March 1984

Limit Cycles of Planar Quadratic Differential Equations

Daniel E. Koditschek
University of Pennsylvania, kod@seas.upenn.edu

K. S. Narendra
Yale University

Follow this and additional works at: http://repository.upenn.edu/ese_papers

Recommended Citation

This is a post-print version. Published in Journal of Differential Equations, Volume 54, March 1984, pages 181-195.

At the time of publication, author Daniel E. Koditschek was affiliated with Yale University. Currently, he is a faculty member in the School of Engineering at the University of Pennsylvania.

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/ese_papers/446
For more information, please contact libraryrepository@pobox.upenn.edu.
Limit Cycles of Planar Quadratic Differential Equations

Abstract
Since Hilbert posed the problem of systematically counting and locating the limit cycle of polynomial systems on the plane in 1900, much effort has been expended in its investigation. A large body of literature - chiefly by Chinese and Soviet authors - has addressed this question in the context of differential equations whose field is specified by quadratic polynomials. In this paper we consider the class of quadratic differential equations which admit a unique equilibrium state, and establish sufficient conditions, algebraic in system coefficients, for the existence and uniqueness of a limit cycle. The work is based upon insights and techniques developed in an earlier analysis of such systems [1] motivated by questions from mathematical control theory.

Comments
This is a post-print version. Published in Journal of Differential Equations, Volume 54, March 1984, pages 181-195.

At the time of publication, author Daniel E. Koditschek was affiliated with Yale University. Currently, he is a faculty member in the School of Engineering at the University of Pennsylvania.

This journal article is available at ScholarlyCommons: http://repository.upenn.edu/ese_papers/446
Limit Cycles of Planar Quadratic Differential Equations

D. E. KOCHTSCHEK AND K. S. NARENDRA

Center for Systems Science,
Yale University, New Haven, Connecticut 06520
Received June 2, 1982; revised March 9, 1983

INTRODUCTION

Since Hilbert posed the problem of systematically counting and locating a limit cycle of polynomial systems on the plane in 1900, much effort has been expended in its investigation. A large body of literature—chiefly by Chinese and Soviet authors—has addressed this question in the context of differential equations whose field is specified by quadratic polynomials. In this paper, we consider the class of quadratic differential equations which admit a unique equilibrium state, and establish sufficient conditions, algebraic in system coefficients, for the existence and uniqueness of a limit cycle. The work is based upon insights and techniques developed in the analysis of such systems [1] motivated by questions from mathematical control theory.

Until the fifties, work on quadratic systems chiefly concerned the existence of a center. In 1952, Bautin [2] showed that a given equilibrium state can support as many as but no more than three limit cycles under a quadratic field. Three years later, a paper by Petrovskii and Landis [3] purported to show that a quadratic system could support no more than three cycles on the plane. Although this result was called into question by subsequent (and the authors later acknowledged an error in the proof [4]) it apparently inspired a number of attempts to complete the Hilbert program by quadratic differential equations [5–7]. A useful survey of the general literature was given by Coppel [8] in 1966, and Ye Yanqian [15] has recently summarized the last decade’s contributions to the quadratic limit cycle problem. Notably, Shi Songling [10] has presented a quadratic differential equation with four limit cycles, finally demonstrating the necessity of the result in [3]. Thus, the Sixteenth Hilbert Problem remains

by a “quadratic system” we mean the differential equation

\[ \dot{x} = Ax + x^T G x, \]  

(1)
where \( A, G, H \in \mathbb{R}^{2 \times 2} \) (and \( x^T G x \) denotes the scalar product of the vectors \( x \) and \( Gx \in \mathbb{R}^2 \)). We adopt the convention

\[
B(x) = \begin{pmatrix} x^T G x \\ x^T H x \end{pmatrix}, \quad \text{and} \quad f(x) = Ax + B(x),
\]

and will assume, throughout the paper, that neither \( A \) nor \( B \) is identically zero. The presentation is organized as follows. In Section 2 we state the central result and include an example to illustrate the conditions listed. Section 3 provides a brief review of some algebraic results in \( \mathbb{R}^2 \) which will be very helpful throughout this investigation. Section 4 establishes the existence of limit cycles as a geometric interpretation of the techniques from the preceding section. Finally, uniqueness is proven in Section 5, and a brief conclusion follows in Section 6.

2. Statement of the Main Result

For ease of exposition, it is helpful to introduce some notation and terminological conventions along with the central theorem. To begin with, we will show (Lemma 2, in Section 3) that "almost all" quadratic systems which admit a unique equilibrium state at the origin may be written in the form

\[
s = Ax + c^T x Dx,
\]

where \( c \in \mathbb{R}^2 \) and \( D \in \mathbb{R}^{2 \times 2} \). Thus, we will find it often necessary to refer to the affine line \( \{ A + \mu D \mid \mu \in \mathbb{R} \} \), which we will call the pencil \((A,D)\) [16]. A linear transformation of the plane is nodal if it has two real eigenvalues, critical if it has a unique eigenspace, and focal if its eigenvalues are complex conjugates. We may now state the main result.

**Theorem 1.** System \( (2) \) has one and only one limit cycle \( \mathcal{L} \) if

(i) \( A \) is focal, with non-zero trace;

(ii) the pencil \((A,D)\) includes bounded nodal values whose eigenvalues have opposite sign to the real part of the eigenvalues of \( A \), and no other nodal values.

To convince the reader that these conditions are algebraic and can be computed, we require some more notation. Denote the skew symmetric matrix \( J \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \), and the symmetric part of any matrix \( A \), by \( A_s = \frac{1}{2}(A + A^T) \). We will see that no quadratic transformation with a unique singularity which may be written as \( (2) \) can give rise to limit cycles.

**Theorem 2.** The conditions of Theorem 1 are equivalent to the following:

(i) \( |AE| \) is sign definite, \( \text{tr} \{ A \} \neq 0 \), and either

(a) \( \text{tr} \{ A \} > 0 \), and \( |JD| \) agrees in sign, or

(b) \( |JD| < 0 \), and \( |(D^T J)A| \) agree in sign.

A few remarks are now in order. In order to be very helpful throughout this investigation, Section 4 establishes the existence of limit cycles as a geometric interpretation of the techniques from the preceding section. Finally, uniqueness is proven in Section 5, and a brief conclusion follows in Section 6.

**Theorem 3.** System \( (1) \) is globally asymptotically stable if and only if

(i) the eigenvalues of \( A \) have non-positive real parts;

(ii) there exist \( c \in \mathbb{R}^2 \) and \( D \in \mathbb{R}^{2 \times 2} \) such that \( B(x) = c^T x Dx \);

(iii) the pencil \((A,D)\) includes nodal values with bounded eigenvalues, and no other nodal values; any singular value of the pencil has a second root in \( \mathbb{C} \) if and only if \( A \) is non-singular.

In fact, according to [13], conditions (ii) and (iii) of this theorem are essentially necessary for the boundedness of solutions to any quadratic system \( (1) \) as well.

In the sequel, we will confine our attention to quadratic systems of the form \( (2) \), and specifically to those shown below (Corollary 2.3) to have a

**Theorem 3.** System \( (1) \) is globally asymptotically stable if and only if

(i) the eigenvalues of \( A \) have non-positive real parts;

(ii) there exist \( c \in \mathbb{R}^2 \) and \( D \in \mathbb{R}^{2 \times 2} \) such that \( B(x) = c^T x Dx \);

(iii) the pencil \((A,D)\) includes nodal values with bounded eigenvalues, and no other nodal values; any singular value of the pencil has a second root in \( \mathbb{C} \) if and only if \( A \) is non-singular.

In fact, according to [13], conditions (ii) and (iii) of this theorem are essentially necessary for the boundedness of solutions to any quadratic system \( (1) \) as well.

In the sequel, we will confine our attention to quadratic systems of the form \( (2) \), and specifically to those shown below (Corollary 3.2) to have a

**Theorem 3.** System \( (1) \) is globally asymptotically stable if and only if

(i) the eigenvalues of \( A \) have non-positive real parts;

(ii) there exist \( c \in \mathbb{R}^2 \) and \( D \in \mathbb{R}^{2 \times 2} \) such that \( B(x) = c^T x Dx \);

(iii) the pencil \((A,D)\) includes nodal values with bounded eigenvalues, and no other nodal values; any singular value of the pencil has a second root in \( \mathbb{C} \) if and only if \( A \) is non-singular.

In fact, according to [13], conditions (ii) and (iii) of this theorem are essentially necessary for the boundedness of solutions to any quadratic system \( (1) \) as well. In the sequel, we will confine our attention to quadratic systems of the form \( (2) \), and specifically to those shown below (Corollary 3.2) to have a

**Theorem 3.** System \( (1) \) is globally asymptotically stable if and only if

(i) the eigenvalues of \( A \) have non-positive real parts;

(ii) there exist \( c \in \mathbb{R}^2 \) and \( D \in \mathbb{R}^{2 \times 2} \) such that \( B(x) = c^T x Dx \);

(iii) the pencil \((A,D)\) includes nodal values with bounded eigenvalues, and no other nodal values; any singular value of the pencil has a second root in \( \mathbb{C} \) if and only if \( A \) is non-singular.

In fact, according to [13], conditions (ii) and (iii) of this theorem are essentially necessary for the boundedness of solutions to any quadratic system \( (1) \) as well.
single equilibrium state. It should be noted that the conditions of Theorems 1, 2, and 3 specify open sets in the space of coefficients of system (2).

We conclude this overview of the main result with an example. Consider the system

\[ \begin{align*}
  x_1 &= x_1 - x_3 + x_2(x_1 - x_2) \\
  x_2 &= x_1 + \alpha x_1 + x_2(x_1 + x_2)
\end{align*} \]

which may be written as (2) with \( A = \alpha I + J, D = I + J, \) and \( c = [1] \). We note that \( A \) is focal for all values of \( \alpha \), \( \text{tr} [A] = 2\alpha \), \( |JD| = -1 \), and \( |D^TJA| = (\alpha - 1)J \). When \( \alpha < 0 \) then the system satisfies the conditions of Theorem 3, and, hence, is globally asymptotically stable. When \( \alpha = 0 \) the system still satisfies the conditions of Theorem 3, even though the linear part of the field has pure imaginary eigenvalues. Systems of this nature, whose linearized equations are critical, necessitated a separate proof in [8].

3. SOME ALGEBRA OF THE PLANE

We will use the following notation throughout the paper. If \( x, y \in \mathbb{R}^2 \) then \( x^T y \) denotes the scalar product of \( x \) and \( y \). \( |x^T y| \) denotes the determinant of the array formed by the coordinates of \( x \) and \( y \). \( x^T J x \) denotes the orthogonal complement of \( x \), \( x^T J x = 0 \) if and only if \( x^T \phi \) is a linear equation with \( c \neq 0 \) and \( \phi \) a linear equation with \( c = 0 \). The following relation between inner products, determinants, and quadratic forms in \( \mathbb{R}^2 \) will be used extensively:

\[ |x^T y| = y^T x^T y = x^T J x. \]

**Lemma 1.** The linear transformation of the plane, \( A \), is focal, critical, nodal if and only if \( |JA| \) is sign definite, semi-definite, or indefinite, respectively.

**Proof.** Since \( x \) is an eigenvector of \( A \) if and only if \( |AX| = 0 \), \( A \) has no eigenvectors if \( x^T J A x \) never vanishes for \( x \neq 0 \). A unique eigenspace \( x^T J A x \) vanishes on a unique line, and two eigenvectors if the quadratic form \( x^T J A x \) vanishes on two lines. These are equivalent to the conditions that \( |JA| \) is definite, semi-definite, or indefinite, respectively.

**Corollary 3.1.** The conditions of Theorem 1 and Theorem 2 are equivalent.

**Proof.** According to Lemma 1 the conditions labelled (i) in each theorem
are equivalent. Condition (ia) of Theorem 2 guarantees that \( \lambda(x) \) is bounded and always negative. This implies that the pencil \( (A, D) \) has bounded and stable and only bounded and stable nodal values according to Lemma 1. Since the eigenvalues of \( A \) have positive real part, this satisfies condition (i) of Theorem 1. Similarly (iib) implies that the pencil has bounded and unstable and only bounded and unstable nodal values whose eigenvalues have opposite sign to be real part of the eigenvalues of \( A \). Thus (ia) and (iib) both imply (ii) of Theorem 1.

Conversely, if \( |JD| \) is not sign definite then the pencil \( (A, D) \) has arbitrarily large nodal values violating (ii) of Theorem 2, while if \( |D'JA| \) is not definite, a nodal value of the pencil has a zero eigenvalue, violating that condition as well. The necessity of the sign agreement and opposition condition is now evident.

**Corollary 3.2.** The conditions of Theorem 1 or 2 guarantee that (2) has a unique equilibrium state at the origin.

**Proof.** \( f \) cannot vanish at \( y \neq 0 \) unless \( |Ax, B(x)| = 0 \) on the line \( \langle c \rangle \). Since \( |Ax, B(x)| = c^T x |Ax, Dx| = c^T x x^T D^T JAx \), and the quadratic form is sign definite under the hypothesis, \( f \) could only vanish on \( \langle c \rangle \). However, \( B(c_1) = 0 \) while \( Ac_1 \neq 0 \), so this is impossible.

**4. Existence of Limit Cycles**

We now put the algebra of the preceding section to good geometric use. As shown in the proof of Corollary 3.1, the assumption that \( L(x) \) is bounded (and that \( A \) is focal) immediately implies that \( D \) is focal. The sign agreement condition may be interpreted to show that the spiral curve defined by a single loop of the linear trajectory, \( e^\delta y \), defines a positive-invariant region in the plane for arbitrarily large values of \( y \).

**Lemma 4.** Condition (ia) of Theorem 2 implies that all trajectories of system (2) remain bounded.

**Proof.** Choose a point \( y \), on \( \langle c \rangle \) whose sign is opposite to the sign of the real part of the eigenvalues of \( D \), say, on the positive ray. Let \( D \approx \{e^\delta y \mid x \in [0, r^*] ; \; e^\delta y = \gamma y ; \; 0 < \gamma < 1 \} \) be a complete spiral loop and let \( A \approx \{x \mid x \in [y, 1] \} \) join its end-points as depicted in Fig. 1.

In the sequel, we will denote the quadratic ratios in \( x \) defined by Lemma 3 as \( p(x) \) and \( A(x) \).

**Note** that there can be no cancellation of factors in \( p \) since \( A \) is focal.

**Lemma 5.** Condition (iib) of Theorem 2 implies that system (2) has bounded solutions for every initial condition outside a compact neighborhood of the origin.

The normal to the curve at any point \( x \in A \) lies in \( (JDx) \) and since \( x^T JDx = |Dx, x| \), \( JDx \) is either interior directed or exterior directed, depending upon whether \( |Dx, x| \) is negative or positive, respectively. With no loss of generality, we assume \( |Dx, x| < 0 \), hence \( JDx \) is the interior directed normal to \( A \) at \( x \). Similarly, \( Jy \) is the interior directed normal to \( A \) for any \( y \in A \). We must now show that \( f^T(x)JDx > 0 \) for \( x \in A \), and \( f^T(y)Jy > 0 \) for \( y \in A \). This will imply that any trajectory originating inside the spiral bounded region must remain within that region for all time. Since the region may be constructed arbitrarily far from the origin, that demonstration completes the proof.

Expanding the first inequality, we have

\[
f^T JDx = x^T [A^T + c^T x D^T ] JDx = x^T A^T JDx = |Dx, Ax| = -|Ax, Dx| > 0
\]

for all \( x \in A \). Expanding the second inequality, we have

\[
f^T Jy = -\gamma^2 Jy = -\gamma^2 Jy - c^T y [Jy, y]
\]

hence, because \( c^T y > 0 \) for \( y \in A \), and \( |Dy, y| < 0 \), the desired inequality holds when the second term dominates the first term far enough away from the origin.
Proof. Let $y$ be a point on $(c)$ whose sign is the same as the sign of the real part of the eigenvalues of $D$, say, on the negative ray. Let $d$ and $d'$ be as in the proof of Corollary 2.2, depicted in Fig. 1. Assume again with no loss of generality that $JDx$ is the interior directed normal to $d$ at $x$ and $Jy$ the interior normal to $A$ for $y \in A$. We need to show that $f^2JDx < 0$ for $x \in A$ and $f^2Jy < 0$ for $y \in A$. Since $[Ax, Dx]$ has opposite sign to $[Dx, y]$ under the assumption that the pencil has positive real eigenvalues, the first inequality follows for every spiral loop $d$. The second inequality holds on $A$ outside of the last loop for which $|y^T y|$ is less than the constant $|y^T Jy|/|y^T JDy|$. 

Having elucidated the geometric implications of the apparatus developed in Section 3, we are now able to show that a limit cycle must exist under the conditions of Theorem 1 or 2. According to the results of Lyapunov, the total stability behavior of system (2) is entirely determined by the spectrum of $A$. According to Lemmas 4 and 5, and Corollary 3.1, the global boundedness of system (2) is determined by the spectrum of the pencil $(A, D)$ in its nodal range. The following result depends crucially on the special nature of limit sets of planar dynamical systems established by the Poineer-Bendixson Theorem.

**Proposition 1.** The conditions (i) and (iiia) of Theorem 2 guarantee the existence of a stable limit cycle of system (2). The conditions (i) and (iiib) guarantee that an unstable limit cycle exists.

Proof. Assume that (i) holds, and the eigenvalues of $A$ have positive real parts. Then the origin is totally unstable, hence for some positive definite symmetric matrix $P$, $\mathbb{R}^2 = \{x | x^T Px < y\}$ for any $y > 0$ is a positive invariant set of system (2). If either version of (ii) holds, then the origin is the sole critical point of system (2), according to Corollary 3.2. By Lemma 4, if condition (iiia) of Theorem 2 holds, then all solutions of (2) are bounded; in particular, the Jordan Curve $J \cup A$ bounds a positive invariant set, $\mathcal{F}$, containing the origin. Thus $\mathcal{F} = \{x | x^T Px < y\}$ is a compact positive invariant set, free of critical points. In consequence of the Poineer-Bendixson Theorem, the positive limit set of a trajectory in $\mathcal{F} = \{x | x^T Px < y\}$ must be a limit cycle [14].

If the eigenvalues of $A$ have negative real part and condition (iiib) holds, then an identical argument concerning negative limit sets using Lemma 5 will establish the existence of a limit cycle.

While the question of necessity is not formally addressed in this paper, it is useful to remark upon the existence of limit cycles of (2) when the conditions of Theorem 2 are not met. Assuming (ii), condition (i) of Theorem 1 or 2 is certainly necessary according to the results of Coppel [3].

Note that when $A$ has purely imaginary eigenvalues and nodal values of the pencil $(A, D)$ have negative eigenvalues, Theorem 3 guarantees global asymptotic stability, while a similar argument establishes that all non-zero solutions of (2) grow without bound when the pencil $(A, D)$ has positive eigenvalues in this case (see the example). If (i) holds and $A + u(x)D$ has an eigenvalue for some $x \in \mathbb{R}^2$ then system (2) has at least one critical point distinct from the origin. On the other hand, given condition (i), there is a case where (ii) is violated due to a nodal matrix $D$, while the plane is left free of additional equilibrium states, and the possibility of a limit cycle remains. As will be seen below, there is good reason to suspect that system (i) cannot support a limit cycle unless $D$ has complex conjugate eigenvalues. Indeed, this would imply that the conditions of Theorem 2 are both necessary and sufficient for a quadratic system (1) with a single critical point to support a limit cycle.

5. **Uniqueness**

We finally show that the limit cycle established by Theorems 1 and 2 is indeed unique. Along the way we will restate the conditions of that theorem (Lemma 6, below) and provide a better intuitive sense of the mechanism underlying the isolated periodic solution. This is achieved by a transformation to polar coordinates.

Assuming $A$ has complex conjugate eigenvalues we may always find a coordinate system (under linear transformation of the state) such that $A = \alpha I + cuI$—where $I = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, and $\alpha$, $\omega \in \mathbb{R}$—and $\xi = \{\xi, \eta\}$. Then, defining the polar coordinate transformation $p \equiv |x^T + x_1^T|^2$.

We finally show that the limit cycle established by Theorems 1 and 2 is indeed unique. Along the way we will restate the conditions of that theorem (Lemma 6, below) and provide a better intuitive sense of the mechanism underlying the isolated periodic solution. This is achieved by a transformation to polar coordinates.

The conditions of Theorem 2 are both necessary and sufficient for a quadratic system (1) with a single critical point to support a limit cycle.

**Lemma 6.** The following conditions are equivalent to those stated in...
Theorem 2, and hence are sufficient for the existence of a limit cycle of system (3): either

(a) \( \sigma > 0 \) and \( \eta < 0 \) or, \( \eta > 0 \)
(b) \( \sigma < 0 \) and \( \eta > 0 \)

Proof. Since \( A = \sigma I + \omega \mathbf{a} I \), condition (i) of Theorem 2 is equivalent to one of the sign conditions on \( \sigma \). From Lemma 2, the eigenvalues of \( A + \mu(x)I \) are given by

\[
\lambda(x) = -\frac{\langle Ax, Dx \rangle}{\langle Dx, x \rangle} = -\frac{1}{\|Dx, x\|} (\sigma \langle x, Dx \rangle + \omega \langle Jx, Dx \rangle) = \frac{\eta}{d}.
\]

Thus, for \( \theta \in [-\pi/2, \pi/2] \), the sign conditions on \( \bar{d} \) and \( \eta \) are equivalent to condition (ii) of Theorem 2.

As reported in [8], limit cycles of quadratic differential equations enclose convex regions, hence, any periodic solution of (3) must have an angular derivative, \( \theta \), of constant sign; no limit cycle may leave the region \( \mathcal{U} \) by reflecting a point in the line through the origin containing the original point. This may be justified as follows: since \( \theta \) is sign definite, for every \( t \in [0, t_1] \) there exists a unique \( \theta \in [0, t_1 - t] \) and \( \eta > 0 \) such that

\[
\psi(x; t_1) = -\psi(x; t).
\]

For convenience we shall denote points on the right-hand curve, \( \Gamma_1 \), by \( r(t) \), and on the left-hand curve, \( \Gamma_2 \), by \( l(t) \), letting \( \rho \overset{\mathcal{U}}{\rightarrow} (\rho) \) and \( \lambda \overset{\mathcal{U}}{\rightarrow} (\lambda) \).

The chief advantage of this map is the induced functional dependence of \( r \) on \( \theta \); hence the ability to write a differential equation for \( \rho \) and \( \lambda \) using the same angular interval. From (3) and the above, we have, for fixed initial conditions,

\[
\frac{d}{dt} \ln \rho = \frac{\sigma - \nu \dot{\psi}}{\omega + \nu \dot{\psi}} \quad \eta \in [-\pi/2, \pi/2].
\]

\[
\frac{d}{dt} \ln \lambda = \frac{\sigma - \nu \dot{\psi}}{\omega + \nu \dot{\psi}} \quad \theta \in [-\pi/2, \pi/2].
\]

\[\text{(4)}\]

The restatement of Theorem 2 in Lemma 3 lends added insight into the mechanism by which \( x(t; p_0) \) grows and decays on \( \Gamma_1 \cup \Gamma_2 \). Considering case (i) of Lemma 3, since \( \bar{d} > 0 \) on \( [-\pi/2, \pi/2] \), the condition \( \eta < 0 \) necessitates \( d > 0 \) on that interval. Hence, from (4), while \( \rho \) must increase on \( \Gamma_2 \), \( \lambda \) becomes negative when \( \Gamma_2 \) enters the region \( \mathcal{W} = \{ x \in \mathbb{R}^2 | x < \psi(t) \} \) in the left half plane. Moreover, \( \mathcal{W} \) has a boundary, \( \partial \mathcal{W} \), in the left half plane and \( \eta \) implies \( \partial \mathcal{W} = \mathcal{W} \); i.e., that certain trajectories contained in \( \mathcal{W} \) must enter \( \mathcal{W} \). Since \( \dot{(\omega/\psi)} \rightarrow \infty \) as \( t \rightarrow \infty \), the growth of a trajectory on \( \Gamma_1 \) is countered with increasing effect on a portion of \( \Gamma_2 \), resulting in a limit cycle. Notice that if \( D \) has real eigenvalues then \( \bar{d} \) is no longer sign definite, hence \( \bar{d} \) may not be sign definite, and these remarks are no longer valid, underscoring the importance of the requirement that \( D \) be localizable.

The differential equations in (4) define two families of functions, \( \rho(\theta; p_0) \) and \( \lambda(\theta; p_0) \), parameterized by initial condition on the negative and positive \( t \)-axes, respectively. Observing that \( \lambda \sim \rho(\pi/2; p_0) \) -- i.e., that \( \lambda \) depends upon \( p_0 \) -- and that the vector fields in (4) are smooth when \( x \in \mathcal{W} \), we may implicitly regard \( \rho \) and \( \lambda \) as functions of \( \theta \) and \( p_0 \), continuously differentiable in both arguments. Since distinct integral curves of autonomous systems defined by smooth fields remain distinct over all time, we have \( \langle \dot{\psi}/\psi, \psi \rangle > 0 \) and \( \langle \dot{\lambda}/\lambda, \lambda \rangle > 0 \) for all \( \theta \in [-\pi/2, \pi/2] \). Hence, the function

\[
\psi(\rho_0) \overset{\mathcal{U}}{\rightarrow} \frac{\dot{\lambda}(\pi/2, \rho_0)}{\dot{\rho}(\pi/2, \rho_0)} \quad \rho_0
\]

\[\text{(5)}\]
which represents the ratio of the magnitudes of the end-points of the curve $q_{1} \cup q_{2}$ (both on the negative $x_{2}$-axis), is a continuously differentiable function of $p_{0}$. Evidently, $q_{1} \cup q_{2}$ is the integral curve of a limit cycle if and only if $\omega = 1$. The proof of uniqueness involves a demonstration that $\psi_{1}$ is monotone in $p_{0}$ over an interval of interest, and hence may pass through the origin at most once. That demonstration depends upon the following computation.

**Lemma 7.** Conditions (a) and (b) of Lemma 6, respectively, imply

(a) $\frac{d}{dp_{0}} \ln \left( \frac{\partial \psi_{1}}{\partial p_{0}} \right) < 2 \left( \frac{\partial \psi_{1}}{\partial \theta} \right) \left( \ln \lambda - \sigma/\omega \right)$.

(b) $\frac{d}{dp_{0}} \ln \left( \frac{\partial \psi_{1}}{\partial p_{0}} \right) > 2 \left( \frac{\partial \psi_{1}}{\partial \theta} \right) \left( \ln \lambda - \sigma/\omega \right)$.

**Proof.**

From (4) we have

$$\frac{\partial}{\partial p_{0}} \ln \left( \frac{\partial \psi_{1}}{\partial \psi_{0}} \right) = \frac{\partial}{\partial \psi_{0}} \ln \lambda + \frac{\partial}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial p_{0}}$$

and

$$\frac{\partial}{\partial p_{0}} \ln \left( \frac{\partial \psi_{1}}{\partial p_{0}} \right) = \frac{\partial}{\partial \psi_{0}} \ln \lambda + \frac{\partial}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial p_{0}}$$

Since $\dot{\psi}_{1} > 0$, we have $\omega/(\omega - \lambda\dot{\psi}_{1}) > 1$, and $\omega/(\omega + \dot{\psi}_{1}) < 1$, for all $\theta \in [-\pi/2, \pi/2]$. Hence, if condition (i) of Lemma 3 holds then

$$\lambda - \sigma/\omega \leq \frac{\partial}{\partial \psi_{0}} \ln \lambda - \sigma/\omega$$

and

$$\frac{\partial}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial p_{0}} \leq \frac{\partial}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial \psi_{0}} \frac{\partial \psi_{1}}{\partial p_{0}}$$

since $\eta < 0$ on $[-\pi/2, \pi/2]$ and substituting from (4).

We may now state the second principal result of this paper.

**Proposition 2.** Under the conditions of Theorem 2, system (2) has only one limit cycle.

**Proof.** $x(t; p_{0})$ is a limit cycle of (2) if and only if $\psi_{1}(p_{0}) = 1$ in system (3). According to Theorem 2, $\psi_{1}(p_{0}) < 1$ is non-empty, and bounded away from the origin, hence $p_{0} \triangleq \inf \psi_{1}(p_{0})$ exists and $p_{0} > 0$. We will show that $(d/dp_{0})\psi_{1}$ is sign definite for all $p_{0} > p_{0}^{*}$, hence $x(t; p_{0}^{*})$ is the only limit cycle of (2).

Note that

$$\frac{d}{dp_{0}} \psi_{1} = \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) - \psi_{1}^{1/2} e^{-\omega \omega_{0}}$$

We will show below that under condition (a) of Lemma 6,

$$\frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) < \psi_{1}^{1/2} e^{-\omega \omega_{0}}$$

and hence

$$\frac{d}{dp_{0}} \psi_{1} < \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) - \psi_{1}^{1/2} e^{-\omega \omega_{0}}$$

Since $\omega/\omega_{0} > 0$ and $\psi(\pi/2) = 1$, this is clearly negative for $p_{0} > p_{0}^{*}$. Similarly, under condition (b) of Lemma 6 the inequalities are reversed, and $\omega/\omega_{0} < 0$ so that $(d/dp_{0})\psi_{1}$ is sign definite for all $p_{0} > p_{0}^{*}$.

To obtain the bound on $(d/dp_{0})\lambda(\pi/2, p_{0})$ we recall that

$$\frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) = \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) - \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0})$$

$$- \ln \left[ \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) - \frac{\partial}{\partial p_{0}} \lambda(\pi/2, p_{0}) \right]$$

$$+ \int_{-\pi/2}^{\pi/2} \frac{d}{d\theta} \left[ \frac{\partial}{\partial \psi_{0}} \lambda(\theta, p_{0}) - \frac{\partial}{\partial \psi_{0}} \lambda(\theta, p_{0}) \right] d\theta,$$
Applying Lemma 7 to case (a) yields
\[ \ln \frac{\partial}{\partial \theta} \lambda(\pi/2, \rho_0) < \int_{\pi/2}^{\pi/2} 2 \left( \frac{\partial}{\partial \theta} \ln \rho \right) d\theta \]
\[ = \ln \left( \frac{\lambda(\pi/2, \rho_0)}{\lambda(-\pi/2, \rho_0)} \right)^2 = -2\pi/\omega, \]
hence \( (\partial/\partial \theta) \lambda(\pi/2, \rho_0) < \omega^2 e^{-2\pi/\omega} \) as claimed. Case (b) proceeds identically with the signs reversed.

6. Conclusions

This paper presents sufficient conditions for the existence of limit cycles of quadratic systems with a unique equilibrium state. The conditions guarantee that the limit cycle is unique. The results are based upon insights and techniques developed during an earlier investigation of the global stability properties of (1) [1], facilitated by the expression of that system in the form (2). They strongly suggest that these conditions are necessary as well, hence, that no quadratic system with a unique equilibrium state can support more than one limit cycle. That result, the uniqueness of a limit cycle around any equilibrium state of (2), the relation of limit cycles of (2) to those supported by general quadratic systems (1), all remain to be rigorously established.

Acknowledgments

This research was supported in part by Office of Naval Research Contract NOO014-76-C-0017 and in part by an NSF graduate research fellowship for the first author.

References

2. N. N. BAUTIN, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. (W.S.) 30 (52) (1952), 181-186; AMS Transl. No. 100 (1955).
3. I. G. PETROVSKII AND E. M. LANDO, On the number of limit cycles of the equation d/dt = P(x, y)Q(x, y) where P and Q are polynomials of the second degree, Mat. Sb. 37 (79) (1955), 209-230; AMS Transl. Series 2, 10 (1958).