$ is:

$$
Q(i) = \frac{1}{2}(g(z(t_i), y(t_i)) - g(z(t_i - \Delta t(i)), y(t_i - \Delta t(i))))
$$

Adams-Bashforth

$$
Q(i) = \frac{1}{2}(3g(z(t_i), y(t_i)) - g(z(t_i - \Delta t(i)), y(t_i - \Delta t(i))))
$$

Table 1: Function $G$ for some well known numerical integration algorithms

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>$z(t_{i+1}) = z(t_i) + \Delta t(i)g(z(t_i), y(t_i))$</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>$z(t_{i+1}) = z(t_i) + \frac{\Delta t(i)}{2}(g(z(t_i), y(t_i)) - g(z(t_i - \Delta t(i)), y(t_i - \Delta t(i))))$</td>
</tr>
<tr>
<td>Adams-Bashforth</td>
<td>$z(t_{i+1}) = z(t_i) + \frac{\Delta t(i)}{2}(3g(z(t_i), y(t_i)) - g(z(t_i - \Delta t(i)), y(t_i - \Delta t(i))))$</td>
</tr>
</tbody>
</table>

Following structure

$$
\mathcal{M}_C : \begin{cases}
\dot{z}(t) = B_i y(t) \\
u(t) = K_P y(t) + K_I z(t) + K_D \dot{y}(t) + L_C u(t) \\
z(0) = 0
\end{cases}
$$

where $B_C \in \mathbb{R}^{n \times p}$, $K_P \in \mathbb{R}^{m \times q}$, $K_I \in \mathbb{R}^{m \times q}$, and $L_C \in \mathbb{R}^{m \times m}$. The controller model $\mathcal{M}_C = (B_i, B_1, \ldots, B_m)$ consists of one control block $B_i$ for integrating all the internal variables $z_i$, and one control block $B_i$ for every variable $u_i$. The challenge is to compute the implementation error, defined over all $t_i$ such that the implementation error defined by equation (12) is:

$$
\epsilon_{\mathcal{M}}(\rho, \tau, \delta, x(0)) = \sum_{i=0}^{t_i=\infty} \psi_t(t_i) Q_{\psi}(t_i)
$$

where vector $\psi(t)$ is defined as $\psi(t) = [x(t), \dot{x}(t), \ddot{x}(t)]^T$.

Theorem 1 has effectively replaced a continuous integration with an infinite sum. The computation of matrix $Q$ is described in the proof of the Appendix. Given $Q$, we need for all $t_i$ are the values of $x(t_i)$, $z(t_i)$ from the model-level semantics, and the values of $\dot{x}(t_i)$ and $\ddot{x}(t_i)$ from the implementation level semantics. We will now define two discrete-time dynamical systems, that will generate these desired sequences.

**Model-Level Semantics**

Using equations (14), (15), and notice that $\dot{y}(t) = C_P \dot{x}(t)$, the evolution of the closed loop feedback system for the model-level semantics can be described as:

$$
\begin{bmatrix}
x(t_i) \\
y(t_i)
\end{bmatrix} = A \begin{bmatrix}
\hat{x}(t_i) \\
\hat{y}(t_i)
\end{bmatrix}
$$

where $M = (I - B_P(I - L_m)^{-1}K_D C_P)^{-1}$. Given the timing sequence $t_i = t$, we can directly generate the desired sequences $x(t_i)$, $z(t_i)$ by considering successive iterations of the following discrete-time, linear, time-invariant system:

$$
\begin{bmatrix}
x(t_{i+1}) \\
z(t_{i+1})
\end{bmatrix} = E \begin{bmatrix}
x(t_i) \\
z(t_i)
\end{bmatrix}
$$

where $E = e^{\delta A}$ is simply the matrix exponential of $\delta A$. Thanks to Theorem 1, the model level semantics that are relevant for the computation of the implementation error are all captured in model (18).

**Implementation-Level Semantics**

Our goal is now to develop a similar discrete-time model for the implementation semantics. The main idea is that the execution of any particular control block can be modeled as a discrete-time linear system. However, as the dispatch sequence switches control blocks, this results in a switching discrete-time linear system. Furthermore, since the sequence of executions of the control blocks is periodic, the implementation semantics will be captured by a periodic, linear, time-varying (PLTV) system.

**Differentiation Computation**

Since each control variable $u_i$ depends on the derivatives $\hat{y}(t_i)$ of plant outputs, we assume that in the execution of each control block $B_i$, the derivatives $\hat{y}(t_i) = \hat{u}(t_{i+1})$ are numerically computed and are then used in updating control variable $u_i$. Because in general, the numerical differentiation algorithm requires the old values $\hat{y}(t_i - \Delta t(i))$, we extend the system by adding the memory variables $\hat{y}_m(t_i) = \hat{y}(t_i - \Delta t(i))$ which capture the needed information for the differentiation computation at $t_i$. Consider the computation
of control block \( \rho(i) \) in time interval \([t_i, t_{i+1}]\). If \( \rho(i) = B_0 \) or \( \rho(j) = B_j \), the values of variables \( \tilde{w} \) and \( \tilde{y}_m \) are not changed, that is \( \tilde{w}(t_{i+1}) = \tilde{w}(t_i) \) and \( \tilde{y}_m(t_{i+1}) = \tilde{y}_m(t_i) \). On the other hand, in the execution of control block \( B_j, 1 \leq j \leq m \), the values of \( \tilde{y}(t_i) \) are saved to the memory variables \( \tilde{y}_m \), that is \( \tilde{y}_m(t_{i+1}) = \tilde{y}(t_i) = C_{P_3} \tilde{x}(t_i) \), and the variables \( \tilde{w} \) are updated according to the chosen algorithm for differentiation computation:

\[
\text{Backward Difference} \\
\tilde{w}(t_{i+1}) = \frac{1}{\Delta D(t)} (C_{P_3} \tilde{x}(t_i) - \tilde{y}_m(t_i)) 
\]

\[
\text{Tustin's Approximation} \\
\tilde{w}(t_{i+1}) = \frac{2}{\Delta D(t)} (C_{P_3} \tilde{x}(t_i) - \tilde{y}_m(t_i)) - \tilde{w}(t_i) 
\]

### Modeling integration block \( B_j \)
Let us consider the evolution of the implementation semantics when integration block \( B_j \) is executed. Since the execution of \( B_j \) does not change the control variables, we have that \( \tilde{u}(t_{i+1}) = \tilde{u}(t_i) \). On the time interval \([t_i, t_{i+1}]\), the plant evolves continuously according to equation (34). By integration we thus obtain that

\[
\tilde{x}(t_{i+1}) = e^{\delta A_p} \tilde{x}(t_i) + \alpha_p(\delta) \tilde{u}(t_i) 
\]

where \( \alpha_p(\delta) = \int_0^t e^{\delta A_p} B_p d\tau \), which reduces to \( \alpha_p(\delta) = (e^{\delta A_p} - I) A_p^{-1} B_p \) if \( A_p \) is invertible. Since \( B_j \) is the integration block, the evolution of the internal variables \( z_i \) is captured by the equations describing the numerical algorithm that integrates the equations (15). Since the Trapezoid and Adams-Bashforth methods are two-step methods that depend on \( \tilde{y}(t_i - \Delta I(i)) \), we can extend the system so that all the relevant information is available at \( t_i \) by defining the memory variables \( \tilde{z}_m(t_i) = \tilde{y}(t_i - \Delta I(i)) \). Recalling that \( \tilde{y}(t_i) = C_{P_3} \tilde{x}(t_i) \), we obtain following discrete-time models for the numerical integrators above

#### Euler

\[
\tilde{z}(t_{i+1}) = \tilde{z}(t_i) + \Delta I(i) B_i C_{P_3} \tilde{x}(t_i) 
\]

#### Trapezoid

\[
\tilde{z}(t_{i+1}) = \tilde{z}(t_i) + \frac{\Delta I(i)}{2} B_i C_{P_3} \tilde{x}(t_i) + \tilde{z}_m(t_i)) 
\]

\[
\tilde{z}_m(t_{i+1}) = C_{P_3} \tilde{x}(t_i) 
\]

#### Adams-Bashforth

\[
\tilde{z}(t_{i+1}) = \tilde{z}(t_i) + \frac{\Delta I(i)}{2} B_i C_{P_3} \tilde{x}(t_i) - \tilde{z}_m(t_i)) 
\]

\[
\tilde{z}_m(t_{i+1}) = C_{P_3} \tilde{x}(t_i) 
\]

Note that more precise, higher order (multi-step) methods can be easily considered by using more memory variables.

Let us define the vector

\[
\tilde{\vartheta}(t) = [\tilde{x}(t) \quad \tilde{z}(t) \quad \tilde{z}_m(t) \quad \tilde{w}(t) \quad \tilde{y}_m(t) \quad \tilde{u}(t)]^T 
\]

which consists the variables of the implementation semantics. Collecting all the above equations allows us to construct a discrete-time linear system that models the evolution of the implementation when the integration block is executed. For example, for the two-step Adams-Bashforth scheme, the discrete-time model is

\[
\tilde{\vartheta}(t_{i+1}) = \mathcal{E}_{B_j}(i) \tilde{\vartheta}(t_i) 
\]

where \( \mathcal{E}_{B_j}(i) \) is not fixed but depends on \( \Delta I(i) \). However, since we assume that \( \rho \) is periodic, \( \Delta I(i) \) is periodic after the first period of the dispatch sequence. For Euler, Trapezoid and other numerical schemes we can obtain similar discrete-time models. Note that for the first order Euler method, there is no need for the extended \( \tilde{z}_m(t_i) \) variables.

#### Modeling idle block \( B_0 \)

The execution of the idle block \( B_0 \) does not affect either the internal or control variables. Therefore we have that \( \tilde{u}(t_{i+1}) = \tilde{u}(t_i), \tilde{w}(t_{i+1}) = \tilde{w}(t_i), \tilde{y}_m(t_{i+1}) = \tilde{y}_m(t_i), \tilde{z}(t_{i+1}) = \tilde{z}(t_i), \) and \( \tilde{z}_m(t_{i+1}) = \tilde{z}_m(t_i) \) in case a higher order integration method is used. On the time interval \([t_i, t_{i+1}]\), the plant still evolves continuously according to equation (21). Therefore, the value of the variables are modified by the execution of the control block \( B_0 \) according to the following discrete-time system

\[
\tilde{\vartheta}(t_{i+1}) = \mathcal{E}_{B_0}(i) \tilde{\vartheta}(t_i) 
\]

Note that unlike matrix \( \mathcal{E}_{B_j} \), matrix \( \mathcal{E}_{B_0} \) is fixed.

#### Modeling control blocks \( B_j \)
Let us now consider the control block \( B_j, 1 \leq j \leq m \), executed during the time interval \([t_i, t_{i+1}]\). Clearly \( \tilde{z}(t_{i+1}) = \tilde{z}(t_i), \) and \( \tilde{z}_m(t_{i+1}) = \tilde{z}_m(t_i) \) in case a higher order method is used. For \( k \neq j \), the execution of \( B_j \) does not modify the value of the variable \( \tilde{u}_k \). Thus, the value of the variable \( \tilde{u}_j \) is updated according to

\[
\tilde{u}_j(t_{i+1}) = \tilde{u}_j(t_i), \quad \text{if} \quad k \neq j 
\]

\[
\tilde{u}_j(t_{i+1}) = [K_i] C_{P_3} \tilde{x}(t_i) + [K_i] J \tilde{z}(t_i) + [K_{D_i}] \tilde{w}(t_{i+1}) + [L_{C_i}] \tilde{u}(t_i) 
\]

where \([K_{C_i}], [K_{I_i}], [K_{D_i}] \) and \([L_{C_i}] \) denote the \( j \)th rows of matrices \( K_{C_i}, K_{I_i}, K_{D_i} \) and \( L_{C_i} \) respectively. Note that \( \tilde{u}_j(t_{i+1}) \) depends on \( \tilde{w}(t_{i+1}) \) which is computed from current values at instant \( t_i \) by some formula determined by the differentiation algorithm. For example, for the Tustin's Approximation method (Eq. (20)), \( \tilde{u}_j(t_{i+1}) \) can be written as

\[
\tilde{u}_j(t_{i+1}) = \frac{2}{\Delta D(t)} \left( [K_{i}] C_{P_3} \tilde{x}(t_i) + [K_{i}] J \tilde{z}(t_i) - [K_{D_i}] \tilde{w}(t_{i+1}) - \frac{\Delta D(t)}{2} \left( [K_{D_i}] \tilde{y}_m(t_i) + [L_{D_i}] \tilde{u}(t_i) \right) \right) 
\]

Let \( U_j(i) \) be the matrix whose rows \([U_j(i)]_{k} \) are 0 if \( k \neq j \) and \([U_j(i)]_{j} \) is the coefficient of \( \tilde{z}(t_i) \) in the equation of \( \tilde{u}_j(t_{i+1}) \). Note that \( U_j(i) \) depends on \( \Delta D(i) \) and the differentiation method used. Similarly let \( V_j \) be the constant matrix whose rows \([V_j]_{k} \) are 0 if \( k \neq j \) and \([V_j]_{j} \) is the matrix whose rows \([W_j(i)]_{k} \) are 0 if \( k \neq j \) and \([W_j(i)]_{j} \) is the coefficient of \( \tilde{w}(t_i) \) in the equation of \( \tilde{u}_j(t_{i+1}) \). \( X_j(i) \) is the matrix whose rows \([X_j(i)]_{k} \) are 0 if \( k \neq j \) and \([X_j(i)]_{j} \) is the coefficient of \( \tilde{y}_m(t_i) \), and \( Y_j \) be the matrix such that
For the Tustin's Approximation method. Note that for its diagonal $Z_i$, the matrix whose coefficients are all zero except the $j^{th}$ element of its diagonal $Z_{i,j} = -1$. Then, 

$$\hat{u}(t_{i+1}) = U_j(i) \tilde{z}(t_i) + V_j(i) \tilde{w}(t_i) + X_j(i) \tilde{y}_m(t_i) + (I + Y_j + Z_j) \tilde{u}(t_i)$$

Thus, the value of the variables are modified by the execution of the control block $B_j$ according to the following discrete-time system:

$$\hat{\theta}(t_{i+1}) = E_B(i) \hat{\theta}(t_i)$$

for the Tustin's Approximation method. Note that for $i = 0$, $\Delta_D(0) = 0$ and $\tilde{\theta}(0) = 0$. Also observe that like matrix $E_B(i)$, matrix $E_B(i)$ is not fixed but periodic after the first period of the dispatch sequence.

**From dispatch sequences to PLTV systems**

Let us consider a dispatch sequence $\rho$ defining the order in which the control blocks have to be executed. The sequences $\tilde{x}(t_i), \tilde{z}(t_i), \tilde{z}(t_i), \tilde{w}(t_i), \tilde{y}_m(t_i)$, and $\tilde{u}(t_i)$, for $i \geq 0$, can be determined from the following discrete-time dynamical system:

$$\hat{\theta}(t_{i+1}) = E(i) \hat{\theta}(t_i) \quad (29)$$

where $E(i) = E_{\rho(i)}(i)$ when $\rho(i) \neq B_i$, and $E(i) = E_{B_i}(i)$ when $\rho(i) = B_i$. Therefore the implementation values at instants $t_i$ are captured by a discrete-time linear system that is time-varying. Furthermore, since we assume that $\rho$ is periodic (let $n_{\rho}$ denote its period), this dynamical system is a PLTV system.

### 3.1 Error Computation

Having modeled both the model-level and implementation level semantics at the instants $t_i$, we define a system which describes both the discretized model-level semantics and the discrete-time implementation level semantics. Its state is the vector $\psi(t)$ defined as

$$\psi(t) = [x(t) \ z(t) \ \tilde{z}(t) \ \tilde{z}(t) \ \tilde{w}(t) \ \tilde{y}_m(t) \ \tilde{u}(t)].$$

Then, 

$$\psi(t_{i+1}) = E(t) \psi(t_i) \quad (30)$$

Note that $E(t)$ is periodic after the first period of the dispatch sequence. Thus the composite system is also a PLTV system of period $n_{\rho}$. Using so-called lifting techniques for the analysis of PLTV systems (see for instance [16]), which transform a periodic time-varying system into a higher order linear time-invariant system, we can state the main result of this paper.

**Theorem 2.** The implementation error is exactly equal to

$$e_M(\rho, \tau, \delta, x(t)) = x(t)^T H^T \hat{\Theta} H x(t) \quad (31)$$

where $H = [I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$, and $\hat{\Theta}, \Theta$ are solutions to the following nested matrix Lyapunov equations

$$\hat{\Theta} = \hat{G}^T \hat{Q} \hat{G} + \hat{E}^T \hat{O} \hat{E} \quad (32)$$

$$\Theta = E^T \Theta E + \tilde{G}^T \tilde{Q} \tilde{G} \quad (33)$$

for implementation-dependent matrices $\hat{E}_0, \hat{G}_0, \tilde{E}, \tilde{G}, \tilde{Q}$ (see proof in Appendix).

Theorem 2 states that the implementation error can be computed exactly for this class of systems and controllers, and furthermore, the $L_2$-norm error is a global quadratic function of the initial state $x_0$.
Figure 1: Simulink Model For Error Computation

PID controller that has been designed for this plant is
\[
\begin{bmatrix}
u_1(t) \\ u_2(t)
\end{bmatrix} = K_P \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + K_I \int_0^t \begin{bmatrix} y_1(\tau) d\tau \\ y_2(\tau) d\tau \end{bmatrix} + K_D \frac{dy_1(t)}{dt} \frac{dy_2(t)}{dt},
\]
where
\[
K_P = \begin{bmatrix} -116 & 0 \\ 0 & -250 \end{bmatrix}, \quad K_I = \begin{bmatrix} -480 & 0 \\ 0 & -30 \end{bmatrix}, \quad K_D = \begin{bmatrix} -0.2 & 0 \\ 0 & -20 \end{bmatrix}.
\]
Control variable \(u_1\) regulates the faster subsystem, whereas \(u_2\) regulates the slower subsystem. For initial state \(x_1(0) = x_2(0) = x_3(0) = 2\), Table 2 summarizes the implementation errors for various dispatch sequences and different numerical integration and differentiation algorithms.

For comparison, we include in Table 2 the corresponding results for a PI controller with the same parameters \(K_P\) and \(K_I\). Figure 2 shows the evolution of output \(y_1(t)\) for various implementation choices. From Table 2, it is observable that the implementation errors for the PI controller are generally less than the corresponding errors for the PID controller. This shows the effect of the derivative part on the performance of the software implementation. Though the ideal closed loop system is stable, implementation \((\rho_1, \tau_1, \delta_1)\) destabilizes the plant (for the PI controller), or produces a large error (for the PID controller). By using a better numerical integration algorithm, \((\rho_2, \tau_2, \delta_2)\) avoids instability and reduces the implementation error. It is worth noting that scheduling can have great effect on the overall performance of the system, as clearly illustrated when comparing implementations \((\rho_1, \tau_1, \delta_1)\), \((\rho_3, \tau_3, \delta_3)\), \((\rho_4, \tau_4, \delta_4)\), \((\rho_5, \tau_5, \delta_5)\), and \((\rho_6, \tau_6, \delta_6)\).

With the same numerical integration and differentiation methods and the same time slot length \(\delta\), but a slight change in the dispatch sequence, implementation \((\rho_3, \tau_3, \delta_3)\) outperforms \((\rho_1, \tau_1, \delta_1)\). Moreover, for the PID controller case, it produces a smaller error than those of other implementations, even those on much faster platforms. Implementation \((\rho_4, \tau_4, \delta_4)\) destabilizes the plant with the PID controller while, by allocating more time slots to control block \(B_i\), implementation \((\rho_5, \tau_5, \delta_5)\) destabilizes the plant with the PI controller but greatly improves the performance when the PID controller is used. The above observations show that the performance of a controller can be considerably increased without changing the platform, however care should be taken when choosing the dispatch sequence. Furthermore, even on a slightly faster computer, the performance of \((\rho_7, \tau_7, \delta_7)\) may be worse than \((\rho_6, \tau_6, \delta_6)\) executing on a slower computer but with better dispatch sequence. Nonetheless for sufficiently faster platform \((\rho_6, \tau_6, \delta_6)\), the performance is improved greatly.

Figure 2: Plots of \(y_1\) for the ideal semantics \([M]\) (top), and implementations \([M]_{(\rho_1, \tau_1, \delta_1)}\) (second to top), \([M]_{(\rho_7, \tau_7, \delta_7)}\) (second to bottom), \([M]_{(\rho_6, \tau_6, \delta_6)}\) (bottom).

5. CONCLUSIONS

In this paper, we continue our efforts aimed at understanding and quantifying the gap between model-level timed semantics of embedded controllers and their implementation on time-triggered platforms. For linear plant models and dynamic PID controllers, we have presented a method to exactly compute the \(L_2\)-error of the deviation of the output of the plant in the implementation semantics from the output of the plant in the continuous time semantics. This method gives a criterion to compare different implementations.

Future research includes the extension of our framework to larger classes of plant models, including nonlinear and hybrid systems, as well as more general nonlinear controllers. Whereas exact computation of the error may not be feasible in these general settings, computable error bounds for appropriate norms will still enable the comparison of different implementations. Finally, our approach may enable us to develop a scheduling framework on time-triggered platforms in order to reduce implementation error, while potentially achieving a decomposition between dispatch sequences and timing functions.

6. REFERENCES


APPENDIX

Proof of Theorem 1. First, note that $e_M(\rho, \tau, \delta, x(0)) = \sum_{i=0}^{t_{i+1}} \mathcal{I}_i$ where $\mathcal{I}_i = \int_{t_i}^{t_{i+1}} \| (y(t) - \tilde{y}(t)) \|^2 dt$. On the interval $[t_i, t_{i+1})$, the evolution of the closed loop system for the model-level, continuous time semantics can be described by the differential equations (17). For the implementation, on the same interval $[t_i, t_{i+1})$, the input $\hat{u}(t) = \tilde{u}(t_i)$ since the computed control stays constant until the next computation, and therefore the evolution of the implementation semantics is given by

$$\dot{x}(t) = A_p \hat{x}(t) + B_p \tilde{u}(t_i).$$

(34)

We now define the following system with output variable $y_c(t) = y(t) - \tilde{y}(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \dot{A} & 0 \\ 0 & A_p \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_p \end{bmatrix} \tilde{u}(t_i)$$

$$y_c(t) = C_p \begin{bmatrix} 0 & -C_p \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

Letting $\varphi(t) = \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$, the above equation is a linear system with constant input $\tilde{u}(t_i)$, having the form

$$\begin{cases} \dot{\varphi}(t) = A_p \varphi(t) + B_p \hat{u}(t_i), \\ y_c(t) = C \varphi(t). \end{cases}$$

(35)

By explicitly solving the differential equation, we obtain

$$y_c(t) = C e^{A_p(t-t_i)} \varphi(t_i) + \int_{t_i}^{t} e^{A_p(t-r)} B d\tau \hat{u}(t_i)$$

The integral term can be written as

$$\int_{t_i}^{t} e^{A_p(t-r)} B d\tau = \int_{0}^{t-t_i} e^{A_p r} B d\tau = \alpha(t - t_i)$$
where $\alpha(\cdot)$ is a matrix-valued function mapping $t \in \mathbb{R}$ to a matrix $\alpha(t) \in \mathbb{R}^{(2n_\rho+1)\times m}$. On the interval $[t_i, t_{i+1}]$, we can check that

$$I_i = \int_{t_i}^{t_{i+1}} \|y(t) - \bar{y}(t)\|^2 dt = \int_{t_i}^{t_{i+1}} y_c(t)^T y_c(t) dt$$

$$= \varphi(t_i)^T \left( \int_0^\delta \alpha(t)^T C^T C e^{At} dt \right) \varphi(t_i)$$

$$+ \varphi(t_i)^T \left( \int_0^\delta \alpha(t)^T C^T C \alpha(t) dt \right) \bar{u}(t_i)$$

$$+ \bar{u}(t_i)^T \left( \int_0^\delta \alpha(t)^T C^T C \alpha(t) dt \right) \varphi(t_i)$$

$$+ \bar{u}(t_i)^T \left( \int_0^\delta \alpha(t)^T C^T C \alpha(t) dt \right) \bar{u}(t_i)$$

(36)

Let us define the following matrices

$$Q_{1,1} = \int_0^\delta e^{At_i} C^T C e^{At_i} dt$$

$$Q_{1,2} = Q_{2,1} = \int_0^\delta e^{At_i} C^T C \alpha(t) dt$$

$$Q_{2,2} = \int_0^\delta \alpha(t)^T C^T C \alpha(t) dt$$

Then, we have

$$I_i = \begin{bmatrix} x(t_i) \ T \\ z(t_i) \ T \\ \bar{x}(t_i) \ T \\ \bar{u}(t_i) \end{bmatrix} \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix} \begin{bmatrix} x(t_i) \\ z(t_i) \\ \bar{x}(t_i) \\ \bar{u}(t_i) \end{bmatrix} = \psi(t_i)^T Q \psi(t_i)$$

(37)

Note that matrix $Q$ is constant and computable. In case matrix $A$ is invertible, it is straightforward to see that $\alpha(t) = (e^{At} - I) A^{-1} B$.

The desired result is directly obtained from (37). □

PROOF OF THEOREM 2. Consider the following matrices

$$\hat{E}_0 = \hat{E}(n_\rho - 1) \hat{E}(n_\rho - 2) \ldots \hat{E}(1) \hat{E}(0)$$

(38)

$$\hat{G}_0 = \begin{bmatrix} I \\ \hat{E}(0) \\ \hat{E}(1) \hat{E}(0) \\ \vdots \\ \hat{E}(n_\rho - 2) \ldots \hat{E}(0) \end{bmatrix}$$

(39)

$$\hat{E} = \hat{E}(2n_\rho - 1) \hat{E}(2n_\rho - 2) \ldots \hat{E}(n_\rho + 1) \hat{E}(n_\rho)$$

(40)

$$\hat{G} = \begin{bmatrix} I \\ \hat{E}(n_\rho) \\ \hat{E}(n_\rho + 1) \hat{E}(n_\rho) \\ \vdots \\ \hat{E}(2n_\rho - 2) \ldots \hat{E}(n_\rho) \end{bmatrix}$$

(41)

Then, we have,

$$\begin{cases} \bar{\psi}(t_{i+1}) = \hat{E}_0 \bar{\psi}(0) \\ \bar{\psi}(t_{i+1}) = \hat{G}_0 \bar{\psi}(0) \end{cases}$$

(42)

and for all $l \geq 1$,

$$\begin{cases} \bar{\psi}(t_{(l+1)n_\rho}) = \hat{E} \hat{\psi}(t_{ln_\rho}) \\ \bar{\psi}(t_{ln_\rho}) = \hat{G} \hat{\psi}(t_{ln_\rho}) \end{cases}$$

(43)

Note that we can write vector $\psi(t)$, defined in Theorem 1, as $\psi(t) = F \bar{\psi}(t)$ where

$$F = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

(44)

We define the block diagonal matrix $Q$ composed of $n_\rho$ blocks equal to $F^T Q F$ where matrix $Q$ is given by Theorem 1:

$$Q = \begin{bmatrix} F^T Q F \\ \vdots \\ F^T Q F \end{bmatrix}$$

From Theorem 1, we have

$$e_M(\rho, \tau, \delta, x(0)) = \sum_{l=0}^{l=+\infty} \sum_{i=1}^{i=n_\rho} \psi(t_i)^T Q \psi(t_i)$$

$$= \sum_{l=0}^{l=+\infty} \sum_{i=1}^{i=n_\rho} \bar{\psi}(t_i)^T F^T Q F \bar{\psi}(t_i)$$

$$= \bar{\psi}(0)^T \hat{G}_0 \hat{G}_0 \bar{\psi}(0)$$

$$+ \sum_{l=1}^{l=+\infty} \bar{\psi}(t_{ln_\rho})^T \hat{G}^T \hat{G} \bar{\psi}(t_{ln_\rho})$$

$$= \bar{\psi}(0)^T \hat{G}_0 \hat{G}_0 \bar{\psi}(0)$$

$$+ \bar{\psi}(0)^T \hat{E}_0 \left( \sum_{l=0}^{l=+\infty} (\hat{E}^T \hat{G}^T \hat{G} \hat{E}) \right) \hat{E}_0 \bar{\psi}(0)$$

$$= \bar{\psi}(0)^T \left( \hat{G}_0^T \hat{G} \hat{G} \hat{E} \right) \bar{\psi}(0)$$

$$= \bar{\psi}(0)^T \hat{G} \bar{\psi}(0)$$

where

$$\mathcal{O} = \sum_{l=0}^{l=+\infty} (\hat{E}^T \hat{G}^T \hat{G} \hat{E})$$

From the theory of LTI systems $\mathcal{O}$ is the solution of the Lyapunov equation (33). Moreover, from the continuous time and the implementation semantics, we have $\bar{\psi}(0) = Hx(0)$. □