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# Hadwiger Integration of Definable Functions

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**Abstract**
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# HADWIGER INTEGRATION OF DEFINABLE FUNCTIONS

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ABSTRACT

HADWIGER INTEGRATION OF DEFINABLE FUNCTIONS

Matthew L. Wright

Robert Ghrist, Advisor

This thesis defines and classifies valuations on definable functionals. The intrinsic volumes are valuations on "tame" subsets of $\mathbb{R}^n$, and by easy extension, valuations on functionals on $\mathbb{R}^n$ with finitely many level sets, each a "tame" subset of $\mathbb{R}^n$. We extend these valuations, which we call *Hadwiger integrals*, to definable functionals on $\mathbb{R}^n$, and present some important properties of the valuations. With the appropriate topologies on the set of definable functionals, we obtain dual classification theorems for general valuations on such functionals. We also explore integral transforms, convergence results, and applications of the Hadwiger integrals.

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# Chapter 1

# Introduction

How can we assign the notion of "size" to a functional—that is, a real-valued function—on $\mathbb{R}^n$? Surely the Riemann-Lebesgue integral is one way to quantify the size of a functional. Yet are there other ways? The integral with respect to Euler characteristic gives a very different idea of the size of a functional, in terms of its values at critical points. Between Lebesgue measure and Euler characteristic lie many other pseudo-measures (or more properly, *valuations*) known as the *intrinsic volumes*, that provide notions of the size of *sets* in $\mathbb{R}^n$. Integrals with respect to these intrinsic volumes integrals provide corresponding quantifications of the size of a functional.

In this thesis, we explore the integration of continuous functionals with respect to the intrinsic volumes. The approach is *o-minimal* and *integral.* First, in order to develop results for "tame" objects, while excluding pathologies such as Cantor sets on which the intrinsic volumes might not be well-defined, we frame the discussion in terms of an *o-minimal* structure. The particular o-minimal structure is not so important, though for concreteness the reader can think of it as being comprised of all subanalytic sets, or all semialgebraic sets. Use of an o-minimal structure makes the discussion context-free and applicable in a wide variety of situations. Second, the approach is *integral* in the sense that we are not primarily concerned with valuations of sets, but instead with integrals of functionals over sets. Valuations of sets have been well-studied in the past; much less is known about valuations of functionals.

The setting of this work is in applied topology and integral geometry. Indeed, the motivation for this research is to answer questions that arise in sensor networks, a key area of applied topology. Integral geometry is an underdeveloped, intriguing subject that studies symmetry-invariant integrals associated with geometric objects [6, 21]. The study of such integrals involves important techniques from geometric measure theory, especially the theory of *currents*. Furthermore, the work has important connections to combinatorics: the intrinsic volumes can be studied from a combinatorial perspective, as presented by Klain and Rota [24], and involving a triangular array of numbers known as the *flag coefficients*.

Chapter 2 contains an o-minimal approach to the intrinsic volumes. Beginning with Hadwiger's formula, we establish various equivalent expressions of the intrinsic volumes, all applicable in the o-minimal setting. Perhaps the most intriguing and least-known expression has to do with currents, which we explain in Chapter 3.

In Chapter 4 , we "lift" valuations from sets to functions over sets, providing important properties of integrals with respect to intrinsic volumes. We call such an integral a *Hadwiger integral*. Hadwiger integrals can be expressed in various ways, corresponding with the different expressions of the intrinsic volumes. We encounter a duality of "lower" and "upper" integrals, which are not equivalent, but arise due to the differences in approximating a continuous function by lower- and upper-semicontinuous step functions.

Next, we discuss general valuations on functionals, and classify them in Chapter 5. The duality observed earlier is again present, with "lower" and "upper" valuations that are continuous in different topologies. This leads to our main result:

**Main Theorem.** *Any lower valuation $v$ on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of lower Hadwiger integrals. For $h \in \mathsf{Def}(\mathbb{R}^n)$,*

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h) \ \lfloor d\mu_k \rfloor,$$

*where the $c_k : \mathbb{R} \to \mathbb{R}$ are increasing functions with $c_k(0) = 0$.*

*Likewise, an upper valuation $v$ on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of upper Hadwiger integrals.*

Chapter 6 explores integral transforms, which are important in applications. In particular, we examine convolution, where the convolution integral is with respect to the intrinsic volumes. We also consider integral transforms analogous to the Fourier and Bessel (or Hankel) transforms.

The ability to estimate integrals based on only an approximation of a functional is important in applications. Thus, Chapter 7 provides further results related to estimation and convergence of Hadwiger integrals.

Chapter 8 discusses known and speculative applications of this valuation theory, as well as opportunities for future research. Applications include sensor networks, image processing, and crystal growth and foam dynamics. In order that Hadwiger integrals may be more easily applied, we need further research into index theory, algorithms, and numerical analysis of the integrals. We could also study more general valuations, replacing Euclidean-invariance with invariance under other groups of transformations.

# Chapter 2

# Intrinsic Volumes

The intrinsic volumes are the $n + 1$ Euclidean-invariant valuations on subsets of $\mathbb{R}^n$. This chapter provides the background information necessary to understand the intrinsic volumes in an o-minimal setting.

## 2.1 Valuations

A *valuation* on a collection of subsets $\mathcal{S}$ of $\mathbb{R}^n$ is a function $v : \mathcal{S} \to \mathbb{R}$ that satisfies the additive property:

$$v(A \cup B) + v(A \cap B) = v(A) + v(B) \quad \text{for } A, B \in \mathcal{S}.$$

On "tame" subsets of $\mathbb{R}^n$ there exist $n + 1$ Euclidean-invariant valuations. These often appear in literature by the names *intrinsic volumes* and *quermassintegrale*, which differ only in normalization. Other terminology for the same concept includes

*Hadwiger measures*, *Lipschitz-Killing curvatures*, and *Minkowski functionals.* Here we will primarily refer to these valuations as *intrinsic volumes* to emphasize that the intrinsic volumes of $A \in \mathcal{S}$ are intrinsic to $A$ and do not depend on any higher-dimensional space into which $A$ may be embedded.

The literature defines the intrinsic volumes in various ways. Klain and Rota [24] take a combinatorial approach, defining the intrinsic volumes first on parallelotopes via symmetric polynomials, then extending the theory to compact convex sets and finite unions of such sets. Schneider and Weil [41] define the intrinsic volumes and quermassintegrale on convex bodies as coefficients of the Steiner formula, which we will discuss in Section 6.1. Morvan [32] takes a similar approach via the Steiner formula. Santaló [35] approaches the quermassintegrale as an average of cross-sectional measures. Schanuel [38] and Schröder [42] provide short, accessible introductory papers on the intrinsic volumes.

We will define the intrinsic volumes in a way lends itself to the integration theory that is our goal. Thus, instead of working with sets that are compact or convex, we will begin with an *o-minimal* structure that specifies "tame" subsets of $\mathbb{R}^n$. Van den Dries [43] defines an o-minimal structure as follows:

**Definition 2.1.** An *o-minimal structure* is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that:

1. for each $n$, $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$—that is, a collection of subsets of $\mathbb{R}^n$, with $\emptyset \in \mathcal{S}_n$, and the collection is closed under unions and complements (and thus also intersections);

2. $\mathcal{S}$ is closed under projections: if $A \in \mathcal{S}_n$, then $\pi(A) \in \mathcal{S}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the usual projection map;

3. $\mathcal{S}$ is closed under products: if $A \in \mathcal{S}_n$, then $A \times \mathbb{R} \in \mathcal{S}_{n+1}$;

4. $\mathcal{S}_n$ contains diagonal elements: $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j \text{ for } 1 \leq i < j \leq n\} \in \mathcal{S}_n$;

5. $\mathcal{S}_1$ consists exactly of finite unions of points and (open, perhaps unbounded) intervals.

Examples of o-minimal structures include the semilinear sets, the semialgebraic sets, and many other interesting structures. The definition of an o-minimal structure $\mathcal{S}$ prevents infinitely complicated sets such as Cantor sets from being included in $\mathcal{S}$. Elements of $\mathcal{S}_n$ we call *definable sets*. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ whose graph is a definable subset of $\mathbb{R}^{n+m}$ is a *definable map*. To explain the name *o-minimal*, the "o" stands for *order*, and "minimal" refers to axiom 5 of Definition 2.1, which establishes a minimal collection of subsets of $\mathbb{R}$.

The o-minimal Euler characteristic, denoted $\chi$, is defined so that for any *open* $k$-simplex $\sigma$, $\chi(\sigma) = (-1)^k$, and to satisfy the additive property. Since any definable set is definably homeomorphic to a disjoint union of open simplices, Euler characteristic is defined on $\mathcal{S}$. The o-minimal Euler characteristic coincides with the usual topological Euler characteristic on compact sets, but not in general. In particular, the usual topological Euler characteristic is not additive. The o-minimal Euler characteristic is

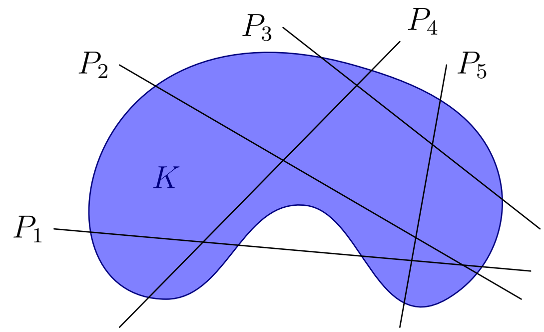



**Figure 2.1:** The intrinsic volume $\mu_k$ of subset $K \subset \mathbb{R}^n$ is defined as the integral over all affine $(n-k)$-planes $P$ of the Euler characteristic $\chi(K \cap P)$, as in Definition 2.2.

that which arises from the Borel-Moore homology, but it is not a homotopy invariant.

## 2.2 Definition via Hadwiger's formula

In this paper, $G_{n,k}$ denotes the Grassmanian of $k$-dimensional linear subspaces of $\mathbb{R}^n$, and $A_{n,k}$ denotes the affine Grassmanian of $k$-dimensional affine subspaces of $\mathbb{R}^n$.

**Definition 2.2.** For a definable set $K \in \mathcal{S}_n$, and $k = 0, 1, \ldots, n$, define the $k^{\text{th}}$ *intrinsic volume*, $\mu_k$, of $K$ as

$$\mu_k(K) = \int_{A_{n,n-k}} \chi(K \cap P) \; d\lambda(P) \tag{2.1}$$

where $\lambda$ is the measure on $A_{n,n-k}$ described below.

Equation (2.1) is known as *Hadwiger's formula*. Figure 2.1 illustrates the definition of the intrinsic volumes in terms of integrals over affine planes.

Each affine subspace $P \in A_{n,n-k}$ is a translation of some linear subspace $L \in$

$G_{n,n-k}$. That is, $P$ is uniquely determined by $L$ and a vector $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{x} \perp P$, such that $P = L + \mathbf{x}$. Thus, we can integrate over $A_{n,n-k}$ by first integrating over $G_{n,n-k}$ and then over $\mathbb{R}^k$. Equation (2.1) is equivalent to

$$\mu_k(K) = \int_{G_{n,n-k}} \int_{\mathbb{R}^k(L)} \chi(K \cap (L + \mathbf{x}))\, d\mathbf{x}\, d\gamma(L) \tag{2.2}$$

where $L \in G_{n,n-k}$, $\mathbf{x} \in \mathbb{R}^k$ is orthogonal to $L$. Let $\gamma$ be the Haar measure on the Grassmanian, scaled so that

$$\gamma(G_{n,m}) = \binom{n}{m} \frac{\omega_n}{\omega_m \omega_{n-m}} \tag{2.3}$$

with $\omega_n$ denoting the $n$-dimensional volume of the unit ball in $\mathbb{R}^n$. We can express $\omega_n$ in terms of the Gamma function:

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$

We scale the measure of the Grassmanian to satisfy equation (2.3) because this makes the intrinsic volume of a set $K$ independent of the dimension of the space in which $K$ is embedded. For example, if $K$ is a 2-dimensional set in $\mathbb{R}^3$ (so $K$ is contained in a 2-dimensional plane), then $\mu_2(K)$ is the area of $K$. The valuations $\mu_k$ are *intrinsic* in the sense that they depend only on the sets on which they are defined, and not on the dimension of the ambient space.

Observe that $\mu_0$ is Euler characteristic and $\mu_n$ is Lebesgue measure on $\mathbb{R}^n$:

$$\mu_0(K) = \int_1 \chi(K \cap \mathbb{R}^n)\, d\lambda = \chi(K) \qquad \text{and} \qquad \mu_n(K) = \int_{\mathbb{R}^n} K\, d\mathbf{x}.$$

The $k^{\text{th}}$ intrinsic volume, $\mu_k$, provides a notion of the $k$-dimensional size of a set. For example, $\mu_1$ gives an idea of the "length" of a set, as Schanuel describes in his classic paper, "What is the length of a potato?" [38].

It follows from equation (2.2) that the intrinsic volume $\mu_k$ is *homogeneous of degree* $k$. That is, $\mu_k(aK) = a^k \mu_k(K)$, for all $a \geq 0$ and definable $K$. Also note that any intrinsic volume of the empty set is zero. By definition, $\chi(\emptyset) = 0$, and equation (2.1) implies that also $\mu_k(\emptyset) = 0$.

The numbers $\binom{n}{m}\frac{\omega_n}{\omega_m \omega_{n-m}}$ in equation (2.3) are called *flag coefficients* and are analogous to the binomial coefficients [24]. As the binomial coefficient $\binom{n}{k}$ counts the number of $k$-element subsets of an $n$-element set, the flag coefficient $\binom{n}{m}\frac{\omega_n}{\omega_m \omega_{n-m}}$ gives the measure of $k$-dimensional linear subspaces of $\mathbb{R}^n$. That is, we scale the Haar measure on the grassmanian $G_{n,m}$ so that equation (2.3) holds. This is precisely the scaling necessary to make the intrinsic volumes *intrinsic*. For more about the flag coefficients, see Appendix A.

The *quermassintegrale* differs from the intrinsic volumes only in terms of normalization. For definable $K \subset \mathbb{R}^n$ and integer $0 \leq k \leq n$, the quermassintegrale $W_{n,k}(K)$ is defined

$$W_{n,k}(K) = \omega_k \binom{n}{k}^{-1} \mu_{n-k}(K). \tag{2.4}$$

Unlike the intrinsic volumes, the quermassintegrale depends on the dimension of the ambient space in which $K$ is embedded, so $n$ properly appears as a subscript.

## 2.3 Important Properties

The intrinsic volumes enjoy the important properties of additivity and Euclidean invariance, as in the following proposition.

**Proposition 2.1.** *For definable sets $A, B \subset \mathbb{R}^n$, and $k = 0, 1, \ldots, n$, the following properties hold:*

- Additivity*: $\mu_k(A \cup B) + \mu_k(A \cap B) = \mu_k(A) + \mu_k(B)$.*
- Euclidean invariance*: $\mu_k(A) = \mu_k(\phi(A))$ for $\phi \in \mathcal{O}_n$, the group of orthogonal transformations on $\mathbb{R}^n$.*

*Proof.* Additivity follows from the fact that Euler characteristic is additive:

$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B).$$

Euclidean invariance follows from the fact that the integral over the affine Grassmanian is invariant under rigid motions of $\mathbb{R}^n$. □

Indeed, additivity is the key property that allows us to call $\mu_k$ a *valuation*. By induction, the intrinsic volumes satisfy the *inclusion-exclusion principle*,

$$\mu_k(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le m} \mu_k(K_{i_1} \cap \cdots \cap K_{i_r})$$

for $K_1, \ldots, K_m \in \mathcal{S}$.

The intrinsic volumes are *continuous* in the sense that if $J$ and $K$ are convex sets that are close in the Hausdorff metric, then $\mu_k(J)$ is close to $\mu_k(K)$. Intuitively,

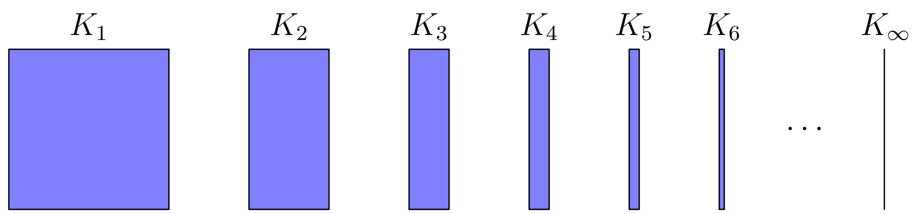



**Figure 2.2:** The intrinsic volumes of the $K_n$ converge to those of $K_\infty$.

the Hausdorff distance between $J$ and $K$ is the smallest $\epsilon$ such that no point in $J$ is farther than $\epsilon$ from some point in $K$, and vice-versa. Formally, Hausdorff distance between $J$ and $K$ can be written

$$d_H(J, K) = \max\{\sup_{x \in J} \inf_{y \in K} d(x, y), \sup_{y \in K} \inf_{x \in J} d(x, y)\}.$$

As an example of continuity, let $\{K_j\}$ be a sequence of $n$-dimensional sets that converge in the Hausdorff metric to an $(n-1)$-dimensional set $K_\infty$, as illustrated in Figure 2.2. Then,

$$\lim_{j \to \infty} \mu_k(K_j) = \mu_k(K_\infty).$$

This is one justification why $\mu_{n-1}(K)$ is equal to half the surface area of $K$.

The intrinsic volumes are not continuous for definable sets in general with respect to the Hausdorff metric. For example, a bounded convex set can be approximated arbitrarily closely in the Hausdorff metric by a large discrete set. However, a compact convex set has Euler characteristic one, while the Euler characteristic of a discrete set equals its cardinality.

The intrinsic volumes are continuous on definable sets with respect to a topology defined in terms of conormal cycles, which we will discuss in Section 3.5.

It is a well-known theorem of Hugo Hadwiger [23] that any continuous valuation on convex subsets of $\mathbb{R}^n$ is a linear combination of the intrinsic volumes:

**Hadwiger's Theorem.** *If $v$ is a Euclidean-invariant, additive functional on subsets of $\mathbb{R}^n$, continuous on convex subsets with respect to the Hausdorff metric, then*

$$v = \sum_{k=0}^{n} c_k \mu_k$$

*for some real constants $c_0, \ldots, c_n$. Furthermore, if $v$ is homogeneous of degree $k$, then $v = c_k \mu_k$.*

We will not reproduce the proof of Hadwiger's Theorem here, but it may be found in a variety of sources [12, 24, 41].

Definition 2.2 allows us to express the intrinsic volumes $\mu_k(K)$ in terms of "slices" of $K$ along affine $(n-k)$-dimensional planes. Recall equation (2.1),

$$\mu_k(K) = \int_{A_{n,n-k}} \chi(K \cap P)\, d\lambda(P).$$

We can also express $\mu_k(K)$ in terms of projections of $K$ onto $k$-dimensional planes. Instead of integrating $\chi(K \cap P)$ for all affine $(n-k)$-planes $P$, we can change our perspective and project $K$ onto linear $k$-subspaces $L$. In particular, let $\pi_L : K \to L$ be the projection map onto $L \in G_{n,k}$. For any $\mathbf{x} \in L$, $\pi_L^{-1}(\mathbf{x})$ is the fiber over $\mathbf{x}$, that is, the set of all points in $K$ that are projected to $\mathbf{x}$. We then have:

$$\mu_k(K) = \int_{A_{n,n-k}} \chi(K \cap (P))\, d\lambda(P) = \int_{G_{n,k}} \int_L \chi(\pi_L^{-1}(\mathbf{x}))\, d\mathbf{x}\, d\gamma(L).$$

In summary, we have the *projection formula*:

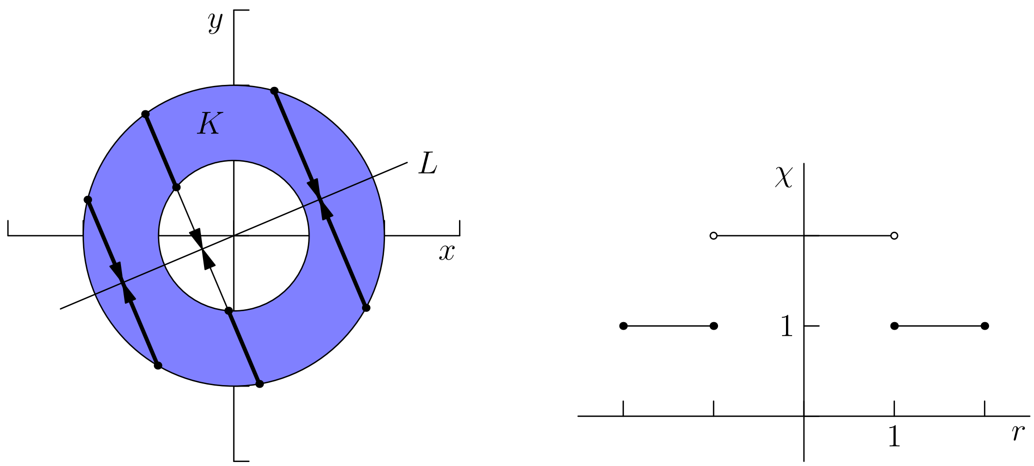



**Figure 2.3:** At left, the annulus $K \subset \mathbb{R}^2$ is projected orthogonally onto an arbitrary linear subspace $L \in G_{2,1}$. At right, the graph of $\chi(\pi_L^{-1}(r))$, the Euler characteristic of the fibers of the projection of $K$ onto $L$.

**Theorem 2.1** (Projection Formula)**.** *For any definable set $K$ in $\mathbb{R}^n$ and $0 \leq k \leq n$,*

$$\mu_k(K) = \int_{G_{n,k}} \int_L \chi(\pi_L^{-1}(\mathbf{x}))\ d\mathbf{x}\ d\gamma(L)$$

*where $\pi_L^{-1}(\mathbf{x})$ is the fiber over $\mathbf{x} \in L$ of the orthogonal projection map $\pi_L : K \to L$.*

*Example.* Consider the annulus $K \subset \mathbb{R}^2$ in Figure 2.3, with inner radius 1 and outer radius 2. We compute $\mu_1(K)$ via the projection formula.

Let $L \in G_{2,1}$ be an arbitrary line through the origin. Several fibers of the projection map $\pi_L$ onto $L$ are indicated at left in Figure 2.3. The Euler characteristic $\chi(\pi_L^{-1}(r))$ is graphed at right in Figure 2.3 as a function of $r$, the position along $L$, measured from the origin. By rotational symmetry about the origin, $\chi(\pi_L^{-1}(r))$ is the same for all $L \in G_{2,1}$.

Integrating, we find that $\int_L \chi(\pi_L^{-1}(r))\ dr = 6$. Then,

$$\mu_1(K) = \int_{G_{2,1}} \int_L \chi(\pi_L^{-1}(r))\ dr\ d\gamma(L) = \int_{G_{2,1}} 6\ dr = 6 \cdot \frac{\pi}{2} = 3\pi. \tag{2.5}$$

This computation agrees with the previous assertion that $\mu_{n-1}$ equals half the surface area of an $n$-dimensional set. Here, $\mu_1(K) = 3\pi$, which is half the (combined inner and outer) perimeter of the annulus.

The situation is simpler if $K$ is compact and convex: In this case, $\chi(\pi_L^{-1}(\mathbf{x})) = 1$ for all $L \in G_{n,l}$ and $\mathbf{x} \in L$. Thus, the projection formula, Theorem 2.1, reduces to the standard *mean projection formula* [24]:

**Theorem 2.2** (Mean Projection Formula)**.** *For $0 \leq k \leq n$ and compact convex subset $K$ of $\mathbb{R}^n$,*

$$\mu_k(K) = \int_{G_{n,k}} \mu_k(K|L)\ d\gamma(L)$$

*where the integrand is the $k$-dimensional volume of the projection of $K$ onto a $k$-dimensional subspace $L$ of $\mathbb{R}^n$.*

*Proof.* For any $P \in G_{n,n-k}$, the intersection $K \cap P$ is also compact convex, so $\chi(K \cap P) = 1$. Accordingly, for $L \in G_{n,k}$, every nonempty fiber $\pi_L^{-1}(\mathbf{x})$ is also compact convex, so $\chi(\pi_L^{-1}(\mathbf{x})) = 1$. Thus, $\int_L \chi(\pi_L^{-1}(\mathbf{x}))\ d\mathbf{x}$ is the $k$-dimensional (Lebesgue) volume of the projection of $K$ onto $L$. Let $K|L$ denote the projection of $K$ onto $L$. Then,

$$\mu_k(K) = \int_{G_{n,k}} \int_L \chi(\pi_L^{-1}(\mathbf{x}))\ d\mathbf{x}\ d\gamma(L) = \int_{G_{n,k}} \mu_k(K|L)\ d\gamma(L). \qquad \square$$

The mean projection formula gives another justification of why $\mu_{n-1}(K)$ is half the surface area of $K \subset \mathbb{R}^n$. First, let $K$ be a convex polyhedron in $\mathbb{R}^n$. For each face $f_i$ of $K$, $\mu_{n-1}(f_i|L)$ is the area of the projection of $f_i$ onto $L \in G_{n,k}$. Since the projection map of $K$ onto $L$ covers each point in its image twice, we have

$$\sum_i \mu_{n-1}(f_i|L) = 2\mu_{n-1}(K|L).$$

Integrating over the Grassmanian $G_{n,n-1}$ and applying the Mean Projection Formula, we obtain

$$\begin{aligned}\sum_i \mu_{n-1}(f_i) = \int_{G_{n,n-1}} \sum_i \mu_{n-1}(f_i|L)\ d\gamma(L) \\ = \int_{G_{n,n-1}} 2\mu_{n-1}(K|L)\ d\gamma(L) = 2\mu_{n-1}(K).\end{aligned} \tag{2.6}$$

Now $\sum_i \mu_{n-1}(f_i)$ is the surface area of $K$, so equation (2.6) implies that the surface area of $K$ is $2\mu_{n-1}(K)$. Since any convex subset is a limit of convex polyhedra, the result holds for all convex subsets of $\mathbb{R}^n$. By additivity, it holds for definable subsets of $\mathbb{R}^n$.

We can express an intrinsic volume of a direct product in terms of the intrinsic volumes of its factors:

**Theorem 2.3** (Product Theorem)**.** *For $J, K \in \mathcal{S}$,*

$$\mu_k(K \times J) = \sum_{i=0}^{k} \mu_i(K)\mu_{k-i}(J). \tag{2.7}$$

Klain and Rota prove the Product Theorem using Hadwiger's Theorem [24]. Schneider and Weil prove it for polytopes, and by extension for general convex bodies

[41]. Representing the intrinsic volumes in terms of conormal cycles, we exhibit a more elegant proof of the Product Theorem, as we will discuss in Section 3.4.

One implication of the Product Theorem is that $\mu_k(K \times J)$ can be computed by integrating over only those affine $(n-k)$-planes that are themselves direct products of affine planes in the factor subspaces containing $K$ and $J$. Furthermore, the Product Theorem extends to direct products of many definable sets:

**Corollary 2.1.** *For $K_1, \dots, K_r \in \mathcal{S}$,*

$$\mu_k(K_1 \times \cdots \times K_r) = \sum_{i_1+\cdots+i_r=k} \mu_{i_1}(K_1) \cdots \mu_{i_r}(K_r). \tag{2.8}$$

*Proof.* Identity (2.8) follows by induction from Theorem 2.3. □

## 2.4 Intrinsic Volumes of Common Subsets

We can now compute the intrinsic volumes of closed balls and rectangular prisms.

*Example.* Let $B_n$ be the closed $n$-dimensional unit ball, and $\omega_n$ its volume. We will compute $\mu_k(B_n)$:

$$\begin{aligned}
\mu_k(B_n) &= \int_{G_{n,n-k}} \int_{\mathbb{R}^k(L)} \chi(B_n \cap (L + \mathbf{x}))\, d\mathbf{x}\, d\gamma(L) \\
&= \int_{G_{n,n-k}} \int_{B_k} 1\, d\mathbf{x}\, d\gamma(L) \\
&= \int_{G_{n,n-k}} \omega_k\, d\gamma(L) = \binom{n}{k} \frac{\omega_n}{\omega_{n-k}}.
\end{aligned}$$

*Example.* Let $K$ be a closed $n$-dimensional rectangular prism in $\mathbb{R}^n$ with side lengths $x_1, \dots, x_n$. The product theorem allows us to compute the intrinsic volumes of $K$, as follows:

$$
\begin{aligned}
\mu_k(K) &= \mu_k([0,x_1]\times\cdots\times[0,x_n]) \\
&= \sum_{i_1+\cdots+i_n=k} \mu_{i_1}([0,x_1])\cdots\mu_{i_r}([0,x_n]) \\
&= \sum_{i_1+\cdots+i_n=k} x_1^{i_1}\cdots x_n^{i_n} \qquad \text{where } i_1,\ldots,i_n\in\{0,1\} \\
&= \sum_{1\le j_1<\cdots<j_k\le n} x_{j_1}\cdots x_{j_k}
\end{aligned}
$$

and this is the elementary symmetric polynomial of degree $k$ on the $n$ variables $x_1,\ldots,x_n$.

## 2.5 Open Sets

Our o-minimal approach to the intrinsic volumes prompts us to consider the intrinsic volumes of non-compact sets, and in particular, open sets. Indeed, in applications we will need to be able to compute the intrinsic volumes for open sets. To begin, recall that if $\sigma$ is an open $k$-dimensional simplex, then $\chi(\sigma) = (-1)^k$.

A *regular open* set is equal to the interior of its closure, and a *regular closed* set is equal to the closure of its interior. That is, $K$ is regular open if $K = \mathrm{Int}(\overline{K})$, and $J$ is regular closed if $J = \overline{\mathrm{Int}J}$.

We have the following lemma:

**Lemma 2.1.** *Let $K$ be a definable, regular closed set in $\mathbb{R}^d$. Then $\chi(\mathrm{Int}K) = (-1)^d\chi(K)$.*

We defer the proof of Lemma 2.1 until Section 4.2.

The lemma leads to a similar result for the intrinsic volumes:

**Theorem 2.4.** *Let $K$ be definable in $\mathbb{R}^n$, such that $K = \overline{\text{Int}K}$ and $K$ is not contained in any $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. Then,*

$$\mu_k(\text{Int}K) = (-1)^{n+k}\mu_k(K).$$

*Proof.* For any $P \in A_{n,n-k}$, $K \cap P$ is a subset, equal to the closure of its interior, of dimension $n-k$. Note that $\text{Int}(K \cap P) = (\text{Int}K) \cap P$, so by the lemma we have:

$$\chi(\text{Int}(K \cap P)) = (-1)^{n-k}\chi(K \cap P).$$

Integrating over $A_{n,n-k}$, we obtain:

$$\begin{aligned} \mu_k(\text{Int}K) &= \int_{A_{n,n-k}} \chi((\text{Int}K) \cap P)\ d\lambda(P) \\ &= (-1)^{n-k}\chi(K \cap P)\ d\lambda(P) = (-1)^{n-k}\mu_k(K). \quad \square \end{aligned}$$

While the Lebesgue measure of a definable set and its closure are the same, the intrinsic volumes are, in general, very sensitive to boundary points. When computing the intrinsic volumes of a set, it is essential to note whether the set contains some or all of its boundary. An understanding of this detail will be important to the integration theory that follows.

# Chapter 3

# Currents and Cycles

The intrinsic volumes also arise from integration of differential forms over normal and conormal cycles of sets. Normal and conormal cycles are examples of *currents*, which are continuous linear functionals on the spaces of differential forms. This chapter gives a brief introduction to currents, providing only information relevant to our applications. For more details, see chapter 7 of Krantz and Parks [25], chapter 12 of Morvan [32], or the exhaustive and technical chapter 4 of Federer [16].

## 3.1 Currents

First we must establish some notation. Let $\Omega_c^k(U)$ denote the space of compactly-supported differential $k$-forms on some $U \subseteq \mathbb{R}^N$. Its dual space, the space of continuous linear functionals on $\Omega_c^k(U)$, we denote as $\Omega_k(U)$, or simply as $\Omega_k$ if $U$ is understood. We call an element of $\Omega_k(U)$ a *$k$-dimensional current on $U$*. The *bound-*

*ary* of a current $T \in \Omega_k$ is the current $\partial T \in \Omega_{k-1}$ defined by $(\partial T)(\omega) = T(d\omega)$ for all $\omega \in \Omega_c^{k-1}$. A *cycle* is a current with zero boundary. Similarly to differential forms, $\partial(\partial T) = 0$.

Currents are naturally associated with submanifolds and geometric subsets [25, 32]. Let $M^n$ be a $C^1$ oriented $n$-dimensional submanifold of $\mathbb{R}^N$. Let $dv_{M^n}$ be the volume form on $M^n$. For any $\omega \in \Omega_c^k(\mathbb{R}^N)$, the restriction of $\omega$ to $M^n$ equals $f_\omega dv_{M^n}$ for some function $f_\omega$ on $M^n$. Define a current $[\![M^n]\!]$ associated to $M^n$ by

$$[\![M^n]\!](\omega) = \int_{M^n} f_\omega dv_{M^n}.$$

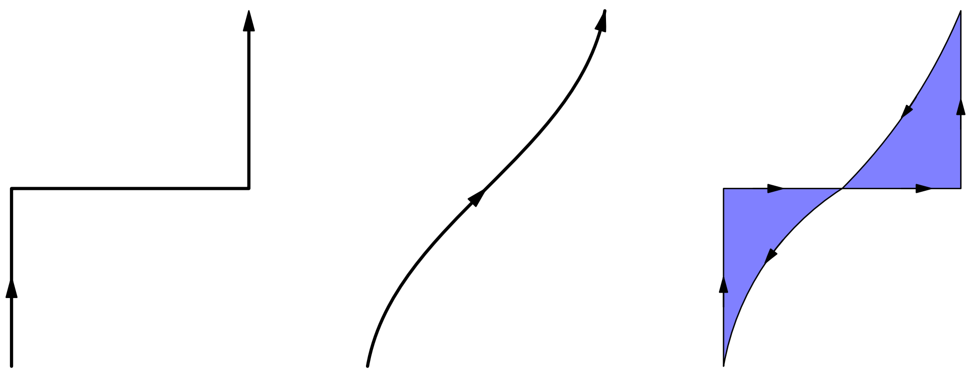
For our work with currents, we need a norm on the space of currents. First, the *mass* $\mathbf{M}(T)$ of a current $T \in \Omega_k(U)$ is the real number defined by:

$$\mathbf{M}(T) = \sup\{\ T(\omega) \mid \omega \in \Omega_c^k(U) \text{ and } \sup|\omega(x)| \leq 1\ \forall\ x \in U\}.$$

The mass of a current generalizes the volume of a submanifold: the mass of a current supported on a tame set is equal to the volume of the set. Second, let $|T|_\flat$ denote the *flat norm* of the current $T \in \Omega_k(U)$, which is the real number defined by:

$$|T|_\flat = \inf\{\mathbf{M}(R) + \mathbf{M}(S) \mid T = R + \partial S, R \in \Omega_k(U), S \in \Omega_{k+1}(U)\}. \tag{3.1}$$

We can think of the flat norm as quantifying the minimal-mass decomposition of a $k$-current $T$ into a $k$-current $R$ and the boundary of a $(k+1)$-current $S$, as illustrated in Figure 3.1. The flat norm is an excellent tool for measuring the distance between shapes, or between their associated currents, as Morgan and Vixie explain in [31].



$T = R + \partial S \in \Omega_1$ $R \in \Omega_1$ $S \in \Omega_2$

**Figure 3.1:** The 1-current $T$ is decomposed as the sum of a 1-current $R$ and the boundary of a 2-current $S$. Intuitively, the flat norm of $T$ is the decomposition that minimizes the length of $R$ plus the area of $S$.

In this context, the word "flat" is not referring to a lack of curvature, but simply to Hassler Whitney's musical notation $\flat$ used to denote this norm [25, 30].

## 3.2 Normal and Conormal Cycles

We are interested in particular currents, known as the normal cycle and conormal cycle, that are associated to compact definable sets. The normal cycle is a generalization of the unit normal bundle of a manifold. The definition of the normal cycle is long and technical; for that we refer the reader to Bernig [7], Fu [20], or Nicolaescu [34]. Let $A$ be a compact definable set in $\mathbb{R}^N$. We denote the normal cycle of $A$ as $\mathbf{N}^A$. Formally, $\mathbf{N}^A$ is an $(N-1)$-current on the unit cotangent bundle $\mathbb{R}^N \times \mathbb{S}^{N-1}$. The normal cycle is Legendrian, which means that its restriction to the canonical

1-form $\alpha$ on the tangent bundle $T^*\mathbb{R}^N$ is zero:

$$\mathbf{N}^A\big|_\alpha = 0.$$

The key property for our purposes is that the normal cycle is additive, that is:

$$\mathbf{N}^{A\cup B} + \mathbf{N}^{A\cap B} = \mathbf{N}^A + \mathbf{N}^B. \tag{3.2}$$

Intuitively, we regard $\mathbf{N}^A$ as the collection of unit tangent vectors to $A$, though this intuition is inadequate if $A$ is not convex. More precisely, if $A$ is a convex set, then the *support* of $\mathbf{N}^A$ is the hypersurface of unit tangent vectors to $A$, with orientation given by the outward normal.

*Example* (Normal cycle of a simplicial complex)*.* If $\sigma$ is a (closed) $k$-simplex in $\mathbb{R}^N$, then $\mathbf{N}^\sigma$ is the current whose support is the surface of unit tangent vectors to $\sigma$, with outward orientation. We can then construct the normal cycle of a simplicial complex via equation (3.2).

Figure 3.2 illustrates the normal cycle of a simplicial complex. The simplicial complex $K$ in $\mathbb{R}^2$ consists of the union of the two closed intervals $ab$ and $bc$. The normal cycle of a closed interval in $\mathbb{R}^2$ is supported on an oriented path at unit distance around the interval. The intervals intersect at point $b$, whose normal cycle is supported on an oriented unit circle at $b$. By equation (3.2),

$$\mathbf{N}^K = \mathbf{N}^{ab} + \mathbf{N}^{bc} - \mathbf{N}^b,$$

which is the normal cycle supported on the oriented path shown in Figure 3.2.

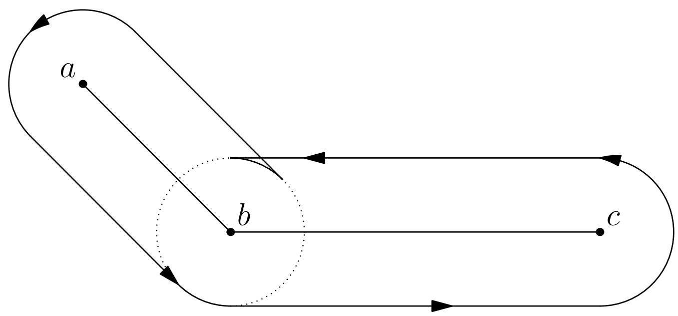



**Figure 3.2:** Normal cycle of a simplicial complex.

Dual to normal cycles are conormal cycles. For details on the construction of the conormal cycle, see Nicolaescu [33]. The conormal cycle of $A$, denoted $\mathbf{C}^A$, is also an $(N-1)$-current on $\mathbb{R}^N \times \mathbb{S}^{N-1}$, and it is the cone over $\mathbf{N}^A$. The conormal cycle is Lagrangian, which means that its restriction to the standard symplectic 2-form $\omega$ on $T^*\mathbb{R}^N$ is zero:

$$\mathbf{C}^A\Big|_{\omega} = 0.$$

For example, the conormal cycle of an interval $[a, b]$ in $\mathbb{R}$ is illustrated by the dark path in Figure 3.3.

Pioneers in the study of normal and conormal cycles were Wintgen [46] and Zähle [47]. Fu gives a detailed treatment of these cycles of subanalytic sets in [19]. Nicolaescu shows in [33] the existence and uniqueness of normal and conormal cycles for definable sets.

## 3.3 Lipschitz-Killing Curvature Forms

The intrinsic volumes arise from integrating certain differential forms over normal and conormal cycles. These differential forms are called the *Lipschitz-Killing curvature forms*, named after Rudolf Lipschitz and Wilhelm Killing. These forms are invariant under rigid motions, as they must be since the intrinsic volumes are invariant under such motions.

Since normal and conormal cycles are $(N-1)$-currents on $\mathbb{R}^N \times \mathbb{S}^{N-1}$, we need differential $(N-1)$-forms on $\mathbb{R}^N \times \mathbb{S}^{N-1}$ invariant under rigid motions. Let $x_1, \ldots, x_N$ be the standard orthonormal basis for $\mathbb{R}^N$, and $\rho_1, \ldots, \rho_{N-1}$ an orthonormal frame for $\mathbb{S}^{N-1}$. Define the following differential $(N-1)$-form on $\mathbb{R}^N \times \mathbb{S}^{N-1}$:

$$\mathcal{V}(t) = (x_1 + t\rho_1) \wedge \cdots \wedge (x_{N-1} + t\rho_{N-1}).$$

Intuitively, if $t = 0$, this is the volume form on $\mathbb{R}^{N-1}$, which is invariant under rigid motions. Morvan explains in [32, ch. 19] that the form $\mathcal{V}(t)$ is invariant under rigid motions of $\mathbb{R}^N$, extended to $\mathbb{R}^N \times \mathbb{S}^{N-1}$, for all $t$. Fu arrives at the same result via a pushforward of an exponential map [21].

We can view the differential form $\mathcal{V}(t)$ as a polynomial in $t$, whose coefficients are the invariant forms that we seek:

**Definition 3.1.** For $0 \leq k \leq N-1$, let $\mathcal{W}_{N-1,k}$ be the coefficient of $t^{N-k-1}$ in $\mathcal{V}(t)$. The form $\mathcal{W}_{N-1,k}$ is called the $k^{th}$ *Lipschitz-Killing curvature form* of degree $N-1$.

The $\mathcal{W}_{N-1,k}$ are exactly the forms we need to integrate (with appropriate scalar multiples) over normal or conormal cycles of sets to obtain the intrinsic volumes. By invariance, they do not depend on the orthonormal frame $x_1, \ldots, x_N, \rho_1, \ldots, \rho_{N-1}$. Indeed, Morvan states that the vector space of invariant differential $(N-1)$-forms on $\mathbb{R}^N \times \mathbb{S}^{N-1}$ is spanned by the $\mathcal{W}_{N-1,k}$ and, if $N$ is odd, by a power of the standard symplectic form. However, the conormal cycle vanishes on the standard symplectic form, so the only invariant forms relevant to our discussion are the $\mathcal{W}_{N-1,k}$.

## 3.4 Back to the Intrinsic Volumes

We can express the intrinsic volumes in terms of normal or conormal cycles, and the Lipschitz-Killing curvature forms.

**Theorem 3.1.** *Let $K \in \mathsf{Def}(\mathbb{R}^n)$. The integrals*

$$\int_{\mathbf{N}^K} \mathcal{W}_{n,k} \quad \text{and} \quad \int_{\mathbf{C}^K} \mathcal{W}_{n,k} \tag{3.3}$$

*are, up to a constant multiple, the intrinsic volume $\mu_k(K)$.*

*Proof.* Since the normal and conormal cycles are additive, and the Lipschitz-Killing curvature forms are Euclidean-invariant, the integrals in (3.3) are valuations on $\mathsf{Def}(\mathbb{R}^n)$. By definition of the Lipschitz-Killing curvature forms, these expressions are homogeneous of degree $k$. The integrals are also continuous on convex sets with respect to the Hausdorff metric. Therefore, by Hadwiger's Theorem, these integrals are $\mu_k(K)$, up to a constant multiple. □

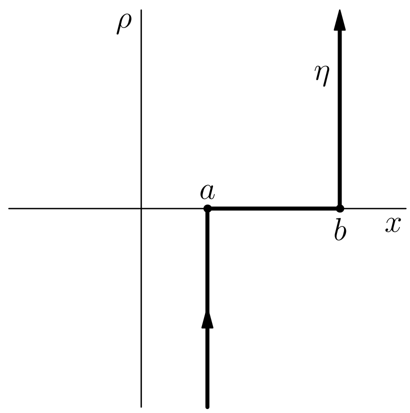



**Figure 3.3:** Conormal cycle of the interval $[a, b]$.

Some low-dimensional examples will help illustrate this "current" approach to the intrinsic volumes.

*Example.* Consider the interval $[a, b] \subset \mathbb{R}$. Its conormal cycle $\mathbf{C}^{[a,b]}$ can be represented by the dark path in Figure 3.3. The space of invariant 1-forms on $\mathbb{R}^2 \times \mathbb{S}^1$ is spanned by the two forms $\mathcal{W}_{1,0} = d\rho$ and $\mathcal{W}_{1,1} = dx$. We obtain the intrinsic volumes of $[a, b]$ by the integrals:

$$\mu_0([a, b]) = \int_{\mathbf{C}^{[a,b]}} \frac{1}{2\pi} \, d\rho = 1$$

$$\mu_1([a, b]) = \int_{\mathbf{C}^{[a,b]}} dx = b - a.$$

*Example.* Let $K$ be a definable subset of $\mathbb{R}^2$. The space of invariant 2-forms of $\mathbb{R}^3 \times \mathbb{S}^2$

contains the following forms:

$$\mathcal{W}_{2,0} = d\rho_1 \wedge d\rho_2$$

$$\mathcal{W}_{2,1} = d\rho_1 \wedge dx_2 + dx_1 \wedge d\rho_2$$

$$\mathcal{W}_{2,2} = dx_1 \wedge dx_2$$

We obtain the intrinsic volumes of $K$ by integrating these forms over $\mathbf{C}^K$:

$$\begin{aligned}
\mu_0(K) &= \iint_{\mathbf{C}^K} \frac{1}{4\pi}\, d\rho_1 \wedge d\rho_2 \\
\mu_1(K) &= \iint_{\mathbf{C}^K} \frac{1}{2\pi}\, d\rho_1 \wedge dx_2 + \frac{1}{2\pi}\, dx_1 \wedge d\rho_2 \\
\mu_2(K) &= \iint_{\mathbf{C}^K} dx_1 \wedge dx_2.
\end{aligned}$$

In the above examples, we use normalizations such as $\frac{1}{2\pi}$ to scale the integrals properly, as is necessary for computations.

Representing the intrinsic volumes in terms of integrals of the Lipschitz-Killing curvature forms, we can now prove the Product Theorem (Theorem 2.3). The key observation is that by Definition 3.1, for integers $0 < m < n$,

$$\mathcal{W}_{n,k} = \sum_{i=0}^{k} \mathcal{W}_{m,i} \wedge \mathcal{W}_{n-m,k-i}.$$

Therefore,

$$\sum_{i=0}^{k} \mu_k(K)\mu_{k-i}(J) = \sum_{i=0}^{k} \int_{\mathbf{C}^K} \mathcal{W}_{m,i} \int_{\mathbf{C}^J} \mathcal{W}_{n-m,k-i} = \int_{\mathbf{C}^{K\times J}} \mathcal{W}_{n,k} = \mu_k(K \times J)$$

which proves the Product Theorem.

## 3.5 Continuity

The flat norm on conormal cycles is the key ingredient of a topology on definable sets, with respect to which the intrinsic volumes are continuous.

Let $A$ and $B$ be definable subsets of $\mathbb{R}^n$. Define the *flat metric* in terms of the flat norm as follows:

$$d(A,B) = \left|\mathbf{C}^A - \mathbf{C}^B\right|_\flat . \tag{3.4}$$

Call the topology on definable subsets of $\mathbb{R}^n$ induced by the flat metric the *flat topology*. The flat topology is a useful generalization of the Hausdorff topology on convex sets. Indeed, if a sequence of convex sets converges in the Hausdorff topology, then the corresponding sequence of their normal cycles converges in the flat topology [17], but the same is not true for non-convex sets. On the other hand, convergence of normal cycles of definable sets implies convergence in the Hausdorff topology. Furthermore, we have the following theorem:

**Theorem 3.2.** *The intrinsic volumes are continuous with respect to the flat topology.*

*Proof.* Let $K \in \mathsf{Def}(\mathbb{R}^n)$ be bounded and $\epsilon > 0$. Let $B$ be a large ball in $\mathbb{R}^n$ containing a neighborhood of $K$.

From the definition of flat norm, we have for any $T \in \Omega_n$ and $\omega \in \Omega_c^n$, both supported on $B$:

$$|T(\omega)| \leq |T|_\flat \cdot \max\left\{\sup_{x\in B}|\omega(x)|, \sup_{x\in B}|d\omega(x)|\right\}. \tag{3.5}$$

Suppose $J \in \mathsf{Def}(\mathbb{R}^n)$ is contained in $B$. Let $T = \mathbf{C}^K - \mathbf{C}^J$ be the difference between conormal cycles of $K$ and $J$, and let $\omega = \mathcal{W}_{n,k}$. Equation (3.5) becomes:

$$\begin{aligned} |\mu_k(K) - \mu_k(J)| &= \left| \int_{\mathbf{C}^K} \mathcal{W}_{n,k} - \int_{\mathbf{C}^J} \mathcal{W}_{n,k} \right| \\ &\leq \left|\mathbf{C}^K - \mathbf{C}^J\right|_\flat \cdot \max\left\{ \sup_{x\in B} |\mathcal{W}_{n,k}(x)|, \sup_{x\in B} |d\mathcal{W}_{n,k}(x)| \right\}. \end{aligned} \tag{3.6}$$

The forms $\mathcal{W}_{n,k}$ and $d\mathcal{W}_{n,k}$ are bounded on $B$, so we can let

$$\delta = \epsilon \cdot \left( \max\left\{ \sup_{x\in B} |\mathcal{W}_{n,k}(x)|, \sup_{x\in B} |d\mathcal{W}_{n,k}(x)| \right\} \right)^{-1},$$

and we have $|\mu_k(K) - \mu_k(J)| < \epsilon$ for all $J \in \mathsf{Def}(\mathbb{R}^n)$ such that $d(K, J) = \left|\mathbf{C}^K - \mathbf{C}^J\right|_\flat < \delta$, which proves continuity. $\square$

# Chapter 4

# Valuations on Functionals

We now "lift" valuations from sets to functionals over sets. This results in the *Hadwiger integrals*—integrals with respect to the intrinsic volumes. For functionals with finite range, integration is straightforward. Integration of continuous functionals is more complicated, resulting in a duality of lower and upper integrals. We discuss properties and equivalent expressions of these Hadwiger integrals of continuous functionals.

## 4.1 Constructible Functions

Having established the basic properties of intrinsic volumes, we now explore their use as a measure for integration. We begin with *constructible functions*, which are integer-valued functions with definable level sets. Moreover, if $f$ is a constructible function, then its domain has a locally finite triangulation such that $f$ is constant on each simplex. Thus, if $f$ has compact support, then $f$ is bounded. Integration of constructible functions with respect to intrinsic volumes is straightforward.

**Definition 4.1.** Let $X \in \mathbb{R}^n$ be compact and $h : X \to \mathbb{Z}$ be a constructible function. So $h = \sum_i c_i \mathbf{1}_{A_i}$, where $c_i \in \mathbb{Z}$ and $\mathbf{1}_{A_i}$ is the characteristic function on a definable set $A_i$. We may assume the $A_i$ are disjoint. Then the *Hadwiger integral* of $h$ with respect to $\mu_k$ is

$$\int_X h \; d\mu_k = \int_X \sum_i c_i \mathbf{1}_{A_i} \; d\mu_k = \sum_i c_i \mu_k(A_i).$$

When $k = 0$, the integral is also called an *Euler integral* and denoted $\int_X h \; d\chi$ [3, 44]. When $k = n$, the integral is the usual Lebesgue integral.

## 4.2 Duality

Schapira [37] defines a useful duality on constructible functions. Let the dual of a function $h \in \mathsf{CF}(\mathbb{R}^n)$, be the function $Dh$ whose value at $x \in \mathbb{R}^n$ is given by

$$Dh(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} h \cdot \mathbf{1}_{B(x,\epsilon)} \; d\chi,$$

where $B(x, \epsilon)$ is the ball of radius $\epsilon$ centered at $x$.

The properties of Schapira's duality that are important for our purposes are summarized in the following theorem.

**Theorem 4.1.** *Let $h \in \mathsf{CF}(\mathbb{R}^n)$. Then:*

1. *The dual $Dh$ is a constructible function,*

2. *Duality is an involution:* $D^2 h = h$*, and*

3. *Duality preserves integrals:* $\int_X h \, d\chi = \int_X Dh \, d\chi$*.*

For proofs, see Schapira [37].

With Euler integrals and duality of constructible functions, we can now prove Lemma 2.1, restated from Section 2.5:

**Lemma 4.1.** *Let* $K$ *be a definable, regular closed subset of* $\mathbb{R}^n$*. Then* $\chi(\mathrm{Int}K) = (-1)^n \chi(K)$*.*

*Proof.* For a characteristic function of a regular closed subset $K$ of $\mathbb{R}^n$, $D(\mathbf{1}_{\mathrm{Int}K}) = (-1)^n \mathbf{1}_K$.

Schapira's duality implies:

$$\chi(\mathrm{Int}K) = \int \mathbf{1}_{\mathrm{Int}K} \, d\chi = \int D(\mathbf{1}_{\mathrm{Int}K}) \, d\chi = \int (-1)^n \mathbf{1}_K d\chi = (-1)^n \chi(K). \qquad \square$$

## 4.3 Extending to Continuous Functions

A *definable function* is a bounded real-valued function on a compact set $X \in \mathbb{R}^n$ whose graph is a definable subset of $\mathbb{R}^{n+1}$. Similar to the real-valued Euler integrals of Baryshnikov and Ghrist in [4], we can integrate a definable function with respect to intrinsic volumes.

**Definition 4.2** (Hadwiger Integral)**.** For definable function $h : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$, the

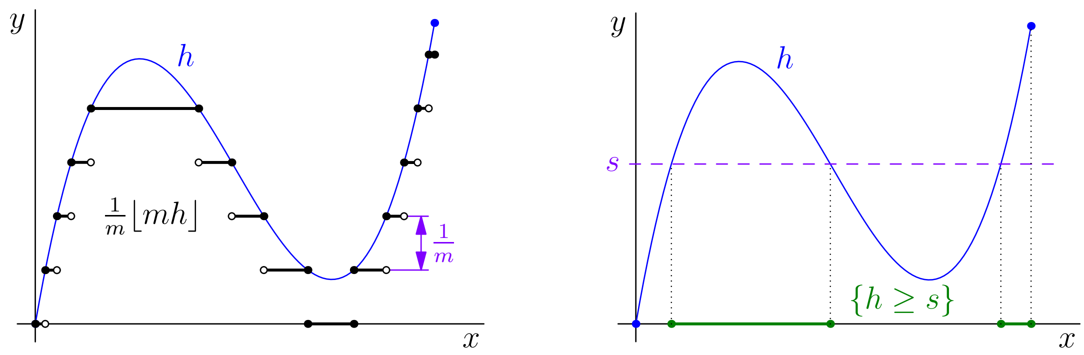



**Figure 4.1:** The lower Hadwiger integral is defined as a limit of lower step functions (left), as in Definition 4.2. It can also be expressed in terms of excursion sets (right), as in Theorem 4.2, equation (4.3).

*lower* and *upper Hadwiger integrals* of $h$ are:

$$\int_X h \, \lfloor d\mu_k \rfloor = \lim_{m\to\infty} \frac{1}{m} \int \lfloor mh \rfloor \, d\mu_k \quad \text{and} \tag{4.1}$$

$$\int_X h \, \lceil d\mu_k \rceil = \lim_{m\to\infty} \frac{1}{m} \int \lceil mh \rceil \, d\mu_k. \tag{4.2}$$

When $k = 0$, we obtain the real-valued Euler integrals of Baryshnikov and Ghrist, and when $k = n$, both of the integrals in Definition 4.2 are in fact Lebesgue integrals. Existence of the limits in Definition 4.2 is a consequence of Theorem 4.2.

The integrals in Definition 4.2 are written in terms of step functions, but they can be expressed in several different ways. We can write the integrals in terms of *excursion sets*, which are sets on which the functional takes on values in a particular interval. For example, $\{h \geq s\} = \{x \mid h(x) \geq s\}$. Figure 4.1 illustrates step functions and excursion sets of a definable function. We can also write the Hadwiger integrals in terms of Euler integrals along affine slices, or projections onto linear subspaces.

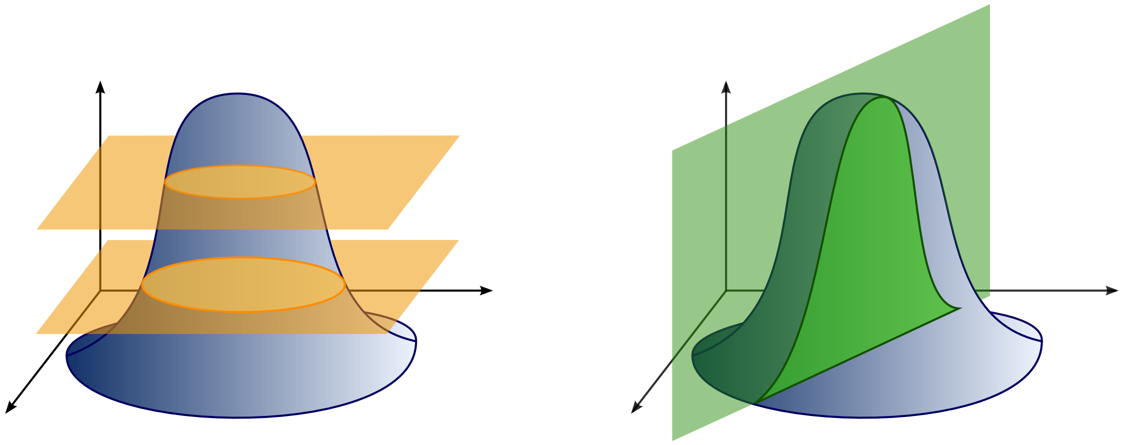


**Figure 4.2:** The Hadwiger integral of a function $h : \mathbb{R}^2 \to \mathbb{R}$ can be expressed in terms of level sets of $h$ (left) or slices of $h$ by planes perpendicular to the domain (right), as in Theorem 4.2.

Illustrated in figure 4.2 for a bump function, these equivalent expressions are stated in the following theorem.

**Theorem 4.2** (Equivalent Expressions)**.** *For a definable function $h : \mathbb{R}^n \to \mathbb{R}$, the lower Hadwiger integral can be written*

$$\int_X h \, \lfloor d\mu_k \rfloor = \int_{s=0}^{\infty} \left(\mu_k\{h \geq s\} - \mu_k\{h < -s\}\right) \, ds \qquad \textit{excursion sets} \tag{4.3}$$

$$= \int_{A_{n,n-k}} \int_{X \cap P} h \, \lfloor d\chi \rfloor \, d\lambda(P) \qquad \textit{slices} \tag{4.4}$$

$$= \int_{G_{n,k}} \int_L \int_{\pi_L^{-1}(\mathbf{x})} h \, \lfloor d\chi \rfloor \, d\mathbf{x} \, d\gamma(L) \qquad \textit{projections} \tag{4.5}$$

*and similarly for the upper Hadwiger integral.*

*Proof.* To express the integral in terms of excursion sets, first let

$$T = \max(\sup(h), -\inf(h))$$

and let $N = mT$. Then,

$$\begin{aligned}\int h \lfloor d\mu_k \rfloor &= \lim_{m\to\infty} \frac{1}{m} \int \lfloor mh \rfloor \, d\mu_k = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{\infty} \mu_k\{mh \geq i\} - \mu_k\{mh < -i\} \\ &= \lim_{N\to\infty} \frac{T}{N} \sum_{i=1}^{N} \mu_k \left\{ h \geq \frac{iT}{N} \right\} - \mu_k \left\{ h < -\frac{iT}{N} \right\} \\ &= \int_0^T \mu_k\{h \geq s\} - \mu_k\{h < -s\} \, ds,\end{aligned}$$

which proves equation (4.3).

For the expression involving affine slices, note that

$$\begin{aligned}&\int_0^\infty \mu_k\{h \geq s\} - \mu_k\{h < -s\} \, ds \\ &\qquad = \int_0^\infty \int_{A_{n,n-k}} \chi(\{h \geq s\} \cap P) - \chi(\{h < -s\} \cap P) \, d\lambda(P) \, ds.\end{aligned}$$

Since the excursion sets $\{h \geq s\}$ and $\{h < -s\}$ are definable, they have finite Euler characteristic, and the integrand is finite. Since $h$ has compact support, the Grassmanian integral is actually over a bounded subset of $A_{n,n-k}$. Moreover, $h$ is bounded, so the real integral is over a bounded subset of $\mathbb{R}$. Thus, the double integral is in fact finite. Since the integral is finite and $\mathbb{R}$ and $A_{n,n-k}$ are $\sigma$-finite measure spaces, Fubini's theorem allows us to change the order of integration. The integral then becomes

$$\begin{aligned}&\int_{A_{n,n-k}} \int_0^\infty \chi(\{h \geq s\} \cap P) - \chi(\{h < -s\} \cap P) \, ds \, d\lambda(P) \\ &\qquad = \int_{A_{n,n-k}} \int_{X \cap P} h \lfloor d\chi \rfloor \, d\lambda(P),\end{aligned}$$

and this proves equation (4.4).

To express the integral in terms of projections, fix an $L \in G_{n,k}$. Let $\pi_L : X \to L$ be the orthogonal projection map on to $L$. Then the affine subspaces perpendicular to $L$ are the fibers of $\pi_L$; that is,

$$\{P \in A_{n,n-k} : P \perp L\} = \{\pi_L^{-1}(\mathbf{x}) : \mathbf{x} \in \pi(X)\}.$$

Instead of integrating over $A_{n,n-k}$, we can integrate over the fibers of orthogonal projections onto all linear subspaces of $G_{n,k}$. That is,

$$\int h \; \lfloor d\mu_k \rfloor = \int_{A_{n,n-k}} \int_{X \cap P} h \; \lfloor d\chi \rfloor \; d\lambda(P) = \int_{G_{n,k}} \int_L \int_{\pi_L^{-1}(\mathbf{x})} h \; \lfloor d\chi \rfloor \; d\mathbf{x} \; d\gamma(L)$$

which is equation (4.5). □

The lower and upper Hadwiger integrals are not linear in general.

*Example.* A simple example that illustrates the nonlinearity of the Euler integral was given by Baryshnikov and Ghrist in [4]:

$$\int_{[0,1]} x \; \lfloor d\chi \rfloor + \int_{[0,1]} (1-x) \; \lfloor d\chi \rfloor = 1 + 1 = 2 \neq 1 = \int_{[0,1]} 1 \; \lfloor d\chi \rfloor.$$

The reader can find similar examples of the nonlinearity of the other Hadwiger integrals, except for the Lebesgue integral.

The lower and upper Hadwiger integrals are dual in the following sense.

**Corollary 4.1** (Duality)**.** *The lower and upper Hadwiger integrals exhibit a duality: for $h \in \mathsf{Def}(\mathbb{R}^n)$,*

$$\int h \; \lfloor d\mu_k \rfloor = -\int -h \; \lceil d\mu_k \rceil.$$

*Proof.* The upper Hadwiger integral can be written in a form similar to equation (4.3):

$$\int h \lceil d\mu_k \rceil = \int_{s=0}^{\infty} \mu_k\{h > s\} - \mu_k\{h \leq -s\} \, ds$$

Duality then follows. $\square$

## 4.4 Hadwiger Integrals as Currents

Just as we can express the intrinsic volumes of subsets in terms of the normal and conormal cycles of the subsets, we can express the Hadwiger Integrals as currents.

Let $h \in \mathsf{Def}(\mathbb{R}^n)$. Writing the Hadwiger integral $\int h \lfloor d\mu_k \rfloor$ in terms of intrinsic volumes of excursion sets via equation (4.3) and expressing these intrinsic volumes via conormal cycles (3.3), we have:

$$\begin{aligned} \int_{\mathbb{R}^n} h \lfloor d\mu_k \rfloor &= \int_{s=0}^{\infty} \left(\mu_k\{h \geq s\} - \mu_k\{h < -s\}\right) \, ds \\ &= \int_{s=0}^{\infty} \left( \int_{\mathbf{C}^{\{h \geq s\}}} \mathcal{W}_{n,k} - \int_{\mathbf{C}^{\{h < -s\}}} \mathcal{W}_{n,k} \right) \, ds. \qquad (4.6) \end{aligned}$$

In this way, we can represent the integrals not only as currents, but in fact as cycles. For a differential $n$-form $\omega$, define:

$$T(\omega) = \int_{s=0}^{\infty} \left( \mathbf{C}^{\{h \geq s\}}(\omega) - \mathbf{C}^{\{h < -s\}}(\omega) \right) \, ds.$$

Then $T$ is a continuous linear functional on the space of differential forms, so it is a current. Furthermore, $T$ is a cycle, as it has no boundary because the conormal cycles have no boundary:

$$\partial T(\omega) = T(\partial\omega) = \int_{s=0}^{\infty} \left(\mathbf{C}^{\{h\geq s\}}(\partial\omega) - \mathbf{C}^{\{h<-s\}}(\partial\omega)\right) = \int_{s=0}^{\infty} 0 \; ds = 0. \tag{4.7}$$

In summary, we have the following proposition.

**Proposition 4.1.** *The lower Hadwiger integrals of $h \in \mathsf{Def}(\mathbb{R}^n)$ can be expressed in terms of the cycle*

$$T(\omega) = \int_{s=0}^{\infty} \left(\mathbf{C}^{\{h\geq s\}}(\omega) - \mathbf{C}^{\{h<-s\}}(\omega)\right) \; ds, \tag{4.8}$$

*evaluated on the Lipschitz-Killing curvature forms, and similarly for the upper Hadwiger integrals.*

## 4.5 Summary of Representations

In summary, we have the following equivalent expressions of the lower Hadwiger integral for a function $h \in \mathsf{Def}(\mathbb{R}^n)$ and integer $0 \leq k \leq n$:

$$\int h \; \lfloor d\mu_k \rfloor = \lim_{m\to\infty} \frac{1}{m} \int \lfloor mh \rfloor \; d\mu_k \qquad \text{step functions (4.1)}$$

$$\int h \; \lfloor d\mu_k \rfloor = \int_{s=0}^{\infty} \mu_k\{h \geq s\} - \mu_k\{h < -s\} \; ds \qquad \text{excursion sets (4.3),}$$

$$\int h \; \lfloor d\mu_k \rfloor = \int_{A_{n,n-k}} \int_{X\cap P} h \; \lfloor d\chi \rfloor \; d\lambda(P) \qquad \text{slices (4.4),}$$

$$\int h \; \lfloor d\mu_k \rfloor = \int_{G_{n,k}} \int_{L} \int_{\pi_L^{-1}(\mathbf{x})} h \; \lfloor d\chi \rfloor \; d\mathbf{x} \; d\gamma(L) \qquad \text{projections (4.5), and}$$

$$\int h \; \lfloor d\mu_k \rfloor = \int_{s=0}^{\infty} \left(\mathbf{C}^{\{h\geq s\}}(\mathcal{W}_{n,k}) - \mathbf{C}^{\{h<-s\}}(\mathcal{W}_{n,k})\right) \; ds \qquad \text{currents (4.8).}$$

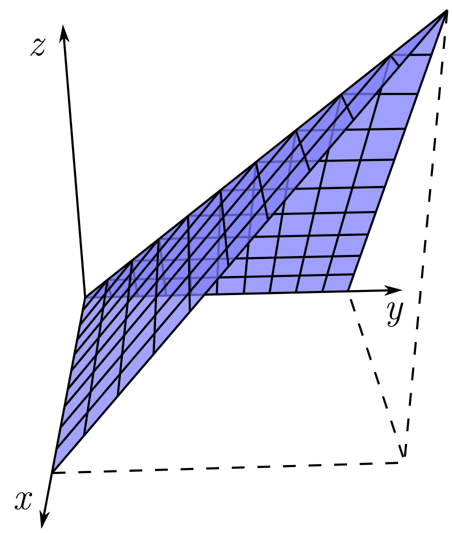



**Figure 4.3:** Function $h : [0,1]^2 \to \mathbb{R}$, defined $h(x,y) = \min(x,y)$, illustrates the non-equivalence of lower and upper Hadwiger integrals.

To obtain expressions of the upper Hadwiger integral, replace the "floor" function $\lfloor \cdot \rfloor$ by the "ceiling" function $\lceil \cdot \rceil$, replace the excursion set $\{h \geq s\}$ by $\{h > s\}$, and replace the excursion set $\{h < -s\}$ by $\{h \leq -s\}$.

## 4.6 Properties of Hadwiger Integration

The lower and upper Hadwiger integrals with respect to $\mu_k$ are not equal in general. The following example illustrates this lack of equality.

*Example.* Define $h : [0,1]^n \to \mathbb{R}$ by $h(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$, illustrated in Figure 4.3 for $n = 2$.

For $s \in [0,1]$, the excursion set $\{h \geq s\}$ is a closed $n$-dimensional cube with side lengths $1 - s$. By Section 2.4,

$$\mu_{n-1}\{h \geq s\} = \mu_{n-1}([s,1]^n) = n(1-s)^{n-1}.$$

The strict excursion set $\{h > s\}$ is also an $n$-dimensional cube with side lengths $1-s$, closed along half of its $(n-1)$-dimensional faces, and open on the other half of such faces. Thus,

$$\mu_{n-1}\{h > s\} = \mu_{n-1}([s,1]^n) - n\mu_{n-1}((s,1)^{n-1}) = n(1-s)^{n-1} - n(1-s)^{n-1} = 0.$$

Thus, the Hadwiger integrals of $h$ with respect to $\mu_{n-1}$ are different:

$$\begin{aligned}
\int_{[0,1]^n} h \; \lfloor d\mu_{n-1} \rfloor &= \int_0^\infty \mu_{n-1}\{h \geq s\} \; ds = \int_0^\infty n(1-s)^{n-1} \; ds = 1, \quad \text{but} \\
\int_{[0,1]^n} h \; \lceil d\mu_{n-1} \rceil &= \int_0^\infty \mu_{n-1}\{h > s\} \; ds = \int_0^\infty 0 \; ds = 0.
\end{aligned}$$

Having established that the lower and upper Hadwiger integrals are different, we would like to know conditions on a functional $h$ that guarantee the equality of its lower and upper integrals. Note that if we modify the functional $h$ in the above example so that $h$ is uniformly zero outside $[0,1]^n$, then $h$ is not continuous on $\mathbb{R}^n$. However, if $f$ is a continuous functional on $\mathbb{R}^n$, then its lower and upper integrals differ only by a minus sign.

**Theorem 4.3.** *Let $f \in \mathsf{Def}(\mathbb{R}^n)$ be a continuous function on $\mathbb{R}^n$. Then,*

$$\int_X f \; \lfloor d\mu_k \rfloor = (-1)^{n+k} \int_X f \; \lceil d\mu_k \rceil. \tag{4.9}$$

*Proof.* The key idea is that a definable function is only constant on finitely many sets with positive ($n$-dimensional) Lebesgue measure. Let on $X \subset \mathbb{R}^n$ be the support of $f$.

By the o-minimal cell decomposition theorem [43], $X$ can be partitioned into finitely many cells, such that $f$ is either constant or affine on each cell. Only if $f$ is constant (say, $f = s$) on a cell $C \subset X$ with positive Lebesgue measure can it be the case that $\mathrm{Int}\{f \geq s\} \neq \{f > s\}$ or $\mathrm{Int}\{f \leq -s\} \neq \{f < -s\}$.

That is, for all but finitely many $s \in [0, \infty)$, $\mathrm{Int}\{f \geq s\} = \{f > s\}$ and $\mathrm{Int}\{f \leq -s\} = \{f < -s\}$. Theorem 2.4 then says that

$$\mu_k\{f \geq s\} = (-1)^{n+k}\mu_k\{f > s\} \quad \text{and} \quad \mu_k\{f < -s\} = (-1)^{n+k}\mu_k\{f \leq -s\}$$

for all but finitely $s \in [0, \infty)$. Thus,

$$\begin{aligned}\int_X f \; \lfloor d\mu_k \rfloor &= \int_0^\infty \mu_k\{f \geq s\} - \mu_k\{f < -s\} \; ds \\ &= (-1)^{n+k} \int_0^\infty \mu_k\{f > s\} - \mu_k\{f \leq -s\} \; ds = (-1)^{n+k} \int_X f \; \lceil d\mu_k \rceil,\end{aligned}$$

which is equation (4.9). □

We have an analog of the inclusion-exclusion property for real-valued Hadwiger integrals:

**Theorem 4.4.** *Let $f \vee g$ and $f \wedge g$ denote the (pointwise) maximum and minimum, respectively, of functions $f$ and $g$ in $\mathsf{Def}(\mathbb{R}^n)$. Then:*

$$\int_{\mathbb{R}^n} f \vee g \; \lfloor d\mu_k \rfloor + \int_{\mathbb{R}^n} f \wedge g \; \lfloor d\mu_k \rfloor = \int_{\mathbb{R}^n} f \; \lfloor d\mu_k \rfloor + \int_{\mathbb{R}^n} g \; \lfloor d\mu_k \rfloor \tag{4.10}$$

*and similarly for the upper integral.*

*Proof.* Since $f$ and $g$ are definable, so are the sets $\{f \geq g\}$ and $\{f < g\}$. We can partition the domain $\mathbb{R}^n$ into these two sets. The proof then amounts to rewriting and recombining the integrals:

$$\int_{\mathbb{R}^n} f \vee g \ \lfloor d\mu_k \rfloor + \int_{\mathbb{R}^n} f \wedge g \ \lfloor d\mu_k \rfloor$$
$$= \int_{\{f \geq g\}} f \ \lfloor d\mu_k \rfloor + \int_{\{f<g\}} g \ \lfloor d\mu_k \rfloor + \int_{\{f \geq g\}} g \ \lfloor d\mu_k \rfloor + \int_{\{f<g\}} f \ \lfloor d\mu_k \rfloor$$
$$= \int_{\mathbb{R}^n} f \ \lfloor d\mu_k \rfloor + \int_{\mathbb{R}^n} g \ \lfloor d\mu_k \rfloor,$$

which is equation (4.10). □

An alternate proof of Theorem 4.4 involves writing the integrals in the form of equation (4.3) and applying the inclusion-exclusion property to excursion sets $\{f \geq s\}$, $\{f < -s\}$, $\{g \geq s\}$, and $\{g < -s\}$.

**Theorem 4.5.** *For $h \in \mathsf{Def}(\mathbb{R}^n)$, we can write the Hadwiger integrals as limits of Lebesgue integrals as follows:*

$$\int_{\mathbb{R}^n} h \ \lfloor d\mu_k \rfloor = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} s \ \mu_k\{s \leq h < s + \epsilon\} \ ds, \ \textit{and} \tag{4.11}$$
$$\int_{\mathbb{R}^n} h \ \lceil d\mu_k \rceil = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} s \ \mu_k\{s < h \leq s + \epsilon\} \ ds. \tag{4.12}$$

*Proof.* We will prove the lower integral. The proof for the upper integral is analogous.

First,

$$\int_{\mathbb{R}^n} h \ \lfloor d\mu_k \rfloor = \lim_{m \to \infty} \frac{1}{m} \int_{\mathbb{R}^n} \lfloor mh \rfloor \ d\mu_k = \lim_{m \to \infty} \frac{1}{m} \sum_i i \ \mu_k \left\{ \frac{i}{m} \leq h < \frac{i}{m} + \frac{1}{m} \right\}.$$

The sum above is a finite sum since $h$ is bounded. Now let $\epsilon = 1/m$. By the o-minimal "conic theorem" [43, Thm. 9.2.3] we can rearrange the limit to obtain:

$$\lim_{m\to\infty} \frac{1}{m} \sum_i i \; \mu_k \left\{ \frac{i}{m} \le h < \frac{i}{m} + \frac{1}{m} \right\} \\ = \lim_{\epsilon\to 0^+} \lim_{m\to\infty} \frac{1}{\epsilon m} \sum_i \frac{i}{m} \; \mu_k \left\{ \frac{i}{m} \le h < \frac{i}{m} + \frac{1}{m} \right\}.$$

Letting $m \to \infty$ and recognizing the Riemann sum,

$$\int_{\mathbb{R}^n} h \; \lfloor d\mu_k \rfloor = \lim_{\epsilon\to 0^+} \lim_{m\to\infty} \frac{1}{\epsilon m} \sum_i \frac{i}{m} \; \mu_k \left\{ \frac{i}{m} \le h < \frac{i}{m} + \frac{1}{m} \right\} \\ = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} s \; \mu_k\{s \le h < s + \epsilon\} \; ds,$$

which proves equation (4.11). □

## 4.7 Product Theorem

The Product Theorem (Theorem 2.3) extends to a theorem for integrals of constructible functions, but equality does not hold for definable functions in general.

**Theorem 4.6.** *For $f \in \mathsf{CF}(\mathbb{R}^m)$, $g \in \mathsf{CF}(\mathbb{R}^n)$, and integer $0 \le k \le m+n$,*

$$\int_{\mathbb{R}^{m+n}} fg \; d\mu_k = \sum_{\ell=0}^{k} \int_{\mathbb{R}^m} f \; d\mu_\ell \int_{\mathbb{R}^n} g \; d\mu_{k-\ell} \tag{4.13}$$

*Proof.* Since $f$ and $g$ are constructible, we can write

$$f = \sum_{i=1}^{p} a_i \mathbf{1}_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{q} b_j \mathbf{1}_{B_j}$$

for some $p, q \in \mathbb{Z}$, and $a_i, b_j \in \mathbb{R}$ and definable sets $A_i$, $B_j$, for all $i$ and $j$ in the appropriate index sets.

By linearity of the Hadwiger integral of constructible functions,

$$\int_{\mathbb{R}^{m+n}} fg \; d\mu_k = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \int_{\mathbb{R}_{m+n}} \mathbf{1}_{A_i} \mathbf{1}_{B_j} \; d\mu_k.$$

By the Product Theorem for sets, we have

$$\sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \int_{\mathbb{R}_{m+n}} \mathbf{1}_{A_i} \mathbf{1}_{B_j} \; d\mu_k = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \sum_{\ell=0}^{k} \int_{\mathbb{R}_m} \mathbf{1}_{A_i} \; d\mu_\ell \int_{\mathbb{R}_m} \mathbf{1}_{B_j} \; d\mu_{k-\ell}.$$

We use linearity again to bring the sums back inside the integrals, and we have

$$\sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \sum_{\ell=0}^{k} \int_{\mathbb{R}_m} \mathbf{1}_{A_i} \; d\mu_\ell \int_{\mathbb{R}_m} \mathbf{1}_{B_j} \; d\mu_{k-\ell} = \sum_{\ell=0}^{k} \int_{\mathbb{R}^m} f \; d\mu_\ell \int_{\mathbb{R}^n} g \; d\mu_{k-\ell},$$

which proves the theorem. $\square$

We would like to say that equation (4.13) holds for definable functionals in general, but this is not so. In general, the product of two definable functions has excursion sets that are not rectangles. The Product Theorem fails on such sets, which breaks the equality of equation (4.13).

For a simple example, let $f(x) = x$ and $g(y) = y$, each defined on the interval $[0, 1]$. Then the excursion sets of $fg = xy$ are convex but not squares, and a simple numerical estimation shows that

$$\int_{[0,1]\times[0,1]} fg \; \lfloor d\mu_1 \rfloor < 0.8.$$

On the other hand,

$$\sum_{\ell=0}^{1} \int_{[0,1]} f \; \lfloor d\mu_\ell \rfloor \int_{[0,1]} g \; \lfloor d\mu_{1-\ell} \rfloor = 1.$$

Of course, equality does hold for the Euler and Lebesgue cases. For $f \in \mathsf{Def}(\mathbb{R}^m)$ and $g \in \mathsf{Def}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^{m+n}} fg \ \lfloor d\chi \rfloor = \int_{\mathbb{R}^m} f \ \lfloor d\chi \rfloor \int_{\mathbb{R}^n} g \ \lfloor d\chi \rfloor \quad \text{and}$$
$$\int_{\mathbb{R}^{m+n}} fg \ dx\, dy = \int_{\mathbb{R}^m} f \ dx \int_{\mathbb{R}^n} g \ dy.$$

The Euler result is due to index theory, and the Lebesgue result is the Fubini Theorem of calculus.

# Chapter 5

# Hadwiger's Theorem for Functionals

This chapter is the core of the thesis. With the theory of Hadwiger integration, we are now able to generalize Hadwiger's Theorem, "lifting" the theorem from sets to functionals over sets. Recall Hadwiger's Theorem from Section 2.3:

**Hadwiger's Theorem.** *Any Euclidean-invariant, continuous, additive valuation $v$ on convex subsets of $\mathbb{R}^n$ is a linear combination of the intrinsic volumes:*

$$v(A) = \sum_{k=0}^{n} c_k \mu_k(A)$$

*for some constants $c_k \in \mathbb{R}$. If $v$ is homogeneous of degree $k$, then $v = c_k \mu_k$.*

Note that the valuations classified in Hadwiger's Theorem are only continuous on the class of convex subsets. As previously mentioned, the intrinsic volumes are not continuous for definable sets in general with respect to the Hausdorff metric. Thus, we first define appropriate topologies on $\mathsf{Def}(\mathbb{R}^n)$, and then we offer a classification theorem for valuations on functionals.

## 5.1 General Valuations on Functionals

**Definition 5.1.** A *valuation* on $\mathsf{Def}(\mathbb{R}^n)$ is an *additive* map $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$. The additive condition means that $v(f \vee g) + v(f \wedge g) = v(f) + v(g)$ for any $f, g \in \mathsf{Def}(\mathbb{R}^n)$, with $\vee$ and $\wedge$ denoting the (pointwise) maximum and minimum, respectively, of $f$ and $g$. So that the valuation is independent of the support of a function, we require that $v(\mathbf{0}) = 0$, where $\mathbf{0}$ is the zero function.

Valuation $v$ is *Euclidean invariant* if $v(f) = v(f \circ \phi)$ for any $f \in \mathsf{Def}(\mathbb{R}^n)$ and any Euclidean motion $\phi$ on $\mathbb{R}^n$. We will define topologies on $\mathsf{Def}(\mathbb{R}^n)$ so that the valuation can be *continuous* as a map between topological spaces, with the standard topology on $\mathbb{R}$.

The additivity condition can be alternately stated as follows:

**Proposition 5.1.** *Let $v$ be an additive valuation on $\mathsf{Def}(\mathbb{R}^n)$, so*

$$v(f \vee g) + v(f \wedge g) = v(f) + v(g), \tag{5.1}$$

*and $v(0) = 0$. This is the case if and only if*

$$v(f) = v(f \cdot \mathbf{1}_A) + v(f \cdot \mathbf{1}_{A^c}) \tag{5.2}$$

*for all definable subsets $A$ of $\mathbb{R}^n$, where $A^c$ denotes the complement of $A$.*

*Proof.* If $v$ satisfies equation (5.1), then for any definable function $f : \mathbb{R}^n \to \mathbb{R}^+$ and any definable set $A$,

$$v(f \cdot \mathbf{1}_A) + v(f \cdot \mathbf{1}_{A^c}) = v(f \cdot \mathbf{1}_A \vee f \cdot \mathbf{1}_{A^c}) + v(f \cdot \mathbf{1}_A \wedge f \cdot \mathbf{1}_{A^c}) = v(f) + v(0) = v(f).$$

Thus, $v$ satisfies equation (5.2).

To prove the other direction, assume $v$ satisfies equation (5.2). Let $f, g \in \mathsf{Def}(\mathbb{R}^n)$. Let $A = \{x \in \mathbb{R}^n \mid f(x) \geq g(x)\}$. Since $f$ and $g$ are definable functions, $A$ is a definable set. Then,

$$\begin{aligned}
v(f) + v(g) &= v(f \cdot \mathbf{1}_A) + v(f \cdot \mathbf{1}_{A^c}) + v(g \cdot \mathbf{1}_A) + v(g \cdot \mathbf{1}_{A^c}) \\
&= v((f \vee g) \cdot \mathbf{1}_A) + v((f \wedge g) \cdot \mathbf{1}_{A^c}) + v((f \wedge g) \cdot \mathbf{1}_A) + v((f \vee g) \cdot \mathbf{1}_{A^c}) \\
&= v(f \vee g) + v(f \wedge g),
\end{aligned}$$

so $v$ also satisfies equation (5.1). □

By induction on Proposition 5.1, an additive valuation $v$ has the property that

$$v(f) = \sum_i v(f \cdot \mathbf{1}_{A_i}),$$

where $\{A_i\}_{i \in \mathbb{Z}}$ is any finite collection of disjoint definable subsets of $\mathbb{R}^n$ whose union is $\mathbb{R}^n$.

In order to discuss continuous valuations, we need an appropriate topology on $\mathsf{Def}(\mathbb{R}^n)$. This topology ought to have the property that any open set containing $r \cdot \mathbf{1}_A$ contains $(r + \epsilon) \cdot \mathbf{1}_A$ for small enough $\epsilon$. Also, if definable sets $A$ and $B$ are close in the flat topology, then $v(r \cdot \mathbf{1}_A)$ should be close to $v(r \cdot \mathbf{1}_B)$ for any continuous valuation $v$. With such a topology, the notion of a continuous valuation on $\mathsf{Def}(\mathbb{R}^n)$ properly extends the notion of a continuous valuation on definable subsets of $\mathbb{R}^n$.

We present two useful topologies with these properties. Recall that $|\cdot|_\flat$ denotes flat norm on currents, defined in equation (3.1).

**Definition 5.2.** Let $f, g, \in \mathsf{Def}(\mathbb{R}^n)$. The *lower* and *upper flat metrics* on definable functions, denoted $\underline{d}_\flat$ and $\overline{d}_\flat$, respectively, are defined as follows:

$$\underline{d}_\flat(f,g) = \int_{-\infty}^{\infty} \left|\mathbf{C}^{\{f\geq s\}} - \mathbf{C}^{\{g\geq s\}}\right|_\flat \, ds \quad \text{and} \tag{5.3}$$

$$\overline{d}_\flat(f,g) = \int_{-\infty}^{\infty} \left|\mathbf{C}^{\{f> s\}} - \mathbf{C}^{\{g> s\}}\right|_\flat \, ds. \tag{5.4}$$

The topologies induced by the lower and upper flat metrics are the *lower* and *upper flat topologies* on definable functions.

Note that the integrals in equations (5.3) and (5.4) may also be written with finite bounds, as it suffices to integrate between the minimum and maximum values of $f$ and $g$. These metrics extend the flat metric on definable sets, for they reduce to equation (3.4) when $f$ and $g$ are characteristic functions.

**Theorem 5.1.** *The lower and upper Hadwiger integrals are continuous in the lower and upper flat topology on definable functions, respectively.*

*Proof.* The proof follows from Theorem 3.2. Let $f, g \in \mathsf{Def}(\mathbb{R}^n)$, supported on $X \subset \mathbb{R}^n$.

For lower Hadwiger integrals:

$$\left|\int f\lfloor d\mu_k\rfloor - \int g\lfloor d\mu_k\rfloor\right| = \left|\int_0^\infty (\mu_k\{f\geq s\} - \mu_k\{g\geq s\})\ ds\right|$$
$$\leq \int_0^\infty \left|\mathbf{C}^{\{f\geq s\}} - \mathbf{C}^{\{g\geq s\}}\right|_\flat \cdot \max\left\{\sup_{x\in X}|\mathcal{W}_{n,k}(x)|, \sup_{x\in X}|d\mathcal{W}_{n,k}(x)|\right\}\ ds$$
$$= \underline{d}_\flat(f,g)\cdot \max\left\{\sup_{x\in X}|\mathcal{W}_{n,k}(x)|, \sup_{x\in X}|d\mathcal{W}_{n,k}(x)|\right\}$$

The inequality above is due to equation (3.6). By finiteness (o-minimality), $\mathcal{W}_{n,k}(x)$ and $d\mathcal{W}_{n,k}(x)$ are bounded for $x \in X$. Thus, if $f$ and $g$ are close in the lower flat topology, then their lower Hadwiger integrals are also close.

The proof for the upper integrals is analogous. □

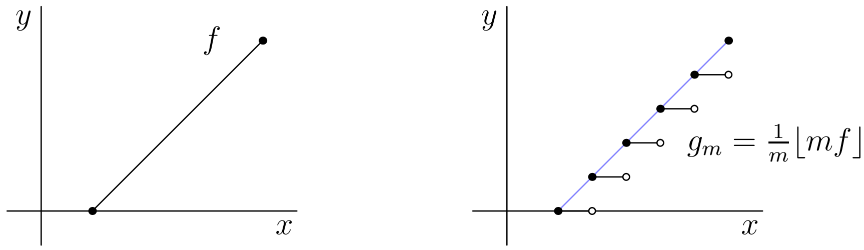
It would be convenient for classifying valuations if the lower and upper flat topologies were in fact the same. Unfortunately, this is not the case.

**Theorem 5.2.** *The lower and upper flat topologies on definable functions are different topologies.*

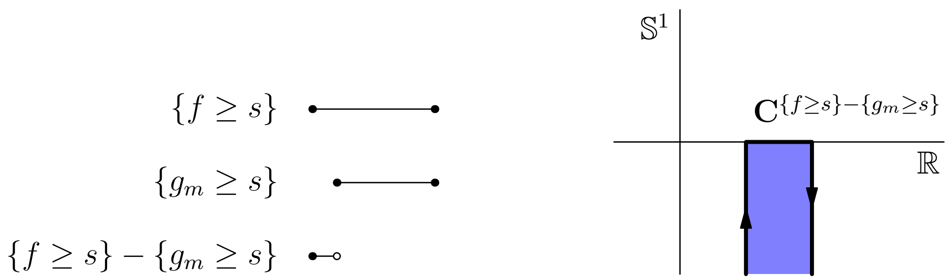
*Proof.* It suffices to find a sequence of functions that converge to a different limit in the lower and upper flat topologies.

Consider the linear function $f : \mathbb{R} \to \mathbb{R}$, linear on a closed interval, as depicted in Figure 5.1(a). For $m > 0$, let $g_m = \frac{1}{m}\lfloor mf\rfloor$, the lower step function of $f$ with step size $\frac{1}{m}$. As $m \to \infty$, $g_m \to f$ in the lower flat topology. This is because, for general $s > 0$, the difference in upper excursion sets $\{f \geq s\}$ and $\{g_m \geq s\}$ is a half-open interval, as illustrated in Figure 5.1(b). As $m \to \infty$, the length of this interval decreases to

(a) A function $f : \mathbb{R} \to \mathbb{R}$ and a sample lower step function $g_m$:




(b) Sample upper excursion sets of $f$ and $g_m$, and the conormal cycle of their difference:




(c) Sample *strict* upper excursion sets of $f$ and $g_m$, and the conormal cycle of their difference:

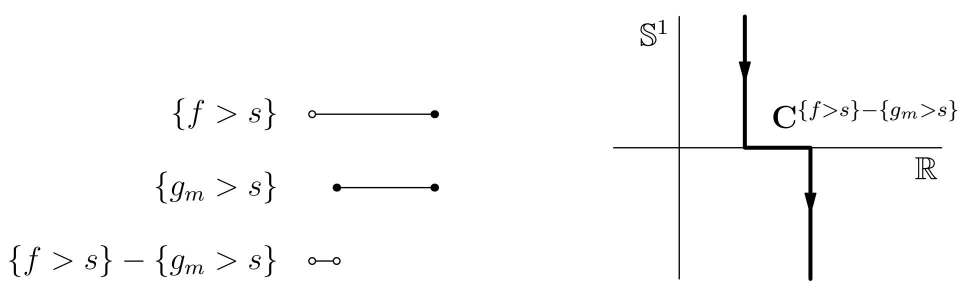



**Figure 5.1:** Illustrations of functions, excursion sets, and conormal cycles for the proof of Theorem 5.2.

zero. The current $\mathbf{C}^{\{f \geq s\} - \{g_m \geq s\}}$, represented by the dark path in Figure 5.1(b), is bounded in flat norm by a constant multiple of the area of the blue region. That is, $\left|\mathbf{C}^{\{f \geq s\} - \{g_m \geq s\}}\right|_\flat \leq cm$ for some constant $c$. Therefore,

$$\lim_{m \to \infty} \underline{d}_\flat(f, g_m) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left|\mathbf{C}^{\{f \geq s\} - \{g_m \geq s\}}\right|_\flat \, ds \leq \lim_{m \to \infty} cm = 0,$$

so $g_m$ converges to $f$ in the lower flat topology.

However, the sequence $g_m$ does *not* converge to $f$ in the upper flat topology For general $s > 0$, the difference in *strict* excursion sets $\{f > s\}$ and $\{g_m > s\}$ is a closed interval, illustrated in Figure 5.1(c). The flat norm of the current $\mathbf{C}^{\{f > s\} - \{g_m > s\}}$ is bounded from below by the length of $\mathbb{S}^1$, which implies that $\overline{d}_\flat(f, g_m)$ does not approach zero.

Dually, the sequence of upper step functions $h_m = \frac{1}{m}\lceil mf \rceil$ converges to $f$ in the upper flat topology, but not in the lower flat topology. The reasoning is analogous to that for the lower step functions, with similar pictures to those in Figure 5.1. $\square$

For functions $f$, $g_m$, and $h_m$ as in the proof of Theorem 5.2, the lower Euler integrals of $g_m$ and $f$ agree, but the upper Euler integrals do not. In general, upper Hadwiger integrals are *not* continuous in the lower flat topology. Likewise, the upper Euler integrals of $h_m$ and $f$ agree, but the lower Euler integrals do not, for lower Hadwiger integrals are generally *not* continuous in the upper flat topology.

The two topologies provide a means of classifying general valuations on functionals, by the following definition.

**Definition 5.3.** Let $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation. We say $v$ is a *lower valuation* if $v$ is continuous in the lower flat topology. Likewise, we say $v$ is an *upper valuation* if $v$ is continuous in the upper flat topology.

Not surprisingly, lower and upper Hadwiger integrals are lower and upper valuations, respectively (Theorem 5.1). Lebesgue integrals are both lower and upper valuations. Indeed, we will see in Section 5.2 that Lebesgue integrals are the only valuations that are both lower and upper.

## 5.2 Classification of Valuations

We now turn to the problem of classifying an arbitrary valuation $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ in terms of Hadwiger integrals. For constructible functions, the classification is a straightforward application of Hadwiger's Theorem.

**Lemma 5.1.** *If $v : \mathsf{CF}(\mathbb{R}^n) \to \mathbb{R}$ is a valuation on constructible functions, then $v$ is a linear combination of constructible Hadwiger integrals. That is, for $h \in \mathsf{CF}(\mathbb{R}^n)$,*

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h)\ d\mu_k.$$

*for some coefficient functions $c_k : \mathbb{R} \to \mathbb{R}$ with $c_k(0) = 0$.*

*Proof.* For a characteristic function, the situation is simple. Let $h = r \cdot \mathbf{1}_A$ for $r \in \mathbb{Z}$ and a definable subset $A$ of $\mathbb{R}^n$. Hadwiger's Theorem for sets implies that

$$v(r \cdot \mathbf{1}_A) = \sum_{k=0}^{n} c_k(r)\mu_k(A), \tag{5.5}$$

where $c_k(r)$ are constants that depend only on $v$, not on $A$.

Now suppose $h$ is a finite sum of characteristic functions of disjoint definable subsets $A_1, \ldots, A_m$ of $\mathbb{R}^n$:

$$h = \sum_{i=1}^{m} r_i \mathbf{1}_{A_i}$$

for some integer constants $r_1 < r_2 < \cdots < r_m$. By equation (5.5) and additivity,

$$v(h) = \sum_{k=0}^{n} \sum_{i=1}^{m} c_k(r_i) \mu_k(A_i). \tag{5.6}$$

We can rewrite equation (5.6) in terms of excursion sets of $h$. Let $B_i = \bigcup_{j \geq i} A_j$. That is, $B_i = \{h \geq r_i\}$ and $B_i = \{h > r_{i-1}\}$. Then the valuation $v(h)$ can be expressed as:

$$v(h) = \sum_{k=0}^{n} \sum_{i=1}^{m} \left(c_k(r_i) - c_k(r_{i-1})\right) \mu_k(B_i), \tag{5.7}$$

where $c_k(r_0) = 0$. Thus, a valuation of a constructible function can be expressed as a sum of finite differences of valuations of its excursion sets. Equivalently, equation (5.7) can be written in terms of constructible Hadwiger integrals:

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h) \; d\mu_k. \tag{5.8}$$

Since we require that a valuation of the zero function is zero, it must be that $c_k(0) = 0$ for all $k$. $\square$

In fact, Lemma 5.1 holds for functions of the form $h = \sum_{i=1}^{m} r_i \mathbf{1}_{A_i}$ where the $r_i \in \mathbb{R}$ are not necessarily integers and the $A_i$ are definable sets.

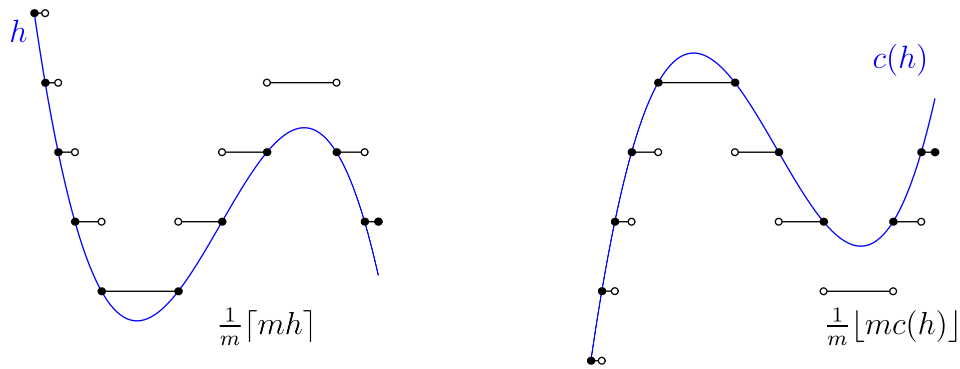



**Figure 5.2:** An upper step function of $h$, depicted at left, composed with a decreasing function $c$, becomes a lower step function of $c(h)$, depicted at right. As the step size approaches zero, we obtain Proposition 5.2.

In writing an arbitrary valuation on definable functionals as a sum of Hadwiger integrals, the situation becomes complicated if the coefficient functions $c_k$ are decreasing on any interval. The following proposition illustrates the difficulty:

**Proposition 5.2.** *Let $c : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly decreasing function. Then,*

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} c\left(\frac{1}{m}\lceil mh \rceil\right) \, d\mu_k = \lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{1}{m}\lfloor mc(h) \rfloor \, d\mu_k. \tag{5.9}$$

*Proof.* The intuition is that both sides of the equality are the same limits of step functions, as illustrated in Figure 5.2.

On the left side of equation (5.9), we integrate $c$ composed with upper step functions of $h$:

$$\int_{\mathbb{R}^n} c\left(\frac{1}{m}\lceil mh \rceil\right) \, d\mu_k = \sum_{i\in\mathbb{Z}} c\left(\tfrac{i}{m}\right) \cdot \mu_k \left\{\tfrac{i-1}{m} < h \leq \tfrac{i}{m}\right\}$$

On the right side of equation (5.9), we integrate lower step functions of the composition $c(h)$:

$$\int_{\mathbb{R}^n} \frac{1}{m} \lfloor mc(h) \rfloor \, d\mu_k = \sum_{t \in \mathbb{Z}} \tfrac{t}{m} \cdot \mu_k \left\{ \tfrac{t}{m} \leq c(h) < \tfrac{t+1}{m} \right\}$$

Since $c$ is strictly decreasing, $c^{-1}$ exists. There exists a discrete set

$$\mathcal{S} = \left\{ c^{-1}\left(\tfrac{t}{m}\right) \;\middle|\; t \in \mathbb{Z} \right\} \cap \{\text{neighborhood around range of } h\}.$$

We can then rewrite the above sum as:

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{1}{m} \lfloor mc(h) \rfloor \, d\mu_k = \sum_{s \in \mathcal{S}} c(s) \cdot \mu_k \{ c(s) \leq c(h) < c(s - \epsilon) \} \\ = \sum_{s \in \mathcal{S}} c(s) \cdot \mu_k \{ s - \epsilon < h \leq s \}, \end{aligned}$$

where $\epsilon \to 0$ as $m \to \infty$ by continuity of $c$.

In the limit, both sides are equal:

$$\lim_{\epsilon \to 0} \sum_{s \in \mathcal{S}} c(s) \cdot \mu_k \{ s - \epsilon < h \leq s \} \} = \lim_{m \to \infty} \sum_{i \in \mathbb{Z}} c\left(\tfrac{i}{m}\right) \cdot \mu_k \left\{ \tfrac{i-1}{m} < h \leq \tfrac{i}{m} \right\}$$

which proves Proposition 5.2. □

Proposition 5.2 implies that if $c : \mathbb{R} \to \mathbb{R}$ is increasing on some interval and decreasing on another, then the maps $v, u : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ defined

$$v(h) = \int_{\mathbb{R}^n} c(h) \lfloor d\mu_k \rfloor \quad \text{and} \quad u(h) = \int_{\mathbb{R}^n} c(h) \lceil d\mu_k \rceil$$

are not continuous in either the lower or the upper flat topology.

Lemma 5.1 and Proposition 5.2 allow us further to generalize Hadwiger's Theorem to express lower and upper valuations in terms of Hadwiger integrals.

**Theorem 5.3.** *Any lower valuation $v$ on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of lower Hadwiger integrals. For $h \in \mathsf{Def}(\mathbb{R}^n)$,*

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h) \ \lfloor d\mu_k \rfloor, \tag{5.10}$$

*where the $c_k : \mathbb{R} \to \mathbb{R}$ are increasing functions with $c_k(0) = 0$.*

*Likewise, an upper valuation $v$ on $\mathsf{Def}(\mathbb{R}^n)$ can be written as a linear combination of upper Hadwiger integrals.*

*Proof.* Let $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ be a lower valuation, and $h \in \mathsf{Def}(\mathbb{R}^n)$.

First approximate $h$ by lower step functions. That is, for $m > 0$, let $h_m = \frac{1}{m}\lfloor mh \rfloor$. In the lower flat topology, $\lim_{m\to\infty} h_m = h$.

On each of these step functions, Lemma 5.1 implies that $v$ is a linear combination of Hadwiger integrals:

$$v(h_m) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h_m) \ d\mu_k. \tag{5.11}$$

for some $c_k : \mathbb{R} \to \mathbb{R}$ with $c_k(0) = 0$, depending only on $v$ and not on $m$. By Proposition 5.2, the $c_k$ must be increasing functions since we are approximating $h$ with lower step functions in the lower flat topology.

We can alternately express equation (5.11) as

$$v(h_m) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h_m) \ \lfloor d\mu_k \rfloor, \tag{5.12}$$

where we choose lower rather than upper integrals since $v$ is continuous in the lower flat topology. Continuity of $v$, and convergence of $h_m$ to $h$, in the lower flat topology imply that $v(h_m)$ converges to $v(h)$ as $h \to \infty$. More specifically,

$$v(h) = \lim_{m\to\infty} v\left(h_m\right) = \sum_{k=0}^{n} \lim_{m\to\infty} \int_{\mathbb{R}^n} c_k\left(h_m\right)\ \lfloor d\mu_k \rfloor. \tag{5.13}$$

By continuity of the lower Hadwiger integrals and the $c_k$, equation (5.13) becomes

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k\left(\lim_{m\to\infty} h_m\right)\ \lfloor d\mu_k \rfloor = \sum_{k=0}^{n} \int_{\mathbb{R}^n} c_k(h)\ \lfloor d\mu_k \rfloor. \tag{5.14}$$

Thus, $v(h)$ is a linear combination of lower Hadwiger integrals.

The proof for the upper valuation is analogous. □

We can now prove a statement from Section 5.1, that only Lebesgue integrals are both lower and upper valuations.

**Corollary 5.1.** *If $v : \mathsf{Def}(\mathbb{R}^n) \to \mathbb{R}$ is both a lower valuation and an upper valuation, then $v$ is Lebesgue integration.*

*Proof.* Since $v$ is both a lower and upper valuation, we have

$$v(h) = \sum_{k=0}^{n} \int_{\mathbb{R}^n} \underline{c}_k(h)\ \lfloor d\mu_k \rfloor = \sum_{k=0}^{n} \int_{\mathbb{R}^n} \overline{c}_k(h)\ \lceil d\mu_k \rceil$$

for some functions $\underline{c}_k$ and $\overline{c}_k$.

Let $A_0$ be a point. By evaluating $v$ on test functions of the form $h = r \cdot \mathbf{1}_{A_0}$, we find that $\underline{c}_0(r) = \overline{c}_0(r)$ for any $r$, and thus $\underline{c}_0 = \overline{c}_0$. Now let $A_1$ be a line segment. Evaluating $v$ on test functions $h = r \cdot \mathbf{1}_{A_1}$, we find that

$$\underline{c}_0(r)\mu_0(A_1) + \underline{c}_1(r)\mu_1(A_1) = \overline{c}_0(r)\mu_0(A_1) + \overline{c}_1(r)\mu_1(A_1).$$

Since $\underline{c}_0 = \overline{c}_0$, it follows that $\underline{c}_1 = \overline{c}_1$. By induction on $k$, we have $\underline{c}_k = \overline{c}_k$ for all $k = 0, 1, \ldots, n$.

From Section 4.6, we know that lower and upper Hadwiger integrals with respect to $\mu_k$ are not the same on $\mathsf{Def}(\mathbb{R}^n)$ for $k = 0, 1, \ldots, n-1$. This implies that $\underline{c}_k = \overline{c}_k = 0$ for $k = 0, 1, \ldots, n-1$. Since all excursion sets of functions $h \in \mathsf{Def}(\mathbb{R}^n)$ are no larger than $n$-dimensional, the lower and upper Hadwiger integrals with respect to $\mu_n$ are in fact Lebesgue integrals, and so they are equal.

Therefore,

$$v(h) = \int_{\mathbb{R}^n} c(h)\ d\mathcal{L}$$

for some continuous function $c : \mathbb{R} \to \mathbb{R}$, and with $d\mathcal{L} = \lfloor d\mu_n \rfloor = \lceil d\mu_n \rceil$ denoting Lebesgue measure. $\square$

Thus, we have a dual generalization of Hadwiger's Theorem, classifying lower and upper valuations in terms of lower and upper Hadwiger integrals. It remains to be seen if, perhaps, there is a topology on $\mathsf{Def}(\mathbb{R}^n)$ that would allow us to combine the dual statements of Theorem 5.3. In particular, we would like a topology that allows any Euclidean-invariant valuation $v$ on $\mathsf{Def}(\mathbb{R}^n)$ to be written in the form

$$v(h) = \sum_{k=0}^{n} \left( \int_{\mathbb{R}^n} \underline{c}_k(h)\ \lfloor d\mu_k \rfloor + \int_{\mathbb{R}^n} \overline{c}_k(h)\ \lceil d\mu_k \rceil \right)$$

for some continuous functions $\underline{c}_k, \overline{c}_k : \mathbb{R} \to \mathbb{R}$. The union of lower and upper flat topologies does not seem to be a reasonable choice, because it enlarges the set of continuous valuations on $\mathsf{Def}(\mathbb{R}^n)$ by too much.

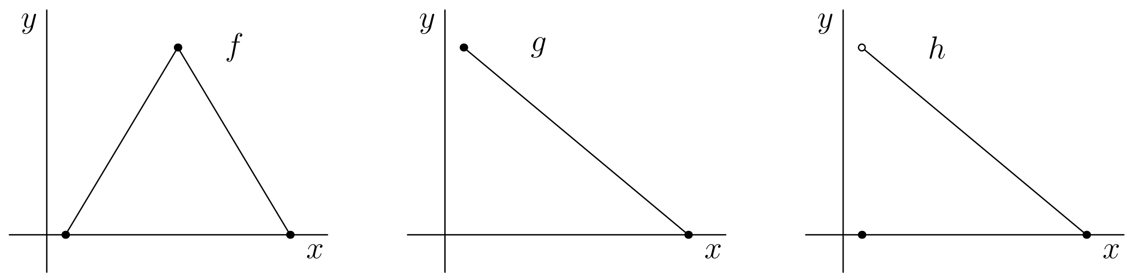



**Figure 5.3:** Functions $f$ and $g$ have congruent upper excursion sets $\{f \geq s\}$ and $\{g \geq s\}$, but *not* congruent strict excursion sets $\{f > s\}$ and $\{g > s\}$. Functions $f$ and $h$ have congruent strict excursion sets, but not congruent excursion sets.

## 5.3 Cavalieri's Principle

Euclidean invariance implies a sort of Cavalieri's principle for valuations: a valuation $v$ cannot distinguish between two functions that have congruent excursion sets at each height $s$ in their range. Functions $f$ and $g$ have congruent upper excursion sets if for any $s \in \mathbb{R}$, $\{f \geq s\} \cong \{g \geq s\}$ as subsets of the domain. Also, $f$ and $g$ have congruent *strict* upper excursion sets if $\{f > s\} \cong \{g > s\}$. Note that $f$ and $g$ may have congruent excursion sets without having congruent strict excursion sets, and vice-versa, as illustrated in Figure 5.3.

**Proposition 5.3** (Cavalieri's Principle)**.** *Let $v$ be a lower valuation on $\mathsf{Def}(\mathbb{R}^n)$, let $u$ be an upper valuation on $\mathsf{Def}(\mathbb{R}^n)$, and let $f, g \in \mathsf{Def}(\mathbb{R}^n)$. If $f$ and $g$ have congruent upper excursion sets $\{f \geq s\} \cong \{g \geq s\}$ for all $s \in \mathbb{R}$, then $v(f) = v(g)$. Likewise, if $f$ and $g$ have congruent strict upper excursion sets $\{f > s\} \cong \{g > s\}$ for all $s \in \mathbb{R}$, then $u(f) = u(g)$.*

*Proof.* The proposition follows directly from Theorem 5.3 and the fact that the lower and upper Hadwiger integrals can be expressed in terms of excursion sets, as in equation (4.3). $\square$

# Chapter 6

# Integral Transforms

Applications of integration often make use of integral transforms such as convolution and the Bessel (or Hankel) and Fourier transforms. Convolution with respect to Euler integration has intriguing connections to the Steiner formula. Ghrist and Robinson have examined topological versions of the Bessel and Fourier transforms in [22]. These transforms can be extended to Hadwiger integrals, and should prove useful in signal processing and other applications.

## 6.1 The Steiner Formula and Convolution

For subsets $K$ and $J$ of $\mathbb{R}^n$, the Minkowski sum (or vector sum) is the set

$$K + J = \{x + y \mid x \in K, y \in J\}.$$

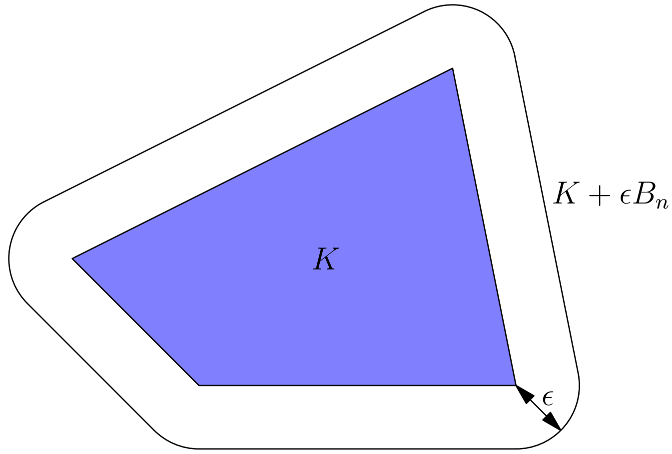



**Figure 6.1:** The $\epsilon$-tube $K + \epsilon B_n$ around a compact convex subset $K$.

If $K$ is closed and $J = B_n$, the closed $n$-dimensional unit ball, then the Minkowski sum $K + B_n$ consists of all the points whose distance from $K$ is not greater than $\epsilon$:

$$K + \epsilon B_n = \{x \mid d(x, K) \leq \epsilon\},$$

which is also known as the $\epsilon$-tube around $K$. An $\epsilon$-tube around a compact convex subset $K$ is illustrated in Figure 6.1

The Steiner Formula is commonly used to express the volume of an $\epsilon$-tube around a compact convex subset as a polynomial in $\epsilon$, whose coefficients involve the intrinsic volumes [32, 35, 36, 40, 41].

**Theorem 6.1** (Steiner Formula)**.** *For compact convex $K \subset \mathbb{R}^n$ and $\epsilon > 0$,*

$$\mu_n(K + \epsilon B_n) = \sum_{j=0}^{n} \epsilon^{n-j} \omega_{n-j} \mu_j(K).$$

When written in terms of characteristic functions, the Steiner Formula is really a statement about convolution. Though denoted with the customary $*$ symbol, here


**Figure 6.2:** Let $K$ be the half-open segment at left. The convolution function $\mathbf{1}_K * \mathbf{1}_{B_2}$ has value one on the (non-compact) blue region and zero elsewhere. It is *not* the characteristic function of the Minkowski sum $K + B_2$, which is depicted in green at right.

we take the convolution integral to be an Euler integral:

**Definition 6.1.** The *Euler convolution* of $f, g \in \mathsf{CF}(\mathbb{R}^n)$ is denoted $f * g$ and is defined

$$(f * g)(x) = \int f(y)g(x - y)\ d\chi(y). \tag{6.1}$$

If we convolve the characteristic function of a compact convex set $K \subset \mathbb{R}^n$ with the characteristic function of the closed $n$-ball of radius $\epsilon$, we obtain the characteristic function of the $\epsilon$-tube about $K$:

$$(\mathbf{1}_K * \mathbf{1}_{\epsilon B_n})(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y)\mathbf{1}_{\epsilon B_n}(x - y)\ d\chi(y) = \mathbf{1}_{K+\epsilon B_n}(x). \tag{6.2}$$

If $K$ is not compact, then equation (6.2) might not hold. Consider the half-open segment depicted in Figure 6.2. If $K$ is this segment, then the convolution $\mathbf{1}_K * \mathbf{1}_{B_2}$ is not $\mathbf{1}_{K+B_2}$.

Alternately, if $K$ is not convex, then equation (6.2) might not hold. For example, if $K$ is a set of two points and $d$ is the distance between the points, then for any

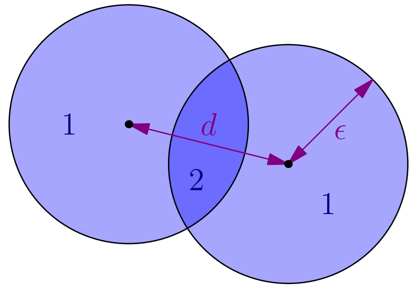



**Figure 6.3:** If $K$ consists of two points, distance $d$ apart, and $\epsilon > \frac{d}{2}$, then the convolution $\mathbf{1}_K * \mathbf{1}_{\epsilon B_2}$, depicted in blue, is *not* the characteristic function of any set, for it attains the value two.

$\epsilon > \frac{d}{2}$ the convolution $\mathbf{1}_K * \mathbf{1}_{\epsilon B_n}$ is not a characteristic function at all, for it attains the value 2, as shown in Figure 6.3.

The concept of *reach* is helpful for understanding this phenomenon. Put simply, the *reach* of a subset $K$ is the supremum of all distances $r$ such that every point in the tube around $A$ of radius $r$ has a unique orthogonal projection onto $A$ [15, 32]. The reach of a convex set is infinity since every point has a unique projection onto a convex set. The reach of the two-point set $K$ of Figure 6.3 is $\frac{d}{2}$. See Figure 6.4 for an example of a set with positive reach and a set with no reach.

Now we can formally connect convolution of characteristic functions and the Minkowski sum for a compact set $K$ of positive reach and the closed $\epsilon$-ball.

**Proposition 6.1.** *Let $K \subset \mathbb{R}^n$ be a subset with reach $r$. For any $0 \le \epsilon < r$, we have*

$$(\mathbf{1}_K * \mathbf{1}_{\epsilon B_n})(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y)\mathbf{1}_{\epsilon B_n}(x-y)\ d\chi(y) = \mathbf{1}_{K+\epsilon B_n}(x). \tag{6.3}$$


**Figure 6.4:** The set $J$ has reach $r$ (the radius of the curve) since every point within $r$ of $J$ has a unique orthogonal projection onto $J$. The set $K$ has reach 0 because points arbitrarily close to $K$ (on the blue dotted line) have no unique orthogonal projection onto $K$.

*Proof.* Each of the three functions of equation (6.3) evaluates to 1 if $x$ is within $\epsilon$ of $K$, and 0 otherwise. □

The Steiner Formula holds for closed subsets of positive reach, as long as the reach $r$ of the subset is greater than the radius $\epsilon$ of the tube [15, 32]. We can generalize the Steiner Formula to express $\mu_k(K + \epsilon B_n)$ in terms of $\mu_0(K), \ldots, \mu_k(K)$, as follows [35].

**Proposition 6.2.** *For $K \subset \mathbb{R}^n$ of reach $r$, $0 \leq \epsilon < r$, and integer $0 \leq k \leq n$,*

$$\mu_k(K + \epsilon B_n) = \sum_{j=0}^{k} \binom{n-j}{n-k} \frac{\omega_{n-j}}{\omega_{n-k}} \epsilon^{k-j} \mu_j(K). \tag{6.4}$$

*Proof.* Let $0 < \rho < r - \epsilon$. The key observation is that

$$K + (\epsilon + \rho)B_n = (K + \epsilon B_n) + \rho B_n.$$

We use the Steiner Formula to write the volume of the $(\epsilon + \rho)$-tube around $K$ in two equivalent ways:

$$\mu_n((K+\epsilon B_n)+\rho B_n) = \sum_{i=0}^{n} \rho^{n-i}\omega_{n-i}\mu_i(K+\epsilon B_n) \tag{6.5}$$

$$\mu_n(K+(\epsilon+\rho)B_n) = \sum_{j=0}^{n}(\epsilon+\rho)^{n-j}\omega_{n-j}\mu_j(K) \tag{6.6}$$

Matching the coefficients of $\rho$ in equations (6.5) and (6.6), we obtain

$$\omega_{n-i}\mu_i(K+\epsilon B_n) = \sum_{j=0}^{i} \binom{n-j}{n-i}\omega_{n-j}\epsilon^{i-j}\mu_j(K),$$

which is equation (6.4). □

Written in terms of convolution, for $K \subset \mathbb{R}^n$ of reach $r$, and $0 \le \epsilon < r$, the Steiner Formula becomes:

$$\int_{\mathbb{R}^n} \mathbf{1}_K * \mathbf{1}_{\epsilon B_n} \, d\mu_n = \sum_{j=0}^{n} \epsilon^{n-j}\omega_{n-j} \int_{\mathbb{R}^n} \mathbf{1}_K \, d\mu_j, \tag{6.7}$$

or more generally,

$$\int_{\mathbb{R}^n} \mathbf{1}_K * \mathbf{1}_{\epsilon B_n} \, d\mu_k = \sum_{j=0}^{k} \epsilon^{k-j} \binom{n-j}{n-k} \frac{\omega_{n-j}}{\omega_{n-k}} \int_{\mathbb{R}^n} \mathbf{1}_K \, d\mu_j. \tag{6.8}$$

Proposition 6.1 and the Steiner Formula extend naturally to some constructible functions, by linearity of the integrals.

**Proposition 6.3.** *Suppose $f \in \mathsf{CF}(\mathbb{R}^n)$ is such that $f = \sum_i c_i \mathbf{1}_{A_i}$ for some compact sets $A_i$, uniformly with reach at least some $r > 0$. Then for $0 \le \epsilon < r$ and integer $0 \le k \le n$,*

$$f * \mathbf{1}_{\epsilon B_n} = \sum_i c_i \mathbf{1}_{A_i+\epsilon B_n}, \quad \textit{and} \tag{6.9}$$

$$\int_{\mathbb{R}^n} f * \mathbf{1}_{\epsilon B_n} \, d\mu_k = \sum_{j=0}^{k} \epsilon^{k-j} \binom{n-j}{n-k} \frac{\omega_{n-j}}{\omega_{n-k}} \sum_i c_i \int_{\mathbb{R}^n} \mathbf{1}_{A_i} \, d\mu_j. \tag{6.10}$$

*Proof.* Since $f = \sum_i c_i \mathbf{1}_{A_i}$ is constructible, the sum over $i$ is a finite sum. Equation (6.9) then follows from equation (6.3) applied to each level set $A_i$:

$$f * \mathbf{1}_{\epsilon B_n} = \sum_i c_i \mathbf{1}_{A_i} * \mathbf{1}_{\epsilon B_n} = \sum_i c_i \int_{\mathbb{R}^n} \mathbf{1}_{A_i} \mathbf{1}_{\epsilon B_n} \, d\chi = \sum_i c_i \mathbf{1}_{A_i + \epsilon B_n}.$$

Likewise, from equation (6.1) and linearity of the integral:

$$\begin{aligned}\int_{\mathbb{R}^n} f * \mathbf{1}_{\epsilon B_n} \, d\mu_k &= \sum_i c_i \int_{\mathbb{R}^n} \mathbf{1}_{A_i} * \mathbf{1}_{\epsilon B_n} \, d\mu_k \\ &= \sum_{j=0}^{k} \epsilon^{k-j} \binom{n-j}{n-k} \frac{\omega_{n-j}}{\omega_{n-k}} \sum_i c_i \int_{\mathbb{R}^n} \mathbf{1}_{A_i} \, d\mu_j,\end{aligned}$$

which is equation (6.10). □

Since Euler characteristic is integer-valued, the Euler convolution of two constructible functions is another constructible function. Bröcker shows that with convolution as a product and addition as usual, the constructible functions form a commutative ring with unit [9]. The unit is the characteristic function of the origin, $\mathbf{1}_0$.

We desire to extend convolution results to definable functions. Indeed, we may convolve two definable functions, with the convolution integral as any of the definable Hadwiger integrals.

**Definition 6.2.** The *lower* and *upper Hadwiger convolution* of $f, g \in \mathsf{Def}(\mathbb{R}^n)$ are

$$\begin{aligned}(f \underset{k}{*} g)(x) &= \int f(y) g(x-y) \, \lfloor d\mu_k(y) \rfloor \quad \text{and} \\ (f \overset{k}{*} g)(x) &= \int f(y) g(x-y) \, \lceil d\mu_k(y) \rceil.\end{aligned}$$

Note that when $k = 0$ and $f, g \in \mathsf{CF}(\mathbb{R}^n)$, the convolutions in Definition 6.2 reduce to the Euler convolution previously discussed. The following theorem provides some insight into Euler convolution of definable functions. See Figure 6.5 for an illustration.

**Theorem 6.2.** *Let $r > 0$ and $f \in \mathsf{Def}(\mathbb{R}^n)$ be a nonnegative function such that each upper excursion set is compact and has reach at least $r$. Then for $0 < \epsilon < r$, each upper excursion set of $f \underset{0}{*} \mathbf{1}_{\epsilon B_n}$ is the $\epsilon$-tube of the corresponding excursion set of $f$.*

*Proof.* Fix $s > 0$, and let $K = \{f \geq s\}$.

By assumption, $K$ has reach at least $r$. Let $0 < \epsilon < r$.

For $x \in \mathbb{R}^n$, $\epsilon B_n(x)$ intersects $K$ if and only if $x$ is within $\epsilon$ of $K$. Since $\epsilon < r$ and $K \cap \epsilon B_n(x)$ is closed, we have $\chi(K \cap \epsilon B_n(x)) = 1$ if $x \in K + \epsilon B_n$ and zero otherwise.

So $\left(f \underset{0}{*} \mathbf{1}_{\epsilon B_n}\right)(x) \geq s$ if and only if $x \in K + \epsilon B_n$. That is,

$$\left\{f \underset{0}{*} \mathbf{1}_{\epsilon B_n} \geq s\right\} = \{f \geq s\} + \epsilon B_n. \qquad \square$$

The dual statement to Theorem 6.2 is: If $f \in \mathsf{Def}(\mathbb{R}^n)$ is such that the *complement* of each upper excursion set has reach at least $r$, then for $0 < \epsilon < r$ the upper excursion sets of $f \overset{0}{*} \mathbf{1}_{\epsilon B_n}$ are the erosions by $\epsilon$ of the corresponding excursion sets of $f$. Here the *erosion* by $\epsilon$ of a set $K$ means the set of points whose distance from the complement of $K$ is at least $\epsilon$.

By the Fubini theorem, Hadwiger convolution of constructible functionals $f$ and $g$ satisfies

$$\int f \underset{k}{*} g \; d\mu_k = \int f \; d\mu_k \int g \; d\mu_k = \int f \overset{k}{*} g \; d\mu_k.$$

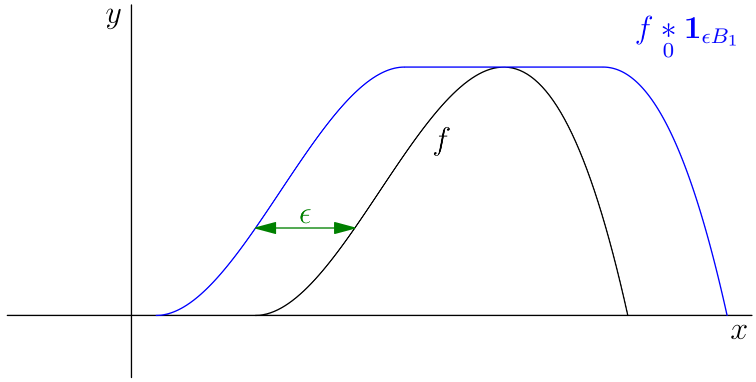



**Figure 6.5:** Lower Euler convolution of a definable function $f$ with the characteristic function of the 1-ball of radius $\epsilon$, as described in Theorem 6.2.

Since the Fubini theorem does not hold for Hadwiger integrals of definable functionals, it is not known whether a similar identity holds in the definable setting. The topic of Hadwiger convolution provides ample opportunities for further investigation of theorems and applications.

## 6.2 Fourier Transform

The basic idea of Hadwiger analogs of the Fourier and Bessel transforms is to first integrate with respect to an intrinsic volume on each member of a family of isospectral sets, then integrate with respect to Lebesgue measure the values obtained over all the isospectral sets. The use of Hadwiger integrals means that these transforms are not purely topological, as in the Euler case, but provide some notion of the geometry of functions over sets.

The Hadwiger generalization of the Fourier transform involves isospectral sets that are parallel hyperplanes orthogonal to some covector $\xi$ in $(\mathbb{R}^n)^*$, the dual space of $\mathbb{R}^n$.

**Definition 6.3.** Let $h \in \mathsf{Def}(\mathbb{R}^n)$ and $\xi \in (\mathbb{R}^n)^*$. Then $\xi^{-1}(s)$ is the $(n-1)$-dimensional hyperplane orthogonal to $\xi$ at distance $s$ from some fixed point. Let $k \in \{0, 1, \ldots, n-1\}$. Define the *lower* and *upper Hadwiger-Fourier transforms*, respectively, of $h$ with respect to $\mu_k$, in the direction of $\xi$:

$$\mathcal{F}_k h(\xi) = \int_{-\infty}^{\infty} \int_{\xi^{-1}(s)} h \lfloor d\mu_k \rfloor \, ds, \tag{6.11}$$

$$\mathcal{F}^k h(\xi) = \int_{-\infty}^{\infty} \int_{\xi^{-1}(s)} h \lceil d\mu_k \rceil \, ds. \tag{6.12}$$

For $k < n$, the Hadwiger-Fourier transform with respect to $\mu_k$ of the characteristic function of a set $A$ gives a *directed* notion of the $(k+1)$-dimensional size of $A$. The Hadwiger integral is gives a $k$-dimensional notion of size, and the Lebesgue integral incorporates one more dimension. The following examples illustrate this concept.

*Example.* If $A$ is a compact convex subset of $\mathbb{R}^n$, then the Euler characteristic of any nonempty slice is 1. Thus, for any $\|\xi\| = 1$, the transform $(\mathcal{F}_0 \mathbf{1}_A)(\xi)$ equals the length of the projection of $A$ onto the $\xi$ axis.

*Example.* Let $A$ be a definable subset of $\mathbb{R}^n$. For any $\|\xi\| = 1$, the transform $(\mathcal{F}_{n-1}\mathbf{1}_A)(\xi)$ integrates the $(n-1)$-dimensional volumes of cross-sections of $A$ orthogonal to the $\xi$-axis. Thus, $(\mathcal{F}_{n-1}\mathbf{1}_A)(\xi)$ equals the $n$-dimensional volume of $A$.

More generally, the Hadwiger-Fourier transform of a functional $h \in \mathsf{Def}(\mathbb{R}^n)$ can be thought of as a *directed* valuation of $h$. The transform $(\mathcal{F}_k f)(\xi)$ provides a notion of the $(k+1)$-dimensional size of $h$ in the direction of $\xi$.

## 6.3 Bessel Transform

Also known as the Hankel transform, the Bessel transform employs isospectral sets consisting of points equidistant from a fixed point. For the usual Euclidean norm on $\mathbb{R}^n$, these sets are concentric spheres. Use of a different norm results in isospectral sets with different geometry, which may be useful in signal processing.

**Definition 6.4.** Let $h \in \mathsf{Def}(\mathbb{R}^n)$. Let $S_r(x) = \{y \mid \|y - x\| = r\}$, which for the Euclidean norm denotes the sphere of radius $r$ centered at $x$. Let $k \in \{0, 1, \ldots, n-1\}$. Define the *lower* and *upper Hadwiger-Bessel transforms*, respetively, of $h$ with respect to $\mu_k$:

$$\mathcal{B}_k h(x) = \int_0^\infty \int_{S_r(x)} h \lfloor d\mu_k \rfloor \, dr, \tag{6.13}$$

$$\mathcal{B}^k h(x) = \int_0^\infty \int_{S_r(x)} h \lceil d\mu_k \rceil \, dr. \tag{6.14}$$

As the sample point of the Hadwiger-Bessel transform moves far from the origin along a fixed ray, the transform converges to the Hadwiger-Fourier transform along the ray's direction. Since any $h \in \mathsf{Def}(\mathbb{R}^n)$ has compact support, the intersection of concentric spheres with the support of $h$ converge to parallel hyperplanes as the radius increases towards infinity. That is, for nonzero $x \in \mathbb{R}^n$ with dual covector $x^*$,

$$\lim_{\lambda \to \infty} (\mathcal{B}_k h)(\lambda x) = (\mathcal{F}_k h)\left(\frac{x^*}{\|x^*\|}\right),$$

and similarly for the upper transforms.

In the context of sensor networks, the Hadwiger-Bessel transform with respect to $\mu_0$ (also known as the Euler-Bessel transform) is useful for target localization [22]. If a functional $h(x)$ counts the number of targets at each point $x$ in the domain, the Euler-Bessel transform highlights the centers of the targets. Likewise, in this situation the Hadwiger-Bessel transforms could offer information about the size and shape of the targets.

Ghrist and Robinson provide index-theoretic interpretations of the Euler-Fourier and Euler-Bessel transformations [22]. Since computation of Hadwiger integrals cannot be reduced to the critical points of a functional, such index-theoretic results seem elusive for the transforms described above.

# Chapter 7

# Convergence

Understanding the convergence of Hadwiger integrals of a sequence of functionals is tricky business, since pointwise convergence of functionals is not enough to guarantee convergence of their integrals. For example, a functional $f$ may have values close to zero, but lots of tiny oscillations in $f$ will make $\int f \lfloor d\chi \rfloor$ arbitrarily large. If $f$ is smooth, then lots of tiny oscillations will cause its derivatives to be very large. In this chapter, we explore ideas related to convergence and estimation of Hadwiger integrals.

## 7.1 Explanation of the Difficulty

Often, applications present an unknown functional $h$ that we can sample at discrete points, constructing an approximate functional by affine interpolations between sample points. We will call such an approximation a *triangulated approximation*. By sampling greater numbers of points and refining the triangulation, we can produce a sequence of approximations $h_1, h_2, \ldots$ such that successive approximations more closely match the functional $h$. However, even if the approximations converge pointwise to $h$, it might not be the case that $\int h_i \lfloor d\mu_k \rfloor$ converges to $\int h \lfloor d\mu_k \rfloor$ as $i$ increases to infinity. For an example of this (unexpected) behavior, we refer to the following example by Baryshnikov:

*Example.* Let $h : [0,1]^2 \to \mathbb{R}$ be defined $h(x, y) = -2\left|x - \frac{1}{2}\right| + 1$. Intuitively, the graph of $h$ looks like a tent, with minimum value $h = 0$ along $x = 0$ and $x = 1$, and maximum value $h = 1$ along $x = \frac{1}{2}$, as illustrated in Figure 7.1. The Euler integral $\int h \lfloor d\chi \rfloor$ evaluates to 1.

By carefully choosing the sample points used to create the triangulated approximations $h_i$, we can cause the Euler integrals of the $h_i$ to diverge to infinity as we refine the approximations. Specifically, refinements of the approximation may possess increasingly many peaks and valleys along the maximum ridge of $h$, as illustrated in Figure 7.1. Even if the $h_i$ converge pointwise to $h$, the Euler integrals $\int h_i \lfloor d\chi \rfloor$ may increase without bound.

The above example is similar to the *Lantern of Schwarz*, a sequence of triangulated surfaces that converge in the Hausdorff topology to a cylinder, whose areas do not converge to the area of the cylinder [32].

In this chapter, we discuss several ideas that provide conditions on a sequence of functions $h_1, h_2, \ldots$ converging to $h$, to guarantee that the Hadwiger integrals

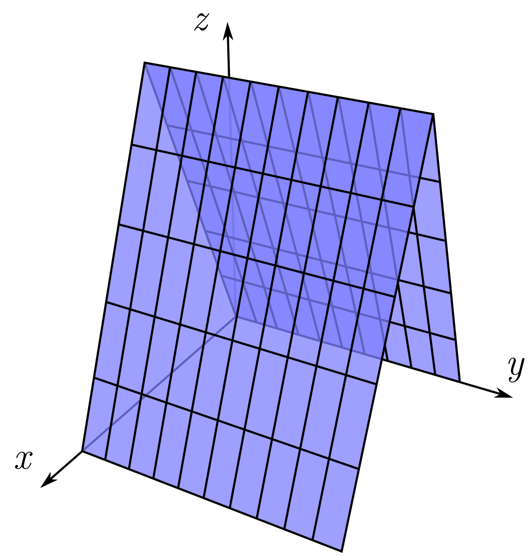
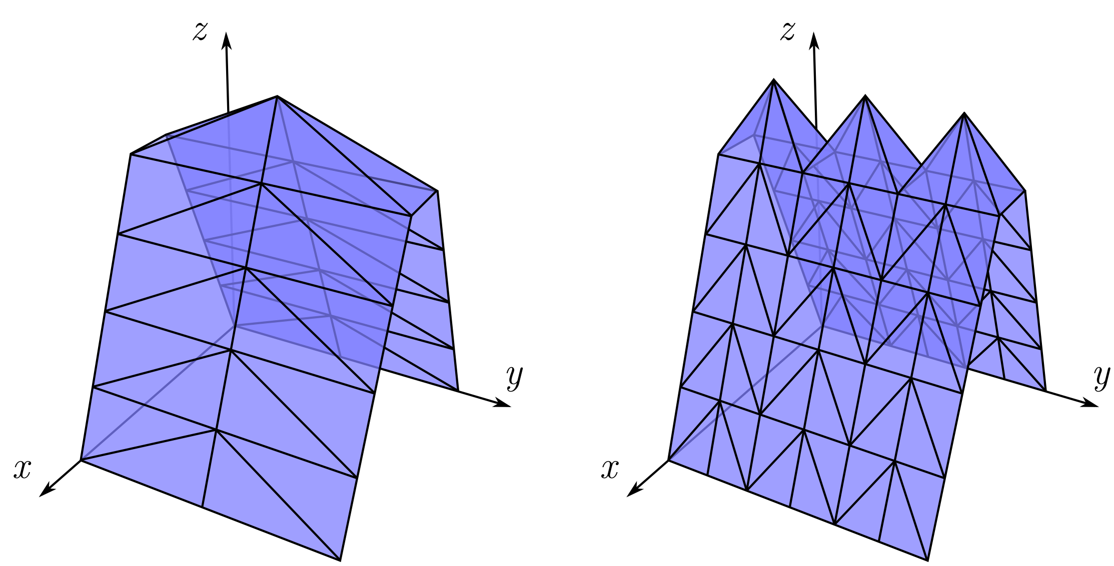



**Figure 7.1:** Above, the graph of $h : [0,1]^2 \to \mathbb{R}$, defined $h(x,y) = -2\left|x - \frac{1}{2}\right| + 1$. Below are two triangulated approximations of $h$; the approximation at right is a refinement of that at left. The Hadwiger integrals of the approximations do not necessarily converge to the corresponding integrals of $h$. Indeed, the approximations may converge pointwise to $h$, but their Euler integrals may increase toward infinity.



$\int h_i \lfloor d\mu_k \rfloor$ converge to $\int h \lfloor d\mu_k \rfloor$ as $i$ increases.

## 7.2 Convergence by Bounding Derivatives

Suppose $f$ is a definable function with compact support. If we have a bound on enough derivatives of $f$, then we can also bound the Hadwiger integrals of $f$, proportional to the area of the support of $f$. We begin with a lemma:

**Lemma 7.1.** *Let $B \subset \mathbb{R}^n$ be a $n$-dimensional ball of radius $r$, and let $f \in \mathsf{Def}(\mathbb{R}^n)$ be supported on $B$ and such that its first $n$ derivatives, $Df, D^2 f, \ldots, D^n f$ exist on $\mathbb{R}^n$ and are bounded in operator norm by some $C > 0$. Then the maximum value of $\int_B f \lfloor d\chi \rfloor$ is proportional to $Cr^n$.*

*Proof.* Since $f = 0$ on the boundary of $B$, the maximum value of $\int f \lfloor d\chi \rfloor$ is attained if $f$ increases as steeply as possible from the boundary, with its absolute maximum value at the center of $B$. Since the derivatives of $f$ are bounded, this maximum value is proportional to $Cr^n$, with constant of proportionality depending only on $n$. Therefore,

$$\int_{\mathbb{R}^n} f \lfloor d\chi \rfloor \leq k_n C r^n$$

where $k_n$ is a constant depending only on the dimension $n$. $\square$

The lemma leads to a similar result for Hadwiger integrals:

**Theorem 7.1.** *Let $B \subset \mathbb{R}^n$ be a $n$-dimensional ball, and let $f \in \mathsf{Def}(\mathbb{R}^n)$ be supported on $B$ and such that its first $n-k$ derivatives $Df, D^2 f, \ldots, D^{n-k} f$ exist on $\mathbb{R}^n$ and are* *bounded in operator norm by some $C > 0$. Then the maximum value of $\int_B f \lfloor d\mu_k \rfloor$ is proportional to $C\mu_n(B)$.*

*Proof.* Let $P \in A_{n,n-k}$, and let $f_P$ be the restriction of $f$ to $P$ So $f_P \in \mathsf{Def}(\mathbb{R}^{n-k})$, and its partial derivatives of order up to $n-k$ are bounded by $C$. By Lemma 7.1,

$$\int_{B\cap P} f_P \lfloor d\chi \rfloor \leq k_n C r^{n-k}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} f \lfloor d\mu_k \rfloor &= \int_{A_{n,n-k}} \int_{B\cap P} f \lfloor d\chi \rfloor \, d\lambda(P) \\ &\leq \int_{A_{n,n-k}} k_n C r^{n-k} \, d\lambda(P) = k_n C r^{n-k} \int_{A_{n,n-k}} d\lambda(P) \\ &= k_n C r^{n-k} \cdot \gamma(G_{n,n-k}) r^k = j_{n,k} C r^n,\end{aligned}$$

where $j_{n,k}$ is a constant depending on $n$ and $k$. Since $B$ is an $n$-ball of radius $r$, $j_{n,k} C r^n$ is proportional to $C\mu_n(B)$. □

Theorem 7.1 provides a convergence result:

**Corollary 7.1.** *Let $B$ be an $n$-dimensional ball in $\mathbb{R}^n$, and let $c_1, c_2, \ldots$ be a sequence of real numbers converging to zero. Let $f_1, f_2, \ldots$ be a sequence of definable functionals supported on $B$, such that the first $n-k$ derivatives of $f_i$ exist on $\mathbb{R}^n$ and are bounded by $c_i$. Then,*

$$\lim_{i\to\infty} \int_B f_i \lfloor d\mu_k \rfloor = 0.$$

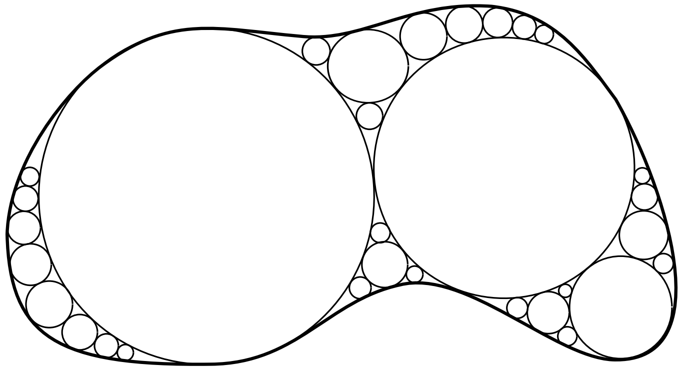


**Figure 7.2:** A function on a domain in $\mathbb{R}^2$ with bounded first and second derivatives and maximum Euler integral can be constructed via a process of circle packing.

*Proof.* The proof follows immediately from Theorem 7.1:

$$\lim_{i\to\infty}\int_B f_i \,\lfloor d\mu_k\rfloor \leq \lim_{i\to\infty} c_i j_{n,k} r^n = 0,$$

where $r$ is the radius of $B$ and $j_{n,k}$ is a constant as before. □

The previous theorem and corollary extend to more general domains via a process of circle packing. Suppose $U$ is a compact, definable region in $\mathbb{R}^n$. We will construct a function $f$, supported on $U$, with the first $n$ derivatives of $f$ bounded in operator norm by $C$. Let $C_1$ be the largest disc inscribed in $U$, let $C_2$ be the largest disc inscribed in $U\backslash C_1$, let $C_3$ be the largest disc inscribed in $U\backslash(C_1 \cap C_2)$, and so on, as illustrated in Figure 7.2. The union of all the $C_i$ fills $U$; that is,

$$\lim_{m\to\infty}\bigcup_{i=1}^{m} C_i = U.$$

Define $f$ on each $C_i$ to be a bump function, zero on the boundary and as large at the center as allowed by the derivative condition. We claim that this $f$ has the greatest possible Euler integral of all functions supported on $U$ and satisfying the derivative condition. Of course, $f$ is not definable in general, for it may have infinitely many discrete critical points. Many different definable functionals may be constructed with Euler integral arbitrarily close to that of $f$. Similarly, we can construct functions with maximal Hadwiger integrals on $U$.

Our desire is for a convergence result for the Hadwiger integrals a sequence of functions with bounded derivatives that converge to an arbitrary (nonzero) function. If the definable Hadwiger integrals were linear, this would be a straightforward application of Corollary 7.1 and the circle packing idea. Unfortunately, the integrals are not linear, and at present we have only the following conjecture.

**Conjecture 7.1.** *Let $c_1, c_2, \ldots$ be a sequence of real numbers converging to zero. Let $f \in \mathsf{Def}(\mathbb{R}^n)$ be supported on a compact subset $U$ of $\mathbb{R}^n$. Let $f_1, f_2, \ldots$ be a sequence of definable functionals supported on $U$, such that the first $n-k$ derivatives of $f_i$ are bounded by $c_i$. Then,*

$$\lim_{i\to\infty} \int_U f_i \ \lfloor d\mu_k \rfloor = \int_U f \ \lfloor d\mu_k \rfloor.$$

## 7.3 Integral Currents

We now turn back to the machinery of currents, which are powerful tools for proving convergence results in integral geometry. Such convergence results generally employ a specific type of current known as *integer-multiplicity currents*. For more details on integer-multiplicity currents see Federer [16], Krantz and Parks [25], or Morvan [32].

In order to discuss integer-multiplicity currents, we need the concept of $m$-rectifiable sets, which we think of intuitively as being almost everywhere the image of $\mathbb{R}^m$ under a Lipschitz map.

**Definition 7.1.** Let $1 \leq m \leq n$ be integers. A set $S \subset \mathbb{R}^n$ is *$m$-rectifiable* if

$$S = S_0 \bigcup \left( \bigcup_{j=1}^{\infty} F_j(S_j) \right)$$

where $\mathcal{H}^m(S_0) = 0$, $S_j \subseteq \mathbb{R}^m$, and $F_j : S_j \to \mathbb{R}^n$ is a Lipschitz function.

We now define integer-multiplicity currents, which are more general than the currents associated with submanifolds. Such a current is associated with a rectifiable set $S$ and possesses integer-valued multiplicity and an orientation in the tangent space of $S$.

**Definition 7.2.** Let $1 \leq m \leq n$ be integers. Let $T \in \Omega_m(U)$ for some open subset $U$ of $\mathbb{R}^n$. $T$ is an *integer-multiplicity $m$-current* if it can be written for all $\omega \in \Omega_c^m$ as

$$T(\omega) = \int_S \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^m(x)$$

where $S$ is a $\mathcal{H}^m$-measurable and $m$-rectifiable subset of $\mathbb{R}^n$; $\theta$ is a locally $\mathcal{H}^k$-integrable, nonnegative, integer-valued function; and $\xi : S \to \bigwedge_m(\mathbb{R}^n)$ is an $\mathcal{H}^m$-measurable function such that, for $\mathcal{H}^m$-almost every point $x \in S$, $\xi(x)$ is a simple unit $m$-vector in $T_xS$. In this notation, we call $\theta$ the *multiplicity* and $\xi$ the *orientation* of $T$. We denote the space of integer-multiplicity $m$-currents supported on $U$ by $\mathcal{I}_m(U)$, or simply by $\mathcal{I}_m$ if $U$ is understood.

The normal and conormal cycles of a definable set are examples of integer-multiplicity currents.

We can also define the *slice* of an integer-multiplicity current [25]. Intuitively, a slice of an integer-multiplicity current $T \in \mathcal{I}_m$ is an current $R \in \mathcal{I}_{m-1}$ obtained by intersecting $T$ with a Lipschitz function.

The primary convergence result for integer-multiplicity currents is the compactness theorem, one version of which follows:

**Theorem 7.2** (Compactness Theorem for Currents)**.** *Let $\mathcal{T}$ be the set of integer-multiplicity currents supported on a compact subset $K$ of $\mathbb{R}^n$ such that*

$$\sup_{T \in \mathcal{T}} (\mathbf{M}(T) + \mathbf{M}(\partial T)) < \infty$$

*Then $\mathcal{T}$ is compact in the flat topology.*

For a proof of the Compactness Theorem, see [16] or [25]. The Compactness Theorem implies any sequence of currents in $\mathcal{T}$ has a subsequence converging to some $T \in \mathcal{T}$.

## 7.4 Convergence of Subgraphs

By expressing the Hadwiger integrals via currents, we obtain convergence results. We employ the *subgraph* of a function $h \in \mathsf{Def}(\mathbb{R}^n)$, the set of points in $\mathbb{R}^{n+1}$ between the domain and the graph of $h$. Formally, the subgraph of $h$ is the set $H \subset \mathbb{R}^{n+1}$ defined by:

$$H = \{(x_1, \ldots, x_n, x_{n+1}) \mid 0 \leq x_{n+1} \leq h(x_1, \ldots, x_n)\}.$$

A key idea is that the convergence of conormal cycles of the subgraphs implies convergence of the integrals:

**Theorem 7.3.** *Let $h_1, h_2, \ldots \in \mathsf{Def}(\mathbb{R}^n)$ be a sequence of nonnegative functions with subgraphs $H_1, H_2, \ldots$, respectively. If the conormal cycles of the subgraphs $\mathbf{C}^{H_i}$ converge to the conormal cycle $\mathbf{C}^H$ of the subgraph of some function $h \in \mathsf{Def}(\mathbb{R}^n)$, then the Hadwiger integrals of the $h_i$ also converge to the corresponding Hadwiger integrals of $h$.*

*Proof.* Consider the slice of $\mathbf{C}^{H_i}$ on a level set at some height $s \geq 0$. Call this slice $T_{i,s} \in \mathcal{I}_n$. Now $T_{i,s}$ is a Lagrangian current supported on the excursion set $\{h_i \geq s\}$. By uniqueness of the conormal cycle of a set [33], $T_{i,s}$ is the conormal cycle $\mathbf{C}^{\{h_i \geq s\}}$.

Since the $\mathbf{C}^{H_i}$ converge to the conormal cycle $\mathbf{C}^H$, the slices also converge. That is, $\mathbf{C}^{\{h_i \geq s\}}$ converges to $\mathbf{C}^{\{h \geq s\}}$ as $i \to \infty$.

Therefore, by the Fubini theorem,

$$\begin{aligned}\lim_{i\to\infty} \int_{\mathbb{R}^n} h_i \, \lfloor d\mu_k \rfloor &= \lim_{i\to\infty} \int_0^\infty \mathbf{C}^{\{h_i \geq s\}}(\mathcal{W}_{n,k}) \, ds \\ &= \int_0^\infty \lim_{i\to\infty} \mathbf{C}^{\{h_i \geq s\}}(\mathcal{W}_{n,k}) \, ds = \int_0^\infty \mathbf{C}^{\{h \geq s\}}(\mathcal{W}_{n,k}) \, ds = \int_{\mathbb{R}^n} h \, \lfloor d\mu_k \rfloor. \quad \square\end{aligned}$$

## 7.5 Triangulated Approximations

With Theorem 7.3, our task simplifies to finding conditions on the sequence of functions $\{h_i\}_{i\in\mathbb{N}}$ that guarantee convergence of the conormal cycles of their subgraphs. In the case where the $h_i$ are triangulated approximations, we would like to require that the simplicies in the triangulation are "fat," that is, they do not approach degeneracy as the triangulation is refined. For instance, we want to require that triangles have a large area relative to the lengths of their sides. Fu and Morvan quantify this concept as *fatness* in [18] and [32], respectively.

**Definition 7.3.** For a $k$-simplex $\sigma$, let $\varepsilon(\sigma)$ be the length of the longest edge of $\sigma$. Let $\mathcal{S}_\sigma^j$ be the set of all $j$-simplicies in $\sigma$. The *fatness* of $\sigma$ is the dimensionless real number

$$\Theta(\sigma) = \min_{j\in\{0,\ldots,k\}} \left\{ \frac{\mu_j(\tau)}{\varepsilon(\sigma)^j} \;\middle|\; \tau \in \mathcal{S}_\sigma^j \right\}.$$

For a simplicial complex $P$, the *fatness* of $P$ is the minimum fatness over all simplices of $P$.

Intuitively, the fatness of a triangle $\tau$ is the quotient of the area of $\tau$ by the square of its longest edge. By requiring a positive lower bound on the fatness of the triangulation, we can obtain a convergence idea.

In applications where $h$ is unknown, it might seem difficult to require that the fatness of the $H_i$, the subgraphs of the triangulated approximations, be uniformly bounded above zero. However, if we make the reasonable assumption that $h$ is Lipschitz, then it suffices to ensure that the fatness of the triangulation produced by sample points in the *domain* has bounded fatness.

**Lemma 7.2.** *Let $h \in \mathsf{Def}(\mathbb{R}^n)$ be a Lipschitz functional. Let $T$ be a simplicial complex in $\mathbb{R}^n$ containing the support of $h$. Let $g$ be a triangulated approximation of $h$, such that $g = h$ on each 0-simplex of $T$, $g$ is affine on each higher-dimensional simplex of $T$, and $g$ is continuous. Let $\Gamma$ denote the graph of $g$. So $\Gamma$ is a simplicial complex that approximates the graph of $h$. If the fatness of $T$ is bounded from below by a positive constant, then so is the fatness of the graph of $g$.*

*Proof.* Let $\ell$ be the Lipschitz constant of $h$. Let $\varepsilon(T)$ denote the length of the longest edge in $T$. For any edge $e \in T$, the corresponding edge $g(e) \subset \Gamma$ satisfies

$$\mu_1(g(e)) \leq \mu_1(e)\sqrt{1+\ell^2} \leq \varepsilon(T)\sqrt{1+\ell^2}.$$

Thus, the length of the longest edge of $\Gamma$ satisfies

$$\varepsilon(g(e))^j \leq \varepsilon(T)^j(1+\ell^2)^{j/2}.$$

For any $j$-simplex $\tau \in T$, the volume of the corresponding simplex $g(\tau) \subset \Gamma$ satisfies $\mu_j(g(\tau)) \geq \mu_j(\tau)$. Therefore,

$$\Theta(\Gamma) \geq \Theta(T)(1+\ell^2)^{-n/2}.$$

Since $\Theta(T)$ is larger than a positive constant, so is $\Theta(\Gamma)$. □

We had hoped to employ some ideas from Morvan and Fu, along with Lemma 7.2 and Theorem 7.3, to show that bounded fatness of triangulated approximations of a functional implies convergence of Hadwiger integrals of the approximations. For now, however, we have the following conjecture.

**Conjecture 7.2.** *Let $h$ be a Lipschitz functional supported on compact $X \subset \mathbb{R}^n$. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of triangulations of $X$, such that the fatness of the $T_i$ is uniformly bounded from below by a positive constant, and the volumes of the simplices of the $T_i$ decrease to zero. Let $h_i$ be the triangulated approximations of $h$ corresponding to $T_i$. Then the Hadwiger integrals of the $h_i$ converge to those of $h$ as $i \to \infty$.*

# Chapter 8

# Applications and Further Research

We now consider areas in which Hadwiger integrals are useful or might prove useful. Still, many open questions may impede the adoption of this integration theory in applied fields. We highlight some of these areas for future research, in which progress could yield much applied fruit.

## 8.1 Algorithms and Numerical Analysis

In order to use Hadwiger integrals in applications, we must be able to efficiently compute the integrals of functionals. Since real-world data is noisy, we would like theoretical bounds on the possible error of Hadwiger integrals computed from approximations of functionals. Thus, we need further numerical analysis results about convergence and estimation of Hadwiger integrals.

The ideas in Chapter 7 provide a starting place for numerical analysis and convergence results. Still, we desire a better understanding of the convergence of Hadwiger integrals. In the case where a functional $h$ is approximated by sampling values at discrete points, how can the sampling be done to ensure convergence of the Hadwiger integrals of the approximations to those of $h$? Can we supply error bounds, even in a probabilistic way, for a particular approximation?

Various algorithms are known to compute the intrinsic volumes of subsets. For example, Meschenmoser and Spodarev present two methods for computing the intrinsic volumes of subsets of an $n$-dimensional digital image, with certain assumptions [29]. Klenk, Schmidt, and Spodarev implement an algorithm that computes, with high precision, intrinsic volumes of polygonal approximations of subsets of $\mathbb{R}^2$ [26]. Schladitz, Ohser, and Nagel describe a method of computing intrinsic volumes of subsets of three-dimensional images [39].

Algorithms for computing Hadwiger integrals, however, are unexplored. Naive approaches involve using the equivalent expressions in Section 4.5 to reduce the computation of a Hadwiger to computing the intrinsic volumes of excursion sets, or Euler integrals of slices or projections. Yet these approaches seem computationally intensive and possibly imprecise. We would like to study such algorithms in terms of computational complexity and error bounds for approximations of functionals.

## 8.2 Image Processing

Image analysis is a central problem in computer science today. With the proliferation of devices such as digital cameras, it has never been easier to collect vast amounts of graphical data. If such data is to be useful, one must extract information from the data. A major challenge of image processing is to extract interesting features from large, often noisy data sets.

Research into geometric data processing employs techniques from topology and differential geometry, including some similar to the content of this thesis. For example, Cohen-Steiner and Morvan use normal cycles to estimate the curvature of a smooth surface based on polyhedral approximations [13]. Vixie, Clawson, and Asaki employ a multiscale flat norm to produce scale-dependent "signatures" of shapes that aid in their classification and recognition [45]. Carlsson and others use topological methods to extract qualitative information from graphical data [11]. Donoho and Huo describe methods of using multiscale "beamlets" to identify lines and curves in images [14].

The intrinsic volumes are alredy of some utility in image processing [29, 39]. At present, this is mostly limited to computing the intrinsic volumes of subsets of binary images—that is, arrays of black and white pixels. If an object of interest appears as a collection of black pixels on a white background, the intrinsic volumes provide some size data about the object. This amounts to computing the Hadwiger integrals of the characteristic function of the object.

Our integration theory could extend such image processing from binary images to grayscale images. Suppose we view a grayscale image as a functional over a domain—ordinarily $\mathbb{R}^2$ or $\mathbb{R}^3$, though higher-dimensional spaces for images that change with time or other parameters—with the domain partitioned into pixels. The value of the functional is constant on each pixel, indicating the darkness of lightness of that pixel of the image, with a convention such as lower or upper semicontinuity to determine values on pixel boundaries. Such a functional is constructible, though we could smooth out noise to produce a continuous functional via convolution with a bump function or some other transform. The Hadwiger integrals of this functional then provide information about the image, possibly allowing for its classification.

With a construction as described, the various Hadwiger integrals provide statistics about the image with varying degrees of scale-dependence. Euler integrals are independent of scale, and thus provide information without regard to size of the image. The other Hadwiger integrals do depend on scale, the dependence increasing with the subscript of $\mu_k$. Taken together, the Hadwiger integrals could reveal information about the size and shape of features of the image. Importantly, all the integrals are independent of *orientation*, which is useful for detecting features of unknown orientation in images.

We could further extend the integration theory to aid in processing color and hyperspectral images. A color pixel is often described by a triplet—of red, green, and blue intensities; or of hue, saturation, and lightness values. Thus, processing color images could employ the unexplored topic of Hadwiger integration of functions with values in $\mathbb{R}^3$. Hadwiger integrals may also aid in hyperspectral imaging, which involves recording a much larger range of wavelengths than simply visual light. A goal of hyperspectral imaging is to identify the materials of which objects are made, in addition to the shape of the objects. The functions corresponding to such images are likely to be very high-dimensional. Still, the Hadwiger integrals of such functionals may provide useful quantitative information about the image.

## 8.3 Sensor Networks

Euler integration is useful in counting targets in sensor networks, as demonstrated by Baryshnikov and Ghrist in [3]. In general, we wish to extract useful information about objects of interest, given data from a network of sensors [5]. For instance, we may wish to know about the size, shape, or density of targets. Euler integration is particularly useful since, as an additive topological invariant, it can easily recover the total number of targets detected by the network, given some modest assumptions on network density and target shape.

Beyond Euler integrals, the Hadwiger integrals could provide information about target size and shape. Since only Euler integrals are completely scale-independent, computing the other Hadwiger integrals would require a metric on the sensor network, which is not necessary in the Euler case. Computing the Hadwiger integrals based on sensor information from a sparse network might also be difficult and imprecise. Yet integral transforms have helped resolves such problems in the Euler case, and might also be useful in the more general Hadwiger case. The application of Hadwiger integrals to sensor networks is an area worthy of study.

## 8.4 Crystal Growth and Foam Dynamics

The intrinsic volumes appear in equations describing dynamics of cellular structures. Examples of cellular structures include crystals found in metals and minerals, gas-filled bubbles in foams, and biological cells. Such cell structures are often dynamic– elements the structure move and change in order to decrease the total energy level of the system. In 1952, von Neumann found a formula for the growth of cells in a two-dimensional structure. In 2007, MacPherson and Srolovitz generalized von Neumann's formula to describe the dynamics cellular structures in three and higher dimensions [28].

Let $C = \bigcup_{i=0}^{n} C_i$ be a closed $n$-dimensional cell, with $C_i$ denoting the union of all $i$-dimensional features of the cell. That is, $C_0$ is the set of vertices, $C_1$ the set of edges, and so on. MacPherson and Srolovitz found that when the cell structure changes by a process of *mean curvature flow*, the volume of the cell changes according to

$$\frac{d\mu_n}{dt}(C) = -2\pi M\gamma \left( \mu_{n-2}(C_n) - \frac{1}{6}\mu_{n-2}(C_{n-2}) \right) \tag{8.1}$$

where $M$ and $\gamma$ are constants determined by the material properties of the cell structure.

While the intrinsic volumes provide information about the size of a cell, Hadwiger

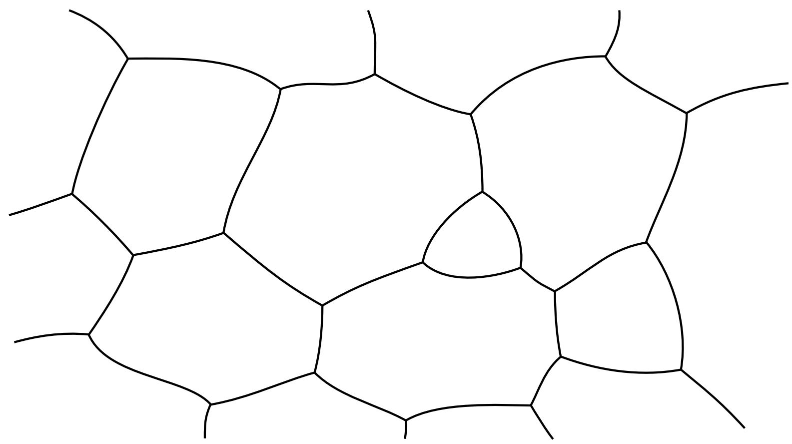


**Figure 8.1:** In this two-dimensional cell structure, three cell walls meet at each vertex, with each pair of walls at uniform angles. The change in area of each cell is described by equation (8.1).

integrals provide a method of measuring functionals defined on the cell. Such functionals could indicate cell temperature, density, or other properties of the cell. For a functional $f$ defined on cell $C$, equation (8.1) may generalize to:

$$\frac{d}{dt}\int_C f\ d\mu_n = -2\pi M\gamma\left(\int_{C_n} f\ d\mu_{n-2} - \frac{1}{6}\int_{C_{n-2}} f\ d\mu_{n-2}\right). \tag{8.2}$$

MacPherson and Srolovitz assume that three (co-dimension 1) cell walls meet at each (co-dimension 2) junction, and that the angles between cell walls at the junction are uniform. This assumption is natural, but not necessary. Le and Du extended MacPherson and Srolovitz's work to generalize the cell junction conditions [27]. The Hadwiger integrals may also play a role in this more general theory of cell dymanics.

## 8.5 Connection to Morse Theory

Euler integrals have a natural connection to Morse Theory, as demonstrated in [4]. These integrals depend only on the topology, not the geometry, of level sets of a functional. The topology of the level sets changes only at critical points of the function. Thus, the integrals are determined by the function at its critical points.

For instance, suppose $h$ is a Morse function on a closed $n$-manifold $\mathcal{M}$. Let $\mathcal{C}$ be the set of critical points of $h$, and let $\iota(p)$ be the Morse index of $p \in \mathcal{C}$. Then,

$$\int_{\mathcal{M}} h \lfloor d\chi \rfloor = \sum_{p \in \mathcal{C}} (-1)^{n-\iota(p)} h(p) \qquad \text{and} \tag{8.3}$$

$$\int_{\mathcal{M}} h \lceil d\chi \rceil = \sum_{p \in \mathcal{C}} (-1)^{\iota(p)} h(p) = (-1)^n \int_{\mathcal{M}} h \lfloor d\chi \rfloor. \tag{8.4}$$

Thus, for the Euler integrals, the behavior of the function between critical points is insignificant. The Morse-theoretic properties of Euler integrals allow for simple computations and elegant theorems. For more details, see [4].

Aside from Euler characteristic, the other intrinsic volumes are not topological invariants; thus the other Hadwiger integrals cannot be computed simply with knowledge of the critical points of a function. As $k$ increases from 0 to $n$, the degree to which $\int h \lfloor d\mu_k \rfloor$ and $\int h \lceil d\mu_k \rceil$ depend on the *geometry* of the level sets of $h$ increases.

Still, there could be an important index-theoretic approach to the general Hadwiger integrals. Perhaps microlocal index theory could provide insight into this area; a good starting place might be the paper by Bröcker and Kuppe [10].

## 8.6 More General Valuations

We have defined a valuation on $\mathsf{Def}(\mathbb{R}^n)$ to be invariant under Euclidean motions of $\mathbb{R}^n$. We could modify the definition to require that the valuation instead to be invariant under the action of some other group on $\mathbb{R}^n$, and thus obtain a more general valuation theory.

Alesker has studied general, invariant with respect to the action of some group [1, 2]. For instance, he provides the following theorem:

**Theorem 8.1** (Alesker)**.** *Let $G$ be a compact subgroup of the orthogonal group. Then the space of continuous (in the Hausdorff metric) valuations on convex subsets of $\mathbb{R}^n$, invariant with respect to the action of $G$, is finite-dimensional if and only if $G$ acts transitively on the unit sphere in $\mathbb{R}^n$.*

We would like to "lift" such ideas from sets to functionals over sets. Is there a similar theorem about valuations on $\mathsf{Def}(\mathbb{R}^n)$ invariant with repect to the action of a compact subgroup of the orthogonal group?

Alesker has also studied valuations on compact submanifolds. Accordingly, we could generalize our theory to consider valuations on functionals on other manifolds besides $\mathbb{R}^n$.

Furthermore, Bernig and Fu have studied convolution of *valuations* (distinct from convoltion of sets) on Euclidean space and connections to the Minkowski sum [8]. We would like to consider a similar convolution of valuations on functionals.

# Appendix A

# Flag Coefficients

The flag coefficients are the numbers $\binom{n}{m}\frac{\omega_n}{\omega_m\omega_{n-m}}$ from Section 2.2, analogous to the binomial coefficients [24]. In this section we will denote the flag coefficients as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}\frac{\omega_n}{\omega_m\omega_{n-m}},$$

where $\omega_n$ denotes the $n$-dimensional volume of the unit ball in $\mathbb{R}^n$, alternately expressed in terms of the gamma function, $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$. As the binomial coefficient $\binom{n}{k}$ counts the number of $k$-element subsets of an $n$-element set, the flag coefficient $\binom{n}{m}\frac{\omega_n}{\omega_m\omega_{n-m}}$ gives the total measure of $k$-dimensional linear subspaces of $\mathbb{R}^n$.

Like the binomial coefficients, the flag coefficients can be written in a triangular array, as in Figure A.1. The interested reader can find many interesting patterns in this array. For instance, consecutive integers appear in two diagonals of the array. Rational multiples of $\pi$ occur at $\begin{bmatrix} n \\ m \end{bmatrix}$ exactly when $n$ is even and $m$ is odd. Each row is unimodal and symmetric. Klain and Rota prove so-called "continuous" analogs

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & \frac{\pi}{2} & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 2 & & 1 & & & & & \\
 & & & & 1 & & \frac{3\pi}{4} & & 3 & & \frac{3\pi}{4} & & 1 & & & & \\
 & & & 1 & & \frac{8}{3} & & 4 & & 4 & & \frac{8}{3} & & 1 & & & \\
 & & 1 & & \frac{15\pi}{16} & & 5 & & \frac{15\pi}{8} & & 5 & & \frac{15\pi}{16} & & 1 & & \\
 & 1 & & \frac{16}{5} & & 6 & & 8 & & 8 & & 6 & & \frac{16}{5} & & 1 & \\
1 & & \frac{35\pi}{32} & & 7 & & \frac{105\pi}{32} & & \frac{35}{3} & & \frac{105\pi}{32} & & 7 & & \frac{35\pi}{32} & & 1 \\
\iddots & & & & & & & & \vdots & & & & & & & & \ddots
\end{array}
$$

**Figure A.1:** The flag coefficient triangle.

of combinatorial theorems, replacing binomial coefficients with flag coefficients [24]. They also provide explicit formulae for the flag coefficients, which are easily computed from the definitions. For positive integers $n$ and $m$,

$$
\begin{aligned}
\begin{bmatrix} 2n \\ 2m \end{bmatrix} &= \binom{2n}{2m}\binom{n}{m}^{-1}, \\
\begin{bmatrix} 2n \\ 2m+1 \end{bmatrix} &= \frac{\pi}{4^n}\frac{(2n)!}{n!m!(n-m-1)!}, \\
\begin{bmatrix} 2n+1 \\ 2m \end{bmatrix} &= 4^m\binom{n}{m}\binom{2m}{m}^{-1}, \quad \text{and} \\
\begin{bmatrix} 2n+1 \\ 2m+1 \end{bmatrix} &= 4^{n-m}\binom{n}{n-m}\binom{2(n-m)}{n-m}^{-1}.
\end{aligned}
$$

We present here two interesting identities that arose from our study of the flag coefficients.

**Proposition A.1.** *The flag coefficients satisfy a recurrence within each row:*

$$
\begin{bmatrix} n \\ m+2 \end{bmatrix} = \frac{n-m-1}{m+1}\begin{bmatrix} n \\ m \end{bmatrix}. \tag{A.1}
$$

*Proof.* First observe that $(m+2)\omega_{m+2}\omega_{n-m-2} = (n-m)\omega_{n-m}\omega_m$:

$$\begin{aligned}
(m+2)\omega_{m+2}\omega_{n-m-2} &= (m+2)\cdot\frac{\pi^{(m+2)/2}}{\Gamma\left(\frac{m+2}{2}+1\right)}\cdot\frac{\pi^{(n-m-2)/2}}{\Gamma\left(\frac{n-m-2}{2}+1\right)} \\
&= \frac{(m+2)\pi^{n/2}}{\frac{m+2}{2}\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{n-m}{2}\right)} = \frac{(n-m)\pi^{n/2}}{\Gamma\left(\frac{m}{2}+1\right)\frac{n-m}{2}\Gamma\left(\frac{n-m}{2}\right)} \\
&= (n-m)\cdot\frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}\cdot\frac{\pi^{(n-m)/2}}{\Gamma\left(\frac{n-m}{2}+1\right)} = (n-m)\omega_m\omega_{n-m}.
\end{aligned}$$

Writing out the flag coefficients and using the above substitution in the denominator, we have:

$$\begin{aligned}
\begin{bmatrix} n \\ m+2 \end{bmatrix} &= \frac{n!}{(m+2)!(n-m-2)!}\cdot\frac{\omega_n}{\omega_{m+2}\omega_{n-m-2}} \\
&= \frac{(n-m-1)n!}{(m+1)m!(n-m)!}\cdot\frac{\omega_n}{\omega_m\omega_{n-m}} = \frac{n-m-1}{m+1}\begin{bmatrix} n \\ m \end{bmatrix}. \qquad \square
\end{aligned}$$

The recurrence in Proposition A.1 is handy because it allows one to write out a row of the triangle more easily than computing each entry via factorials. Next, we give an alternate method of expressing the $\begin{bmatrix} n \\ 1 \end{bmatrix}$ coefficients.

**Proposition A.2.** *The coefficients* $\begin{bmatrix} n \\ 1 \end{bmatrix}$ *can be expressed in terms of integrals of powers of the sine function:*

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{\pi}{s_{n-1}}, \quad \text{where} \quad s_j = \int_0^\pi \sin^j x\, dx. \tag{A.2}$$

*Proof.* We use the reduction formula

$$s_j = \int_0^\pi \sin^j x\, dx = \frac{j-1}{j}\int_0^\pi \sin^{j-2} x\, dx.$$

First suppose $n$ is even. The reduction formula implies that

$$s_{n-1} = \frac{(n-2)(n-4)\cdots 4\cdot 2}{(n-1)(n-3)\cdots 3\cdot 1}\cdot 2.$$

Thus,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{\pi n!}{2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2}-1\right)!} = \frac{\pi(n-1)(n-3)\cdots 3\cdot 1}{2(n-2)(n-4)\cdots 4\cdot 2} = \frac{\pi}{s_{n-1}}.$$

Now suppose $n$ is odd. The reduction formula implies that

$$s_{n-1} = \frac{(n-2)(n-4)\cdots 3\cdot 1}{(n-1)(n-3)\cdots 4\cdot 2}\cdot \pi.$$

Thus,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = 2^{n-1}\binom{n-1}{\frac{n-1}{2}}^{-1} = \frac{(n-1)(n-3)\cdots 4\cdot 2}{(n-2)(n-4)\cdots 3\cdot 1} = \frac{\pi}{s_{n-1}}. \qquad \square$$

Together, the two propositions allow us to recursively generate each row of the flag coefficient triangle. For those who desire all the coefficients in a particular row, recursive generation is more efficient than computing each coefficient via explicit formulae.

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