More on the Reliability Function of the BSC

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Disciplines
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I. INTRODUCTION

Let $C(n, M = 2^M) \subseteq \{0, 1\}^n$ be a code of rate $R$ used over a binary symmetric channel with crossover probability $p$. Denote by $P_e(C, p)$ the average error probability of maximum likelihood decoding of $C$. The best attainable exponent $E(R, p)$ of that probability (optimized over the choice of codes for a given channel) is called the reliability function of the channel. The best known lower bounds on $E(R, p)$ were derived by Elias and Gallager. In particular, for $R$ between the critical rate of the channel $R_{cr} = \sqrt{p(1-p)}$ and the channel capacity $C = 1-h(p)$ the function $E(R, p)$ is known exactly (here $h(p)$ is the binary entropy function).

Sequential improvements of the upper bounds on $E(R, p)$ for low rates were obtained in [2], [3], [4]. The purpose of this paper is to present a new, tighter upper bound on $E(R, p)$.

II. THE RESULTS

We will need the following notation:

$$G(\alpha, \tau) = 2^{-2(1-\alpha) - \tau(1-\tau)}$$

$$h(\tau) = 1 - R - h(\alpha), \delta_{LP}(R) := \min_{0 \leq \tau \leq R} \{G(\alpha, \tau)\}$$

$$A(\omega) = \omega \log 2 \sqrt{p(1-p)}.$$

The results of [1] and [3] for low rates can be stated as follows:

$$-A(\delta_{LP}(R)) \leq E(R, p) \leq -A(\delta_{LP}(R)).$$

Recent improvements of error exponents for the BSC and the Gaussian channel [4, 5, 6] were obtained based on estimates of the distance distribution of an arbitrary code of a given rate $R$. Let $B_u, w = 0, 1, \ldots, n$ be the average distance distribution of the code $C$. In [4] it is proved that for any family of codes of sufficiently large length $n$ and rate $R$ and any $\alpha \in [0, 1/2]$ there exists a value $0 \leq \omega \leq G(\alpha, \tau)$ such that

$$n^{-1} \log B_u, w \geq \mu(R, \alpha, \omega) - o(1) \quad (1)$$

(the exact expression for $\mu$ is rather cumbersome and is omitted). We rely on the bound (1) together with a version of the estimation method of [6] to prove the following result.

**Theorem 1**

$$E(R, p) \leq -A(\delta_{LP}(R)) - R + 1 - h(\delta_{LP}(R)) \quad 0 \leq R \leq R^*_p \quad (2)$$

where $R^*_p$ is a certain value of the code rate, depending on $p$. For $R \geq R^*_p$

$$E(R, p) \leq \max_{0 \leq \lambda \leq \delta_{LP}(R)} \max_{\lambda \leq \omega \leq \delta_{LP}(R)} B(\omega, \lambda) - A(\lambda) \quad (3)$$

where

$$B(\omega, \lambda) = - \omega - (1-\omega)h(\omega) + \max_{\alpha \notin \frac{1}{2} \bmod \frac{1}{2}, \omega(1-\omega) \geq \omega(1-\omega)} \left( \omega \left( \frac{2\omega}{\lambda} \right) + (1-\omega)h \left( \frac{\omega}{\lambda} \right) \right),$$

+ $(1-\omega - \frac{\omega}{\lambda})h \left( \frac{\omega(1-\omega)}{1 - \omega - \frac{\omega}{\lambda}} \right)$.

Remarks 1. The bound (2) simply states that the error probability $P_e(C, p)$ for any code $C$ of large length $n$ cannot be smaller than the probability of incorrect decoding to a codeword at a distance $n\delta_{LP}$ from the transmitted codeword, multiplied by the number of such codewords in a random code.

2. An improvement of Theorem 1 over the results of [4] is in the range of code rates where the bound (2) can be claimed to be true. For instance, for $p = 0.01$ analysis of the results in [4] shows that (2) holds for $0 \leq R \leq 0.271$. Theorem 1 extends that range to $0 \leq R \leq R^*_p \approx 0.388$.

We can also apply the same estimation technique to codes with the binomial weight distribution: $B_w = \binom{n}{w} 2^{n-w}$. This question is of interest because almost all codes in the ensemble of all linear codes of rate $R$ for large $n$ have the weight distribution $B_w$. The result is as follows: there exists some value $R^*_w$, a function of $p$, such that for $R \leq R^*_w$ the lower bound on the error exponent of such a code coincides with the expurgation exponent $-A(\delta_{GV}(R))$. This complements the result of [7].

REFERENCES