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Jason DeVito
University of Pennsylvania, thedevitos@gmail.com

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The classification of simply connected biquotients of dimension at most 7 and 3 new examples of almost positively curved manifolds

Abstract

We classify all compact 1-connected manifolds M^n for $2 \leq n leq 7$ which are diffeomorphic to biquotients. Further, given that M is diffeomorphic to a biquotient, we classify the biquotients it is diffeomorphic to. Finally, we show the homogeneous space $Sp(3)\Sp(1) \tines Sp(1)$ and two of its quotients $Sp(3)\Sp(1) \times Sp(1) \times S^1$ and $\delta S^1 \backslash Sp(3)/Sp(1)\times Sp(1)$ admit metrics of almost positive curvature.
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Degree Type
Dissertation

Degree Name
Doctor of Philosophy (PhD)

Graduate Group
Mathematics

First Advisor
Wolfgang Ziller

Subject Categories
Geometry and Topology

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THE CLASSIFICATION OF SIMPLY CONNECTED BIQUOTIENTS OF DIMENSION AT MOST 7 AND 3 NEW EXAMPLES OF ALMOST POSITIVELY CURVED MANIFOLDS

Jason DeVito

A Dissertation in Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2011

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Wolfgang Ziller
Supervisor of Dissertation

____
Jonathan Block
Graduate Group Chairperson

Dissertation Committee:

Herman Gluck (Chair), Professor of Mathematics
Wolfgang Ziller, Professor of Mathematics, and
Stephen Shatz, Professor of Mathematics Emeritus

Acknowledgments

Where do I even begin? There are so many people without whom this thesis would never have come into existence. Let me start with the primary mathematical contributors. This list must, of course, begin with my amazing advisor Wolfgang Ziller. Between the weekly individual meetings and the department's worst kept secret, the "secret seminar", I had just the right impetus to keep making progress not only on my thesis, but in my understanding of geometry as a whole. To the secret seminar crowd, both current and past - Lee, Marco, Ricardo, Chenxu, John, Martin, Corey, Will, Jeffrey, Luigi, and all the other more sporadic attenders, I am sure I have plagued you with more questions than I should have and I thank you for both your answers and your patience. To the rest of the math department, I thank you as well for you ear at pizza seminar and your help, especially during my first and second year classes.

To the office staff, Janet, Monica, Paula, and Robin, without you all, the department would cease functioning. I don't think we really appreciate how hard it is to put up with all of us "crazy mathematicians". I also want to thank you for our daily office chats - they were always quite hilarious and relaxing.

To my family, I could not have made it here without your emotional support. To my beautiful wife, Sara, you have borne the brunt of this. You have bent over backwards to accommodate my weird mathematical schedule and the numerous trains I missed due to losing track of time. Thank you from the bottom of my heart. To my beautiful daughter Ava, you came at the worst of times, but you also came at the best of times. Thank you for being such an easy baby and I hope my mathematical stories continue to put you to sleep as quickly as they currently do.

To my parents, Ben and Lyn, your constant support while I was growing up and your drive for me to get as much education as possible are the primary reason why I'm where I am today. To my sister Jaimie, thank you for constantly showing me that every task is worth as much effort as it takes and to Jenna, thank you for allowing me to plant such weird math in your head at such a young age. Not too many 11 years olds know how to compute $\sqrt{-4}$.

To my extended family, thank you for your constant support and not ragging on me too much for being married while in school. Grandma, I'm sorry you're not here to celebrate the end of this journey, but I'm sure you're celebrating nonetheless.

Finally, I want to thank God, for whom none of this would be possible or meaningful at the most basic level.

ABSTRACT

The classification of simply connected biquotients of dimension at most 7 and 3 new examples of almost positively curved manifolds

Jason DeVito

Wolfgang Ziller, Advisor

We classify all compact 1-connected manifolds M^n for $2 \leq n \leq 7$ which are diffeomorphic to biquotients. Further, given that M is diffeomorphic to a biquotient, we classify the biquotients it is diffeomorphic to. Finally, we show the homogeneous space $Sp(3)/Sp(1) \times Sp(1)$ and two of its quotients $Sp(3)/Sp(1) \times Sp(1) \times S^1$ and $\Delta S^1 \backslash Sp(3)/Sp(1) \times Sp(1)$ admit metrics of almost positive curvature.

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Chapter 1

Introduction

Geometrically, a biquotient is the quotient of a Riemannian homogeneous space by a free isometric group action. Taking the free isometric group action to be the trivial action by the trivial group, we see that biquotients encompass the whole class of homogeneous spaces. Equivalently, if $U \subseteq G \times G$ is a subgroup, then U naturally acts on G by $(u_1, u_2) * g = u_1 g u_2^{-1}$. When the action is (effectively) free, the orbit space $G/\!\!/U$ naturally has the structure of a smooth manifold with canonical submersion $G \to G/\!\!/U$. If G is given a biinvariant metric, then U acts by isometries so $G/\!\!/U$ inherits a metric from G which is, by the O'Neill formulas [22], nonnegatively curved. When $U = H \times K \subseteq G \times G$, the orbit space is sometimes denoted $H\backslash G/K$.

Biquotients were introduced by Gromoll and Meyer [14] in 1974 when they showed that an exotic 7-dimensional sphere Σ^7 is diffeomorphic to a biquotient

$Sp(2)/\!\!/Sp(1)$. In fact, in this example, the biinvariant metric on $Sp(2)$ induces a metric on Σ^7 with quasi-positive curvature - nonnegative sectional curvature and a point $p \in \Sigma^7$ for which all sectional curvatures are positive. This was the first example of metric on an exotic sphere of nonnegative sectional curvature and the first nontrivial example of a metric with quasi positive curvature. Recently, it was shown by Petersen and Wilhelm [24] that this sphere in fact has a metric of positive sectional curvature. However, it was also shown by Totaro [29] as well as Kapovitch and Ziller [18], that the Gromoll-Meyer sphere is the unique exotic sphere of any dimension which is diffeomorphic to a biquotient.

Biquotients have also been very fruitful in the search for new examples of positively curved manifolds. In fact, every known example of a positively curved manifold, with the exception of one, is diffeomorphic to a biquotient. The one exception, due to Grove, Verdiani, and Ziller [15] and independently Dearricott [7] is known to be homeomorphic but not diffeomorphic to a biquotient. Further, every known examples of an almost positively curved manifold, a manifold which has an open dense set of points of positive curvature, and every known example of a quasipositively curved manifold is diffeomorphic to a biquotient.

Further, it turns out that biquotients share two of the key properties which make homogeneous spaces so useful: their topology is computable and often their geometry is as well. Also, as mentioned above, the O'Neill formulas [22] imply every biquotient has a metric of nonnegative sectional curvature. This makes them ideal for study as a means of creating and testing conjectures about the relationship between the topology and geometry of manifolds, especially for nonnegatively and positively curved manifolds.

Because of all of this, it seems useful to have a classification of the low dimensional simply connected biquotients. Further, since each description of a manifold as a biquotient gives rise to a new Riemannian metric on it, it is also worth classifying all the ways a given manifold can be written as a biquotient. Unfortunately, given a biquotient $G/\!\!/U$ the biquotient $\Delta G\backslash(G\times G)/U$ is canonically diffeomorphic to $G/\!\!/U$ via the map sending (g_1, g_2) to $g_1^{-1}g_2$. By iterating this, we see that, unlike in the homogeneous case, there are always infinitely many biquotients diffeomorphic to a given one and the dimension of groups involved go to infinity.

However, according to Totaro [29] every simply connected biquotient is diffeomorphic to one of the form $G/\!\!/U$ where G is simply connected, U is connected, and no simple factor of U acts transitively on any simple factor of G. It turns out that when restricted to this setting, only finitely many pairs of groups (G, U) can lead to the same diffeomorphism type of biquotient, though infinitely many different embeddings of U into $G\times G$ may still give the same diffeomorphism type. When we refer to classifying biquotients, we will mean in this seemingly stricter sense, where we further insist that no simple factor of U acts trivially.

We now summarize our results. There are no simply connected compact 1 dimensional manifolds. In dimension 2, there is a unique simply connected compact manifold, S^2 which is diffeomorphic only to the biquotient S^3/S^1. In dimension 3, there is again a unique simply connected compact manifold, S^3, which is diffeomorphic only to the biquotient $S^3/\{e\}$.

In dimension 4, we have

Theorem 1.0.1. *The simply connected compact 4-dimensional biquotient are precisely S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, or $\mathbb{C}P^2\sharp \pm \mathbb{C}P^2$. The manifolds $S^2 \times S^2$ and $\mathbb{C}P^2\sharp - \mathbb{C}P^2$ are diffeomorphic to infinitely many biquotients, while the other 3 manifolds are diffeomorphic to only finitely many biquotients.*

Of course, the homogeneous spaces S^4, $\mathbb{C}P^2$, and $S^2 \times S^2$ are classically known. Totaro [29] showed that the other two manifolds $\mathbb{C}P^2\sharp\pm\mathbb{C}P^2$ are each diffeomorphic to a biquotient. We show that this list is complete and classify all actions giving rise to these quotients.

In dimension 5, Pavlov [23] has already completed the classification of diffeomorphism types:

Theorem 1.0.2. *(Pavlov) The simply connected compact 5-dimensional biquotients are precisely S^5, the Wu manifold $SU(3)/SO(3)$, $S^2 \times S^3$, and $S^3\hat{\times}S^2$, the unique nontrivial S^3 bundle over S^2 with structure group $SO(4)$.*

We show that both $S^3 \times S^2$ and $S^3\hat{\times}S^2$ are diffeomorphic to infinitely many different biquotients while the other 2 manifolds are diffeomorphic to only finitely many.

Before stating the results for dimension 6 and 7, we need the following definition.

Definition 1.0.3. We say a biquotient $G/\!\!/U$ is naturally the total space of a fiber bundle if $G = G_1 \times G_2$ with both G_1 and G_2 nontrivial, $U = U_1 \times U_2$, U_1 acts trivially on $\{e\} \times G_2$ and U_2 acts freely on $\{e\} \times G_2$.

When $G/\!\!/U$ is naturally the total space of a fiber bundle the projection map $\pi : G_1 \times G_2 \to G_2 \to G_2/\!\!/U_2$ gives $G/\!\!/U$ the structure of a fiber bundle over $G_2/\!\!/U_2$ with fiber $G_1/\!\!/U_1$.

For example, suppose $G = G_1 \times S^3$ and $U = S^1$ which acts arbitrarily on the first factor and as the Hopf action on the S^3 factor. Then $G/\!\!/U$ naturally is the total space of a fiber bundle over S^2 with fiber G_1.

More generally, if one has two biquotient $G_1/\!\!/U_1$ and $G_2/\!\!/U_2$, then one can form a $G_1/\!\!/U_1$ bundle over $G_2/\!\!/U_2$ by considering $(G_1/\!\!/U_1) \times_{U_2} G_2$ where U_2 acts diagonally with an arbitrary action on $G_1/\!\!/U_1$ and the given effectively free action on G_2.

With this definition in hand, we can state

Theorem 1.0.4. *Suppose M is a simply connected 6 dimensional biquotient. Then one of the following holds:*

a) M is homogeneous or Eschenburg's inhomogeneous flag manifold.

b) M is naturally the total space of a fiber bundle.

c) M is diffeomorphic to $\mathbb{C}P^3 \sharp \mathbb{C}P^3$, $S^5 \times_{T^2} S^3$, or $S^3 \times S^3 \times S^3/\!\!/T^3$.

The manifolds in b) include both S^4 bundles over S^2, all 3 $\mathbb{C}P^2$ bundles over S^2, and many fiber bundles $S^2 \to G/\!\!/U \to B^4$ with B^4 a 4 dimensional biquotient. When $B^4 \neq S^4$, we show that every such bundle with structure group a circle is a biquotient. There are only 4 free isometric T^2 actions on $S^5 \times S^3$ which do not naturally give $G/\!\!/U$ the structure of a fiber bundle. There are only 12 free isometric T^3 actions on $\left(S^3\right)^3$ which do not naturally give $G/\!\!/U$ the structure of a fiber bundle.

We also classify which actions give rise to which quotients.

Finally, in dimension 7, we have

Theorem 1.0.5. *Suppose M is a simply connected 7 dimensional biquotient. Then one of the following holds:*

a) M is homogeneous or an Eschenburg Space.

b) M is either the Gromoll-Meyer sphere or M is naturally the total space of fiber bundle with fiber S^3 and base either S^4 or $\mathbb{C}P^2$.

c) M is diffeomorphic to $S^5 \times_{S^1} S^3$ or $(SU(3)/SO(3)) \times_{S^1} S^3$.

d) M is diffeomorphic to a biquotient of the form $S^3 \times S^3 \times S^3/\!\!/T^2$.

Unlike in dimension 6, there are infinitely many examples in c) and d) which are not naturally the total space of a fiber bundle.

One of the goals of this thesis was to find new examples of almost positively or quasipositively curved manifolds. While this particular hope did not come to fruition, studying these examples led to

Theorem 1.0.6. *The manifold $Sp(3)/Sp(1) \times Sp(1)$ has a metric with almost positive curvature. In this metric, there are precisely 4 free isometric actions by non-trivial connected compact Lie groups, 2 by S^3 and the other 2 coming from the circle* *subgroups in each S^3. Hence, the 4 quotient manifolds also inherit metrics of almost positive curvature.*

The manifold $Sp(3)/Sp(1) \times Sp(1)$ was shown by Tapp [27] to have a metric of quasipositive curvature. The manifold $Sp(3)/Sp(1)^3$ was shown by Wallach [1] to have a homogeneous metric of positive curvature. Wilking [32] has shown $\Delta Sp(1)\backslash Sp(3)/Sp(1) \times Sp(1)$ has a metric of almost positive curvature. It is not known whether the metric we endow $\Delta Sp(1)\backslash Sp(3)/Sp(1) \times Sp(1)$ is isometric to his, nor whether or not this metric has any 0 curvature planes.

This paper is organized as follows. The Chapter 2 will cover preliminaries on biquotients, including the computation of their cohomology rings and characteristic classes. Chapter 3 will cover the rational homotopy theory which will lead to a short list of the possible structures of $\pi_*(M) \otimes \mathbb{Q}$ when M is a low dimensional simply connected biquotient. By making repeated use of a theorem of Totaro's, chapter 4 will translate the list of possible rational homotopy groups into a list of pairs of groups (G, U) so that any low dimensional simply connected biquotient is diffeomorphic to $G/\!\!/U$ for some (G, U) on the list. We will also rule out many of the potential cases by elementary representation theory. Chapter 5 will, dimension by dimension, carry out the classification of all embeddings of U into $G \times G$ inducing effectively free actions and will characterize the diffeomorphism type of the quotients. Chapter 6 will talk about the three new examples of almost positively curved manifolds.

Chapter 2

Preliminaries

2.1 Basics of biquotient actions

The basics of biquotients are covered in detail in Eschenburg's Habilitation [10] (in German). We recall some of it here.

Definition 2.1.1. We say the action of a group G on a set X is effectively free if for all $g \in G$, if there is an $x \in X$ so that $gx = x$, then for all $y \in X$, $gy = y$.

If we let $G' = \{g \in G | gx = x \ \forall x \in X\}$, then G' is a normal subgroup of G and G/G' acts freely on X.

Proposition 2.1.2. *A biquotient action defined by $U \subseteq G \times G$ is effectively free iff for all $(u_1, u_2) \in U \subseteq G \times G$, if u_1 is conjugate to u_2 in G, then $u_1 = u_2 \in Z(G)$.*

We have an immediate corollary:

Corollary 2.1.3. *A biquotient action is free iff the action is free when restricted to the maximal torus of U.*

We also have

Proposition 2.1.4. *Consider the inner automorphism $C = C_{k_1,k_2} : G\times G \to G\times G$, conjugation by (k_1, k_2). Suppose $U \subseteq G \times G$. Then U acts effectively freely on G iff $C(U)$ acts effectively freely on G and the quotients are canonically diffeomorphic.*

Proof. Suppose initially that U acts effectively freely on G. Suppose $k_1u_1k_1^{-1}$ is conjugate to $k_2u_2k_2^{-1}$ in G, so there is a $g \in G$ with $gk_1u_1k_1^{-1}g^{-1} = k_2u_2k_2^{-1}$. Rearranging gives $u_2 = (k_2^{-1}gk_1)u_1(k_2^{-1}gk_1)^{-1}$ so that u_1 and u_2 are conjugate. Hence, we conclude $u_1 = u_2 \in Z(G)$ so $k_iu_ik_i^{-1} = u_i \in Z(G)$. Thus, the action of $C(U)$ is also effectively free.

The map sending gU to $k_1gk_2^{-1}C(U)$ is easily seen to be well defined diffeomorphism between the quotients. □

In particular, for any biquotient, we may assume without loss of generality that $T_U \subseteq T_{G\times G}$, where T_K denotes the maximal torus of the compact Lie group K.

Note that as a simple corollary, we have that if $G/\!\!/U$ is a biquotient, then $rk(U) \leq rk(G)$. For, if U acts freely on G, then so does T_U. But the action of T_U preserves a maximal torus T_G of G which must be free. But then $0 \leq dim(T_G/\!\!/T_U) = dim(T_G) - dim(T_U) = rk(G) - rk(U)$.

The profit of these facts is that, for a given pair (G, U), we can reduce the classification of all biquotients $G/\!\!/U$ to a representation theory problem.

That is, a homomorphism $f : G \to SU(n)$ is the same as a complex representation, a homomorphism $G \to SO(n)$ is a real representation, and a homomorphism $G \to Sp(n)$ is a symplectic representation. Further, if two embeddings are conjugate, then the representations they define are equivalent, and the converse is almost true.

More specifically, we have

Theorem 2.1.5. *(Mal'cev [21]) If two complex representations are equivalent, then the corresponding images in $SU(n)$ are conjugate. If two odd dimensional real representations are equivalent, then the corresponding images in $SO(2n+1)$ are conjugate. Likewise, if two symplectic representations are equivalent, then the corresponding images in $Sp(n)$ are conjugate. If two even dimensional real representations are equivalent, then they are conjugate in $O(2n)$, but not necessarily in $SO(2n)$. In fact, they will be conjugate in $SO(2n)$ unless every irreducible subrepresentation is even dimensional.*

2.2 Topology of Biquotients

In this section, we explain a method due to Eschenburg [11] for computing the cohomology rings of biquotients and one due to Singhoff [25] for computing characteristic classes. We will compute the cohomology rings of biquotients from a spectral sequence associated to a fibration given as the pullback of a canonical fibration. This will allow us to compute differentials in the spectral sequence.

Theorem 2.2.1. *(Eschenburg) Assume $U \subseteq G \times G$ defines a free biquotient action. Then there is a map $\phi_G : G/\!\!/U \to B(\Delta G)$ and commutative diagram of fibrations*

$$\begin{tikzcd}
G \arrow[d] & G \arrow[d] \\
G/\!\!/U \arrow[r, "\phi_G"] \arrow[d, "\phi_U"'] & B\Delta G \arrow[d] \\
BU \arrow[r, "Bf"] & BG \times BG
\end{tikzcd}$$

where the map from $G/\!\!/U$ to BU is the classifying map of the U principal bundle $U \to G \to G/\!\!/U$, the map from $B\Delta G$ to $BG \times BG$ is induced from the inclusion $\Delta G \to G \times G$, and the map from BU to $BG \times BG$ is induces from the inclusion of U in $G \times G$.

Remark 2.2.2. Eschenburg's theorem unfortunately doesn't directly apply when the action of U on G is merely effectively free. In these cases, we'll have to use a trick to apply this.

Thus, if we can understand both the differential in the Serre spectral sequence for the fibration $G \to BG \to BG \times BG$, and if we can understand the map Bf on cohomology, we can compute the differentials in the Serre spectral sequence for the fibration $G \to G/\!\!/U \to BU$. To carry this out, we must make a further assumption: we fix a coefficient ring R so that $H^*(G; R)$ is a free exterior algebra in generators $x_1, ..., x_n$ of degree $|x_k| = 2r_k - 1$. For example, $R = \mathbb{Q}$.

The next two propositions show how to compute the differential in the Serre spectral sequence for $G \to BG \to BG \times BG$.

Proposition 2.2.3. *Suppose $H^*(G; R) = \Lambda_R[x_1, ...x_k]$ with $|x_k| = 2r_k - 1$. Then cohomology ring $H^*(BG; R)$ is isomorphic to $R[\overline{x_1}, ..., \overline{x_k}]$ with $|\overline{x_i}| = |x_i| + 1$. The generators $\overline{x_i}$ can be chosen so that $dx_i = \overline{x_i}$ in the spectral sequence of the fibration $G \to EG \to BG$.*

Theorem 2.2.4. *(Eschenburg) Consider the fibration $G \to B_{\Delta G} \to B_G \times B_G$ where the second map is induced from the inclusion of ΔG into $G \times G$. Then in the spectral sequence for this fibration the elements x_i are totally transgressive and $dx_i = \overline{x_i} \otimes 1 - 1 \otimes \overline{x_i}$ with notation as in the previous lemma.*

We now try to understand the pullback f^* on cohomology. First note that for a torus $T = T^n$, we have $H^1(T; R) \cong Hom(\pi_1(T), R)$. Letting $\mathfrak{t}$ denote the Lie algebra to T, we set $\Gamma = exp^{-1}(e)$, where $exp : \mathfrak{t} \to T$ is the group exponential. Because every loop in T is homotopic to one of the form $t \to exp(t(a_1, .., a_n))$ uniquely, we have $\Gamma \cong \pi_1(T)$. Hence, we may think of $H^1(T) = Hom(\Gamma, R)$. By using the transgressions of elements of $H^1(T; R)$ as generators for $H^2(BT; R)$, we can think of every root and weight of T as an element of $H^2(BT)$.

The next lemma is useful for actually computing the pullback f^*:

Lemma 2.2.5. *Suppose $T_G \subseteq G$ is the maximal torus and let $i : T_G \to G$ be the inclusion map. Then the map $i^* : H^*(BG; R) \to H^*(BT; R)$ is 1-1 with image precisely $H^*(BT; R)^W$, the Weyl group invariant elements.*

Specializing this to the classical groups we have

Corollary 2.2.6. *Suppose $T^n \subseteq G$ where*

$$G \in \{U(n), SU(n+1), Sp(n), SO(2n), SO(2n+1)\}.$$

Let $x_1, ..., x_n$ be the generators of $H^1(T^n; R)$ and $\overline{x_i}$ be the corresponding generators of $H^2(BT^n; R)$. Then

if $G = U(n)$,

$$H^*(BG; R) = R[\sigma_1(\overline{x_i}), ..., \sigma_n(\overline{x_i})].$$

If $G = SU(n+1)$,

$$H^*(BG; R) = R[\sigma_2(\overline{x_i}), ..., \sigma_{n+1}(\overline{x_i})].$$

If $G = Sp(n)$,

$$H^*(BG; R) = R[\sigma_1(\overline{x_i}^2), ..., \sigma_n(\overline{x_i}^2)].$$

If $G = SO(2n)$,

$$H^*(BG; R) = R[\sigma_1(\overline{x_i}^2), ..., \sigma_{n-1}(\overline{x_i}^2), \sigma_n(\overline{x_i})].$$

And if $G = SO(2n+1)$,

$$H^*(BG; R) = R[\sigma_1(\overline{x_i}^2), ..., \sigma_n(\overline{x_i}^2)].$$

Here, $\sigma_i(\overline{x_i})$ denotes the ith elementary symmetric polynomial in the variables $\overline{x_i}$.

Remark 2.2.7. Note that the final generator of the cohomology ring of $BSO(2n)$ is not in terms of the square of the $\overline{x_i}$s.

There is a similar statement for computing the $\mathbb{Z}/2\mathbb{Z}$ cohomology of a biquotient. Here, one replaces the maximal torus with the maximal 2-group. The details can be found in [4].

Because of this and the fact that we can assume that for $U \subseteq G \times G$ that $T_U \subseteq T_G \times T_G$, it follows that we can simply compute $f^* : H^*(B(T_G \times T_G); R) \to H^*(BT_U; R)$ and restrict it to $H^*(B(G \times G); R) \to H^*(BU; R)$.

Finally, note that we can easily compute the map $f^* : H^*(BT_{G\times G}; R) \to H^*(BT_U; R)$ in terms of the induced map $f^* H^*(T_{G\times G}; R) \to H^*(T_U; R)$ by using naturality of

$$\begin{tikzcd}
T_U \arrow[r, "f"] \arrow[d] & T_{G\times G} \arrow[d] \\
ET_U \arrow[r] \arrow[d] & ET_{G\times G} \arrow[d] \\
BT_U \arrow[r, "Bf"] & BT_{G\times G}
\end{tikzcd}$$

In fact, we find that using x_i to denote the generators of $H^1(T_{G\times G}; R)$ and y_i for the generators of $H^1(T_U; R)$, if $f^*(x_i) = \sum a_i y_i$, then $Bf^*\overline{x_i} = \sum a_i \overline{y_i}$ where $\overline{x_i}$ are the transgressions of the x_i in the Serre spectral sequence for $T^l \to ET^l \to BT^l$ and likewise for the $\overline{y_i}$.

Now, to compute characteristic classes, we turn to the work of Singhoff.

Theorem 2.2.8. *(Singhoff)*

For $U \subseteq G \times G$ defining a free biquotient action, the total Pontryagin class of (the tangent bundle to) $G/\!\!/U$ is

$$p(G/\!\!/U) = \phi_G\big(\Pi_{\lambda\in\Delta G}(1+\lambda^2)\big)\phi_U\big(\Pi_{\mu\in\Delta U}(1+\mu^2)\big)^{-1}$$

where ΔK denotes the set of positive roots of K, and where ϕ_H and ϕ_G are the maps in the fibration at the beginning of this section. Likewise, the total Stiefel-Whitney class is given as

$$w(G/\!\!/U) = \phi_G\big(\Pi_{\lambda\in\Delta^2 G}(1+\lambda)\big)\phi_U\big(\Pi_{\mu\in\Delta^2 U}(1+\mu)\big)^{-1}$$

where $\Delta^2 G$ denotes the 2-roots of G.

The roots of the classical groups and G_2 are well known and are catalogued, for example, in [13]. The 2-roots are less well known, but can be found, for example, in [5].

In practice, the map ϕ_U on cohomology is computable as it's the edge homomorphism in the Serre spectral sequence for $G \to G/\!\!/U \to BU$. On the other hand, directly computing ϕ_G is difficult, but it's easy to see that, if $j : G \to G \times G$ is the diagonal map, then $Bj^* : H^*(B(G\times G); R) \to H^*(BG; R)$ is surjective, and the commutativity in the initial diagram shows $\phi_G^* Bj^* = \phi_H^* Bf^*$, so we can compute Bj^* using this.

Both Eschenburg's and Singhoff's results require the action be free, while we are allowing effectively free actions. As an example of where this discrepancy arises, consider the S^1 action on $S^3 \times S^3$ induced by the map $z \to \big((z, z^2), (z, 1)\big) \in (S^3 \times S^3)^2$. This map is clearly injective. However, the map induces the action $z * (p, q) = (zp\overline{z}, z^2 q)$ where $p, q \in S^3$. If $z \notin \{\pm 1\}$, then $z^2 q \neq q$ so these z act freely. If $z = \pm 1$, we see that z fixes every point of $S^3 \times S^3$. Hence the S^1 action on $S^3 \times S^3$ is effectively free, but not free, despite the fact that the embedding defining the action is injective.

The solution is to find a new action defined by $S^1 \to (U(2) \times U(2))^2$ which preserves $SU(2) \times SU(2)$ and has the same orbits through $SU(2) \times SU(2)$ as the original action. In this case, we map z to $(\mathrm{diag}(z, 1), \mathrm{diag}(z, \overline{z})), (\mathrm{diag}(z, 1), \mathrm{diag}(1, 1))$.

For the remainder of this section, let $G = SU(k_1) \times ... \times SU(k_n)$ and $G' = U(k_1) \times ... \times U(k_n)$,

By investigating the commutative diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G' \arrow[r, "="] \arrow[d] & G' \arrow[d] \\
G/\!\!/U \arrow[r] \arrow[d] & G'/\!\!/U \arrow[r] \arrow[d] & B\Delta G' \arrow[d] \\
BU \arrow[r, "="] & BU \arrow[r, "Bf"] & BG' \times BG'
\end{tikzcd}$$

and using naturality, it is easy to see that the differentials for the left most spectral sequence can be computed just as above, in terms of the induced map from $H^*(BT_{G' \times G'}; R)$ to $H^*(BT_U; R)$.

As far as characteristic classes are concerned, we have

Proposition 2.2.9. *The normal bundle of $G/\!\!/U$ in $G'/\!\!/U$ is trivial.*

Proof. First note that every 1 (real) dimensional representation of a compact Lie group is trivial. We know the normal bundle of G in G' is trivial - take a basis of normal vectors at $T_eG \subseteq T_eG'$ and use the group operation in G to translate this around. By the slice theorem, the normal bundle of $G/\!\!/U$ in $G'/\!\!/U$ is $G \times_U \mathbb{R}^n$.

We claim that the U action on $\mathbb{R}^n$ is trivial, so that the normal bundle of $G/\!\!/U$ is trivial in $G'/\!\!/U$. More precisely, $\mathbb{R}^n$ breaks into a sum of trivial one dimensional representations of U. This is because we have a chain of codimension one subgroups

$$\begin{aligned} G &\subseteq S\big(U(k_1) \times U(k_2)\big) \times ... \times SU(k_n) \\ &\subseteq ... \\ &\subseteq S\big(U(k_1) \times ... \times U(k_n)\big) \\ &\subseteq G' \end{aligned}$$

and each one is preserved by the U action, and hence, each (1 -dimensional) normal bundle is preserved. □

As a simple corollary, $i^*(TG'/\!\!/U)$ is stably isomorphic to $TG/\!\!/U$ and hence they have the same characteristic classes. Further, Singhoff's work guarantees we can compute the characteristic classes for $G'/\!\!/U$, and hence we can compute them for $G/\!\!/U$.

Chapter 3

Possibilities for $\bigoplus_k \pi_k(G/\!\!/U)_{\mathbb{Q}}$

In this section, we tabulate a list of all possible rational homotopy groups for all biquotients $G/\!\!/U$ of dimension smaller than or equal to 7 using rational homotopy theory.

Definition 3.0.10. A topological space X is called rationally elliptic if

$$dim(H_*(X) \otimes \mathbb{Q}) < \infty$$

and

$$dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$$

.

We'll use the notation $A_{\mathbb{Q}}$ to denote $A \otimes \mathbb{Q}$ for a $\mathbb{Z}$ module A.

Proposition 3.0.11. *Every biquotient is rationally elliptic.*

Proof. Since biquotients are compact, $dim(H_*(G/\!\!/U)) < \infty$.

Next, given a biquotient $G/\!\!/U$, we know that U acts on G effectively freely. If we let $U' = \{u \in U | \; u * g = g \; \forall g \in G\}$, the ineffective kernel of the action, then U' is a (discrete) normal subgroup of U and U/U' acts freely on G. Hence, we get a U/U' principal fiber bundle $G \to G/\!\!/U$. The long exact sequence of rational homotopy groups, together with the well known fact that all Lie groups are rationally elliptic implies the rational homotopy groups $\pi_k(G/\!\!/U)_{\mathbb{Q}}$ of $G/\!\!/U$ are 0 for sufficiently large k. □

Our chief tool for tabulating the possible rational homotopy groups of biquotients is the following which can be found in Félix, Halperin, and Thomas' book [12]:

Theorem 3.0.12. *Let X be a rationally elliptic topological space. Let n be the greatest integer so that $H^n(X)_{\mathbb{Q}} \neq 0$ (If X is a simply connected manifold, n is simply its dimension). Let x_i be a basis of $\pi_{odd}(X)_{\mathbb{Q}}$ and y_j a basis of $\pi_{even}(X)_{\mathbb{Q}}$. Then*

(1) $\sum deg(x_i) \leq 2n - 1$

(2) $\sum deg(y_j) \leq n$

(3) $\pi_i(X)_{\mathbb{Q}} = 0$ *for* $i \geq 2n$

(4) $dim(\pi_*(X))_{\mathbb{Q}} \leq n$

(5) $n = \sum deg\ (x_i) - \sum (deg\ (y_j) - 1)$

(6) $dim(\pi_{odd}(X))_{\mathbb{Q}} - dim(\pi_{even}(X))_{\mathbb{Q}} \geq 0$

(7) $\chi(X) = \sum(-1)^i dim(H_i(X))_{\mathbb{Q}} \geq 0$

(8) *We have equality in (6) iff we do not have equality in (7)*

Here, the dim refers to the dimension thought of as a $\mathbb{Q}$ *vector space and the degree of* $x_i \in \pi_k(X)_{\mathbb{Q}}$ *is* k*.*

We will also find the rational Hurewicz theorem useful. The proof can be found, for example, in [20].

Theorem 3.0.13. *Let* X *be a simply connected topological space with* $\pi_i(X)_{\mathbb{Q}} = 0$ *for* $0 < i < r$*. Then the Hurewicz map induces an isomorphism* $\pi_i(X)_{\mathbb{Q}} \cong H_i(X)_{\mathbb{Q}}$ *for* $1 \leq i < 2r - 1$ *and is a surjection for* $i = 2r - 1$*.*

We now handle each dimension one at a time.

3.1 n = 2, 3, 4 and 5

Proposition 3.1.1. *If* M *is a rationally elliptic 1-connected 2-manifold, then* M *has the same rational homotopy groups as the 2-sphere.*

Proof. By the Hurewicz theorem, $\pi_2(M)_{\mathbb{Q}} = \mathbb{Q}$. Now, since $\chi(M) = 2 > 0$, we have by (8), that there is precisely one odd degree element x. Then, using (5), we see that $2 = \deg\,(x) - 1$ so that $\deg\,(x) = 3$

□

Proposition 3.1.2. *If* M *is a rationally elliptic 1-connected 3-manifold, then* M

has the same rational homotopy groups as the 3-sphere. In fact, M is a homotopy 3-sphere.

Proof. By Poincare duality and the Hurewicz theorem, M is a simply connected homology sphere, which implies M is a homotopy sphere by Whitehead's theorem.

□

Proposition 3.1.3. *If M is a rationally elliptic 1-connected 4-manifold, them M has the same rational homotopy groups as either S^4, $\mathbb{C}P^2$, or $S^2 \times S^2$.*

Proof. We separate the proof into cases depending on $dim(\pi_2(M)_{\mathbb{Q}}) = k$. First notice that by (2), $0 \leq k \leq 2$. Also, since every simply connected 4-manifold has positive Euler characteristic, by (8), $dim(\pi_{odd}(M)_{\mathbb{Q}}) = dim(\pi_{even}(M)_{\mathbb{Q}})$.

If $k = 0$, then M is a rational homotopy sphere, so by the rational Hurewicz theorem, $\pi_4(M)_{\mathbb{Q}} = H_4(M)_{\mathbb{Q}} = \mathbb{Q}$. By (2), there are no more nonzero even rational homotopy groups and then, by (8), $dim(\pi_{odd}(M)_{\mathbb{Q}} = 1)$ with basis, say, x. Using (5), we can solve for the degree of x, to find that deg $(x) = 4 + 3 = 7$. Thus, M has the same rational homotopy groups as S^4.

If $k = 1$, we have by (2) that there are no other nonzero even homotopy groups. Further, since $\chi(M) > 0$, we again have that the dimension of the odd rational homotopy groups is 1, generated by x again. Then, using (5) we can solve for the degree of x to find that $\deg(x) = 4 + 1 = 5$. Thus M has the same rational homotopy groups as $\mathbb{C}P^2$.

If $k = 2$, we have by (2) that there are no other nonzero even homotopy groups.

Further, since $\chi(M) > 0$, we again have that the dimension of the odd rational homotopy groups is 2, generated by, say, x_1 and x_2. Then, using (5), we see that $\deg(x_1) + deg(x_2) = 4 + 2 = 6$. Since $\deg(x_i) \geq 3$, it follows that $\deg(x_1) = deg(x_2) = 3$. Thus, M has the same rational homotopy groups as $S^2 \times S^2$. $\square$

Proposition 3.1.4. *Let M be a rationally elliptic 1-connected 5-manifold. Then M has the same rational homotopy groups as either S^5 or $S^2 \times S^3$*

Proof. We again break it down into cases depending on $k = dim(\pi_2(M)_{\mathbb{Q}})$. By (2), $0 \leq k \leq 2$. Also, notice that since M is a 5-manifold, its euler characteristic is 0. So, by (8), the dimension of the odd rational homotopy groups is strictly larger than the dimension of the even rational homotopy groups.

If $k = 0$, then by Poincare duality and the rational Hurewicz theorem, we have $\pi_3(M)_{\mathbb{Q}} = \pi_4(M)_{\mathbb{Q}} = 0$. Then by (2), there can be no other nonzero even rational homotopy groups. But then, if x_i is a basis for the odd rational homotopy groups, (5) says that $5 = \sum \deg(x_i)$. Since $\deg(x_i) \geq 3$, it follows that $i = 1$ and the $\deg(x_i) = 5$. Thus, M has the same rational homotopy groups as S^5.

If $k = 1$, then by (2), there can be no other nonzero even rational homotopy groups. But then, if x_i is a basis for the odd rational homotopy groups, (5) says that $6 = \sum \deg(x_i)$. Again, since $\deg(x_i) \geq 3$, it follows that $i = 2$ and $\deg(x_1) = \deg(x_2) = 3$. Thus, we see that M has the same rational homotopy groups as $S^2 \times S^3$.

If $k = 2$, then by (8), $dim(\pi_{odd}(M)_{\mathbb{Q}}) \geq 3$ with basis x_1, x_2, x_3. But then by (1), we need deg(x_1)+ deg(x_2)+ deg(x_3) ≤ 9. Since deg(x_i) ≥ 3, this forces deg(x_i) $= 3$ for each i. But then, (5) tells us that $5 = 9 - 2$, giving a contradiction,

□

3.2 n=6

Proposition 3.2.1. *Let M be a rationally elliptic 1-connected 6-manifold. Then the dimension of the rational homotopy groups of M are given by the following chart, where a blank indicates the dimension is 0.*

2	3	4	5	6	7	8	9	10	11	Example
				1					1	S^6
	2									$S^3 \times S^3$
	2		1	1						M_1^6
1	1	1			1					$S^2 \times S^4$
1		1	2							M_2^6
1					1					$\mathbb{C}P^3$
2	1		1							$S^2 \times \mathbb{C}P^2$
3	3									$S^2 \times S^2 \times S^2$

Table 3.1: Possible dimensions of the rationally homotopy groups of a rationally elliptic 6-manifold

To our knowledge, the manifolds M_1^6 and M_2^6 do not have more common de-

scriptions.

We will later show that no biquotient has the same rational homotopy groups as M_1^6 and M_2^6.

Proof. Again, we break it down into cases depending on $k = dim(\pi_2(M)_{\mathbb{Q}})$. Before doing that, notice that $\chi(M) = 2 + 2dim(H_2(M)_{\mathbb{Q}}) - dim(H_3(M)_{\mathbb{Q}}) = 2 + 2k - dim(H_3(M)_{\mathbb{Q}})$ by Poincare duality and Hurewicz. Also, by (2), $k \leq 3$.

If $k = 0$, then by Hurewicz, $H_3(M)_{\mathbb{Q}} = \pi_3(M)_{\mathbb{Q}}$ so that they have the same dimension, say l. Then (7) implies that $l \leq 2$. We further break into subcases depending on l.

If $k = 0$ and $l = 0$, then we have $H_3(M)_{\mathbb{Q}} = 0$. Then, by Poincare duality and Hurewicz, we find that $\pi_i(M)_{\mathbb{Q}} = 0$ for $1 \leq i \leq 5$. But then $H_6(M)_{\mathbb{Q}} = \mathbb{Q} = \pi_6(M)_{\mathbb{Q}}$. By (2), there can be no other nonzero even rational homotopy groups. But now $\chi(M) > 0$ so (8) tells us the dimension of the odd rational homotopy groups is 1. Using (5), one can solve for the degree of this homotopy group to find it's 11. This gives the first entry in the table.

If $k = 0$ and $l = 1$, then we have $\chi(M) > 0$, so that by (8), the even and odd rational homotopy groups have the same dimension, which is at least 1 since $l = 1$. Since the smallest nontrivial even degree is 4, and since by (2) we have the sum of the even degrees is less than or equal to 6, it follows that the dimension of the evens is less than or equal to 1, so that the dimension of the evens and odds is 1. Then we can solve for the degree of the even piece using (5) to find that it's -2, given us a contradiction. Thus, this case can't happen.

If $k = 0$ and $l = 2$, then $\chi(M) = 0$, so we have more odds than evens. Now, by (1), there can be at most other one other nonzero odd rational homotopy group - π_5. If $\pi_5(M)_{\mathbb{Q}} = 0$, then by using (5), we have $6 = 6 + \sum(y_j - 1)$, so that there are no even degree nonzero rational homotopy groups. Thus, in this case we'd find that $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$ while all others are 0. This fills in the second table entry. If $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$, then by (5) we'll have $6 = 11 - \sum(\deg(y_j) - 1)$, so that $\sum(\deg(y_j) - 1) = 5$. Since each term $deg(y_j) - 1$ is odd and the sum is odd, it follows that there must be an odd number of summands. Since we know the number is less than 3, there must be precisely one. Thus, we can solve for the degree and find that it's 6. This fills in the third entry.

We now move on to the $k = 1$ case. By (2), the only other possible even degree nonzero rational homotopy group is 4. If $\pi_4(M)_{\mathbb{Q}}$, then by (6), we have at least 2 odds. By (5) we can solve for the sum of the degrees to find $\sum deg(x_i) = 6+1+3 = 10$. But for a sum of odd terms to be even, there must be an even number of them. If there are 4 or more, we'll have $\sum\deg(x_i) \geq 4 * 3 = 12 > 11$, contradiction (1). Thus, there are precisely two odd degree pieces. Then, it's clear that they must have degrees 3 and 7, or 5 and 5, filling in the 4th and 5th rows in the table.

Again with $k = 1$, if $\pi_4(M)_{\mathbb{Q}} = 0$, then again using (5) we find that $\sum\deg(x_i) = 7$, but this can only happen if there is a single summand with $\deg(x) = 7$. This fills in the 6th row of the table, and finishes up the $k = 1$ case.

If $k = 2$, then by (2), there are no other even degree pieces. Using (5) to solve for the degrees of the x_i (and noting there's at least 2 by (6)), we find $\sum deg(x_i) = 8$, so they must have degree 3 and 5, filling in the second to last row of the table.

Finally, if $k = 3$, then by (5), we have $\sum \deg(x_i) = 9$. Since there must be at least 3 odd degrees, we find that we have exactly 3, all of dimension 3, filling in the last row.

□

3.3 n=7

Proposition 3.3.1. *Let M be a rationally elliptic 1-connected 7-manifold. Then the rational homotopy groups of M are given in the table at the top of the next page.*

As in the previous case, the manifolds N_i^7 are not commonly known by other names.

As in the $n = 6$ case, we'll see that no biquotient has the same rational homotopy groups as any of the N_i^7.

Proof. Once again, we break it into cases depending on $k = dim(\pi_2(M)_{\mathbb{Q}})$. Since M is odd dimensional, $\chi(M) = 0$, so we know we always have more odd degree pieces then even degree pieces. We'll start with the $k = 0$ piece, and once again let $l = dim(\pi_3(M)_{\mathbb{Q}})$ in order to further break into subcases. By (1), $l \leq 4$.

If $k = 0$ and $l = 0$, then by Poincare duality and Hurewicz, the first nonzero

2	3	4	5	6	7	8	9	Example
					1			S^7
	1	1			1			$S^3 \times S^4$
	1			1			1	N_1^7
	2	2			1			N_2^7
	4			1				N_3^7
1	2	1	1					N_4^7
1	1		1					$S^2 \times S^5$ or $\mathbb{C}P^2 \times S^3$
2	3							$S^2 \times S^2 \times S^3$

Table 3.2: Possible dimensions of the rational homotopy groups of a rationally elliptic 7-manifold

rational homotopy group is $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$. By (1), there can be no other odd degree pieces. Thus, there are no even degree pieces, filling in the first row.

If $k = 0$ and $l = 1$, then by (1) we can either have no other odd nonzero rational homotopy groups, or $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$, $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$, or $\pi_9(M)_{\mathbb{Q}} = \mathbb{Q}$. If both are 0, then we have no even degree nonzero rational homotopy groups. But then by (5), $7 = 3 + 0$ an obvious contradiction. Thus, precisely one of $\pi_5(M)_{\mathbb{Q}}$ and $\pi_7(M)_{\mathbb{Q}}$ is nonzero. If $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$, then by (5), we have that $\sum(\deg(y_j) - 1) = 1$ with the sum having either zero or one term. If 0, we have $0 = 1$, giving us a contradiction. If 1 term, we find $\deg(y) = 2$, contradiction $k = 0$.

If $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$ then by (5), we find $\pi_4(M)_{\mathbb{Q}} = \mathbb{Q}$ and all the rest are 0. If $\pi_9(M)_{\mathbb{Q}} = \mathbb{Q}$, then we see that $\pi_6(M)_{\mathbb{Q}} = \mathbb{Q}$ with all others 0. These fill in the second two entries in the table, completing the $k = 0$, $l = 1$ case.

If $k = 0$ and $l = 2$, by (1), we find that either there are no other odd nonzero rational homotopy groups, or $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$ or $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$. If both are 0, then by (5), we have $\sum(\deg(y_j) - 1) = -1$, which can't happen. If $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$, then we have $\sum(\deg(y_j) - 1) = 4$. But this sum has at most 2 summands by (8) and since the sum is even, we must have exactly two summands. But then the only choices are clearly $\deg(y_1) = 2$ and $\deg(y_2) = 4$. But $k = 0$, so this is a contradiction.

If $k = 0$, $l = 2$, and $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$, then (5) gives us $\sum(\deg(y_j) - 1) = 6$, again with precisely two summands. Thus we find the degrees are both 4, giving the 4th entry in the table.

If $k = 0$ and $l = 3$, we have by (1) that there can be no other odd degree terms. Then (5) gives $2 = \sum(\deg(y_j) - 1)$, with at most two summands. The only way this can happen is if $\deg(y_1) = \deg(y_2) = 2$, but $k = 0$, giving us a contradiction.

If $k = 0$ and $l = 4$, we have by (1) that there are no other odd terms and by (5) $\sum(\deg(y_j) - 1) = 5$, where we sum at most 3 terms. Since the sum is odd, we must have an odd number of summands, so we have 1 or 3 summands. But since they each have degree at least 4, 3 summands is too many. Thus, there is a single summand of degree 6, filling in the 5th entry in the table and concluding the $k = 0$ case.

If $k = 1$, then there at least 2 odd degree nonzero rational homotopy groups. Further, by (2), either $\pi_4(M)_\mathbb{Q} = \mathbb{Q}$ or all other even degree rational homotopy groups are 0. If $\pi_4(M)_\mathbb{Q} = \mathbb{Q}$, then by (5), $\sum \deg(x_i) = 11$, and there are at least 3 summands by (8). From here, it's clear that they summands must have degree 3, 3, and 5, giving the 6th entry.

If $k = 1$ and $\pi_4(M)_\mathbb{Q} = 0$, then we have by (5) that $\sum \deg(x_i) = 8$, with at least 2 summands. Thus we find the summands have degree 3 and 5, filling in the 7th row of the table. This also completes the $k = 1$ case.

If $k = 2$, then by (2), there are no other even degree nonzero rational homotopy groups and thus there are at least 3 odd degree nonzero rational homotopy groups. Then (5) gives $\sum \deg(x_i) = 9$, from which it follows that there are precisely 3 summands each of degree 3, filling in the final entry in the table.

It remains to show that $k = 3$ cannot occur. If $k = 3$, then again by (2), there can be no other nonzero even degree rational homotopy groups. But then by (8) there must be at least 4 nonzero odd degree rational homotopy groups. By (1), the sum of the degrees must be less than 14, so that clearly there are exactly 4 nonzero odd degree rational homotopy groups, each of degree 3. But then (5) says that $7 = 12 - 3$, which is obviously false. Thus $k = 3$ cannot occur.

$\square$

Chapter 4

All possible pairs of groups

In this section, we present the classification of possible pairs of groups (G, U) for which there could possibly be a biquotient $G/\!\!/U$ of dimension less than or equal to 7. The results will only depend on the rational homotopy types of G and U.

Since $Sp(n)$ and $Spin(2n+1)$ have the same rational homotopy type this means that the existence of a pair (G, U) involving $Spin(2n+1)$ automatically implies the existence of a pair involving $Sp(n)$. Since $Spin(3) = Sp(1)$ and $Spin(5) = Sp(2)$, this new pair is actually the same as the old pair. However, $Sp(n)$ and $Spin(2n+1)$ for $n \geq 3$ are distinct compact Lie groups. Note that in many cases where $Spin(7)$ occurs, it's a simple factor of G and G_2 or $Spin(6) = SU(4)$ is a simple factor of U where the G_2 or $SU(4)$ can only act on $Spin(7)$. Since there are no $G_2 \subseteq Sp(3)$ nor $SU(4) \subseteq Sp(3)$, this extra case doesn't really arise. Likewise, $Spin(7)$ also often occurs as a simple factor of U with $Spin(8)$ the only simple factor of G on which

$Spin(7)$ can act. Since there are no $Sp(3) \subseteq Spin(8)$, this extra case doesn't really arise either.

Because of this, when classifying the pairs (G, U), we will leave off some pairs if they cannot be biquotients for the above reason.

Our main tools for the classification of pairs are the following two theorems due to Totaro [29]. Note that in the classification, the group U is only given up to finite cover.

Theorem 4.0.2. *Let M be compact and simply connected and assume $M = G/\!\!/U$. Then there is a 1-connected compact group G', and a connected group U' such that U' doesn't act transitively on any simple factor of G, such that $M = G'/\!\!/U'$.*

Definition 4.0.3. Given $M = G/\!\!/U$, let G_i be a simple factor of G. We say G_i contributes degree k to M if the homomorphism $\pi_{2k-1}(U) \to \pi_{2k-1}(G_i)$ in the long exact sequence of rational homotopy groups is not surjective.

The possible degrees every simple G_i could contribute have been previously tabulated. See, for example, [29]. We will later see that the exceptional groups F_4, E_6, E_7 and E_8 will not come up in the classification, so we'll ignore them in our table.

Proposition 4.0.4. *$SU(n)$ has degrees 2,3,4,...,n,*

$SO(2n+1)$ and $Sp(n)$ have degrees 2,4,6,...,2n,

$SO(2n)$ has degrees 2,4,...,2n-2, n, and

G_2 has degrees 2,6.

Theorem 4.0.5. *Let M be compact and simply connected and assume $M = G/\!\!/U$ with G 1 connected, U connected and such that U doesn't act transitively on any simple factor of G. Let G_i be any simple factor of G. Then at least one of the following 4 occurs.*

(1) G_i contributes its maximal degree.

(2) G_i contributes its second highest degree and there is a simple factor U_i of U such that U_i acts on one side of G_i and G_i/U_i is isomorphic to either $SU(2n)/Sp(n)$ for $n \geq 2$, or $Spin(7)/G_2 = S^7$, $Spin(8)/G_2 = S^7 \times S^7$, or E_6/F_4. In each of the four cases, the second highest degree is $2n-1$, 4, 4, or 9 respectively.

(3) $G_1 = Spin(2n)$ with $n \geq 4$, contributing degree n and there is a simple factor U_1 of U such that $U_1 = Spin(2n-1)$ acts on G_1 on one side in the usual way so that $G_1/U_1 = S^{2n-1}$.

(4) $G_1 = SU(2n+1)$ and there is a simple factor U_1 of U such that $U_1 = SU(2n+1)$ and U_1 acts on G_1 via $h(g) = hgh^t$. In this case, G_1 contributes degrees 2, 4, 6, ..., $2n$ to M.

From this, an easy corollary is:

Corollary 4.0.6. *Let $M = G/\!\!/U$ with G 1-connected, U connected, and U not acting transitively on any simple factor of G. Then each simple factor of G contributes at least one odd degree. In particular, the number of simple factors of G is bounded by $dim(\pi_{odd}(M_{\mathbb{Q}})$.*

From here on, when we write $M = G/\!\!/U$, we will always assume G is 1-connected,

U is connected, and U doesn't act transitively on any simple factor of G.

At this stage, we are ready to complete the biquotient classification in dimension 2 and 3.

Theorem 4.0.7. *Let M be a simply connected two or three dimensional biquotient, $M = G/\!\!/U$. Then M is diffeomorphic to S^2 or S^3, $G = SU(2)$, $U = S^1$ or $\{e\}$, and if $U = S^1$, the action is the Hopf action.*

Proof. Since we've already showed M has exactly one odd homotopy group in degree 3, it follows that $G = G_1$ is simple. It is easy to see that the only case of Totaro's theorem which can apply is (1), so G contributes its highest degree of 1. But this implies $G = SU(2)$. It follows that U is 0 or 1 dimensional, so $U = \{e\}$ or $U = S^1$.

Now, note that the only linear free action of S^1 on S^{2n-1} is, up to equivalence, the Hopf action. For, if we think of $S^{2n-1} \subseteq \mathbb{C}^n$ with complex coordinates $(p_1, ..., p_n)$, then any linear action, up to conjugacy, looks like $z(p_1, ..., p_n) = (z^{k_1}p_1, ..., z^{k_n}p_n)$ where we may assume without loss of generality that $\gcd(k_1, ..., k_n) = 1$. Now, if z is a k_ith root of 1, then z fixes $(0, ..., 1, ..., 0)$ where the 1 is in the ith coordinate and there are 0s elsewhere. Since the action is free, we must have $k_i = \pm 1$. But then, switching coordinates $p_i \to \overline{p_i}$ whenever $k_i = -1$ shows this action is equivalent to the Hopf action.

$\square$

Remark 4.0.8. Since, by Synge's theorem, S^1 cannot act freely and linearly on an even dimensional sphere, we get, as an immediate corollary to the proof, that the only free linear action of S^3 on a sphere (up to reparamaterization) is the Hopf action on S^{4k-1}. This follows because given any linear action of S^3 on a sphere, the action must restrict to the Hopf action of the circle subgroup. Hence, all the weights of the representation are ± 1, so the representation of S^3 is the direct sum of copies of the standard representation, i.e., the S^3 action is the Hopf action on $\mathbb{C}^{2k}$.

4.1 (G, U) for $dim(G /\!\!/ U) = 4$

Proposition 4.1.1. *Suppose M^4 is simply connected and has the same rational homotopy groups as S^4 and suppose further that $M = G/\!\!/U$ is a biquotient. Then G and U are one of the pairs in the following table.*

G	U
$SU(4)$	$SU(2) \times SU(3)$
$Sp(2)$	$SU(2) \times SU(2)$
$Spin(7)$	$G_2 \times SU(2)$
$Spin(8)$	$G_2 \times Sp(2)$
$Spin(8)$	$Spin(7) \times SU(2)$

Table 4.1: Potential pairs (G, U) with the same rational homotopy groups as S^4

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, it follows from Totaro's theorem that G is simple. Then, since M is 3-connected and $\pi_4(M)_{\mathbb{Q}} = \mathbb{Q}$, it follows from the long exact sequence of rational homotopy groups that $U = U_1 \times U_2$ is simply connected, consisting of precisely two simple factors.

Now, assume initially that G contributes it's maximum degree of 4 to M. Then $G = SU(4)$ or $G = Sp(2)$. If $G = SU(4)$, then by the long exact sequence in rational homotopy groups, we have $\pi_7(U)_{\mathbb{Q}} = 0$, $\pi_5(U)_{\mathbb{Q}} = \mathbb{Q}$, and $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q}+\mathbb{Q}$, so that $U = SU(3) \times SU(2)$. If instead, $G = Sp(2)$, then we find that $\pi_7(U)_{\mathbb{Q}} = \pi_5(U)_{\mathbb{Q}} = 0$, so that $U = SU(2) \times SU(2)$. This completes the first two entries in the table.

Now, assume that G doesn't contribute its highest degree (so we're in case 2 or 3 of Totaro's theorem). Then $G = SU(5), Spin(7), Sp(3)$, or $Spin(8)$ (but the case of $Sp(3)$ can't happen). Further, we know that one simple factor, say U_1 acts on G on one side and $G/U = S^7$ or $S^7 \times S^7$. First notice that unless $G = Spin(8)$, $G/U \neq S^7 \times S^7$ by looking at the long exact sequence of rational homotopy groups at degree 4.

If $G = SU(5)$, then if $G/U_1 = S^7$, we find from the long exact sequence of rational homotopy groups that $\pi_9(U_1)_{\mathbb{Q}} = \mathbb{Q}$ while $\pi_7(U_1)_{\mathbb{Q}}$, but this is impossible, so $G \neq SU(5)$.

If $G = Spin(7)$, we find that $U_1 = G_2$, the exceptional lie group of dimension 14. Then since $dim(G) - dim(U_1) - dim(U_2) = 4$, it follows that $U_2 = SU(2)$. Thus, $G = Spin(7)$ and $U = G_2 \times SU(2)$ works. This fills the next entry in the table

Finally, let $G = Spin(8)$. If $G/U_1 = S^7 \times S^7$, then it follows that $U_1 = G_2$. Then, again from dimension count, we have that $dim(U_2) = 10$ so that $U_2 = Sp(2)$.

If $G/U_1 = S^7$, it follows that $U_1 = Spin(7)$. Then again from dimension count, $U_2 = SU(2)$, filling in the last entry in the table.

□

Proposition 4.1.2. *Let M^4 be simply connected with the same rational homotopy groups as $\mathbb{C}P^2$. Suppose $M = G/\!/U$ is a biquotient. Then either $G = SU(3)$ and U is $SU(2) \times S^1$, or $G = SU(4)$ and U is $Sp(2) \times S^1$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, it follows that G is simple. Since $\pi_2(M)_{\mathbb{Q}} = \mathbb{Q}$, it follows that $U = S^1 \times U'$, where U' is a product of simple factors. Further, since $\pi_3(M)_{\mathbb{Q}} = \pi_4(M)_{\mathbb{Q}} = 0$, it follows that U' is simple.

Now, suppose G contributes is maximum degree of 3, so that $G = SU(3)$. Then from the long exact sequence of rational homotopy groups we see that $\pi_k(U)_{\mathbb{Q}} = 0$ for $k > 3$, so that $U' = SU(2)$. Thus, if $G = SU(3)$, $U = S^1 \times SU(2)$.

If G does not contribute its highest degree, then we must be in case 2. Thus, $G = SU(4)$ so the simple factor U' is a group acting on one side of G such that $G/U = SU(4)/Sp(2)$, i.e., $U' = Sp(2)$. □

Proposition 4.1.3. *Let M^4 be simply connected with the same rational homotopy groups as $S^2 \times S^2$. Suppose $M = G/\!/U$. Then $G = SU(2) \times SU(2)$ and $U = S^1 \times S^1$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}} = 2$, it follows that G has at most two simple factors. Since $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, it follows that G has at least two simple factors. Thus $G = G_1 \times G_2$. Then it again follows from the long exact sequence of rational homotopy groups that $\pi_3(U)_{\mathbb{Q}} = 0$, so that U is a product of circles. In particular, U has no simple factors so only case 1) of Totaro's theorem can occur. Thus, G_1 and G_2 contribute their maximum degree of 3, so that $G_1 = G_2 = SU(2)$. From dimension count, we see that $U = S^1 \times S^1$. □

4.2 (G, U) for $dim(G/\!\!/U) = 5$

Proposition 4.2.1. *Let M^5 be simply connected with the same rational homotopy groups as S^5. Assume $M = G/\!\!/U$ is a biquotient. Then we have either $G = SU(3)$ and $U = SU(2)$ or $G = SU(4)$ and $U = Sp(2)$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, G is simple. Since $\pi_3(M)_{\mathbb{Q}} = 0$, it follows that U has precisely one simple factor. Further since M is 4-connected, U is simple. Now, first assume G contributes is max degree of 3, so that $G = SU(3)$. Then, from dimension count, it follows that $U = SU(2)$. If G does not contribute its highest degree , then we're in the first part of the second case of Totaro's theorem. Thus, U acts on one side of $G = SU(4)$ and by the long exact sequence of rational homotopy groups, $U = Sp(2)$. □

Proposition 4.2.2. *Let M be simply connected with the same rational homotopy groups as $S^2 \times S^3$. Suppose $M = G/\!\!/U$ is a biquotient. Then $G = SU(2) \times SU(2)$ and $U = S^1$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}} = 2$, it follows that G has at most two simple factors.

Since $dim(\pi_3(M)_{\mathbb{Q}} = 2$, it follows that G has at least two simple factors so that $G = G_1 \times G_2$. Then, since $\pi_4(M)_{\mathbb{Q}} = 0$, it follows that U has no simple factors. Thus, we can only be in case (1) of Totaro's theorem. Thus, $G_1 = G_2 = SU(2)$. Then, by dimension count, $U = S^1$. □

4.3 (G, U) for $dim(G/\!\!/U) = 6$

In this subsection, we again catalogue which G and U are possible strictly from rational homotopy and dimension concerns.

Proposition 4.3.1. *Let M be simply connected with the same rational homotopy groups as S^6. Suppose $M = G/\!\!/U$ is a biquotient. Then either $G = G_2$ with $U = SU(3)$ or $G = Spin(7)$ with $U = Spin(6)$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, G is simple. Since $\pi_2(M)_{\mathbb{Q}} = \pi_3(M)_{\mathbb{Q}} = 0$, U is simple.

Now, assume G contributes its highest degree of 6. Then $G = G_2, SU(6), Spin(7)$ or $Spin(8)$ (the case $G = Sp(3)$ can't happen). If $G = G_2$, then from the long exact sequence in rational homotopy groups we have $\pi_5(U)_{\mathbb{Q}} = \pi_3(U)_{\mathbb{Q}} = Q$, so that $U = SU(3)$.

If $G = SU(6)$, we find that $\pi_9(U)_{\mathbb{Q}} = \mathbb{Q}$ while $\pi_7(U)_{\mathbb{Q}} = 0$, but this is impossible, so that $G \neq SU(6)$.

If $G = Spin(7)$ (or $Sp(3)$), we find that $\pi_7(U)_{\mathbb{Q}} = \pi_5(U)_{\mathbb{Q}} = \mathbb{Q}$ so that $U =$

$SU(4) = Spin(6)$.

Finally, if $G = Spin(8)$, we find $\pi_7(U)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$ while $\pi_5(U)_{\mathbb{Q}} = \mathbb{Q}$. But this is impossible. This handles the case of G contributing its maximal degree.

Note however, G cannot contribute a nonmaximal degree of 6 - if G contributes as in (2) of Totaro's theorem, then its second highest degree is of the form $2n - 1$ or is 4, so this can't happen. If G contributes as in (3) of Totaro's theorem, then $G = Spin(6)$ and $G/U = S^1 1$, giving an obvious contradiction. Further G can't contribute as in 4) since there is only a single simple factor. Thus, all examples come from case 1). $\square$

Proposition 4.3.2. *Let M be simply connected with the same rational homotopy groups of $S^3 \times S^3$. Suppose that $M = G /\!\!/ U$. Then $G = SU(2) \times SU(2)$ and U is trivial.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$, G has at most 2 simple factors. Since $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, G must have at least 2 simple factors. Thus $G = G_1 \times G_2$. Further, since $\pi_2(M)_{\mathbb{Q}} = 0$ it follows that U is simply connected with no simple factors. Thus $U = \{e\}$ as claimed. But cases (2),(3), and (4) of Totaro's theorem require U to contain a simple factor, so we see that both G_1 and G_2 contribute their maximum degree of 2. Thus $G_1 = G_2 = SU(2)$. $\square$

Proposition 4.3.3. *Let M be simply connected with the same rational homotopy groups as the M_1^6. That is, $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, $\pi_5(M)_{\mathbb{Q}} = \pi_6(M)_{\mathbb{Q}} = \mathbb{Q}$ with all other rational homotopy groups trivial. Further, assume $M = G /\!\!/ U$ is a biquotient.*

Then $G = SU(3) \times SU(2) \times SU(2)$ *and* $U = SU(3)$*.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 3$, we have that G has at most 3 simple factors. Since $dim(\pi_3(M)_{\mathbb{Q}}) = 2$, we have that G has at least 2 simple factors. Further, if G has 2 simple factors, U has none, while if G has 3 simple factors, U has precisely 1. Also, since M is rationally 2-connected, U is simply connected.

So, initially assume that $G = G_1 \times G_2$, that is, G has 2 simple factors. Since in this case U has no simple factors, but since $\pi_6(M)_{\mathbb{Q}} = \mathbb{Q}$, we must have that $dim(\pi_5(U)_{\mathbb{Q}}) \geq 1$, so that U contains at least one simple factor. This gives us a contradiction so this case cannot occur.

Now, assume $G = G_1 \times G_2 \times G_3$ and thus that U is simple. Now, since $\pi_6(M)_{\mathbb{Q}} = \mathbb{Q}$, it follows that $dim(\pi_5(U)_{\mathbb{Q}}) \geq 1$, so that $U = SU(n)$ for some $n \geq 3$. Now, assume without loss of generality that G_1 contributes degree 3 to M, so we must be in case (1) or (2) of Totaro's theorem. Then, G_1 contributes either its maximum degree or second highest, so that $G_1 = SU(3)$ or $SU(4)$. If $G = SU(4)$ contributes it's second highest degree of 5, then we're in the first part of the second case of Totaro's theorem. Thus, $U = Sp(2)$, contradicting the fact that $U = SU(n)$ for some n. Thus, $G_1 = SU(3)$.

Now, G_2 contributes a degree 2 and this is either the highest or second highest degree of G_2. If second highest, G_2 is coming from case 4 of Totaro's theorem, so that $G_2 = U = SU(3)$. But then $dim(M) = dim(SU(3)) + dim(SU(3)) + dim(G_3) - dim(SU(3)) > 8$, giving a contradiction. This similarly applies to G_3. Thus we find that $G_2 = G_3 = SU(2)$. Then, from dimension count, we find that $U = SU(3)$. $\square$

We have as an easy corollary

Corollary 4.3.4. *M_1^6 cannot be diffeomorphic to a biquotient.*

Proof. If $M_1^6 = G/\!/U$ were a biquotient, we have $G = SU(3) \times SU(2) \times SU(2)$ and $U = SU(3)$. But $SU(3)$ cannot act on $SU(2) \times SU(2)$ and the only free action on itself is transitive.

$\square$

Proposition 4.3.5. *Let $M = G/\!/U$ be a simply connected biquotient with the same rational homotopy groups as $S^2 \times S^4$. Then G and U are on the table on the following page:*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$, G has at most 2 simple factors. First assume that G is simple (so we can't be in case 4 of Totaro's theorem). By looking at the long exact sequence in rational homotopy groups, we see that U must contain precisely one simple factor U'. Now, if G contributes its max degree of 4 to M, then $G = SU(4)$ or $G = Sp(2)$. Then, from the long exact sequence in rational homotopy groups, we find that in the first case, $U = SU(3) \times S^1$ and in the second that $U = SU(2) \times S^1$. If G doesn't contribute its highest degree, then $G = SU(5), Spin(7), Sp(3)$, or $Spin(8)$ from case (2) and (3) of Totaro's theorem which implies $G/U' = S^7$ or $S^7 \times S^7$, ruling out $Sp(3)$. However, in either case, since there are no more simple factors in U, we'll find that, for example, $\pi_4(M)_{\mathbb{Q}} = \pi_4(G/U')_{\mathbb{Q}} = 0$, giving a contradiction.

G	U
$SU(4)$	$SU(3) \times S^1$
$Sp(2)$	$SU(2) \times S^1$
$SU(4) \times SU(2)$	$SU(3) \times SU(2) \times S^1$
$Sp(2) \times SU(2)$	$SU(2) \times SU(2) \times S^1$
$Spin(7) \times SU(2)$	$G_2 \times SU(2) \times S^1$
$Spin(8) \times SU(2)$	$Spin(7) \times SU(2) \times S^1$
$Spin(8) \times SU(2)$	$G_2 \times Sp(2) \times S^1$
$SU(4) \times SU(3)$	$SU(3) \times SU(3) \times S^1$
$Sp(2) \times SU(3)$	$SU(3) \times SU(2) \times S^1$
$Spin(7) \times SU(3)$	$G_2 \times SU(3) \times S^1$
$Spin(8) \times SU(3)$	$Spin(7) \times SU(3) \times S^1$

Table 4.2: Potential pairs (G, U) with the same rational homotopy groups as $S^2 \times S^4$

Next, assume $G = G_1 \times G_2$, and assume without loss of generality that G_1 contributes the degree 4 piece while G_2 contributes the degree 2 piece. From the long exact sequence in rational homotopy groups, we see that $U = U_1 \times U_2 \times S^1$, that is U contains precisely two simple factors. If both G_1 and G_2 contribute their maximal degrees, then $G_2 = SU(2)$ while $G_1 = SU(4)$ or $Sp(2)$. If $G_1 = SU(4)$, then we see that $\pi_5(U)_{\mathbb{Q}} = \mathbb{Q}$ and $\pi_3(U)_{\mathbb{Q}} = \mathbb{Q}+\mathbb{Q}$, so that $U = SU(3)\times SU(2)\times S^1$. We also see that if $G_1 = Sp(2)$, then U is $SU(2) \times SU(2) \times S^1$.

Now, assume that G_1 does not contribute its maximal degree while $G_2 = SU(2)$ contributes its maximal degree of 2. Then recalling we've already ruled out $G_1 = SU(4)$, we see $G_1 = Spin(7), Sp(3)$, or $Spin(8)$, though, as usual, the $Sp(3)$ case can't happen. Further, Totaro's theorem tells us $G_1/U_1 = S^7$ or $S^7 \times S^7$. Thus, if $G_1 = Spin(7)$, we see that $U_1 = G_2$, the exception Lie group. Then, from dimension count, we have that $dim(U_2) = 3$ so that $U_2 = SU(2)$. Thus, we have $G = Spin(7) \times SU(2)$ while $U = G_2 \times SU(2) \times S^1$.

If $G_1 = Spin(8)$, then $U_1 = Spin(7)$(if $G_1/U_1 = S^7$) or G_2 (if $G_1/U_1 = S^7 \times S^7$. If $U_1 = Spin(7)$, then again by dimension count, we have $dim(U_2) = 3$ so that $U_2 = SU(2)$. If instead, $U_1 = G_2$, then we have that $dim(U_2) = 10$ so that $U_2 = Sp(2)$.

If G_1 does not contribute from case (1) or (2) of Totaro's theorem, it must contribute by case (3) and then we'll have $G_1 = Spin(8)$ with $U_1 = Spin(7)$, and this has already been handled.

Now, if G_2 doesn't contribute its maximal degree, then G_2 must come from case 4 of Totaro's theorem. Thus, $G_2 = SU(3)$ and $U_1 = SU(3)$. We now figure out what G_1 can be. If G_1 contributes its highest degree of 4, then $G_1 = SU(4)$ or $Sp(2)$. Thus we find that $U_2 = SU(3)$ or $SU(2)$ respectively.

If G_1 does not contribute its maximal degree, then we see that G_1 is either $SU(5), Spin(7)$, or $Spin(8)$. Further, if G_1 comes from case 2 of Totaro's theorem, we have that there is a simple factor U' such that $G_1/U' = S^7$ or $S^7 \times S^7$. Notice that if $G_1/U' = S^7 \times S^7$, then $dim(\pi_7(M)_{\mathbb{Q}}) = 2$, giving a contradiction. We've also already seen that if $G_1 = SU(5)$, then there is no U' such that $G_1/U' = S^7$. If $G_1 = Spin(7)$, then we must have $U' = G_2$, the exceptional Lie group. In particular, in this case we have $U = SU(3) \times G_2 \times S^1$. Finally, if $G_1 = Spin(8)$, then we have $U' = Spin(7)$, so that $U = SU(3) \times Spin(7) \times S^1$.

Next, assume G_1 comes from case 3 of Totaro's theorem. Then we see that $G = Spin(8)$, and we've already handled this case. Since G_1 doesn't contribute degree 3, G_1 can't come from case 4 of Totaro's theorem and so we're done.

□

Proposition 4.3.6. *Let M be simply connected with the same rational homotopy groups as M_2^6 - that is $\pi_2(M)_{\mathbb{Q}} = \pi_4(M)_{\mathbb{Q}} = \mathbb{Q}$, $\pi_5(M)_{\mathbb{Q}} = Q + Q$, and all other rational homotopy groups are trivial. Suppose $M = G/\!\!/U$ is a biquotient. Then either $G = SU(3) \times SU(3)$ with $U = SU(2) \times SU(2) \times SU(2) \times S^1$ or $G = SU(4) \times SU(3)$ with $U = Sp(2) \times SU(2) \times SU(2) \times S^1$.*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$ while $dim(\pi_3(M)_{\mathbb{Q}}) = 0$, we conclude that either G is simple or $G = G_1 \times G_2$. However, if G is simple, then we have $dim(\pi_5(G)_{\mathbb{Q}}) \geq 5$, which is impossible for simple G. Thus, we have $G = G_1 \times G_2$ is a product of 2 simple factors. We also conclude from the long exact sequence in rational homotopy groups that U contains precisely 3 simple factors and so $U = U_1 \times U_2 \times U_3 \times S^1$.

Now, if both G_1 and G_2 contribute their maximum degrees of 3, then we conclude that $G_1 = G_2 = SU(3)$. Then from dimension count, we have $dim(U_1) + dim(U_2) +$

$dim(U_3) = 9$, so that $U_1 = U_2 = U_3 = SU(2)$.

If one, say G_1 doesn't contribute its maximal degree, then we have $G_1 = SU(4)$ and $G_2 = SU(3)$. Further, in order to contribute a nonmaximal degree of 5, this case must be the first part of case 2 of Totaro's theorem. Thus, we know a simple factor, say $U_1 = Sp(2)$. Then we find $dim(U_2) + dim(U_3) = 6$, so that $U_1 = U_2 = SU(2)$.

If both G_1 and G_2 fail to contribute their maximal degree, then we know that $G_1 = G_2 = SU(4)$. Further, we'd again be in the first part of case 2) of Totaro's theorem, so that $U_1 = Sp(2)$. But then we find that $\pi_5(U_1 \times U_2)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$ while $\pi_k(U_1 \times U_2)_{\mathbb{Q}} = 0$ for all higher k so that $U_2 = U_3 = SU(3)$. But then computing dimension gives $6 = dim(M) = 2dim(SU(5)) - dim(Sp(2)) - 2dim(SU(3)) - dim(S^1) = 3$, so we have a contradiction. $\square$

Corollary 4.3.7. *M_2^6 cannot be a biquotient.*

Proof. By the previous proposition, if $M = G/\!\!/U$, then $G = SU(3) \times SU(3)$ and $U = SU(2)^3 \times S^1$ or $G = SU(4) \times SU(2)$ and $U = Sp(2) \times SU(2)^2 \times S^1$. We will show none of these gives rise to free actions, even forgetting the S^1 factor of U.

Assume initially that $G = SU(3) \times SU(3)$ and $U = SU(2)^3$. There are precisely 2 nontrivial embeddings of $SU(2)$ into $SU(3)$, the block embedding and the map $SU(2) \to SO(3) \subseteq SU(3)$. It follows that 2 of the $SU(2)$s in U must act only on one $SU(3)$ each. If they act on the same one, then we either must have a free action of $SU(2)$ or $SO(3)$ on $SU(2)\backslash SU(3) = S^5$, but we've seen that this can't happen. Hence, we may assume the first two $SU(2)$s in U act on the left of $SU(3) \times SU(3)$

and the other $SU(2)$ in U acts diagonally on the right. But this can't happen either just by checking cases.

Next, assume that $G = SU(4) \times SU(2)$ and $U = Sp(2) \times SU(2)^2$. We already know that $Sp(2)$ can only act on one side of $SU(4)$ and can't act on $SU(2)$, so the biquotient must look like $S^5 \times_{SU(2)\times SU(2)} SU(2)$. At most one $SU(2)$ can act on the $SU(2)$ factor of G by conjugation (because it can't act transitively by convention), so the other $SU(2)$ factor of U must act freely on S^5. But we've already seen this is impossible.

□

Proposition 4.3.8. *Let M be simply connected with the same rational homotopy groups as $\mathbb{C}P3$. Assume $M = G/\!\!/U$ is written according to our conventions. Then G and U are given, up to finite cover, in the following table:*

G	U
$SU(4)$	$SU(3) \times S^1$
$Sp(2)$	$SU(2) \times S^1$
$Spin(7)$	$G_2 \times S^1$
$Spin(8)$	$Spin(7) \times S^1$

Table 4.3: Potential pairs (G, U) with the same rational homotopy groups as $\mathbb{C}P^3$

Proof. Since $\dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, G is simple. Then from the long exact sequence of rational homotopy groups we conclude U has exactly one simple factor.

If G contributes its maximum degree of 4, then $G = SU(4)$ or $Sp(2)$ and the long exact sequence of rational homotopy groups shows that U is $SU(3)$ or $SU(2)$ respectively.

If G does contribute its maximal degree, then $G = SU(5), Spin(7)$ or $Spin(8)$ and the simple factor U' of U acts only on one side of G with $G/U = S^7$ or $S^7 \times S^7$. Since there are no other simple factors, we can't have $G/U = S^7 \times S^7$, since otherwise $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}+\mathbb{Q}$, giving a contradiction. We have now handled this problem several times, so we know that $G = Spin(7)$ with $U = G_2$, or $G = Spin(8)$ with $U = Spin(7)$. Case 3) of Totaro's theorem is subsumed in the previous case, and case 4) can't happen as there is no degree 3 term.

$\square$

Proposition 4.3.9. *Let M be simply connected with the same rational homotopy groups as $S^2 \times \mathbb{C}P^2$. Assume $M = G/\!/U$ written according to our conventions. Then G and U are given in the following table:*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$, either G is simple or $G = G_1 \times G_2$. We also see from the long exact sequence of rational homotopy groups that U has one less simple factor than G. If G is simple, then U is a product of two circles. But then $\pi_5(G)_{\mathbb{Q}} = Q$ and $\pi_k(G)_{\mathbb{Q}} = 0$ for larger k, so that $G = SU(3)$. G cannot contribute via case 2), 3), or 4) of Totaro's theorem since in this case U has no simple factors. Thus, we're done with the case of G being simple.

So, assume $G = G_1 \times G_2$ with G_1 contributing degree 3 and G_2 contributing

G	U
$SU(3)$	$S^1 \times S^1$
$SU(3) \times SU(2)$	$SU(2) \times S^1 \times S^1$
$SU(4) \times SU(2)$	$Sp(2) \times S^1 \times S^1$
$SU(3) \times SU(3)$	$SU(3) \times S^1 \times S^1$

Table 4.4: Potential pairs (G, U) with the same rational homotopy groups as $S^2 \times \mathbb{C}P^2$.

degree 2. If they both contribute their maximal degrees, then $G_1 = SU(3)$ and $G_2 = SU(2)$. Then by dimension count (since U has precisely 1 simple factor) we have $U = SU(2) \times S^1$, up to cover.

If G_1 doesn't contribute its maximal degree, then $G = SU(4)$ and we're in the first part of case 2) of Totaro's theorem. Thus, the simple factor U' of U is equal to $Sp(2)$. Then, we have $dim(G_2) = 6 - dim(G_1) + dim(U) = 6 - 15 + 12 = 3$, so that $G_2 = SU(2)$.

If G_2 does not contribute its maximal degree, then $G_2 = SU(3)$ and the simple factor U' of U is equal to $SU(3)$ also. Notice that if G_1 doesn't contribute its highest degree, we have $U' = Sp(2)$ so that G_1 must contribute it's highest degree of 5. Then $G_1 = SU(3)$ $\square$

Proposition 4.3.10. *Let M be simply connected with the same rational homotopy groups as $S^2 \times S^2 \times S^2$. Assume $M = G/\!\!/U$ is a biquotient. Then $G = SU(2) \times$*

$SU(2) \times SU(2)$ and $U = S^1 \times S^1 \times S^1$.

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = dim(\pi_3(M)_{\mathbb{Q}}) = 3$, we have that $G = G_1 \times G_2 \times G_3$ contains precisely three simple factors. Further, U must contain none. Thus, case 2), 3), and 4) of Totaro's theorem can't happen, so that each G_i contributes its maximum degree. Thus, $G_i = SU(2)$ for each i, and so by dimension count, or by $dim(\pi_2(M)_{\mathbb{Q}})$, we have that $U = S^1 \times S^1 \times S^1$. □

4.4 (G, U) for $dim(G/\!\!/U) = 7$

Proposition 4.4.1. *Let M be simply connected with the same rational homotopy groups as S^7. Assume $M = G/\!\!/U$ is a biquotient. Then G and U appear on the following table:*

G	U
$SU(4)$	$SU(3)$
$Sp(2)$	$SU(2)$
$Spin(7)$	G_2
$Spin(8)$	$Spin(7)$

Table 4.5: Potential pairs (G, U) with the same rational homotopy groups as S^7

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 1$, G is simple. Further, from the long exact sequence of rational homotopy groups we see that U is simple and simply connected.

If G contributes its max degree 0f 4, then we have $G = SU(4)$ or $G = Sp(2)$. Correspondingly, we see that $U = SU(3)$ or $U = SU(2)$.

If instead, G does not contribute its maximal degree, then we know that $G/U = S^7$ and $G = SU(5), Spin(7)$, or $Spin(8)$. We done this several times now, giving the above result. $\square$

Proposition 4.4.2. *Let M be simply connected with the same rational homotopy groups as $S^3 \times S^4$. Assume $M = G/\!/U$ is a biquotient. Then G and U are given in the following table:*

G	U
$SU(4)$	$SU(3)$
$Sp(2)$	$SU(2)$
$SU(4) \times SU(2)$	$SU(3) \times SU(2)$
$Sp(2) \times SU(2)$	$SU(2) \times SU(2)$
$Spin(7) \times SU(2)$	$G_2 \times SU(2)$
$Spin(8) \times SU(2)$	$Spin(7) \times SU(2)$
$Spin(7) \times SU(3)$	$G_2 \times SU(3)$
$Spin(8) \times SU(3)$	$Spin(7) \times SU(3)$
$Spin(8) \times SU(2)$	$G_2 \times Sp(2)$

Table 4.6: Potential pairs (G, U) with the same rational homotopy groups as $S^3 \times S^4$

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$, we have that either G is simple or $G = G_1 \times G_2$.

Notice that U has the same number of simple factors as G.

If G is simple, assume first that G contributes its maximum degree of 4. Then $G = SU(4)$ or $Sp(2)$. From the long exact sequence of rational homotopy groups, we'll have that $U = SU(3)$ or $SU(2)$, respectively (with Dynkin index 0). Case 2) and 3) can't happen, because then we'd have $G/U = S^7$ or $S^7 \times S^7$, so that M would have the same rational homotopy groups as S^7, giving a contradiction. Finally, case 4) can't happen because G is simple.

Now, assume $G = G_1 \times G_2$ and $U = U_1 \times U_2$., with G_1 contributing degree 4 and G_2 contributing degree 2. If both G_1 and G_2 contribute their maximum degrees, we have that $G_1 = SU(4)$ or $Sp(2)$ and $G_2 = SU(2)$. Then we see that $U = SU(3) \times SU(2)$ or $U = SU(2) \times SU(2)$ respectively. If instead, G_2 does not contribute its maximal degree, then we find that $G_2 = SU(3)$ (only case 4 of Totaro's theorem allows a degree 2 contribution). In this case, U must contain $SU(3)$ as a simple factor as well as an S^1. By dimension count, the remaining factor must be either $SU(3)$ or $SU(2)$ for $G_1 = SU(4)$ or $Sp(2)$, respectively.

Next assume that G_1 does not contribute its maximal degree. We've seen several times then that $G_1 = Spin(7)$ with $U_1 = G_2$, the exceptional Lie group or $G_1 = Spin(8)$ with $U_1 = Spin(7)$ or G_2. In every case except $Spin(8)/G_2 = S^7 \times S^7$, we have that $dim(G_2) = dim(U_2)$. If G_2 contributes its highest degree, then $G_2 = SU(2) = U_2$. If instead, G_2 does not contribute its maximal degree, then by case 4) of Totaro's theorem, $G_2 = SU(3) = U_2$. In the $Spin(8)/G_2 = S^7 \times S^7$ case, we must have $\pi_7(U_2)_{\mathbb{Q}} = \mathbb{Q}$, so that we rule out case 4) for G_2. Thus, $G_2 = SU(2)$. Then by dimension considerations, we see that $U_2 = Sp(2)$. $\square$

Proposition 4.4.3. *Let M be simply connected with the same rational homotopy groups as N_1^7, that is, with $\pi_3(M)_{\mathbb{Q}} = \pi_6(M)_{\mathbb{Q}} = \pi_9(M)_{\mathbb{Q}} = \mathbb{Q}$ with all other rational homotopy groups trivial. Then M cannot be diffeomorphic to a biquotient.*

Proof. Assume for a contradiction that $M = G/\!\!/U$. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$ while $\dim(\pi_3(M)_{\mathbb{Q}}) = 1$, we have that G is either simple or $G = G_1 \times G_2$, and U is simply connected with one less simple factor than G. Thus, if G is simple, then U is trivial. But then we'll have $\pi_6(M)_{\mathbb{Q}} = 0$, giving a contradiction. Thus, $G = G_1 \times G_2$ and U is simple. Assume without loss of generality that G_1 contributes degree 5 while G_2 contributes degree 2.

If G_1 contribute its maximal degree, then $G_1 = SU(6)$. But then we'll have $\pi_5(G)_{\mathbb{Q}} = \mathbb{Q}$ so that we'll need $\pi_5(U)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, but this is impossible since U is simple. Thus G_1 must contribute a nonmaximal degree of 5 so that $G_1 = SU(6)$. In order to contribute degree 5, we must be in the first part of case 2 of Totaro's theorem, so that U has the same rational homotopy groups as $Sp(3)$, so $U = Sp(3)$ or $Spin(7)$. Then by dimension count, we have $dim(G_2) = 7 - 35 + 21 < 0$, giving a contradiction. $\square$

Proposition 4.4.4. *Let $M = G/\!\!/U$ be simply connected with the same rational homotopy groups as N_2^7 - that is $\pi_3(M)_{\mathbb{Q}} = \pi_4(M)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, $\pi_7(M)_{\mathbb{Q}} = \mathbb{Q}$, and all others are trivial. Then G and U are given, up to finite cover, in the following* *table:*

G	U
$SU(4) \times SU(2)$	$SU(3) \times SU(2)$
$Sp(2) \times SU(2)$	$SU(2) \times SU(2)$
$SU(4) \times SU(3)$	$SU(3) \times SU(3)$
$Sp(2) \times SU(3)$	$SU(2) \times SU(3)$
$Spin(7) \times SU(2)$	$G_2 \times SU(2)$
$Spin(8) \times SU(2)$	$Spin(7) \times SU(2)$
$Spin(7) \times SU(3)$	$G_2 \times SU(3)$
$Spin(8) \times SU(3)$	$Spin(7) \times SU(3)$
$Spin(8) \times SU(2)$	$G_2 \times Sp(2)$

Table 4.7: Potential pairs (G, U) with the same rational homotopy groups as N_2^7

Proof. Again, from $dim(\pi_{odd}(M)_{\mathbb{Q}})$ and $dim(\pi_3(M)_{\mathbb{Q}})$, we conclude that G is a product of either 2 or 3 simple groups. So, assume initially that $G = G_1 \times G_2$. It follows that $U = U_1 \times U_2$ is 1- connected. Assume without loss of generality that G_1 contributes degree 4 while G_2 contributes degree 2.

Now, assume initially that G_1 contributes its maximal degree of 4 so that $G_1 = SU(4)$ or $Sp(2)$. If G_2 contributes its maximal degree, then $G_2 = SU(2)$. Then we find $\pi_7(U)_{\mathbb{Q}} = 0$ (for either choice of G_1), and $\pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$ or 0 for $G_1 = SU(4)$ or $Sp(2)$ respectively. Thus we find that $U = SU(3) \times SU(2)$ or $SU(2) \times SU(2)$ respectively. If G_2 doesn't contribute its maximal degree, then we're in case 4 of

Totaro' theorem so $G_2 = SU(3)$ and $U_1 = SU(3)$. Then we find $dim(U_2) = 8$ so that $U_2 = SU(3)$ or $dim(U_2) = 3$, so that $U_2 = SU(2)$.

If instead G_1 does not contribute its maximal degree, then (as we've seen several times now), $G_1 = Spin(7)$ or $Spin(8)$ and there is a simple factor U_1 of U such that $G_1/U_1 = S^7$ or $S^7 \times S^7$. Thus, if $G_1 = Spin(7)$, then $U_1 = G_2$, the exceptional Lie group. If $G_1 = Spin(8)$, then $U_1 = Spin(7)$ or G_2. So, assume initially that $G_1/U_1 = S^7$. It follows from this that $dim(G_2) = dim(U_1)$. Since G_2 contributes degree 2, we must have $G_2 = U_1 = SU(2)$ or $SU(3)$.

If $G_1/U_1 = S^7 \times S^7$, then we have $G_1 = Spin(8)$ and $U_1 = G_2$, the exceptional Lie group. If G_2 does not contribute its maximal degree, then we'll have $U_2 = SU(3) = G_2$, so that $dim(M) = 14$, a contradiction. Thus $G_2 = SU(2)$. Then, from dimension count, we see that $U_2 = Sp(2)$.

□

Corollary 4.4.5. *The manifold N_2^7 is not a biquotient.*

Proof. Because in each of the possibilities above, G and U both consist of 2 simple factors, the only way to have $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q} = \mathbb{Q}$ is if the map $\pi_3(U)_{\mathbb{Q}} \to \pi_3(G)_{\mathbb{Q}}$ is the 0 map, but this map is given as the differences in the Dynkin indices of the left and right embeddings. Since, in every case it is easy to see that there is a simple factor of U which acts only one side of one factor of G, it follows that the difference in the Dynkin indices cannot be 0. □

Proposition 4.4.6. *Let M be simply connected with the same rational homotopy* *groups as N_3^7 - $\pi_3(M)_{\mathbb{Q}} = \mathbb{Q}^4$, $\pi_6(M)_{\mathbb{Q}} = \mathbb{Q}$ with all others trivial. Then M is not diffeomorphic to a biquotient.*

Proof. Assume that $M = G/\!\!/U$. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = dim(\pi_3(M)_{\mathbb{Q}}) = 4$, we see that $G = G_1 \times G_2 \times G_3 \times G_4$, with each G_i simple. Further, from the long exact sequence in rational homotopy groups, it follows that U contains no simple factors. Thus, $\pi_6(M)_{\mathbb{Q}} \neq \mathbb{Q}$, a contradiction. $\square$

Proposition 4.4.7. *Let M be simply connected with the same rational homotopy groups as N_4^7 - $\pi_2(M)_{\mathbb{Q}} = \pi_4(M)_{\mathbb{Q}} = \pi_5(M)_{\mathbb{Q}} = \mathbb{Q}$ and $\pi_3(M)_Q = \mathbb{Q} + \mathbb{Q}$, with all others trivial. Assume $M = G/\!\!/U$. Then G and U appear in the following table:*

G	U
$SU(3) \times SU(2)$	$SU(2) \times S^1$
$SU(4) \times SU(2)$	$Sp(2) \times S^1$
$SU(3) \times SU(3)$	$SU(3) \times S^1$
$SU(3) \times SU(2) \times SU(2)$	$SU(2) \times SU(2) \times S^1$
$SU(4) \times SU(2) \times SU(2)$	$Sp(2) \times SU(2) \times S^1$
$SU(3) \times SU(3) \times SU(2)$	$SU(3) \times SU(2) \times S^1$
$SU(3) \times SU(3) \times SU(3)$	$SU(3) \times SU(3) \times S^1$
$SU(4) \times SU(3) \times SU(2)$	$Sp(2) \times SU(3) \times S^1$

Table 4.8: Potential pairs (G, U) with the same rational homotopy groups as N_4^7

Proof. We have that $G = G_1 \times G_2$ or $G = G_1 \times G_2 \times G_3$ with each G_i simple. Further, U has one less simple factor than G.

So, assume initially that $G = G_1 \times G_2$ so that $U = U_1 \times S^1$ up to finite cover. Assume without loss of generality that G_1 contributes degree 5 and G_2 contributes degree 3. Then if both contribute their top degree, we have $G_1 = SU(3)$ and $G_2 = SU(2)$. Then it follows from dimension count that $U_1 = SU(2)$. If instead G_1 contributes it's second highest degree, then we are in the first part of case 2 of Totaro's theorem so that $G_1 = SU(4)$ and $U_1 = Sp(2)$. Notice in this case, G_2 must contribute its top degree (or else $U_1 = SU(3)$ coming from case 4 of Totaro's theorem, a contradiction). Conversely, assume G_2 contributes its second highest degree (and thus, G_1 contributes its highest) so we're in case 4 of Totaro's theorem. Thus, $G_2 = U_1 = SU(3)$. This concludes the case where $G = G_1 \times G_2$ is a product of 2 simple factors.

So, now assume that $G = G_1 \times G_2 \times G_3$, with G_1 contributing only degree 5 and G_2 and G_3 contributing only degree 3. Write $U = U_1 \times U_2 \times S^1$. If all three contribute their maximum degree, then we have $G_1 = SU(3)$, $G_2 = G_3 = SU(2)$. Then we find $dim(U_1) + dim(U_2) = 6$, so that $U_1 = U_2 = SU(2)$. If G_1 contributes its second highest degree, then we have $G_1 = SU(4)$ and $U_1 = Sp(2)$. If G_2 and G_3 contribute their max, then $G_2 = G_3 = SU(2)$ and by dimension count, we find $U_2 = SU(2)$.

If G_2 contributes its second highest degree, then we find that $G_2 = U_1 = SU(3)$.

If G_1 contributes its highest, $G_1 = SU(3)$. But then we have that $dim(G_3) = dim(U_2)$. Thus we find that $G_3 = U_2$ and these are equal to either $SU(2)$ or $SU(3)$. If G_2 contributes its second highest degree while G_1 contributes its highest, we find that $U = Sp(2) \times SU(3) \times S^1$, so by dimension count $G_3 = SU(2)$. □

Corollary 4.4.8. *The manifold N_4^7 cannot be a biquotient.*

Proof. Since $\pi_3(N_4^7)_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}$, if G has precisely two simple factors, then the map $\pi_3(U)_{\mathbb{Q}} \to \pi_3(G)_{\mathbb{Q}}$ must be the 0 map. But just as in the case of N_2^7, this can't happen.

Now we handle the case where G has three simple factors. For the first two entries, there is always a simple factor of U isomorphic to $SU(2)$ which must act freely on $SU(3)/SU(2)$ or $SU(3)/SO(3)$, but it's easy to see there are no such free action of $SU(2)$ on these spaces. For the remaining three entries, there is always a simple factor $SU(3)$ which either has to act freely on $SU(3)$ or $SU(4)/Sp(2) = S^5$, but in the first case the only such free actions are transitive and in the second, there are no free actions by dimension counting. □

Proposition 4.4.9. *Let M be simply connected with the same rational homotopy groups as $S^2 \times S^5$ (or $S^3 \times \mathbb{C}P^2$). Assume $M = G/\!\!/U$. Then G and U are in the table on the following page:*

Proof. Since $dim(\pi_{odd}(M)_{\mathbb{Q}}) = 2$ while $dim(\pi_3(M)_{\mathbb{Q}}) = 1$, we have that G is simple or $G = G_1 \times G_2$. Further U has one less simple factor than G.

Manifold Model: $S^3 \times \mathbb{C}P^2$ or $S^2 \times S^5$

G	U
$SU(3)$	S^1
$SU(3) \times SU(2)$	$SU(2) \times S^1$
$SU(3) \times SU(3)$	$SU(3) \times S^1$
$SU(4) \times SU(2)$	$Sp(2) \times S^1$

Table 4.9: Potential pairs (G, U) with the same rational homotopy groups as $S^2 \times S^5$ or $S^3 \times \mathbb{C}P^2$

If G is simple, then $U = S^1$ has no simple factors and hence we must be in case 1 of Totaro's theorem. Then G contributes its maximal degree of 3 so $G = SU(3)$.

If instead $G = G_1 \times G_2$, then $U = U_1 \times S^1$. Assume without loss of generality that G_1 contributes degree 3 while G_2 contributes degree 2. Notice that if G_1 does not contribute its maximal degree, then $U_1 = Sp(2)$ and likewise, if G_2 doesn't contribute its maximal degree, then $U_1 = SU(3)$. Thus, at least one must contribute its maximal degree.

So assume initially that G_1 contributes its max degree of 3 so that $G_1 = SU(3)$. It follows then that $dim(U_1) = dim(G_2)$. Then $G_2 = U_1 = SU(2)$ or $SU(3)$ depending on whether or not G_2 contributes its maximal degree.

If G_1 does not contribute its maximum degree, then it falls under the first part of case 2 of Totaro's theorem so that $G_1 = SU(4)$ and $U_1 = Sp(2)$. Then G_2 must contribute its maximal degree so $G_2 = SU(2)$. $\square$

Proposition 4.4.10. *Suppose M is simply connected with the same rational homotopy groups as $S^2 \times S^2 \times S^3$. Assume that $M = G /\!\!/ U$ is a biquotient. Then $G = SU(2) \times SU(2) \times SU(2)$ and $U = S^1 \times S^1$.*

Proof. Since $\dim(\pi_{odd}(M))_{\mathbb{Q}} = \dim(\pi_3(M)_{\mathbb{Q}}) = Q$, $G = G_1 \times G_2 \times G_3$ is a product of 3 simple groups. From the long exact sequence in rational homotopy groups it follows that U has no simple factors. Thus each G_i contributes its maximal degree of 2, so each $G_i = SU(2)$. Then, counting dimensions give $U = S^1 \times S^1$.

□

Chapter 5

Classifying Free Actions and Quotient Diffeomorphism Type

5.1 4 dimensional biquotients

In this section, we'll handle the different G and U in the S^4 chart.

Proposition 5.1.1. *If $G = SU(4)$ and $U = SU(2)\times SU(3)$, then the only biquotient is of the form $SU(2)\backslash SU(4)/SU(3) = S^3\backslash S^7 = S^4$. The embedding of $SU(3)$ into $SU(4)$ is the block embedding and the embedding of $SU(2)$ into $SU(4)$ is the diagonal block embedding.*

Proof. There is no almost faithful $f : SU(2) \times SU(3) \to SU(4)$ since the smallest almost faithful representation of $SU(2) \times SU(3)$ has dimension 5.

Thus, the biquotient will be of the form $SU(2)\backslash SU(4)/SU(3)$.

Now, $SU(3)$ has precisely 2 nontrivial 4-dimensional representations, up to equivalence - the standard plus one trivial, and the conjugate of the standard plus one trivial. Regardless of which representation we use, the quotient is the homogeneous space $SU(4)/SU(3) = S^7$. Choosing a particular left invariant right $SU(3)$ invariant metric on $SU(4)$, we may assume S^7 is round. Then $SU(2)$ (or $SO(3)$) will act isometrically and must act freely on a round S^7. This implies the action is that of $SU(2)$ and is the Hopf action. In particular, there is a unique free $SU(2)$ action. To express this as a biquotient, it is easy to see the action of $SU(2)\times SU(3)$ on $SU(4)$ given by $(A, B) * C = \mathrm{diag}(A, A)C\mathrm{diag}(B, 1)^{-1}$ is easily seen to be free

□

Proposition 5.1.2. *If $G = Sp(2)$ and $U = Sp(1) \times Sp(1)$, then there are precisely two biquotients given by the two maps $f : Sp(1) \times Sp(1) \to Sp(2) \times Sp(2)$ with $f(a, b) = (diag(a, b), I)$ and $g : Sp(1) \times Sp(1) \to Sp(2) \times Sp(2)$ with $g(a, b) = (diag(a, a), diag(b, 1))$. In the first case, we get the homogenous space $Sp(2)/Sp(1)\times Sp(1) = S^4$ and in the second we get a biquotient $\Delta Sp(1)\backslash Sp(2)/Sp(1)$ also diffeomorphic to S^4.*

Proof. We first check for homogeneous spaces. This is the same as asking which 4 dim representations of $Sp(1) \times Sp(1) = SU(2) \times SU(2)$ are symplectic. There are only 2 4 dimensional almost faithful representations of $SU(2)\times SU(2)$ - the standard block embedding and $SU(2) \times SU(2) \to SO(4) \to SU(4)$. The second is an outer tensor product of irreducible representations and hence is irreducible. Since it is clearly orthogonal, it cannot be symplectic. The standard block embedding, on the other hand, is a sum of 2 symplectic representations and is thus symplectic. It is well known that $Sp(2)/Sp(1) \times Sp(1) = S^4$.

We now look for biquotients. To do this, we must first list all the symplectic representations $f : Sp(1) \to Sp(2)$. It's easy to see that the only choices are

1) The unique 4 dimensional representation of $SU(2)$.

2) The map sending $a \in Sp(1)$ to $\mathrm{diag}(a, a)$ in $Sp(2)$.

3) The map sending $a \in Sp(1)$ to $\mathrm{diag}(a, 1)$ in $Sp(2)$.

Thus, we need to figure out for which pairs of 1),2), 3) we get a free biquotient action. The freeness condition in this case is equivalent to asking that no noncentral element in the image of one map be conjugate to any noncentral element in the image of the second map. Notice that two elements in $Sp(2)$ are conjugate iff after conjugating them separately to the standard maximal tori, the eigenvalues agree up to order and complex conjugation.

Thus, it's clear we may restrict to choosing two different maps. Further, to check this condition, it's enough to check where the maximal tori of the images intersect. Eigenvalues on the maximal tori are respectively given by:

1) λ^3, λ

2) μ, μ

3) $\eta, 1$

It easily follows that 1) and 2) together do not give a biquotient action. To see this, let $\lambda = i$. Then the eigenvalues we get for 1) are $-i, i$. If we let $\mu = i$, then we get i, i for the eigenvalues. But these two elements of $T^2 \subseteq Sp(2)$ are conjugate and noncentral.

Likewise, 1) and 3) do not give a biquotient action, which can be seen by choosing $\lambda = \mu = \zeta_3$, a primitive 3rd root of unity. Hence the eigenvalue lists are both $1, \lambda$, giving a noncentral conjugacy.

However, for 2) and 3), do give a biquotient action. This is because if the list μ, μ is the same as the list $\eta, 1$ up to order and complex conjugacy, then we clearly have $\mu = 1$ which implies $\eta = 1$. But the only element of $T^2 \subseteq Sp(2)$ with both eigenvalues 1 is the identity element.

Finally, we note that $\Delta Sp(1) \backslash Sp(2)/Sp(1)$ is diffeomorphic to S^4. In fact, by viewing this as $\Delta Sp(1)[Sp(2)/Sp(1)]$, we see that this is an S^3 action on S^7. By choosing the metric on $Sp(2)$ appropriately, the induced metric on $Sp(2)/Sp(1)$ is round, so the S^3 action is linear. We've already seen this must be the Hopf action and the quotient is $\mathbb{H}P^1 = S^4$.

$\square$

Proposition 5.1.3. *If $G = Spin(7)$ and $U = G_2 \times SU(2)$ up to finite cover, then the only biquotient is given as $G_2 \backslash Spin(7)/SU(2)$ where G_2 embeds into $Spin(7)$ in the standard way and $SU(2)$ is embedded into $Spin(7)$ via the lift of the map $SU(2) \to SO(3) \to SO(7)$ mapping an element B of $SO(3)$ to $diag(B, 1, 1, 1, 1)$). The biquotient is also diffeomorphic to S^4.*

Proof. First note that there is precisely one G_2 in $SO(7)$ since G_2 has a unique nontrivial 7 dimensional representation. It follows that there are no homogeneous spaces of the form $Spin(7)/(G_2 \times SU(2))$. Thus, any biquotients must be of the form $G_2\backslash Spin(7)/SU(2)$.

The unique embedding of G_2 into $Spin(7)$ gives the space $G_2\backslash Spin(7) = S^7$ with a round metric if the metric on $Spin(7)$ is biinvariant. This implies the $SU(2)$ action must be the Hopf action and the quotient diffeomorphic to S^4. In particular the $SU(2)$ action must be unique.

So, despite the fact that there are 6 nontrivial immersions of $SU(2)$ into $SO(7)$, there is a unique one giving a free action. To see that it's the map $SU(2) \to SO(3) \to SO(7)$, the second map being the usual block embedding, we consider the eigenvalues of the maximal torus of the image. For G_2 the eigenvalues of the maximal torus are $\lambda, \overline{\lambda}, \mu, \overline{\mu}, \lambda\mu, \overline{\lambda}\overline{\mu}, 1$ while for $SU(2)$ the eigenvalues of the maximal torus are $\eta^2, \overline{\eta}^2, 1, 1, 1, 1, 1$. Two entries in $SO(7)$ are conjugate iff they have the same eigenvalues up to order. Equating the two lists of eigenvalues, we see that since 5 of the eigenvalues of the $SU(2)$ list are 1, we must have without loss of generality that $\lambda = 1$. This implies that either $\mu = 1$ or $\lambda\mu = 1$ which implies $\mu = 1$. This then implies that $\eta^2 = \pm 1$, so that both elements are the identity.

$\square$

Proposition 5.1.4. *There are no biquotients with $G = Spin(8)$ and $U = G_2 \times Sp(2)$.*

Proof. First note there are no homogeneous spaces of the form $Spin(8)/G_2 \times Sp(2)$ because the minimal representation of $G_2 \times Sp(2)$ has dimension 11.

Now, G_2 only embeds in $Spin(8)$ one way, coming from $7dim + 1dim$. Hence, the eigenvalues of the maximal torus of G_2 in $Spin(8)$ are $\lambda, \mu, \lambda\mu, \overline{\lambda}, \overline{\mu}, \overline{\lambda\mu}, 1, 1$.

On the other hand $Sp(2)$ immerses in $Spin(8)$ in precisely two ways: first via $Sp(2) \to SO(5) \to SO(8)$, where the first map is the double cover and the second is the block embedding, and second the embedding induced from the map $\mathbb{H}^2 \to \mathbb{C}^4 \to \mathbb{R}^8$. The first has eigenvalues $\eta^2, \nu^2, \overline{\eta^2}, \overline{\nu^2}, 1, 1, 1, 1$ and the second has eigenvalues $\eta, \eta, \overline{\eta}, \overline{\eta}, \nu, \nu, \overline{\nu}, \overline{\nu}$.

In the first case, choosing $\mu = \overline{\lambda}$ and $\eta = \nu = \mu$, we see the eigenvalues lists agree up to order, and hence we get a noncentral conjugacy, and hence do not get a biquotient action. Likewise, in the second case, setting $\lambda = \overline{\mu} = \eta$ and $\nu = 1$, we get a noncentral conjugacy so do not get a biquotient. □

Proposition 5.1.5. *If $G = Spin(8)$ and $U = Spin(7) \times SU(2)$, then the only biquotient is of the form $Spin(7)\backslash Spin(8)/SU(2) = S^7/SU(2) = S^4$ with the $Spin(7)$ in $Spin(8)$ coming from the standard block embedding of $SO(7)$ into $SO(8)$ and the $SU(2)$ embedded via $SU(2) \to SU(4) \to SO(8)$, the first map sending $A \to diag(A, A)$ and the second the standard inclusion induced from the map $\mathbb{C}^4 \to \mathbb{R}^8$.*

Proof. There are no homogeneous actions since the smallest representation of U is 9 dimensional. Hence, all biquotients must be of the form $Spin(7)\backslash Spin(8)/SU(2)$.

Now, $Spin(7)$ embeds into $Spin(8)$ in the 3 different ways. The first is the lift of the usual embedding of $SO(7)$ into $SO(8)$ and the other two are given by the 2 spin representations. However, $Spin(8)$ is unique in that is has extra outer automorphisms (via triality) so that the three $Spin(7)$ are actually brought into each other via these outer automorphisms. Hence, $Spin(7)\backslash Spin(8) = S^7$ regardless of the embedding of $Spin(7)$ in $Spin(8)$. Thus, as usual, we need a free isometric $SU(2)$ action on a round S^7, so it must be the Hopf action with quotient S^4. It remains to determine which embedding of $SU(2)$ into $Spin(8)$ induces the Hopf action.

We may assume without loss of generality that $Spin(7)$ is embedded into $Spin(8)$ via the lift of the standard block embedding. Hence, the list of eigenvalues of the image maximal torus are $\lambda, \nu, \eta, \overline{\lambda}, \overline{\nu}, \overline{\eta}, 1, 1$. For the proposed embedding of $SU(2)$ into $Spin(8)$, the eigenvalues are μ, μ, μ, μ and their complex conjugates. If these two lists are equal up to order, then clearly $\mu = 1$, so the only conjugacy is at the identity element. Hence, this action gives the free action.

$\square$

Putting this altogether gives the following theorem:

Theorem 5.1.6. *Suppose $M = G/\!\!/U$ has the same rational homotopy groups as S^4. Then M is diffeomorphic to S^4 and we either have U simple with $G/\!\!/U = G/U$ homogeneous or $U = U' \times SU(2)$ with $U'\backslash G = S^7$ homogeneous.*

We now handle the case of biquotients with the same rational homotopy groups as $\mathbb{C}P^2$.

Proposition 5.1.7. *Suppose $M = G/\!\!/U$ with $G = SU(3)$ and $U = SU(2) \times S^1$. There is a unique homogeneous action with $SU(3)/U(2) = \mathbb{C}P^2$ and a unique nonhomogeneous action*

$$(A, z) * B = diag(zA, \overline{z}^2) B \, diag(z^4, z^4, z^{-8})^{-1}.$$

The quotient in the nonhomogeneous case is diffeomorphic to $\mathbb{C}P^2$ just as in the homogeneous case.

Proof. First, we check for homogenous spaces. $SU(2)$ has a 2 nontrivial 3 dimensional representations, the first given as the block embedding $SU(2) \subseteq SU(3)$ and the second given by $SU(2) \to SO(3) \subseteq SU(3)$. It is easy to see that since the second case determines an irreducible 3 dimensional representation, that there is no almost faithful extension of this map to a map $S^1 \times SU(2) \to SU(3)$. Hence, in the homogeneous case, we must use the block embedding. From here, it's clear that the only extension of $SU(2) \subseteq SU(3)$ to $S^1 \times SU(2) \to SU(3)$ is given by $U(2) \subseteq SU(3)$.

Now we check for biquotients. If we use the map $SU(2) \to SO(3) \subseteq SU(3)$, then the eigenvalues of the maximal torus of the image are $\lambda^2, \overline{\lambda}^2, 1$. The circle S^1 must act on the other side via a map $z \to \mathrm{diag}(z^a, z^b, z^c)$ with $a + b + c = 0$ and without loss of generality $\gcd(a, b, c) = 1$. Note that we cannot have $|a| = |b| = |c|$

by the condition on greatest common divisors. Suppose without loss of generality that $|c|$ is the largest. Letting z be a primitive cth root of 1, we see that the image of z is $\mathrm{diag}(z^a, z^b, 1)$ where $z^a\ neq 1$ and the determinant condition implies $z^a = \overline{z^b}$. Setting $\lambda^2 = z^a$, we get a noncentral conjugacy.

Hence, we must use the block embedding of $SU(2)$ in $SU(3)$. Since we know $SU(2)\backslash SU(3) = S^5$ and that by choosing the metric on $SU(3)$ appropriately S^5 is round, it follows that the circle must act as the Hopf action. So, consider the action of $S^1 \times SU(2)$ on $SU(3)$ given by $(z, A) * B = \mathrm{diag}(z^a A, \overline{z}^{2a}) B \mathrm{diag}(z^b, z^c, z^d)^{-1}$ with $\gcd(a, b, c, d) = 1$. We can identify $SU(2)\backslash SU(3)$ with S^5 by taking the last row of $SU(3)$. Using this, the induced circle action is

$$z * (p_1, p_2, p_3) = (z^{-2a-b} p_1, z^{-2a-c} p_2, z^{-2a-d} p_3).$$

Since we already know this must be, up to change of coordinates, the Hopf action, this implies that either $2a + b = 2a + c = 2a + d$ or $2a + b = 2a + c = -2a - d$ (up to reordering b, c, and d.) The first case implies $b = c = d = 0$, so recovers the homogeneous action. For the second case, it is easy to see that $(a, b, c, d) = (1, 4, 4, -8)$ generates all the solutions over $\mathbb{Z}$. The fact that $\gcd(a, b, c, d) = 1$ implies this is the only solution. $\square$

Proposition 5.1.8. *If $M = G/\!\!/U$ and $G = SU(4)$ and $U = Sp(2) \times S^1$, then $G/\!\!/U = Sp(2)\backslash SU(4)/S^1$ is diffeomorphic to $\mathbb{C}P^2$.*

Proof. First note that $Sp(2)$ only has one nontrivial representation of *dim* ≤ 4, given by the usual embedding of $Sp(2)$ in $SU(4)$ induced from the forgetful map

$f : \mathbb{H} \to \mathbb{C}^2$. It follows that there can be no homogeneous actions for there is no extension of this map to an almost faithful map $f : Sp(2) \times S^1 \to SU(4)$. It is well known that $Sp(2)\backslash SU(4) = Spin(5)\backslash Spin(6) = SO(5)\backslash SO(6) = S^5$ which is round. Hence, the circle must act as the Hopf map, so the quotient is diffeomorphic to $\mathbb{C}P^2$.

To see what this S^1 looks like in $SU(4)$, notice first that the maximal torus of $Sp(2)$ in $SU(4)$ has eigenvalues $\lambda, \overline{\lambda}, \eta, \overline{\eta}$. If the circle embeds into $SU(4)$ as $\mathrm{diag}(z, z, z, \overline{z}^3)$, then we'll get a free action. For setting the two lists equal, up to order, we see that, possibly renaming λ and η, that $\lambda = z = \overline{\lambda}$ so that $z = \pm 1$. This implies $\overline{z}^3 = \pm 1$ as well, which implies $\lambda = z = \eta = \pm 1$. The choice of -1 gives to elements which are both in the center, so we merely get an effectively free action, not a free action.

□

Summarizing these results, we see

Theorem 5.1.9. *If $M = G/\!\!/U$ with M having the same rational homotopy groups as $\mathbb{C}P^2$, then M is diffeomorphic to $\mathbb{C}P^2$. Further, up to ineffective kernel, every nonhomogeneous biquotient is of the form $U = U' \times S^1$ with $U'\backslash G = S^5$ an the S^1 acting as the Hopf action.*

We now handle the case of biquotients with the same rational homotopy groups as $S^2 \times S^2$.

Proposition 5.1.10. *If $M = G/\!\!/U$ with $G = SU(2) \times SU(2)$ and $U = S^1 \times S^1$, then the embedding of $S^1 \times S^1 \to SU(2)^4$ is, up to outer equivalence, given by either*

$(z, w) \to (diag(z^l, \overline{z^l}), diag(z^2, \overline{z^2}), diag(w, \overline{w}), Id)$ *with $l > 0$ an arbitrary odd integer or*

$(z, w) \to (diag(z^l, \overline{z^l}), diag(z, \overline{z}), diag(w, \overline{w}), Id)$ *with l an arbitrary integer or*

$(z, w) \to (diag(z, \overline{z}), diag(z\overline{w}, \overline{z}w), diag(w, \overline{w}), diag(zw, \overline{zw}))$. *In the first case, all quotients are diffeomorphic to $\mathbb{C}P^2\# - \mathbb{C}P^2$, the (unique) nontrivial S^2 bundle over S^2. In the second case, all quotients are diffeomorphic to $S^2 \times S^2$. In the last case, the quotient is diffeomorphic to $\mathbb{C}P^2\#\mathbb{C}P^2$.*

Remark 5.1.11. For the first two actions, by taking the quotient with respect to the w coordinate first, we see these look like S^1 actions on $S^3 \times S^2$. Geometrically, in the first case, the S^1 acts on S^3 as the Hopf action while rotating the S^2 l times. In the second case, the S^1 acts on S^3 as the Hopf action while rotation S^2 2l times.

Proof. A general linear action of T^2 on $S^3 \times S^3$ looks like

$$(z, w) * \big((p_1, p_2), (q_1, q_2)\big) = \big((z^a w^b p_1, z^c w^d p_2), (z^e w^f q_1, z^g w^h q_2)\big).$$

Notice that if $det \begin{bmatrix} a & b \\ e & f \end{bmatrix} = 0$, then there are infinitely many solutions to the simultaneous equations

$$au + bv = 0 \qquad (5.1.1)$$

$$eu + fv = 0 \qquad (5.1.2)$$

If, say, $(u, v) = (n_1, n_2)$ is a nontrivial integral solution, then

$$(z^{n_1}, z^{n_2}) * \big((1,0),(1,0)\big) = \big((1,0),(1,0)\big)$$

for all z. Thus, we cannot have an effectively free action in this case.

Hence, we must assume the above determinant is nonzero. It follows by making the action ineffective that we may change coordinates to $(u', v') = (u^a v^b, u^e v^f)$. In these coordinates, the action now looks like

$$(u', v') * \big((p_1, p_2), (q_1, q_2)\big) = \big((u'^a p_1, u'^b v'^c p_2), (v'^d q_1, u'^e v'^f q_2)\big)$$

for some new a, b, c, d, e, and f. We will abuse notation and reuse (u, v) for (u', v'). We may assume without loss of generality that $\gcd(a, b, e) = \gcd(c, d, f) = 1$.

Freeness implies $a = \pm d = \pm 1$, for if $u = \zeta_a$ is an ath root of 1, then $(u, 1)$ fixes $\big((1,0),(1,0)\big)$. By swapping p_1 with $\overline{p_1}$ and likewise for q_1, we may assume that $a = d = 1$. Now, we also see that $b = \pm f \pm 1$ for any bth root of unity u, $(u, 1)$ fixes $\big((0,1),(1,0)\big)$. Again, by replacing p_2 with $\overline{p_2}$ and likewise for q_2, we may assume that $b = f = 1$.

Finally, we claim that such an action is free iff $1 - ec = \pm 1$. This is necessary because this is precisely the condition which guarantees that no (u, v) fixes $\big((0,1),(0,1)\big)$. Conversely, if (u, v) fixes any point, then it must fix one of the 4 pairs of points we've already checked.

By possibly swapping u and v (and thus, e and c), we see the only possible solutions to $1 - ec = \pm 1$ are when $c = 0$ and e is arbitrary or $e = 1$, $c = 2$ or

$e = -1$, $c = -2$. By sending v to $-v$, q_1 to $\overline{q_1}$, and q_2 to $\overline{q_2}$, we may assume c is nonnegative, so our solutions are $c = 0, e \geq 0$ or $e = 1, c = 2$.

Totaro [29] has already shown that the sporadic example is diffeomorphic to $\mathbb{C}P^2 \sharp \mathbb{C}P^2$.

We now focus on the $c = 0$ case. Note that when $e = 0$, $G /\!\!/ U$ is naturally the total space of a fiber bundle coming from projection onto the first factor $SU(2) \times SU(2) \to SU(2) \to SU(2)/S^1 = S^2$, so that $G /\!\!/ U$ has the structure of an S^2 bundle over S^2. Since the fiber S^2 is round, the structure group of this bundle is $SO(3)$ and a simple clutching function argument shows there are precisely two such bundles. These bundles may be identified with $S^2 \times S^2$ and $\mathbb{C}P^2 \sharp - \mathbb{C}P^2$. Note that their cohomology rings distinguish them.

We will compute these cohomology rings using the techniques of chapter 2, but first we must convert these actions into effective biquotient actions. To that end, consider the map $T^2 \to \big(SU(2) \times SU(2)\big) \times \big(SU(2) \times SU(2)\big)$ sending (u, v) to $\big((\mathrm{diag}(u^2, \overline{u}^2), \mathrm{diag}(u^{-e}v, \overline{u^{-e}v})\big), \big(Id, \mathrm{diag}(u^{-e}, \overline{u}^{-e})\big)$. This action has the same orbits but has ineffective kernel because -1 is in the kernel of the map if l is even and -1 is in the kernel of the action if l is odd.

To convert this to an effective biquotient action, multiply the first and third matrices by the element $\mathrm{diag}(u^2, u^2) \in Z(U(2))$ and the second and fourth by $\mathrm{diag}(u^{-e}, u^{-e})$ and then replacing u^2 by u everywhere. One obtains the new map sending (u, v) to

$$\big(\mathrm{diag}(u^2, 1),\ diag(u^{-e}v, \overline{v})\big), \big(\mathrm{diag}(u, u), \mathrm{diag}(u^{-e}, 1)\big).$$

We claim this new map now induces a free action. Since we already know its effectively free, we need only show that the only element which fixes every point is (e, e). So, assume (u, v) fixes every point. Then, it fixes a point of the form (Id, A) which implies that $\mathrm{diag}(u^2, 1)\mathrm{diag}(u, u)^{-1}) = Id$ which, of course, implies $u = 1$. Fixing a point of the form (A, Id) now easily implies $v = 1$.

With the description, we are in a position to apply the formalism from chapter 2. Let u and v denote the coordinates of T^2 and let x_1, x_2, y_1, y_2 be the coordinates on $T^4 \subseteq U(2) \times U(2)$. Since $H^*(U(2); \mathbb{Z}) = \Lambda_{\mathbb{Z}}[r_1, r_3]$ we can compute with $\mathbb{Z}$ coefficients. Here we see that

$$H^*(B(U(2) \times U(2)); \mathbb{Z}) \subseteq H^*(BT_{U(2)\times U(2)}; \mathbb{Z}) = \mathbb{Z}[\overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2}]$$

can be identified with the subalgebra

$$\mathbb{Z}[\overline{x_1} + \overline{x_2}, \overline{y_1} + \overline{y_2}, \overline{x_1 x_2}, \overline{y_1 y_2}].$$

Hence, we see that $H^*(B(SU(2)\times SU(2)); Z)$ can be identified with $\mathbb{Z}[\overline{x_1 x_2}, \overline{y_1 y_2}]$.

Letting $f = (f_1, f_2) : T^2 \to (U(2) \times U(2))^2$ be the two maps defining the biquotient action, we see that $f_1^* x_1 = 2u$, $f_1^* x_2 = 0$, $f_1^* y_1 = -eu + v$, $f_1^* y_2 = -v$ and that $f_2^* x_1 = u$, $f_2^* x_2 = u$, $f_2^* y_1 = -eu$, and $f_2^* y_2 = 0$.

It follows that $Bf^*(\overline{x_1 x_2}\otimes 1 - 1\otimes \overline{x_1 x_2}) = -u^2$, $Bf^*(\overline{y_1 y_2}\otimes 1 - 1\otimes \overline{y_1 y_2}) = euv - v^2$.

Now, if e is even, one can change basis to $\{u, v - e/2u\}$. Then one immediately sees that $(v - e/2u)^2 = v^2 - euv = 0$ so in this basis, we see the cohomology ring is that of $S^2 \times S^2$.

If, instead, e is odd, then use the basis $\{[1+(e-1)/2]u - v, v - [(e-1)/2]u\}$. A computation shows the product of the two basis elements is 0 and that the square of one is minus the square of the other, i.e., this is the cohomology ring of $\mathbb{C}P^2 \sharp = \mathbb{C}P^2$.

$\square$

5.2 5 dimensional biquotients

Proposition 5.2.1. *Assume a 1-connected biquotient $M = G/\!/U$, has the same rational homotopy groups as S^5. Then $M = SU(4)/Sp(2), SU(3)/SU(2)$, or the Wu manifold $SU(3)/SO(3)$. All cases are homogeneous. The first two are diffeomorphic to S^5 while the Wu manifold is not homotopy equivalent to S^5.*

Proof. We've already seen that either $G = SU(3)$ and U is given, up to finite cover, as $SU(2)$ or $G = SU(4)$ and $U = Sp(2)$ given by a homogenous action. In the second case, notice simply that $SU(4)/Sp(2) = Spin(6)/Spin(5) = SO(6)/SO(5) = S^5$ as there is a unique embedding of $Sp(2)$ into $SU(4)$.

Now, assume $G = SU(3)$ and U is $SU(2)$. A biquotient action is given by a map $SU(2) \to SU(3) \times SU(3)$. However, there are, up to conjugacy, only 2 nontrivial homomorphisms $SU(2) \to SU(3)$ given by 1) the block embedding and 2) $SU(2) \to SO(3) \to SU(3)$. A nonhomogeneous biquotient action can only occur by using 1) for the left action and 2) for the right action. However, this won't work because the eigenvalues of the image torus are $\lambda, \overline{\lambda}$, and 1 for the block embedding and $\lambda^2, \overline{\lambda}^2, 1$ for the $SO(3)$ embedding. Choosing $\lambda = \zeta_3$, a 3rd root of unity gives a noncentral conjugacy.

Thus, the only options are homogeneous, giving $SU(3)/SU(2) = S^5$ and the Wu manifold $SU(3)/SO(3)$. The long exact sequence of homotopy groups shows $\pi_2(SU(3)/SO(3)) = \mathbb{Z}/2\mathbb{Z}$, showing this example is distinct up to homotopy from S^5.

$\square$

Proposition 5.2.2. *Suppose M is a 1-connected biquotient and that M has the same rational homotopy groups as $S^3 \times S^2$ so $G = SU(2) \times SU(2)$ and $U = S^1$. The action is, up to equivalence, given by*

$$z \to (diag(z^a, z^{-a}), diag(z^b, z^{-b}), diag(z^c, z^{-c}), diag(z^d, z^{-d}))$$

where we may assume without loss of generality that $\gcd(a, b, c, d) = 1$. *The action is free iff* $\gcd(a^2 - c^2, b^2 - d^2) = 1$ *or* 4. *The quotient $G/\!\!/U$ is diffeomorphic to $S^2 \times S^3$ when this* gcd *is* 1 *and is diffeomorphic to the unique nontrivial S^3 bundle over S^2 when the* gcd *is* 4.

Proof. We first check for effective freeness. So, assume $\mathrm{diag}(z^a, \overline{z}^a)$ is conjugate to $\mathrm{diag}(z^c, \overline{z}^c)$ and that $\mathrm{diag}(z^b, \overline{z}^b)$ is conjugate to $\mathrm{diag}(z^d, \overline{z}^d)$. Then this implies $z^a = z^{\pm c}$ and $z^b = z^{\pm d}$. For now, assume we use the + sign, the other cases being similar. Sice $z^a = z^c$ and $z^b = z^d$, then it follows that $z^{a-c} = z^{b-d} = 1$, so that z must be a $\gcd(a-c, b-d)$th root of unity. Conversely, any $\gcd(a-c, b-d)th$ root of unity satisfies both equations. Note that if $\gcd(a-c, b-d) = 0$, then U fixes the usual maximal torus of $SU(2) \times SU(2)$ so does not act freely. Hence, in order to get an effectively free action, we need $0 < \gcd(a-c, b-d)$ and that for every $\gcd(a-c, b-d)$th root of 1, ζ, that $\zeta^a = \zeta^c = \pm 1 \in Z(SU(2))$ and likewise for b and d. But this implies that $\gcd(a-c, b-d) | \gcd(2a, 2b, 2c, 2d)$ so $\gcd(a-c, b-d)|2$. Doing this for the other choices of plus and minus clearly gives the necessary and sufficient condition for freeness that $\gcd(a \pm c, b \pm d) = 1$ or 2.

Note that the parity of $a-c$ is the same as that of $a+c$. This implies that all 4 of these gcds are equal. Finally, it is easy to see that if the gcd of all of them is 1, then so is $\gcd(a^2-c^2, b^2-d^2)$ and if all the gcds are 2, then $\gcd(a^2-c^2, b^2-d^2) = 4$.

Pavlov [23] has already shown that all of these quotients are diffeomorphic to either $S^2 \times S^3$ or $S^3 \hat{\times} S^2$, the unique nontrivial S^3 bundle over S^2. The idea is that $\pi_2(G/\!\!/U) = \mathbb{Z}$ by the long exact sequence of homotopy groups and so $H_2(G/\!\!/U) = \mathbb{Z}$ by Hurewicz. Poincare duality then shows the ring structure of $H^*(G/\!\!/U)$ is that of $S^2 \times S^3$. One then appeals to the work of Smale [26] and Barden [3] which shows that compact simply connected 5-manifolds are classified up to diffeomorphism by their cohomology rings and second Stiefel-Whitney classes. Hence, our goal is to compute w_2 of all of these biquotients.

We now break into 2 cases depending on whether or not S^1 acts effectively on

$SU(2) \times SU(2)$: the action is effective iff $\gcd(a^2 - c^2, b^2 - d^2) = 1$. If the action is effective, then we can actually apply Singhoff's formula for computing the Stiefel-Whitney classes immediately. Since neither $SU(2)$ nor S^1 have any 2-roots, all the products in the formula are the empty products, giving total Stiefel-Whitney class of 1. Hence, in these cases the biquotient is diffeomorphic to $S^2 \times S^3$.

So, we may now assume we're in the case where $\gcd(a^2 - c^2, b^2 - d^2) = 4$, i.e., when $\gcd(a \pm c, b \pm d) = 2$ independent of the choice of signs. Notice that this implies a and b have different parities, for we know $a \cong c$ (2) and $b \cong d$ (2), so if $a \cong b$ (2), it's easy to see that for some choice of signs $\gcd(a \pm c, b \pm d) = 4$, giving a contradiction. We will thus assume without loss of generality that a is odd and b is even.

We modify the biquotient action to an effective one in the usual way, getting the embedding

$$z \to \operatorname{diag}(z^a, 1), \operatorname{diag}(z^b, 1), \operatorname{diag}(z^{\frac{a+c}{2}}, z^{\frac{a-c}{2}}), \operatorname{diag}(z^{\frac{b+d}{2}}, z^{\frac{b-d}{2}})$$

which is easily seen to be effective having the same orbits as the original action.

For computing Stiefel-Whitney classes, we use the slightly modified 2-group version of the techniques in chapter 2. The maximal 2-group of $U(2) \times U(2)$ is generated by $(\operatorname{diag}(-1,1), Id)$, $(\operatorname{diag}(1,-1), Id)$, $(Id, \operatorname{diag}(-1,1))$, and $(Id, \operatorname{diag}(1,-1)$. Let $\{r_1, r_2, s_1, s_2\}$ denote the dual basis to these 4 elements. Then, since the maximal 2-group of $U(2)$ is contained in the maximal torus of $U(2)$, we see that we can identify $H^*(B(U(2) \times U(2)); \mathbb{Z}/2\mathbb{Z})$ with the subalgebra

$$\mathbb{Z}/2\mathbb{Z}[\sigma_1(r_i^2), \sigma_2(r_i^2), \sigma_1(s_i^2), \sigma_2(s_i^2) \subseteq \mathbb{Z}/2\mathbb{Z}[r_1, r_2, s_1, s_2]$$

, the cohomology ring of the classifying space of the maximal 2-group. By forgetting the σ_1 terms, we get the subalgebra isomorphic to $H^*(BSU(2); \mathbb{Z}/2\mathbb{Z})$. Likewise, we can identify $H^*(BU; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[z^2]$ where z^2 is the dual to $-1 \in S^1$. Note that we already know $H^2(G/\!\!/U; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. It is easy to see that in the spectral sequence, z^2 is the only element of degree 2. It follows that z^2 generates $H^2(G/\!\!/U; \mathbb{Z}/2\mathbb{Z})$.

Now, recalling that $a \cong b+1 \cong 0$ (2), we see that $Bf_1^*(r_1) = z$ while $Bf_1^*(x) = 0$ for any other basis element x. Recall that the 2-roots of $U(2)$ are $r_1 + r_2$ with multiplicity 2.

It follows that

$$\begin{aligned}
w(G/\!\!/U) &= \phi_G\big(\Pi_{\lambda \in \Delta^2 G}(1+\lambda)\big)\phi_U\big(\Pi_{\mu \in \Delta^2 U}(1+\mu)\big)^{-1} \\
&= \phi_G^*\big((1 + r_1^2 + r_2^2)(1 + s_1^2 + s_2^2)\big) \\
&= \phi_U^* Bf^*\big((1 + r_1^2 \otimes 1 + r_2^2 \otimes 1)(1 + s_1^2 \otimes 1 + s_2^2 \otimes 1)\big) \\
&= \phi_U^*(1 + z^2) \\
&= 1 + z^2
\end{aligned}$$

Thus, we see the second Stiefel-Whitney class is nontrivial, so all these biquotients are diffeomorphic to $S^3 \hat{\times} S^2$.

$\square$

Remark 5.2.3. Pavlov [23] previously showed that the nontrivial S^3 bundle over S^2 arises. The action he describes in words corresponds to the above choosing $a = 2$, $c = 0$, $b = d = 1$. A much quicker proof of his result using our result in dimension 4 is this: Notice that this action preserves the equatorial $S^2 \times S^3$. By restricting the action there, we find that $S^2 \times_{S^1} S^3 \subseteq M$. However, using our analysis in dimension 4 we see that $S^2 \times_{S^1} S^3$ is diffeomorphic to $\mathbb{C}P^2 \# - \mathbb{C}P^2$. Then, using Pavlov's trick, it follows that $w_2(M) \neq 0$ since $w_2(\mathbb{C}P^2 \# - \mathbb{C}P^2) \neq 0$.

Finally, the action Pavlov writes down in terms of matrices doesn't actually give the desired biquotient as it rotates the S^2 fiber around too many times.

5.3 6 dimensional biquotients

Proposition 5.3.1. *Suppose M is 1-connected and diffeomorphic to a biquotient $G/\!\!/U$ and suppose M and has the same rational homotopy groups at S^6. Then, G/U is homogeneous with either $G = G_2$ and $U = SU(3)$ or $G = Spin(7)$ and $U = Spin(6)$. In each case, the embedding of U into G is unique up to outer automorphism.*

Proof. From the classification of pairs in chapter 4, we have that $G = G_2$ or $Spin(7)$ and U is given respectively as $SU(3)$ or $Spin(6)$.

Now, there is, up to outer automorphism, only a single almost faithful image of $SU(3)$ in G_2. This is because such a map of $SU(3)$ must land in $SO(7)$, but the only map from $SU(3)$ into $SO(7)$ is given by $\Gamma_{1,0} + \Gamma_{0,1} + \Gamma_{0,0}$, where $\Gamma_{i,j}$ denotes the unique irreducible representation of $SU(3)$ labeled by integers over the root diagram. Hence, there is, at most, one $SU(3)$ inside of G_2. The fact that there is at least one follows from the description of G_2 as the automorphism group of the Cayley numbers. It is well established that G_2 acts transitively on S^6 =unit imaginary Cayley numbers, with stabilizer $SU(3)$. Hence, since there is only one map from $SU(3)$ into G_2, the only possible biquotient action is homogeneous and $G/\!\!/U = G_2/SU(3) = S^6$.

Likewise, there is a unique nontrivial map from $Spin(6) = SU(4)$ into $Spin(7)$. This follows because every map from $SU(4)$ to $Spin(7)$ gives a map from $SU(4)$ to $SO(7)$. However, the smallest orthogonal irreducible representations of $SU(4)$ are $\Gamma_{1,0,1}$ which has dimension 8 and $\Gamma_{0,1,0}$ which has dimension 6. The smallest reducible orthogonal representation of $SU(4)$ is given by $\Gamma_{1,0,0} + \Gamma_{0,0,1}$ which also has dimension 8.

Thus, the only way of mapping $SU(4)$ to $SO(6)$ comes from $\Gamma_{0,1,0}$, but this is the usual identification of $SU(4)$ with $Spin(6)$. Hence, the only embedding of $SU(4)$ into $Spin(7)$ is the usual embedding of $Spin(6)$ into $Spin(7)$ induced from the usual embedding of $SO(6)$ into $SO(7)$. In particular, the only biquotient actions are actually homogeneous actions and $G/\!\!/U = Spin(7)/Spin(6) = SO(7)/SO(6) = S^6$.

□

Proposition 5.3.2. *If $M = G/\!\!/U$ with $G = SU(4)$ and $U = SU(3)\times S^1$, then there are precisely two only non-homogeneous biquotients: one of the form $SU(4)/\!\!/U(3)$*

and the other in the form $SU(4)/\!\!/S^1 \times SU(3)$. *The action of the first is* $A * B = diag(A, \det A^{-1})B(\det(A)^{-1}, \det(A)^{-1}, \det(A)^{-1}, \det(A)^3)^{-1}$ *and the action of the second is* $(z, A) * B = diag(A, 1)B(z^{-1}, z^{-1}, z, z)^{-1}$. *In all cases (including the homogeneous cases), the quotient is diffeomorphic to* $\mathbb{C}P^3$.

Proof. First notice that the only nontrivial map from $SU(3)$ into $SU(4)$ is given by either the block embedding or by first precomposing by complex conjugation and then embedding via the block embedding. In either case, the quotient $SU(3)\backslash SU(4)$ is diffeomorphic to S^7, and by choosing the appropriate metric on $SU(4)$, we may assume S^7 is round. It follows that the circle must act via the Hopf action, so the quotient will always be diffeomorphic to $\mathbb{C}P^3$. Just as in the case of $G = SU(3)$ and $U = SU(2) \times S^1$, we need only find which actions actually look like the Hopf action on the last row of a matrix in $SU(4)$, up to complex conjugation on some of the coordinates. That is, if the last row of a matrix in $SU(4)$ is $[v_1, v_2, v_3, v_4]$, then we're looking for actions which look like

$$z * [v_1, v_2, v_3, v_4] = [z^{k_1}v_1, z^{k_2}v_2, z^{k_3}v_3, z^{k_4}v_4]$$

where each k_i is ± 1. We may assume without loss of generality that the number of $k_i = -1$ is at most 2.

It is easy to verify that when the number of $k_i = -1$ is 0, then the action can only come from the usual homogeneous action. The case where precisely one $k_i = -1$ comes from the action of $SU(3) \times S^1$ on $SU(4)$ given as

$$(A, z) * B = \mathrm{diag}(zA, \overline{z}^3)B\mathrm{diag}(z^3, z^3, z^3, \overline{z}^9)^{-1}.$$

Finally, when precisely two k_i are -1, the action comes from

$$(A, z) * B = \mathrm{diag}(A, 1)B\mathrm{diag}(z, z, \overline{z}, \overline{z})^{-1}.$$

$\square$

Proposition 5.3.3. *If $M = G/\!\!/U$ is nonhomogeneous with $G = Sp(2)$ and U is $S^1 \times SU(2)$, then there is a unique free action given by $(z, q) \in S^1 \times Sp(1) \to$diag(q,1), diag(z,z). Further $G/\!\!/U = \mathbb{C}P^3$.*

Proof. We begin by classifying maps from $S^1 \times Sp(1)$ into $Sp(2)$. The first thing to notice is that there are precisely 4 maps of $SU(2)$ into $Sp(2)$. The only ones which admit extensions to S^1 are given as the inclusion $U(2) \subseteq Sp(2)$ and $Sp(1) \times S^1 \subseteq Sp(2)$ as the block embedding. It follows that all of the nontrivial symplectic 4-dimensional representations of $S^1 \times SU(2)$ are

1)(q) (the unique irreducible 4-d representation of $SU(2)$ is symplectic)

2)$\mathrm{diag}(z^a, z^b)$

3)$\mathrm{diag}(z^a A)$ thought of as $U(2) \subseteq Sp(2)$ and

4)$\mathrm{diag}(q, z^a)$

This will give us several cases to check. But first note that in all of the representations above, the maximal torus is actually embedded in $U(2) \subseteq Sp(2)$. Conjugacy in $Sp(2)$ of elements in $U(2) \subseteq Sp(2)$ is simple: two elements are conjugate (in $Sp(2)$) iff the eigenvalues of each element are correspondingly the same, up to both order and complex conjugation.

For now, we want to rule out 1) paired with anything. To this end, notice that when restricted to maximal tori, we find that the eigenvalues of 1) are λ and λ^3. If we pair with 2), then notice we can assume (a,b)=1. Further, if $b = 3a$, it's clear we don't get a free action. Now, let z be a primitive $b - 3a$th root of unity and set $\lambda = z^a$. Then it's easy to see that this solves

$$\begin{aligned} z^a &= \lambda \\ z^b &= \lambda^3 \end{aligned}$$

To maintain a free action, we need $z^a = z^b = \pm 1$. This forces $b - 3a | a$ and $b - 3a | b$ so we have $b - 3a | \gcd(a, b)$, that is, $|b - 3a| = 1$. Repeating the same argument using z a primitive $3a + b$th root of unity we see $|b + 3a| = 1$. Of course, it follows that $a = 0$ and so, without loss of generality, $b = 1$. Finally, set $\lambda = \zeta_3$ a third root of unity to get another noncentral conjugacy.

Choosing 1) and 3), simply set $z = 1$ and let $\lambda = i$. Then this clearly solves

$$\begin{aligned} \lambda &= \lambda \\ \overline{\lambda} &= \lambda^3 \end{aligned}$$

Hence, we get a conjugacy. But $\mathrm{diag}(i, -i) \notin Z(Sp(2))$ so we won't get a free action.

Finally, choosing 1) and 4). Set $z = 0$ and $\lambda = \zeta_3$ a 3rd root of unity. This clearly also gives a conjugacy which isn't in the center of $Sp(2)$, hence we don't get a free action.

Hence, we only have 3 cases left: 2) and 3), 2) and 4), and 3) and 4). We start with 3) and 4). We may assume $\gcd(a, b) = 1$. Further, we can assume $a \neq 0$, for if $a = 0$, then $b \neq 0$ and setting $z^b = \overline{\lambda}$ gives infinitely many of conjugacies.

On the maximal torus, elements look like $\mathrm{diag}(z^a\lambda, z^a\overline{\lambda}), \mathrm{diag}(\lambda, z^b)$. So, let z be an ath root of unity and set $\lambda = z^{-b}$. This clearly solves

$$\begin{aligned} z^a\lambda &= \lambda \\ z^a\overline{\lambda} &= z^b \end{aligned}$$

In order to have the action be free, we must have $z^b = \pm 1$, i.e., $a|2b$. But since $(a, b) = 1$ this actually forces $a|2$. Further, by precomposing with $z \to -z$, we may assume that $a = 1$ or $a = 2$.

Assume initially that $a = 1$. Now, setting z a $2b-3$th root of unity and $\lambda = z^{b-1}$ solves

$$\begin{aligned} z\overline{\lambda} &= \lambda \\ z\lambda &= z^b \end{aligned}$$

Thus, to maintain a free action, we must have $\pm 1 = \lambda = z^{b-1}$, so we conclude that $2b - 3|2(b - 1)$. But $2b - 3$ is odd, and hence this forces $2b - 3|b - 1$, so in particular, we must have $|2b - 3| \leq |b - 1|$. This clearly forces $|b| \leq 2$. For $|b| \leq 2$, checking whether or not $|2b-3| \leq |b-1|$ shows that we must have b $=1$ or b=2. Hence, when $a = 1$, we must have $b = 1$ or $b = 2$.

Next, assume $a = 2$. Running through a similar computation as above, we conclude that $6 - 2b|2 - b$. Thus, we immediately conclude that $|b| \leq 4$. For each $|b| \leq 4$, computing whether or not $6 - 2b|2 - b$ shows that $b = 2, 3$, or 4. However, $a = 2$ and we have $(a, b) = 1$, so we may assume $b = 3$.

Hence, we've narrowed it down the checking 3 cases: $a = 1$ and $b = 1$ or 2, $a = 2$ and $b = 3$.

For the $a = b = 1$ case, set z a primitive 5th root of unity and set $\lambda = z^2$. Then this clearly solves

$$\begin{aligned} z\lambda &= \overline{\lambda} \\ z\overline{\lambda} &= \overline{z} \end{aligned}$$

But then λ is also a primitive 5th root of unity, and hence $\mathrm{diag}(\lambda, *) \notin Z(Sp(2))$.

Next, we rule out $a = 1$ and $b = 2$. To this end, let z be a primitive 7th root of unity and set $\lambda = z^3$. As above, this element won't be in the center of $Sp(2)$. Now, it's easy to see that this solves

$$\begin{aligned} z\lambda &= \overline{\lambda} \\ z\overline{\lambda} &= \overline{z^2} \end{aligned}$$

Hence we don't get a free action in this case.

Finally, we rule out $a = 2$ and $b = 3$. For any z at all, set $\lambda = \overline{z}$. Then the two matrices look like $\mathrm{diag}(z, z^3)$ and $\mathrm{diag}(\overline{z}, z^3)$ which are clearly conjugate. Thus, there is no choice of a and b which give a free action. This completes the 3) and 4) case.

To handle the remaining cases (2) and 3), as well as 2) and 4)), we again turn to Eschenburg's Habilitation [10] where he classifies all possible maximal tori actions on $Sp(n)$. The key observation, as in the $SU(n)$ case is that a single S^1 acts on the right. In fact, again according to Eschenburg, the action on the right, up to conjugacy, is given as $\mathrm{diag}(1, z)$ or $\mathrm{diag}(z, z)$. Hence, when studying conjugacy of $\mathrm{diag}(z^a, z^b), \mathrm{diag}(z^c\lambda, z^c\overline{\lambda})$ (with $(a, b, c) = 1$) we may assume that either $a = 0$ or $a = b =\neq 0$, simplifying things greatly. Note that if we assume $a = 0$, we may assume $b \neq 0$, since otherwise we get the homogenous action $Sp(2)/U(2)$.

Now, assume $a = 0$. Let z be any $2c-b$th root of unity. Set $\lambda = \overline{z^c}$ (if $2c-b = 0$, this will give infinitely many conjugacies, contradicting freeness). Then this clearly solves

$$\begin{aligned} z^c\lambda &= 1 \\ z^c\overline{\lambda} &= z^b \end{aligned}$$

To maintain a free action, we must then have $z^b = 1$ and $z^{2c} = 1$. Together, this implies $2c - b|(b, 2c)$ so that $2c - b|2$.

If we instead set $z =$ a $(2c+b)$th root of unity and set $\lambda = \overline{z^c}$, this solves

$$\begin{aligned} z^c\lambda &= 1 \\ z^c\overline{\lambda} &= \overline{z}^b \end{aligned}$$

Hence, by the same reasoning, we'll conclude that $2c + b|2$. But then we have $|2c \pm b| \leq 2$ which clearly forces b= 0 (so we're in the homogenous case).

Next, assume that $a = b \neq 0$. Setting $\lambda = 1$, conjugacy implies that z with $z^a = z^{\pm c}$ gives a conjugacy. In order to maintain a free action, we must therefore have $(a \pm c)|2(a, c) = 2$. This immediately implies that $|a| \leq 2$ and we can assume $a = 1 or 2$ by precomposing with $z \to -z$. If $a = 2$, we must clearly have $c = 0$, but this contradicts $(a, c) = 1$. Hence, we may assume $a = 1$. Clearly this implies that $b = -1, 0$, or 1. If $b = \pm 1$, then again by setting $\lambda = 1$, we see that any z gives a conjugacy, and hence does not give a free action.

So, assume $b = 0$. Then by setting $\lambda = z^a$, we see that for any z we get a conjugacy, so this isn't free either. This concludes the case of choosing 2) and 3).

We only have the case of 2) and 4) left. So, consider elements of the form $\text{diag}(z^a, z^b)$ and $\text{diag}(\lambda, z^c)$ where $(a, b, c) = 1$. Again, as can be found in Eschenburg's Habilitation, we may assume either $a = 0$, or $a = b \neq 0$. Assume initially that $a = 0$.

Set $\lambda = z^a$ and $z^c = 1$. This gives a conjugacy. In order to keep a free action, we must have $z^a = 1$, so that $c|a$. But $(a,c) = 1$ so this implies that $a = \pm 1$ (and we can assume $a = 1$) and $c = \pm 1$ or 0. If $c = \pm 1$, the setting $\lambda = 1$ gives a conjugacy for all z, and hence won't give a free action. If $c = 0$, setting $\lambda = z^a$ for any z also gives a conjugacy. Hence, when $a = 0$, there are no free actions.

Finally, assume $a = b \neq 0$ and $(a,c) = 1$. As above, setting z any $(a \pm c)$th root of unity and letting $\lambda = z^a$, we'll get a conjugacy. freeness then requires that $a \pm c|(2a, 2c) = 2$. Hence, we conclude that $a = 2$ and $c = 0$ (contradicting $(a,c) = 1$) or $a = 1$ and $c = \pm 1$ or 0.

For $c = \pm 1$, setting $\lambda = z$ gives a conjugacy for all z, so we won't get a free action in this case. Hence, the only possible free action is given by $a = 1$ and $c = 0$. I claim that this does, in fact, give a free action and the quotient is $\mathbb{C}P^3$.

To see this, notice we're comparing $\operatorname{diag}(z,z)$ with $(\lambda, 1)$. Conjugacy implies that $z = 1$ and hence both matrices are equal to the identity. Thus, we actually get a free action (as opposed to free).

To understand the quotient, simply map $B \in Sp(2)$ to the class of it's last row. Here, if the last row $(b_{2,1}, b_{2,2})$ is expanded as $(z_{2,1} + w_{2,1}j, z_{2,2} + w_{2,2}j)$, then we declare it to be equivalent to $(z_{2,1}\overline{z} + w_{2,1}zj, z_{2,2}\overline{z} + w_{2,2}zj)$. This is clearly equivalent to the Hopf action, and hence the quotient is diffeomorphic to $\mathbb{C}P^3$.

$\square$

Proposition 5.3.4. *Suppose $M = G/\!\!/U$ with $G = SU(4) \times SU(2)$ and $U = S^1 \times SU(3) \times SU(2)$. Then there are infinitely many actions of the form $G/\!\!/S^1 \times SU(3) \times$*

$SU(2)$, $G/\!\!/U(3)\times SU(2)$, $G/\!\!/SU(3)\times U(2)$ or $G/\!\!/S(U(3)\times U(2))$. In all cases the quotient is either diffeomorphic to $S^4\times S^2$ or to the unique nontrivial S^4 bundle over S^2.

Proof. Notice first that there are precisely 2 embeddings of $SU(3)$ into $SU(4)\times SU(2)\times S^1$ given by $A\to(\mathrm{diag}(A,1),Id,1)$ and $A\to(\mathrm{diag}(\overline{A},1),Id,I)$.

There are several embeddings of $SU(2)$ into $SU(4)$ given by Γ_3, $\Gamma_2\oplus\Gamma 0$, $\Gamma_1\oplus 2\Gamma_0, 2\Gamma_1$, and $4\Gamma_0$. The eigenvalues of the embedded maximal torus are given, respectively as, $\lambda,\lambda^3,\lambda^{-3},\lambda^{-1}$; $\lambda^2,\lambda^{-2},1,1$; $\lambda,\lambda{-}1,1,1$; $\lambda,\lambda,\lambda^{-1},\lambda^{-1}$; and $1,1,1,1$.

An embedding of $SU(2)$ into $SU(4)\times SU(2)$ is given by $\Gamma\otimes\Gamma_1$ or $\Gamma\otimes 2\Gamma_0$ where Γ is any entry on the previous list. The fact that no simple factor of U acts transitively on G implies that if $f_1,f_2:U\to G$ define the biquotient action, then when restricted to $SU(2)$, we must have the same representation into the $SU(2)$ factor.

We start by ignoring the S^1 factor and classify possible which of the reps of $SU(2)$ are compatible with the $SU(3)$ reps. The first thing to note is that there is no almost effective map from $SU(2)\times SU(3)\to SU(4)$ since the smallest almost effective rep of $SU(2)\times SU(3)$ is $\Gamma_1\otimes\Gamma_{0,0}\oplus\Gamma_0\otimes\Gamma_{1,0}$ which has dimension 5. Further, if both $f_{1|_{SU(3)}}$ and $f_{2|_{SU(3)}}$ are non trivial, then the element $(A,B)=(\mathrm{diag}(i,-i,1),Id)$ gives a noncentral conjugacy, so we don't get a free action in this case.

Thus, all maps we must consider are of the form

$$(A,B)\to((\mathrm{diag}(A),1),B^\delta,\Gamma,B^\delta)$$

where $\delta = 0$ or 1. For any choice of B, we clearly have B^δ and B^δ conjugate, so checking for conjugacy here amounts to checking in the $SU(4)$ piece.

To this end, I claim that the only way to have a free action is if $\Gamma = 2\Gamma_1$. Now, the eigenvalues of an element in $SU(3)$ look like (ρ, η, ν) with $\rho\eta\nu = 1$. For if $\Gamma = \Gamma_3$, set let $\lambda^3 = 1$ and set $\rho = \lambda$, $\eta = \overline{\lambda}, \nu = 1$. Then this gives a noncentral conjugacy.

If $\Gamma = \Gamma_2 \oplus \Gamma 0$, then $B \subseteq SU(3) \subseteq SU(4)$, so there are infinitely many conjugacies. Likewise, if $\Gamma = \Gamma_1 \oplus 2\Gamma_0$, then $B \subseteq SU(3) \subseteq SU(4)$, so there are infinitely many conjugacies.

If $\Gamma = 4\Gamma_0$, then in order to have an effective embedding of $SU(2)$ which doesn't act transitively on the $SU(2)$ in G, we must have $\delta = 1$ in both cases. But then (Id, B) gives a conjugacy for all B.

Finally, consider $\Gamma = 2\Gamma_1$. Conjugacy implies that λ or $\overline{\lambda} = 1$, so either way we conclude that $\lambda = 1$. This implies $B = Id$ and hence that $A = Id$. Thus, we do get a free action in this case.

We now try to add the S^1 action. The first point to make is that by setting if we find a free action of $S^1 \times SU(3) \times SU(2)$, then by setting $z = 1$, we obtain a free action of $SU(3) \times SU(2)$ on $SU(4)$, but we just classified those above. Then thing to note is that if $\delta = 1$, then we cannot fit S^1 into $SU(2)$ along side $SU(2)$. More precisely, there are no maps $S^1 \times SU(2) \to SU(2)$ which, when restricted to $SU(2)$ are the identity and which are almost effective on the S^1 factor.

But we have $rk(U) > rkSU(4)$, so there are no biquotient actions of U on $SU(4)$. Hence, we are forced to take $\delta = 0$. The most general map satisfying all of this is given by $f : S^1 \times SU(3) \times SU(2) \to SU(4) \times SU(2) \times SU(4) \times SU(2)$ with

$$f(z, A, B) = (\mathrm{diag}(z^a A, \overline{z^{3a}}), \mathrm{diag}(z^c, \overline{z^c}), \mathrm{diag}(z^b B, z^{-b} B), \mathrm{diag}(z^d, \overline{z^d}))$$

Now, since the S^1 action and the $SU(3) \times SU(2)$ action commute, then assume we have a free action, we can write the quotient as $(G /\!\!/ (SU(3) \times SU(2)))/S^1$ where the S^1 acts freely and isometrically.

However, if G is given a round $\times$ round metric, then $G /\!\!/ SU(3) \times SU(2) = \mathbb{H}P^1 \times S^3 = S^4 \times S^3$ with a round $\times$ round metric. To see this up to diffeomorphism, first notice that $SU(3) \times SU(2)$ acts trivially on the $SU(2)$ factor in G, and hence $G /\!\!/ SU(3) \times SU(2) = \Big(SU(4) /\!\!/ SU(3) \times SU(2)\Big) \times SU(2)$. Further, in the 4-dimension, we found an explicit diffeomorphism between $SU(4) /\!\!/ SU(3) \times SU(2)$ and $\mathbb{H}P^1 = S^4$ - send a matrix $C \in SU(4)$ the class of it's last row, where $[w_1, w_2] = [w_1 B, w_2 B]$ for any $B \in SU(2)$. Further, by first dividing by the $SU(3)$ factor, we get the usual Hopf fibration $S^3 \to S^7 \to S^4$. The metric on S^7 induced from a biinvariant metric on $SU(4)$ is not round, but rather a Berger metric. However, $SU(2)$ still acts by isometries in the usual way and the quotient is still a round S^4. The easiest way to see this is simply to notice that with a Berger metric looks like the normal round sphere metric in the directions orthogonal to the (1-dimensional) Hopf fibration. Since the S^1 Hopf fibration is naturally included in the S^3 fibration we get here, the metric looks the same as the round metric normal to the orbits, and hence, round in the base.

When we make these identifications, how does S^1 act on $S^4 \times S^3$? On S^3 it acts just as it did above on $SU(2)$: $z(z_1, z_2) = (z^c z_1 z^{-d}, z^c z_2 z^d) = (z^{c-d} z_1, z^{c+d} z_2)$.

The action on S^4 is takes some work to describe. By writing the last row of an element of $SU(4)$ as a pair of 1x2 complex vectors w_1 and w_2, we see that as elements of $\mathbb{H}P^1$, the $z^{\pm b}$ terms act by complex multiplication. Thus, if $w_1 = [z_1, z_2]$ and we think of this as $z_1 + z_2 j$, then $w_1 z^b = [z_1 z^b, z_2 z^b]$ and this becomes $z_1 z^b + z_2 z^b j$ on the quaternionic level.

Now, since $z * [1:0] = [z^{-3a} z^b, 0] = [1:0]$ and $z * [0:1] = [0 : z^{-3a} z^{-b}] = [0:1]$, we find that the circle fixes these two points (which we'll take to be the north and south poles of S^4). The action is thus determined by it's action on the equatorial S^3, which can be identified with points in $\mathbb{H}P^1$ of the form $[q:1]$ with $q = u + vj$ a unit quaternion (with $u, v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$). Then the S^1 action is given as $z * [q:1] = [z^{-3a} u z^b + z^{-3a} v z^b j : z^{-3a-b}] = [\big(z^{-3a} u z^b + z^{-3a} v z^b j\big) z^{b+3a} : 1] = [z^{2b} u + z^{-6a} vj : 1]$.

Thus, we see the action on S^4 is the suspension of the action of S^3 given by $z * (z_1, z_2) = (z^{2b} z_1, z^{-6a} z_2)$.

Hence, using (t, z_1, z_2) for coordinates on $S^4 = \sum S^3$ and using w_1, w_2 as coordinates on $SU(2)$ (the other factor of G), we see that the S^1 action is $z * (t, z_1, z_2, w_1, w_2) = (t, z^{2b} z_1, z^{-6a} z_2, z^{c+d} w_1, z^{c-d} w_2)$ with $(a, b, c, d) = 1$. We still need to find necessary and sufficient conditions so that the action of S^1 on $S^4 \times S^3$

is free.

To begin with, consider the action on points of the form $(1,0,0,w_1,w_2)$. The S^1 action clearly preserves these. To have a free action on $S^4 \times S^3$, it must restrict to a free action on these points, but that means the action of S^1 on the $S^3 = SU(2)$ factor must be free. Hence, we conclude that the action must be some multiple of the Hopf action, so that either $d = 0$ or $c = 0$. Without loss of generality, we may assume $d = 0$.

Thus, our action must be of the form

$$z * (t, z_1, z_2, w_1, w_2) = (t, z^{2b} z_1, z^{-6a} z_2, z^c w_1, z^c w_2).$$

Now, suppose z is a primitive cth root of unity. Then $z*(1,0,0,1,0) = (1,0,0,1,0)$. Thus, in order for the action to be free, z must fix all points. Thus, we conclude that, for example, $(0,1,0,1,0) = z * (0,1,0,1,0) = (0, z^{2b}, 0, 1, 0)$, so that $z^{2b} = 1$. Since z is a primitive cth root of unity, we conclude that $c|2b$. Likewise, since $(0,0,1,1,0) = z * (0,0,1,1,0)$, we conclude that $c|6a$.

Hence $c|(2b, 6a)$ so that $c|2(b, 3a)$. I claim that this condition is also sufficient to guarantee a free action. For suppose $c|2(b, 3a)$ and $(t, z_1, z_2, w_1, w_2) = z * (t, z_1, z_2, w_1, w_2) = (t, z^{2b} z_1, z^{-6a} z_2, z^c w_1, z^c w_2)$. Since $(w_1, w_2) \in S^3$, we cannot have both w_1 and w_2 equal to 0. Hence, assume without loss of generality that $w_1 \neq 0$. Then, since $w_1 = z^c w_1$ and $w_1 \neq 0$, we conclude that $z^c = 1$.

Thus this implies $z^c = z^{2b} = z^{-6a} = 1$, and hence that z acts trivially on $S^4 \times S^3$. Thus, the action is free.

From here, there are 2 goals: first, for each action, we want to classify what the ineffective kernel of the action is. Second, for each allowable action, we want to classify the quotient, up to diffeomorphism.

Towards the first goal, notice first that we may assume without loss of generality that $c > 0$ (if c $= 0$, then we have free action of $S^1 \times SU(3) \times SU(2)$ on $SU(4)$, which is impossible by rank reasons). Further, since $(b, 3a) \leq 3(b, a) = 3$, we have $c|2(b, 3a)$ so that $c \leq 6$.

Further, notice that $c = 4$ and $c = 5$ cannot occur. For if $c = 4$, then $c|2b$ implies that $2|b$. Likewise, $c|6a$ implies that $2|a$. But then $2|(a, b, c)$, contradicting $(a, b, c) = 1$. Likewise, if $c = 5$, then $5|2b$ implies $5|b$ and $5|6a$ implies $5|a$, and hence $5|(a, b, c)$ so $5|1$, a contradiction.

Thus, we simply go through the cases $c = 1, 2, 3$, or 6.

If $c = 1$, then the action is effective for S^1 acts effectively on $SU(2)$. Thus, in this case U is isomorphic to $S^1 \times SU(3) \times SU(2)$.

If $c = 2$, then, going back to the $G = SU(4) \times SU(2)$ action we see that if b is odd, then $(-1, Id, -Id) \in S^1 \times SU(3) \times SU(2)$ acts ineffectively. If b is even, then only $(-1, Id, Id)$ acts ineffectively. Hence, if b is even, then U is isomorphic to $S^1 \times SU(3) \times SU(2)$ while if b is odd, then U is isomorphic to $SU(3) \times U(2)$.

If $c = 3$, then since 3 doesn't divide 2, we must have $3(b, 3a)$. Since $(a, b, c) = 1$, 3 cannot divide a. Looking at the action of U on $SU(4)$, and using the fact that for any 3rd root of unity z, we must have $z^b = z^{-3a} = 1$, it follows that the ineffective kernel of the action is generated by $(z, \overline{z}Id, Id)$. Hence, in this case, $U = U(3) \times SU(2)$.

Finally, if $c = 6$, then we have $6|2(b, 3a)$ so that $3|(b, 3a)$ and again we conclude that 3 cannot divide a. From here we break into cases depending on the parity of a and b (which cannot both be even since this would violate $(a, b, c) = 1$).

If a and b are both odd, then we find that if $z = \zeta_6$ is a primitive 6th root of unity, z^a is also a primitive 6th root of unity, and hence $-z^a$ is a $3rd$ root of unity. So, setting $A = \overline{-z^a}Id \in SU(3)$ and $B = -Id \in SU(2)$, we find (z, A, B) acts ineffectively. Further, it's clear that this is the only choice. Hence, the kernel of the action is generated by this. It is easy to see that $S^1 \times SU(3) \times SU(2)$ divided by this kernel is $S(U(3) \times U(2))$, for example, by consider the map sending (z, A, B) to $(z^2A, z^{-3}B)$. Hence, in this case, U $= S(U(3) \times U(2))$.

If a is odd and b is even, then the ineffective kernel is generated by $(\zeta_6, \overline{-\zeta_6}Id, Id)$, so again $U = U(3) \times SU(2)$.

Finally, if a is even and b is odd (and divisible by 3), then similar reasoning shows that $U = SU(3) \times U(2)$.

All we have left is determining the diffeomorphism type. The first thing to notice is that we essentially have an ineffective Hopf action on the $SU(2)$ factor of G. Nonetheless, the projection onto the second factor gives $G/\!\!/U$ the structure of an S^4 bundle over S^2. There are precisely 2 S^4 bundles over S^2 corresponding to the choice of clutching function in $\pi_1(SO(5)) = \mathbb{Z}/2\mathbb{Z}$. Further, these two manifolds are distinguished by their second Stiefel-Whitney class, as in the S^3 bundle over S^2 case. However, in this case, computing the Stiefel-Whitney class in terms of a, b, and c will be significantly easier than it was in the 5-dimensional case.

The key observation can be found in Pavlov's paper [23]: the S^1 action on S^4 preserves the equatorial S^3, and hence we see that the S^1 action on $S^4 \times S^3$ preserves an $S^3 \times S^3 \subseteq S^4 \times S^3$. This will give us a codimension 1 submanifold $N = S^3 \times_{S^1} \times S^3 \subseteq M = S^4 \times_{S^1} S^3$. The computation of $w_2(N)$ is accomplished via the section on 5-dimensional biquotients. Then, we'll argue that $w_2(N) = w_2(M)$.

We'll actually start by arguing that $w_2(M) = w_2(N)$. To be more precise, we'll see that $H^2(N, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = H^2(M, \mathbb{Z}/2\mathbb{Z})$ and that $w_2(N) \neq 0$ iff $w_2(M) \neq 0$. The first part of the argument is almost entirely general and due to Pavlov [23] - if N and M are both orientable, and if $w_2(N) \neq 0$ then $w_2(M) \neq 0$. Notice that our N and M are both orientable because they're simply connected.

For the converse, we'll have to do a little more work and actually use the structure of the manifolds a bit. In particular, it's not generally true that if a codimension 1 submanifold has $w_2 = 0$, then the whole manifold must also have $w_2 = 0$. A quick counterexample is provided by the inclusion $\mathbb{R}P^1 \subseteq \mathbb{R}P^2$. $\mathbb{R}P^1 = S^1$ has trivial tangent bundle hence $w_2 = 0$, but $\mathbb{R}P^2$ has $w_2 = a^2$ where $H^*(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]/a^3$.

So, Consider the inclusion of $i : N \to M$ and assume $w_2(N) \neq 0$. The inclusion gives the following: $i^*(TM) \cong TN \oplus \nu$ where ν is a trivial one dimensional bundle (it's trivial because both N and M are orientable). The Whitney sum formula then gives $w_2(i^*(TM)) = w_0(TN) \cup w_2(\nu) + w_1(TN) \cup w_1(\nu) + w_2(TN) \cup w_0(\nu)$. However, $w_2(\nu) = 0$ since ν has rank 1 and $w_1(\nu) = 0$ since ν is a trivial bundle. Hence, we conclude $i^*(w_2(M)) = w_2(i^*(TM)) = w_2(N)$. It's immediate now that if $w_n(N) \neq 0$, then we must have $i^*(w_2(M)) \neq 0$, so that $w_2(M) \neq 0$.

For the converse, we must work harder. First notice that the induced S^1 action on $S^3 \times S^3$ is effectively Hopf on the right S^3 and hence $pi_2|_{S^3 \times_{S^1} S^3}$ is the map defining the bundle structure of the S^3 bundle over S^2. In other words, if we call π the map from $S^3 \times_{S^1} S^3 \to S^2$ defining the bundle structure, then we have $\pi \circ i = \pi_2$. Thus, we'll show i^* is an isomorphism on $H^2(M) \to H^2(N)$ by showing π and π_2 are. From here, if $i^*(w_2) = 0$, of course we must conclude $w_2 = 0$. I'll only work out the proof for π is an isomorphism since the proof that π_2 is almost exactly the same.

Using the Gysin sequence associated to $S^3 \to S^3 \times_{S^1} S^3 \to S^2$, one sees that

$$H^{1-3}(S^2) \to H^2(S^2) \to H^2(N) \to H^{2-3}(S^2)$$

where the map $H^2(S^2) \to H^2(N)$ is π^*. However, clearly the terms on the end are 0, so π is an isomorphism as claimed.

Thus, we have shown that M is diffeomorphic to the nontrivial S^4 bundle over S^2 iff N is diffeomorphic to the nontrivial S^3 bundle over S^2, and likewise M is diffeomorphic to $S^4 \times S^2$ iff N is diffeomorphic to $S^3 \times S^2$.

In order to compute the diffeomorphism type of N, we'll go back to our classification in dimension 5. We must write the action as

$$z * (z_1, z_2, w_1, w_2) = (z^{k_1+k_2} z_1, z^{k_1-k_2} z_2, z^{n_1+n_2} w_1, z^{n_1-n_2} w_2).$$

Solving this system of equations gives $k_1 = b - 3a$, $k_2 = b + 3a$, $n_1 = c$ and $n_2 = 0$. Since $n_2 = 0 \cong 0 \bmod 4$, we see that N is diffeomorphic iff $n_1 \cong 2 \bmod 4$ and k_1 and k_2 are both odd. It is clear that k_1 is odd iff k_2 is, so we'll just check one.

Thus, for $c = 1$ or 3, we have N is diffeomorphic to $S^3 \times S^2$ and hence M is diffeomorphic to $S^4 \times S^2$.

If $c = 2$ or $c = 6$, then if a and b are both odd, we have M is diffeomorphic to the nontrivial S^4 bundle over S^2. If precisely one is odd, then M is diffeomorphic to S^4. Both can't be even, since this would contradict $(a, b, c) = 1$.

□

Proposition 5.3.5. *There are 2 families of actions of $U = SU(2) \times SU(2) \times S^1$ on $G = Sp(2) \times SU(2)$ (where neither $SU(2)$ in U acts transitively on the $SU(2)$ in G). The first action, under the identification of $SU(2) \times SU(2)$ with $Sp(1) \times Sp(1)$ is given by $f(p, q, z) = (diag(p, q), diag(z^a, \overline{z^a}), diag(z^b, z^b), Id)$ where $p, q \in Sp(1)$ and $z \in S^1$. In this case, we have $(a, b, c) = 1$ and $a|(b \pm c)$. The second action is $f(p, q, z) = ((R(a\theta))diag(p, p), diag(z^b, \overline{z^b}), diag(q, z^c), Id)$ with $(a, b, c) = 1$ and $b|(a, 2c)$.*

In the first case, the conditions imply that $a = 1$ or $a = 2$. If $a = 1$, all quotients are diffeomorphic to $S^4 \times S^2$. If $a = 2$, then c and d must both be odd and the quotient is diffeomorphic to the unique nontrivial S^4 bundle over S^2.

In the second case, we must have $b = 1$ or $b = 2$. No matter what, the quotient is diffeomorphic to $S^4 \times S^2$.

Proof. The first thing to realize is that both $SU(2) \times e$ and $e \times SU(2)$ must act almost effectively on $Sp(2)$. For if, say $SU(2) \times e$ doesn't act almost effectively on $Sp(2)$, then it must act on $SU(2)$ by conjugation (point 4 of Totaro's theorem). But then the action, when restricted to $SU(2) \times e$, won't be free. But if the action is free, the restriction of the action to any subgroup of U must also be free.

Biquotients of the form $Sp(2)/Sp(1) \times Sp(1)$ were classified in the 4-dimensional case. There are precisely two: the homogeneous action and

$$(p, q) \to (\mathrm{diag}(p, p), \mathrm{diag}(q, 1)).$$

In either case, the quotient is S^4. Hence, as in the previous case, we must have $G/\!\!/U = S^4 \times_{S^1} S^3$. We'll classify exactly which S^1 actions arise from this construction in each case, and then use the previous case to determine the diffeomorphism type of the quotient.

Assume we're in the homogeneous case. Since $Sp(1) \times Sp(1)$ is maximal, the S^1 can only be mapped into $e \times SU(2) \times Sp(2) \times SU(2)$. Thus, the general map is of the form

$$(p, q, z) \to (\mathrm{diag}(p, q), \mathrm{diag}(z^a, \overline{z}^a), \mathrm{diag}(z^b, z^c), \mathrm{diag}(z^d, \overline{z}^d)).$$

As before, the action of S^1 on the $SU(2)$ factor of G will have to be effectively

Hopf, so we need $d = 0$. We now identify $Sp(2)/Sp(1) \times Sp(1)$ with S^4 and see exactly how the S^1 action looks.

To map $Sp(2)/Sp(1) \times Sp(1)$ to S^4, send $B \in Sp(2)$ to the class o the last row under left multiplication by S^3. Then, we see that S^1 acts on the first column of B by right multiplication with z^{-b} and it acts on the second column by right multiplication with z^{-c}.

Hence, for $[q_1 : q_2] \in \mathbb{H}P^1$, we have $z * [q_1 : q_2] = [q_1 z^{-b} : q_2 z^{-c}]$. Then this action clearly fixes $[1 : 0]$ and $[0 : 1]$, which we identify with the north and south poles of S^4. Then the equatorial S^3 in S^4 is, in $\mathbb{H}P^1$, given by points of the form $[q_1 : 1]$ with $|q_1| = 1$. Then the S^1 action on such points is $z * [q_1 : 1] = [q_1 z^{-b} : z^{-c}] = [z^c q_1 z^{-b} : 1]$. Writing $q_1 = u + vj$ with $u, v \in \mathbb{C}$, we see the action is given by $z * (u, v) = (z^{c-b} u, z^{c+b} v)$. Putting this all together, we conclude that the S^1 action on $S^4 \times S^3$ is given by $z * (t, z_1, z_2, w_1, w_2) = (t, z^{c-b} z_1, z^{c+b} z_2, z^a w_1, z^a w_2)$. Freeness, as in the previous case, is equivalent to $a|(b-c, b+c)$. Note that if $a|b-c$ and $a|b+c$, then $a|2b$ and $a|2c$, so that $a|(2b, 2c) = 2(b, c)$, so that $a|2$. We may assume without loss of generality that $a > 0$ by precomposing with the automorphism $z \to -z$. Hence, $a = 1$ or $a = 2$. If $a = 1$, we recognize, following the argument in the previous case, that the quotient is $S^4 \times S^2$. If $a = 2$, then since $2|c - b$ we must have b and c of the same parity. If they're both even, we contradict $(a, b, c) = 1$, so they're both odd. As in the previous case, we conclude that these quotients are diffeomorphic to the unique nontrivial S^4 bundle over S^2.

We now handle the other map $f(p,q) \to \mathrm{diag}(p,p), \mathrm{diag}(q,1)$. We now try to fit in an S^1 action (which will be significantly messier in this case). An S^1 fits on both sides, but on the left, it must fit in as $R(a\theta)$. Hence, a general action is given by $(p,q,z) \to ((R(a\theta))\mathrm{diag}(p,p), \mathrm{diag}(z^b, \overline{z^b}), \mathrm{diag}(q, z^c), \mathrm{diag}(z^d, overlinez^d))$ with $z = e^{i\theta}$. As above, in order to have a free action, we must have $d = 0$.

Now, we again identify $Sp(2)/Sp(1) \times Sp(1)$ with S^4. This time, it occurs by sending an element $B \in Sp(2)$ to the class of its 2nd column under left quaternionic multiplication. The S^1 action on $[q_1, q_2]$ then looks like

$$z * [q_1, q_2] = [\cos(a\theta)q_1\overline{z}^c + \sin(a\theta)q_2\overline{z^c} : -\sin(a\theta)q_1\overline{z}^c + \cos(a\theta)q_2\overline{z}^c]$$

Identifying this with S^4 and understanding the action will take some work. First note that $[\pm i : 1]$ is fixed by this action:

$$\begin{aligned} z * [i:1] &= [\cos(a\theta)iz^{-c} + \sin(a\theta)z^{-c} : -\cos(a\theta)iz^{-c} + \sin(a\theta)z^{-c}] \\ &= [\cos(a\theta)i + \sin(a\theta) : -\sin(a\theta) + \cos(a\theta)] \\ &= [(\cos(a\theta) + i\sin(a\theta))(\cos(a\theta)i + \sin(a\theta)) : 1] \\ &= [i:1] \end{aligned}$$

Under this identification, it's natural to identify the equatorial S^3 with $[n_1 + n_2 j + n_3 k : n_4]$ for $n_1^2 + n_2^2 + n_3^2 = 1$. We claim that the S^1 action preserves this set. In fact, by linearity, it's enough to show that the S^1 action preserves points of the form $[1 : n_4]$ (or $([0 : 1])$ and $[n_2 j + n_3 k : 1]$ with $n_2^2 + n_3^2 = 1$.

For the first, we compute:

$$\begin{aligned} z * [1 : n_4] &= [\cos(a\theta)z^{-c} + \sin(a\theta)n_4 z^{-c} : -\sin(a\theta)z^{-c} + \cos(a\theta)n_4 z^{-c}] \\ &= [1 : (\cos(a\theta) + \sin(a\theta)n_4)(-\sin(a\theta) + \cos(a\theta)n_4)] \end{aligned}$$

which is clearly of the right form.

For the second, we compute again:

$$\begin{aligned} z * [n_2 j + n_3 k : 1] &= [\cos(a\theta)n_2 j z^{-c} + \cos(a\theta)n_3 k z^{-c} + \sin(a\theta)z^{-c} \\ &\quad : -\sin(a\theta)n_2 j z^{-c} - \sin(a\theta)n_3 k z^{-c} + \cos(a\theta)z^{-c}] \\ &= [\cos(a\theta)n_2 j + \cos(a\theta)n_3 k + \sin(a\theta)z^{-2c} \\ &\quad : -\sin(a\theta)n_2 j - \sin(a\theta)n_3 k + \cos(a\theta)z^{-2c}] \\ &= [z^{2c}(n_2 j + n_3 k) : 1] \end{aligned}$$

Hence, we conclude that the action on S^4 is $z*(t, z_1, z_2) = (t, z^a z_1, z^{2c} z_2)$. Hence the S^1 action on $S^4 \times S^3$ is $z*(t, z_1, z_2, w_1, w_2) = (t, z^a z_1, z^{2c} z_2, z^b w_1, z^b w_2)$. Freeness, as we showed earlier, is equivalent to $b|(a, 2c)$. Since $(a, b, c) = 1$, this implies $b = 1$ or $b = 2$. If $b = 1$, the quotient is $S^4 \times S^2$ and if $b = 2$, then we must have a even, and hence, $a \pm 2c$ is even, and hence, by the above classification, the quotient is diffeomorphic to $S^4 \times S^2$. $\square$

Proposition 5.3.6. *There is a unique family of biquotient actions of $U = G_2 \times SU(2) \times S^1$ on $G = Spin(7) \times SU(2)$, which we can study by using $G = SO(7) \times$*

$SU(2)$ and $U = G_2 \times SO(3) \times S^1$. The G_2 and $SO(3)$ in U only act on the $SO(7)$, so that the biquotient is diffeomorphic to $S^4 \times_{S^1} S^3$. The S^1 acts on S^3 by the Hopf action. The S^1 action on S^4 is determined from a map $S^1 \to SO(7)$ and is parameterized by 2 integers. The quotient $S^4 \times_{S^1} S^3$ is diffeomorphic to $S^4 \times S^2$ when the 2 parameters have the same parity and is diffeomorphic to the unique nontrivial S^4 bundle over S^2 when the two parameters have opposite parity.

Proof. We first show that every biquotient with $G = Spin(7) \times SU(2)$ descends to a biquotient with $G = SO(7)$. The converse follows from Totaro's proof that we can write a biquotient according to the conventions.

Let $\pi : Spin(7) \to SO(7)$ be the canonical projection. Then π clearly induces a covering map $\overline{pi} : \big(Spin(7) \times SU(2)\big)/\!\!/U \to \big(SO(7) \times SU(2)\big)/\!\!/\pi(U)$. Since π is two sheeted, $\overline{\pi}$ is at most two sheeted. We claim that, in fact, $\overline{\pi}$ is single sheeted, i.e., a diffeomorphism. To see this, we will argue that $\ker(\pi)$ is contained in one orbit.

The $SU(2)$ in U cannot act on the $SU(2)$ in G without acting transitively. This is because the only nontransitive nontrivial action of $SU(2)$ on itself is via conjugation. Further, there is no almost effective action of $S^1 \times SU(2)$ on $SU(2)$ extending this action. It follows that if U acts on G freely, then U must act freely on $Spin(7)$, but this is impossible by rank reasons. Likewise, the G_2 factor of U cannot act on $SU(2)$. It follows that $G_2 \times SU(2)$ acts freely on $Spin(7)$.

Since $G_2 \backslash Spin(7) = S^7$ and since the $SU(2)$ must act freely on this, it follows that the $SU(2)$ factor (which is, a priori, only defined up to cover), must actually be isomorphic to $SU(2)$. Finally, since $\pi(G_2) = G_2$, G_2 being centerless, we see the induced quotient is $G_2 \backslash SO(7) = \mathbb{R}P^7$ and since $\pi(SU(2))$ must act freely on this, $\pi(SU(2)) = SO(3)$. It follows that $\ker(\pi) \subseteq SU(2) * e$ for $e \in Spin(7)$ the identity element.

Since the $SU(2)$ has a unique free action on S^7, if we find one free action of $G_2 \times SO(3)$ on $SO(7)$, we've found the only one. The eigenvalues of the maximal torus of G_2 in $SO(7)$ are $\lambda, \overline{\lambda}, \nu, \overline{\nu}, \lambda\nu, \overline{\lambda\nu}$, and 1, and the eigenvalues of the standard block embedding of $SO(3)$ into $SO(7)$ are $\mu^2, \overline{\mu^2}, 1, 1, 1, 1$, and 1, we can easily compare eigenvalues. For G_2, it is easy to see that it cannot have exactly 5 eigenvalues equal to 1, since if $\lambda \neq 1$ and $\nu = 1$, the $\lambda\nu \neq 1$. Thus, a conjugacy can only occur when $\lambda = \nu = \mu^2 = 1$, so this action is effectively free.

All of this together implies that any biquotient of the form $Spin(7) \times SU(2) /\!\!/ G_2 \times SU(2) \times S^1$ is diffeomorphic to $S^4 \times_{S^1} S^3$ for some action of S^1 on S^4. Since every circle action on S^4 has a fixed point, the S^1 must act (effectively) freely on S^3, i.e., it must be (effectively) the Hopf action.

Going back to our previous description, we see that the map $SO(3) \times S^1 \to SO(7)$ is $(A, \theta) \to \operatorname{diag}(A, R(p\theta), R(q\theta))$ where R denotes the standard $2{\times}2$ rotation matrix.

Unfortunately, it seems difficult to determine how p and q relate to the weights of the representation of S^1 into $SO(5)$ defined by a general linear action on S^4. Thus, we must actually compute the diffeomorphism type from the biquotient description itself.

To this end, we notice that every manifold of the form $S^4 \times_{S^1} S^3$ is naturally an S^4 bundle over S^2, and that there are two such bundles, distinguished by their second Stiefel-Whitney classes. Thus, we really only need to determine w_2 for each of our biquotients.

To this end, we note that the previously developed machinery for computing Pontryagin classes can be applied, but instead of using root systems, one must use 2-roots.

For definiteness, we use the map $f_1 : G_2 \times SO(3) \times S^1 \to SO(7) \times SU(2)$ with $f_1(A, B, z) = (A, I)$ where $A \subseteq SO(7)$ is the standard embedding. We also use the map $f_2 : G_2 \times SO(3) \times S^1$ with $f_2(A, B, \theta) = (\mathrm{diag}(B, R(p\theta), R(q\theta), z)$, where $z = e^{i\theta}$.

According to Borel and Hirzebruch [5], the maximal 2-group of G_2 is $\mathbb{Z}/2\mathbb{Z}^3 \subseteq G_2$ generated by $\langle s_1, s_2, s_3 \rangle$ where, letting e_i be the standard basis of $\mathbb{R}^7$, we have

$$s_i(e_j) = \begin{cases} e_j & j = i+1, i+5, \text{ or } i+6 \\ & \\ -e_j & \text{otherwise} \end{cases}.$$

The maximal 2-group of $SO(n)$ is a $\mathbb{Z}/2\mathbb{Z}^{n-1}$ consisting of all diagonal matrices with ± 1 on the diagonal, subject to the determinant being 1.

We let y_i for $i = 1...7$ denote generators of the maximal 2-group of $SO(7)$ corresponding to the diagonal matrix with a -1 in the ith position and 1s elsewhere. They satisfy the relation that $y_1 \cdot y_7 = 1$. We'll let t be the generator for the unique

$\mathbb{Z}/2\mathbb{Z}$ in $SU(2)$, generated by $-Id$. Finally, define p_i for $i = 1, 2, 3$ analogously for $SO(3)$ and let z denote the generator of the 2-group of S^1.

In this notation, we see that $f_1^*(y_1) = s_1$, $f_1^*(y_2) = s_2$, $f_1 * (y_3) = s_1 + s_3$, $f_1^*(y_4) = s_1 + s_2$, $f_1^*(y_5) = s_1 + s_2 + s_3$, $f_1^*(y_6) = s_2 + s_3$, $f_1^*(y_7) = s_3$ and $f_1^*(t) = 0$.

Likewise, we see that $f_2^*(y_1) = p_1$, $f_2^*(y_2) = p_2$, $f_2^*(y_3) = p_3$, $f_2^*(y_4) = f_2^*(y_5) = pz$, $f_2^*(y_6) = f_2^*(y_7) = qz$ and $f_2^*(t) = z$.

Now, we can identify $H^*(\big(BSO(7) \times BSU(2)\big) \times \big(BSO(7) \times BSU(2)\big); \mathbb{Z}/2\mathbb{Z}) \subseteq H^*(B\mathbb{Z}/2\mathbb{Z}^{12}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[y_i \otimes 1, 1 \otimes y_i]$ where i ranges from 1 to 7. It's isomorphic to $\mathbb{Z}/2\mathbb{Z}[\sigma_j(y_i \otimes 1), \sigma_j(1 \otimes y_i)]$ where j ranges from 2 to 7.

Likewise, we can identify

$$H^*(B(G_2 \times SO(3) \times S^1); \mathbb{Z}/2\mathbb{Z}) \subseteq H^*(B\mathbb{Z}/2\mathbb{Z}^7; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[s_1, s_2, s_3, p_1, p_2, p_3, z]$$

as the subalgebra generated by the elements

$[\sigma_2(s_i) + \sigma_1(s_i)^2]^2 + \sigma_1(s_i)\sigma_3(s_i)$,

$[\sigma_3(s_i) + \sigma_1(s_i)\sigma_2(s_i)]^2 + [\sigma_2(s_i) + \sigma_1(s_i)^2]\sigma_1(s_i)\sigma_3(s_i)$,

$\sigma_1(s_i)\sigma_3(s_i)[\sigma_3(s_i) + \sigma_1(s_i)\sigma_2(s_i)]$,

$\sigma_2(p_i)$,

$\sigma_3(p_i)$, and

z^2.

Now, just as with $\mathbb{Z}$ coefficients, this information tells us the differentials in the spectral sequence $G \to G/\!\!/U \to BU$. If x_1 is the generator of $H^1(G; \mathbb{Z}/2\mathbb{Z})$, the we know that $d_2(x_1) = f^*(\sigma_2(y_i) \otimes 1 + 1 \otimes \sigma_2(y_i))$. The first piece is easy to evaluate since $f^*(\sigma_2(y_i) \otimes 1) \subseteq H^2(BG_2; \mathbb{Z}/2\mathbb{Z}) = 0$. For the second, following a computation, we see that $d_2(x_i) = \sigma_2(p_i) + (p+q)z^2$.

Thus, d_2 is injective and we see that

$$H^2(G/\!\!/U; \mathbb{Z}/2\mathbb{Z}) \cong \langle \sigma_2(p_i), z^2 \rangle / \sigma_2(p_i) + (p+q)z^2.$$

Now, to compute the characteristic classes, we first list the nonzero 2-roots of each of the participating groups. The 2-roots of G_2 are s_1, s_2, s_3, s_1+s_2, s_1+s_3, s_2+s_3, and $s_1+s_2+s_3$, all with multiplicity 2. The 2-roots of $SO(3)$ are p_1+p_2, p_2+p_3, and p_1+p_3, and for $SO(7)$ there are all y_i+y_j with $1 \leq i < j \leq 7$, all with multiplicity 1. The other participating groups, $SU(2)$ and S^1, have no nonzero 2-roots.

Following Singhoff [25], the Stiefel-Whitney classes of $G/\!\!/U$ is the pullback of the class $\Pi_{\lambda_i \text{ a 2-root of } G}(1+\lambda_i)$ in BG times the pullback of the class $\big(\Pi_{\mu_i \text{ a 2-root of } U}(1+\mu_i)\big)^{-1}$ in BU. We can write this element as

$$\phi_U^*\big(f_2^*\Pi(1+1\otimes\lambda_i)\Pi(1+1\otimes\mu_i)^{-1}\big)$$

Now, ϕ_U^* is trivial on elements coming from G_2 in H^2, and since we only care about the degree 2 piece of this product, we can ignore the 2-roots of G_2.

Computing the product is made simpler because $f_2^*(1+y_i+y_4)(1+y_i+y_5) = (1+f_2^*y_i+pz)(1+f_2^*y_i+pz) = (1+f_2^*y_1^2+pz^2)$ we see that and similarly replacing y_4 and y_5 by y_6 and y_7 (and if $i=4$, then the sum cancels out). So, all the factors have degree 2, so the degree 2 piece is the sum of elements, which is $(p+q)z^2$.

Finally, recalling that $H^2(G/\!\!/U;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\sigma_2(p_i), z^2]/\sigma_2(p_i) + (p+q)Z^2$ we see that if $(p+q)$ is *even*, then $w_2(G/\!\!/U) = 0$, while if $p+q$ is odd, then $w_2(G/\!\!/H)$ is the unique nontrivial element of $H^2(G/\!\!/U;\mathbb{Z}/2\mathbb{Z})$.

In short, if p and q have the same parity, the biquotient is diffeomorphic to $S^4 \times S^2$, while if p and q have different parities, then the biquotient is diffeomorphic to the unique nontrivial S^4 bundle over S^2. □

Proposition 5.3.7. *There are infinitely man biquotients with $G = Spin(8) \times SU(2)$ and $U = Spin(7) \times SU(2) \times S^1$, which are all diffeomorphic to $S^4 \times_{S^1} S^3$ for some linear S^1 action on S^4.*

Proof. The point is that the $SU(2)$ factor in U can only act on the $Spin(7)$ part. For it either acts transitively on $SU(2)$ or by conjugation. The first case is ruled out by convention while in the second, there is no room for an S^1 action on $SU(2)$. Since the $SU(2)$ must also act on $Spin(8)$ to keep a free action, it follows that U must act freely on $Spin(8)$, but this is impossible by rank reasons. Hence, the $SU(2)$ acts only on $Spin(8)$. Since we know $Spin(7)$ only acts only on one side of $Spin(8)$ and all embeddings of $Spin(7)$ into $Spin(8)$ are outer conjugate to the lift of the usual embedding of $SO(7)$ into $SO(8)$. Hence, the biquotient looks like $((Spin(7)\backslash Spin(8))/SU(2)) \times_{S^1} SU(2) = (S^7/SU(2)) \times_{S^1} SU(2) = S^4 \times_{S^1} SU(2)$. □

Proposition 5.3.8. *There are no biquotients with $G = Spin(8) \times SU(2)$ and $U = G_2 \times Sp(2) \times S^1$.*

Proof. As there are no G_2 nor $Sp(2) \subseteq Spin(8)$, the action of $G_2 \times Sp(2) \times 1$ on $Spin(8)$ must be free, but this is impossible by a previous theorem in the classification when $G/\!\!/U$ has the same rational homotopy groups as S^4. $\square$

Proposition 5.3.9. *There are no biquotients of the form*

$$(SU(4) \times SU(3))/\!\!/SU(3)^2 \times S^1,$$

$$(Sp(2) \times SU(3))/\!\!/(SU(3) \times SU(2) \times S^1),$$

$$(Spin(7) \times SU(3))/\!\!/(G_2 \times SU(3) \times S^1), \text{ nor}$$

$$(Spin(8) \times SU(3))/\!\!/(Spin(7) \times SU(3) \times S^1).$$

Proof. In each case, let $U' \times S^1 = U$. Then in every case we must have U' acting freely on the first factor of G, but this can't happen due to elementary representation theory. $\square$

Proposition 5.3.10. *(Eschenburg) If $G = SU(3)$ and $U = T^2$, there are precisely two biquotients. The first is the homogenous space while the second is a biquotient $SU(3)/\!\!/T^2$ which is not even homotopy equivalent to $SU(3)/T^2$. Interestingly, both of these spaces admit metrics with positive sectional curvature.*

Proposition 5.3.11. *Suppose $G = SU(3) \times SU(2)$ and $U = SU(2) \times T^2$. Then the $SU(2)$ in U must act solely on the $SU(3)$ in G, so that the biquotient is diffeomorphic to $S^5 \times_{T^2} S^3$ for some free T^2 action on $S^5 \times S^3$. Likewise, if $G = SU(4) \times SU(2)$ and $U = Sp(2) \times T^2$, then all free actions are equivalent to free T^2 actions on $S^5 \times S^3$.*

Proof. There are, up to conjugacy, there are only 2 maps $f : SU(2) \to SU(2)$ given by $f(A) = A$ and $f(A) = Id$. We must show that $f(A) = A$ cannot be used to define the biquotient action. The point is that if $f(A) = A$ describes the left side of the action of $SU(2)$ on $SU(2)$, then in order to stay within our conventions, we must also have $f(A) = A$ describe the right side of the action. However, these two maps have maximal images in the $SU(2)$ in G, and so the T^2 in U must act solely on the $SU(3)$ in G. Further, in order to have the $SU(2)$ in U acting freely, it must act freely on $SU(3)$. Hence, assuming $f(A) = A$, we must have U acting freely on $SU(3)$, but this is impossible by rank reasons.

Thus, we know that the $SU(2)$ in U must act freely on $SU(3)$. We've already classified these actions: there are 2 and both are homogeneous. One gives $SU(3)/SU(2) = S^5$ while the other gives $SU(3)/SO(3)$, the Wu manifold. Now we show the Wu manifold can not occur.

The reason the Wu manifold can not occur is that $SO(3)$ is maximal in $SU(3)$, so the T^2 must act on the $SU(2)$ and on the Wu manifold simultaneously. So, parameterizing the torus T^2 by z and w, we may assume without loss of generality that the z factor acts nontrivially on the Wu manifold, for if not, then we must have a free T^2 action on S^3, which is impossible. Set $w = 1$ so as to focus on z. The embedding of z into $SU(3)$ is given as $\mathrm{diag}(z^a, z^b, z^c)$ with $a + b + c = 0$. From this equation, it follows that we cannot have $|a| = |b| = |c|$, so, assume $|a| < |b|$. Let z be an *ath* root of unity. Then the matrix looks like $\mathrm{diag}(1, z^b, z^c)$ with both

$z^b \neq 1$ and $z^c \neq 1$. Such a matrix is conjugate to one in $SO(3)$, which shows that this element fixes a point, but not all points. Thus, there are no free actions of T^2 on $[SU(3)/SO(3)] \times SU(2)$.

Thus, the only possible action we're left with is an action of T^2 on $S^5 \times S^3$. Explicitly, consider the map $SU(2) \times T^2 \to G \times G$ sending (A, z, w) to

$$(\mathrm{diag}(z^a w^b A), \overline{z}^{2a}\overline{w}^{2b}), \mathrm{diag}(z^c w^d, \overline{z}^c \overline{w}^d), \mathrm{diag}(z^e w^f, z^g w^h, z^i w^j), \mathrm{diag}(z^k w^l, \overline{z}^k \overline{w}^l)$$

(with $e + g + i = f + h + j = 0$) then the induced T^2 action on $S^5 \times S^3$ is given by

$$\begin{aligned} (z, w) \quad * \quad & (p_1, p_2, p_3, q_1, q_2) = \\ & (z^{2a-e} w^{2b-f} p_1, z^{2a-g} w^{2b-h} p_2, z^{2a-i} w^{2b-j} p_3, z^{c-k} w^{d-l} q_1, z^{c+k} w^{d+l} q_2). \end{aligned}$$

Here, we're treating $S^5 = \{(p_1, p_2, p_3) \in \mathbb{C}^3 |\ |p_1|^2 + |p_2|^2 + |p_3|^2 = 1\}$ and $S^3 = \{(q_1, q_2) \in \mathbb{C}^2 |\ |q_1|^2 + |q_2|^2 = 1\}$.

It is clear from this description that, possibly by adding ineffective kernel, any linear action of T^2 on $S^5 \times S^3$ can be obtained as a biquotient action.

The case where $G = SU(4) \times SU(2)$ and $U = Sp(2) \times T^2$ is even easier, as we know we must have $SU(4)/Sp(2) = S^5$. The $Sp(2)$ in $SU(4)$ is maximal, so any map from U to G must send (A, z, w) to

$$(\mathrm{diag}(A), \mathrm{diag}(z^a w^b, \overline{z}^a \overline{w}^b), \mathrm{diag}(z^c w^d, z^e w^f, z^g w^h, z^i w^j), \mathrm{diag}(z^k w^l, \overline{z}^k \overline{w}^l))$$

with $c + e + g + i = d + f + h + j = 0$. The induced T^2 action on $S^5 \times S^3$ is

$$\begin{aligned} (z, w) \quad * \quad & (p_1, p_2, p_3, q_1, q_2) = \\ & z^{c+e} w^{d+f} p_1, z^{c+g} w^{d+h} p_2, z^{e+g} w^{f+h} p_3, z^{a-k} w^{b-l} q_1, z^{a+k} w^{b+l} q_2). \end{aligned}$$

Again, it's clear that, possibly by adding ineffective kernel, we may achieve any linear action at all. □

We now classify which T^2 actions on $S^5 \times S^3$ are free and analyze the quotients.

Proposition 5.3.12. *Suppose T^2 acts on $S^5 \times S^3$ linearly and freely. Then the action is equivalent to $(z, w) * (p_1, p_2, p_3, q_1, q_2) = (zp_1, zw^c p_2, zw^e p_3, wq_1, z^g wq_2)$ with the p_i complex coordinates in $\mathbb{C}^3$ and the q_i are complex coordinates in $\mathbb{C}^2$. Further, we have that either $g = 0$, $c = e = 0$, or we have*

$$(g, c, e) \in \{(1, 1, 2), (2, 2, 1), (0, 2, 1), (0, 1, 2)\}.$$

In the case that $c = e = 0$, the projection onto the first factor gives $G/\!/U$ the structure of an S^2 bundle over $\mathbb{C}P^2$. In the case that $g = 0$, the projection onto the second factor gives $G/\!/U$ the structure of a$\mathbb{C}P^2$ bundle over S^2.

Proof. As above, we'll use (p_1, p_2, p_3) as complex coordinates on S^5 (with $|p_1|^2 + |p_2|^2 + |p_3|^2 = 1$) and likewise, we'll use (q_1, q_2) as complex coordinates on S^3.

I first claim that any effective linear action of T^2 on $S^5 \times S^3$, up to reparamaterization, must look like

$$(z, w) * (p_1, p_2, p_3, q_1, q_2) = (z^a p_1, z^b w^c p_2, z^d w^e p_3, w^f q_1, z^g w^h q_2)$$

with $a \neq 0$ and $f \neq 0$. For, if not, then all circle actions must be dependent, which would contradict effectiveness. Further, we can obviously assume without loss of generality that $(a, b, d, g) = (c, e, f, h) = 1$.

From here, notice that if (z, w) fixes some point, then it fixes a point of the form $(1, 0, 0, 1, 0)$, with a 1 in one of the first the slots and a 1 in the last 2 slots and all else 0. This is because if (z, w) fixes $(p_1, p_2, p_3, q_1, q_2)$, then at least one of the $p_i \neq 0$ and at least one $q_j \neq 0$. Assume without loss of generality that it's p_1 and q_1 which are nonzero. Then (z, w) will also fix $(1, 0, 0, 1, 0)$. Thus, in order to check that the action is (effectively) free, it's enough to show that any point (z, w) either moves all points $(1, 0, 0, 1, 0)$, etc. or fixes all points. It will turn out that when written like this, an effectively free action is automatically free.

First, I claim that $a = f = 1$. For, let z be an ath root of 1 and consider the point $(z, 1)$. This will fix $(1, 0, 0, 1, 0)$. To have an effectively free action, we must have $(z, 1)$ fixing all points in $S^5 \times S^3$. In particular, it must fix the point $(0, 1, 0, 1, 0)$ which implies $a|b$. Likewise, arguing with the point $(0, 0, 1, 1, 0)$, one learns that $a|d$. Finally, the point $(1, 0, 0, 0, 1)$ shows that $a|g$. Thus, we have that $a|(a, b, d, g) = 1$ so $a = \pm 1$. By using the automorphism $z \to -z$, one can assume $a = 1$. Carrying out the analogous argument with $(1, w)$ for w an fth root of 1 shows that $f = 1$.

Now that we have pinned down a and f, I claim that $b = d = h = 1$. Let's focus on b, the other arguments being identical. If $b = 0$, then every point of the form $(z, 1)$ fixes $(0, 1, 0, 1, 0)$. Having an effectively free action then requires that every point of the form $(z, 1)$ fix every point. But this contradicts having finite ineffective kernel. Hence $b \neq 0$. Now, let z be a bth root of 1 and consider the point

$(z, 1)$. This fixes $(0, 1, 0, 1, 0)$ an so must fix every point. But it will not fix the point $(1, 0, 0, 1, 0)$ unless $b = \pm 1$. Now, by sending $p_2 \to \overline{p_2}$, we may assume $b = 1$.

At this point, our action must look like

$$(z, w) * (p_1, p_2, p_3, q_1, q_2) = (zp_1, zw^c p_2, zw^e p_3, wq_1, z^g wq_2).$$

This is necessary an sufficient to guarantee that every nontrivial (z, w) moves $(1, 0, 0, 1, 0)$, $(1, 0, 0, 0, 1)$, $(0, 1, 0, 1, 0)$ and $(0, 0, 1, 1, 0)$. Thus, in order to verify freeness, we must only find conditions guaranteeing every nontrivial (z, w) moves both $(0, 1, 0, 0, 1)$ and $(0, 0, 1, 0, 1)$. If (z, w) fixes $(0, 1, 0, 0, 1)$, then the coordinates z and w must satisfy

$$zw^c = z^g w = 1.$$

If we can find a nontrivial (z, w) satisfying this, it will fix $(1, 0, 0, 1, 0)$ giving a contradiction to freeness. Hence, we seek conditions on c and g which guarantee that the only solution is $(z, w) = (1, 1)$. Clearly the only way to guarantee that $(z, w) = (1, 1)$ is if $|1 - gc| = 1$.

Carrying out this same argument on $(0, 0, 1, 0, 1)$ shows that $|1 - ge| = 1$. It is clear that these conditions are also sufficient. Finally, we analyze what possibilities exist from g, e, and c. First note that by simultaneously applying complex conjugation to z, w, all p_i and q_j, we may assume that $g \geq 0$.

If $1 - gc = 1 - ge = 1$, then $gc = ge = 0$. Hence, either $g = 0$ with c and e arbitrary or both $c = e = 0$ with g arbitrary. In the first case, the action looks like

$$(z, w) * (p_1, p_2, p_3, q_1, q_2) = (zp_1, zw^c zp_2 w^e zp_3, wq_1, wq_2).$$

Notice that the z coordinate only acts on the S^5 and it acts via the Hopf map. Hence, this action is equivalent to an S^1 action on $\mathbb{C}P^2 \times S^3$. Since w is acting as the Hopf action on S^3, the projection map $\mathbb{C}P^2 \times S^3 \to S^3 \to S^2$ gives the biquotient the structure of a $\mathbb{C}P^2$ bundle over S^2 associated the Hopf bundle.

Likewise, focusing on the case with $c = e = 0$ and g arbitrary, one can show the projection $S^5 \times S^2 \to S^5 \to \mathbb{C}P^2$ gives the biquotient the structure of an S^2 bundle over $\mathbb{C}P^2$, associated to the Hopf bundle over $\mathbb{C}P^2$.

Next, assume $1 - gc = 1$ but $1 - ge = -1$. In this case, we have $gc = 0$ and $ge = 2$. Thus, $g \neq 0$ so $c = 0$, and g and e have the same sign, one being, up to sign, 1 and the other being, up to the same sign, 2. Reversing the roles of $1 - gc$ and $1 - ge$, we find that $e = 0$ with g and c having the previous relationship/

Finally, assume $1 - gc = 1 - ge = -1$. Then $c = e$ and again, g and c have the same sign with one of them being, up to sign, 1 and the other being, up to the same sign, 2. $\square$

The action can be described as a biquotient action by the embedding $SU(2) \times T^2 \to \big(SU(3) \times SU(2)\big) \times \big(SU(3) \times SU(2)\big)$ by sending (A, z, w) to

$$\mathrm{diag}(\overline{z}\overline{(w)}^{c+e}A, z^2), \mathrm{diag}(z^g w^2, \overline{z}^g \overline{w}^2, \mathrm{diag}(1, \overline{w}^{2c}, \overline{w}^{2e}), \mathrm{diag}(z^g, \overline{z}^g).$$

If we introduce the new variable $z' = zw^{a+b}$ and simplify, thinking of $z'A \in U(2)$ then mapping $U(2) \times S^1$ into $\big(SU(3) \times SU(2)\big) \times \big(SU(3) \times SU(2)\big)$ by sending (A, w)

to

$$\operatorname{diag}(A, \overline{detA}\overline{w}^{c+e}), \operatorname{diag}(\overline{detA}^g w^{1-g(c+e)}), \operatorname{diag}(1, \overline{w}^c, \overline{w}^e), \operatorname{diag}(\overline{detA}^g \overline{w}^{g(c+e)}, 1)$$

gives an action with the same orbits.

It is easy to see that this new embedding is 1-1. The induced action is clearly effective since the right hand matrices can only intersect the center in the identity. Hence, we've found an equivalent description of this action as a free action.

When $(c, e, g) = (1, 1, 2)$, the spectral sequence is hard to compute. As a fix, we can instead use the action defined by $U(2) \times S^1 \to \big(SU(3) \times SU(2)\big)^2$ with

$$(A, w) \to \operatorname{diag}(\overline{w}A, \overline{detA}), \operatorname{diag}(\overline{detA}w, detA\overline{w}), \operatorname{diag}(1, \overline{w}, \overline{w}), \operatorname{diag}(\overline{detA}, detA).$$

Similarly, when $(c, e, g) = (2, 2, 1)$, we instead use

$$(A, w) \to \operatorname{diag}(\overline{w}^2 A, \overline{detA}), \operatorname{diag}(\overline{detA}w, \overline{w}), \operatorname{diag}(1, \overline{w}^2, \overline{w}^2), \operatorname{diag}(\overline{detA}, 1).$$

Using this description and the usual spectral sequence arguments, we now analyze the cohomology rings and characteristic classes of these examples.

Proposition 5.3.13. *Let σ_1 and w be two degree 2 elements. Then, for any allowable (c, e, g), $H^*(G/\!\!/U)$ is isomorphic to a quotient of $\mathbb{Z}[\sigma_2, w]$ by two relations and has first Pontryagin class and total Stiefel-Whitney class as given in the table.*

The rings together with characteristic classes in the first family only depend on (a+b) mod 3. These 3 rings are mutually nonisomorphic and not isomorphic to any of the other rings in this list (except for the case $c = 0$ in the second family).

(c,e,g)	$w^2 =$	σ_1^3	p_1	$w-1$
$(c,e,0)$	0	$-2(c+e)\sigma_1^2 w$	$3\sigma_1^2 + 4(c+e)\sigma_1 w$	$\sigma_1 + \sigma_1^2$
$(0,0,g)$	$c\sigma_1 w$	0	$(g^2+3)\sigma_1^2$	$(g+1)\sigma_1 + (g+1)\sigma_1^2$
$(0,2,1)$	$-w\sigma_1$	0	$4\sigma_1^2$	0
$(0,1,2)$	$-2w\sigma_1$	0	$7\sigma_1^2$	$\sigma_1 + \sigma_1^2$
$(2,2,1)$	$= \sigma_1 w$	$4\sigma_1^2 w$	$4\sigma_1^2 + 28w^2$	0
$(1,1,2)$	$2w\sigma_1$	0	$8\sigma_1 w + 7\sigma_1^2$	$\sigma_1 + \sigma_1^2$

Table 5.1: The topology of $(S^5 \times S^3)/\!\!/T^2$

The rings in the second family only depend on $|c|$ *and are mutually nonisomorphic otherwise.*

There are characteristic class preserving ring isomorphisms between $(0,2,1)$ *and* $(0,0,1)$ *and between* $(0,1,2)$ *and* $(0,0,2)$*. There is an isomorphism between* $(1,1,2)$ *and* $(0,0,2)$ *which preserves the Stiefel-Whitney class, but not one which preserves the first Pontryagin class. There is no isomorphism between* $(2,2,1)$ *and anything else on the list.*

Proof. First, it easy to show that in all rings except in the first family (other than the $c=0$ case in the second family), the only element which squares to 0 is 0. This implies the first family is distinct from all the rest of the rings.

In the first family, it is easy to see that the only elements which square to 0 are multiples of w. This implies that any isomorphism between two such rings must take

w to $\pm w = w'$. This implies the isomorphism must take σ_1 to $\sigma_1' = (-1)^a\sigma_1 + cw$. Now one checks if there is an integer k such that $-2k\sigma_1'^2 w' = \sigma_1'^3$. Computation shows there is such an integer k iff $c = 2l$ is even. Then $k = \pm(e+g) + 3l$.

This implies that the rings in family one are isomorphic to each other iff $2(e+g) = 2(e'+g')$ mod 3, which is iff $e+g = e'+g'$ as claimed.

Now one simply checks that the above isomorphism takes characteristic classes as claimed.

Next we analyze the second family. The map sending w to $-w$ maps the ring with $w^2 = cw\sigma_1$ to the ring with $w^2 = -cw\sigma_1$ and it takes characteristic classes to characteristic classes. This shows the ring only depends on $|c|$. The see that for $|c| \neq |c'|$, these rings are not isomorphic, one first shows that only multiples of σ cube to 0. For if $0 = (bw + a\sigma_1)^2$, then one concludes $b(a^2 - ac + b^2c^2) = 0$, i.e., if $b \neq 0$, then $c = \frac{a \pm a\sqrt{1-4b^2}}{2}$, i.e., that c is NOT real.

This implies that any ring isomorphism must send σ_1 to $\pm\sigma_1$. Writing down the general map as in the previous case, and using that $H^4(G/\!\!/U)$ is freely generated by σ_1^2 and $\sigma_1 w$, we see that for $w' = \pm w + b\sigma_1$ and $\sigma_1' = \pm\sigma$, that $w'^2 = c'\sigma_1 w$ iff $|c'| = |c|$.

Now one simply checks that this isomorphism takes characteristic classes to characteristic classes.

The proof that only multiples of σ_1 cube to 0 clearly goes through for the rings $(0,2,1)$, $(0,1,2)$, and $(1,1,2)$. However, we'll see that no nonzero element of $(2,2,1)$

cubes to 0, showing it is distinct from everything on the list.

To see this, consider $0 = (a\sigma_1 + bw)^3 = (4a^3 + 3a^2b + 3ab^2 + b^3)\sigma_1^2 w$ with $gcd(a,b) = 1$. Now, by reduction mod 2, we see that a and b cannot both be odd. If a is even, then reduction mod 2 shows b^3 is even, contradicting $gcd(a,b) = 1$. Thus, we assume a is odd and b is even. Reduction mod 4 shows $3a^2b = 0$ mod 4 so b is divisible by 4. Now, consider $4a^3 + 3a^2b + 3ab^2 + b^3$ as a polynomial in b. By the rational roots theorem, any rational solution must be integral and divide $4a^3$. So, if b is the solution, then $b|4a^3$. Since we assume that $gcd(a,b) = 1$, this implies $b|4$. Since we already know $4|b$, we conclude that $b = \pm 4$. Finally, one just checks that $4a^3 \pm 12a^2 + 48a \pm 64$ has no integral roots, say, using the rational roots theorem.

For $(0,2,1)$ the map sending w to $-w$ clearly matches $(0,2,1)$up with $(0,0,1)$. Similarly, the map sending w to $-w$ matches $(0,1,2)$ with $(0,0,2)$.

Finally, looking at $(1,1,2)$ we see that the ring itself is clearly isomorphic to $(0,0,2)$. However, since σ_1 must be taken to $\pm\sigma_1$, it's easy to see that, up to sign, the only possible isomorphism taking p_1 to p_1 sends σ_1 to σ_1 and sends w to $-w+2\sigma_1$ (the other isomorphisms are simply flipping the sign on one or both of σ_1 and w), but this isomorphism doesn't map p_1 to p_1. $\square$

Due to the classification of torsion free 6 manifolds [17] [31] [34], it follows that these cohomology rings uniquely determine the diffeomorphism type of the biquotient, with possibly the exception of $(2,0,0)$ and $(1,1,2)$. We do not know if these two examples are homotopy equivalent or not, but they are not homeomorphic.

As a remark, Totaro has shown that $(0,1,2)$ is diffeomorphic to $\mathbb{C}P^3 \sharp \mathbb{C}P^3$. The fact that this is diffeomorphic to $(0,0,1)$ comes from the fact that $\mathbb{C}P^3 \sharp \mathbb{C}P^3$ is diffeomorphic to $\mathbb{C}P^3 \sharp - \mathbb{C}P^3$ and $\mathbb{C}P^3 \sharp - \mathbb{C}P^3$ is an S^2 bundle over $\mathbb{C}P^2$.

As a corollary we have

Theorem 5.3.14. *All 3 bundles of the form $\mathbb{C}P^2 \to M \to S^2$ with finite dimensional structure group and all bundles of the form $S^2 \to M \to \mathbb{C}P^2$ where the structure group reduces to a circle are diffeomorphic to biquotients.*

Proof. For the first statement, there are 3 bundles since the identity component of $Iso(\mathbb{C}P^2)$ is isomorphic to $SU(3)/\mathbb{Z}/3\mathbb{Z}$ which has fundamental group $\mathbb{Z}/3\mathbb{Z}$. So, there are 3 such bundles. Since we found 3, we found them all.

For the second statement, we use a standard trick. If $\pi : G/\!\!/U \to \mathbb{C}P^2$ is the projection giving $G/\!\!/U$ the structure of an S^2 bundle over $\mathbb{C}P^2$, and if $\mathbb{R}^3 \to X \to \mathbb{C}P^2$ is the real vector for which $G/\!\!/U$ is the sphere bundle, then we have $T(G/\!\!/U) \oplus 1 = \pi^*(X \to \mathbb{C}P^2) \oplus \pi^*(T\mathbb{C}P^2 \to \mathbb{C}P^2)$.

Using this fact, we can read off the characteristic classes of the bundle $S^2 \to G/\!\!/U \to \mathbb{C}P^2$ in terms of the characteristic classes of the tangent bundles to $G/\!\!/U$ and $\mathbb{C}P^2$. From this, it follows that $p_1(S^2 \to G/\!\!/U \to \mathbb{C}P^2) = c^2$ and $w(S^2 \to G/\!\!/U \to \mathbb{C}P^2) = 1 + c\sigma_1 + c\sigma_1^2$.

Now, Dold and Whitney [8] proved that for oriented sphere bundles over a 4-manifold with finite dimensional structure group, their characteristic classes classify them. Further, they show that the structure group reduces to a circle precisely when

p_1 is a square. Hence, we get all bundles $S^2 \to G /\!\!/ U \to \mathbb{C}P^2$ where the structure group reduces to a circle.

$\square$

5.3.1 Manifolds diffeomorphic to $(S^3)^3 /\!\!/ T^3$

In this section, we analyze biquotients of the form $(S^3)^3 /\!\!/ T^3$. These manifolds are all distinct from the previous examples because they'll have second Betti number 3.

Totaro [30] has already shown the following.

Proposition 5.3.15. *Consider the matrix* $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ *corresponding to the action*

$$
\begin{aligned}
(u, v, w) \quad * \quad & \big((p_1, p_2), (q_1, q_2), (r_1, r_2)\big) = \\
& \big((up_1, u^{a_1} v^{a_2} w^{a_3} p_2), (vq_1, u^{b_1} v^{b_2} w^{b_3} q_2), (wr_1, u^{c_1} v^{c_2} w^{c_3} r_2)\big).
\end{aligned}
$$

Then this is a free action iff $a_1, b_2,$ and c_3 are ± 1, and the 2 x 2 minors around the diagonal entries have determinant ± 1, and the matrix itself has determinant ± 1.

We now show that Totaro's result is completely general.

Proposition 5.3.16. *Suppose T^3 acts linearly and freely on $(S^3)^3$. Then, up to reparamaterization, the action is as above.*

Proof. The most general linear action of T^3 on $(S^3)^3$ is described by

$$(u, v, w) * \big((p_1, p_2), (q_1, q_2), (r_1, r_2)\big) =$$

$$\big((u^{\alpha_1} v^{\alpha_2} w^{\alpha_3} p_1, u^{a_1} v^{a_2} w^{a_3} p_2), (u^{\beta_1} v^{\beta_1} w^{\beta_1} q_1, u^{b_1} v^{b_2} w^{b_3} q_2), (u^{\gamma_1} v^{\gamma_1} w^{\gamma_1} r_1, u^{c_1} v^{c_2} w^{c_3} r_2)\big).$$

We seek conditions on all of exponents which guarantee the action is free. So, suppose $(u, v, w)\big((1, 0), (1, 0), (1, 0)\big) = \big((1, 0), (1, 0), (1, 0)\big)$. This is equivalent to solving the system of equations

$$\begin{aligned} u^{\alpha_1} v^{\alpha_2} w^{\alpha_3} &= 1 \\ u^{\beta_1} v^{\beta_2} w^{\beta_3} &= 1 \\ u^{\gamma_1} v^{\gamma_2} w^{\gamma_3} &= 1 \end{aligned}$$

This is equivalent to asking that

$$\begin{aligned} \alpha_1 u + \alpha_2 v + \alpha_3 w &= 2\pi k_1 \\ \beta_1 u + \beta_2 v + \beta_3 w &= 2\pi k_2 \\ \gamma_1 u + \gamma_2 v + \gamma_3 w &= 2\pi k_3 \end{aligned}$$

have all solutions $(u, v, w) \cong (0, 0, 0) \bmod 2\pi$ in each factor independently.

Now, assume for a contradiction that

$$det \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = 0.$$

Setting $k_1 = k_2 = k_3$, we find that since there is one solution $(u, v, w) = (0, 0, 0)$, there must be infinitely many solutions $(u, v, w) = (n_1 k, n_2 k, n_3 k)$ for some integers n_1, n_2, n_3 and all real numbers k.

But then this implies that the circle $(z^{n_1}, z^{n_2}, z^{n_3}$ simultaneously solves the first system of equations. This contradicts the freeness of the action. It follows that

$$d = det \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \neq 0.$$

By first making the T^3 action ineffective by replacing (u, v, w) with (u^d, v^d, w^d), we can use the coordinate substitution

$$(u', v', w') = (u^{alpha_1} v^{\alpha_2} w^{\alpha_3}, u^{\beta_1} v^{\beta_2} w^{\beta_3}, u^{\gamma_1} v^{\gamma_2} w^{\gamma_3}).$$

Then, in these new coordinates, the action will look like

$$(u', v', w') * \big((p_1, p_2), (q_1, q_2), (r_1, r_2)\big) =$$

$$\big((u'^{alpha} p_1, u'^{a_1} v'^{a_2} w'^{a_3} p_2), (v^{\beta} q_1, u'^{b_1} v'^{b_2} w'^{b_3} q_2), (w^{\gamma} r_1, u'^{c_1} v'^{c_2} w'^{c_3} r_2)\big)$$

and we may assume without loss of generality that

$$\gcd(\alpha, a_1, b_1, c_1) = \gcd(\beta, a_2, b_2, c_2) = \gcd(\gamma, a_3, b_3, c_3) = 1.$$

Here, we have abused notation by relabeling the new integers as a_1, a_2, etc.

Finally, we again check freeness at $t = \big((1, 0), (1, 0), (1, 0)\big)$. If $(u', v', w') * t = t$, then we must conclude $(u', v', w') = (1, 1, 1)$ But we see, for example, that any

$(\zeta_\alpha, 1, 1)$ fixes t (where ζ_α is a primitive αth root of 1). Freeness implies we must have $\alpha = 1$. Analogously, we see $\beta = \gamma = 1$, so we've reduced the action to the form Totaro has already handled.

$\square$

Remark 5.3.17. Totaro's proof boils down to simply checking freeness at each point (p, q, r) where each of p, q, and r is $(1, 0)$ or $(0, 1)$.

We've now reduced tabulating all free linear T^3 action on $\left(S^3\right)^3$ to a combinatorial question - we need to find all matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

which satisfy the property that all of the diagonal minors have determinant equal to ± 1.

Note that by, say, conjugating the first S^3 factor by j and mapping $u' \to -u'$, this sends $a_{11} \to -a_{11}$. Hence, we can (and do) assume $a_{11} = a_{22} = a_{33} = 1$. Further, by swapping, say, v' and w' and swapping the p coordinates with the q coordinates we see that the matrix $\begin{bmatrix} 1 & a_2 & a_3 \\ b_1 & 1 & b_3 \\ c_1 & c_2 & 1 \end{bmatrix}$ and the matrix $\begin{bmatrix} 1 & a_3 & a_2 \\ c_1 & 1 & c_2 \\ b_1 & b_3 & 1 \end{bmatrix}$ correspond to the same action. Likewise, by doing the appropriate swapping, we can move a_2 to any nondiagonal entry, but the rest of the entries are then determined. Finally, by sending, say, $p_1 \to \overline{p_1}$ and making similar changes elsewhere, we may assume $a_2 \geq 0$.

Proposition 5.3.18. *Assume $A = (a)_{ij}$ is a 3 x 3 matrix with all diagonal entries 1 and all diagonal minors having determinant ± 1. Then, up to the above equivalences, there are three infinite families*

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & a_3 \\ 1 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix}, and \begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$$

and 12 other sporadic examples:

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, and \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Proof. The 2 x 2 determinant conditions are

$$1 - a_2 b_1 = \pm 1 \tag{5.3.1}$$

$$1 - a_3 c_1 = \pm 1 \tag{5.3.2}$$

$$1 - c_2 b_3 = \pm 1 \tag{5.3.3}$$

and the 3 x 3 determinant condition is

$$1 - c_2 b_3 - a_2(b_1 - c_1 b_3) + a_3(b_1 c_2 - c_1) = \pm 1.$$

The 2 x 2 equations imply that $a_2 b_1 = 0$ or 2, and likewise for $a_3 c_1$ and $c_2 b_3$. Assume for now that each of these are 2. We may assume without loss of generality that $a_2 = 2$ and thus $b_1 = 1$. Reducing the 3 x 3 determinant condition mod 2, we see that either a_3 is even or c_2 and c_1 have the same parity. We may assume without loss of generality that $a_3 \geq 0$.

Assume for now that a_3 is even, so $a_3 = 2$ and $c_1 = 1$. Then, again using the 3 x 3 determinant condition, we see that $(c_2, b_3) = (2, 1)$ or $(1, 2)$. But these two actions are equivalent by swapping v and w.

Next, assume a_3 is odd, to $a_3 = 1$ and $c_1 = 2$. Since c_2 has the same parity as c_1, we must have $c_2 = \pm 2$, which forces $b_3 = \pm 1$, with the same sign as c_2. Choosing minus signs for both doesn't give the right 3 x 3 determinant, so we must choose positive signs for both, but this action is also equivalent to the first action.

We next assume precisely one of the 2 x 2 determinant equations is 1, while the other two are -1. We will continue to assume that $a_2 = 2$ and $b_1 = 0$, but that either a_3c_1 or c_2b_3 is 0, but not both.

First, assume $a_3c_1 = 0$ while $c_2b_3 = 2$. The first equation says either a_3 is 0 or c_1 is zero, so we begin by assuming $a_3 = 0$. The 3 x 3 determinant condition, if set equal to -1, together with the fact we can assume $b_3 \geq 0$, gives $b_3 = c_1 = 1$ with $c_2 = 2$. The case where the 3 x 3 determinant is 1 gives either $(b_3, c_1, c_2) = (2, 1, 1)$ or $(1, 2, 2)$.

Next, assume a_3 is nonzero, so that $c_1 = 0$. We can assume $a_3 > 0$ and then we see that from the 3 x 3 determinant condition that a_3c_2 is 2 or 4. Since we must have $c_2b_3 = 2$, we get the following solutions: $(a_3, b_3, c_2) = (4, 1, 2)$, $(2, 1, 2)$, $(2, 2, 1)$, or $(1, 1, 2)$. The action corresponding to $(2, 1, 2)$ is equivalent to the action in the previous paragraph with $(b_3, c_1, c_2) = (1, 2, 2)$ and the action corresponding to $(1, 1, 2)$ is the same as in the previous paragraph when the 3 x 3 determinant is -1. This completes the case where $a_3c_1 = 0$.

So, we now assume $a_3c_1 = 2$ and $c_2b_3 = 0$. Assume further the $c_2 = 0$. Then the 3 x 3 determinant, if -1, gives $(a_3, b_3, c_1) = (2, 1, 1)$. If the 3 x 3 determinant if 1, then we get $(a_3, b_3, c_1) = (2, 2, 1)$ or $(1, 1, 2)$, but this last action is equivalent to the action with $(a_2, a_3, b_1, b_3, c_1, c_2) = (2, 0, 1, 2, 1, 1)$ we say previously. This completes the case when $c_2 = 0$.

We now assume $c_2 \neq 0$, so that $b_3 = 0$. As above, we get $(a_3, c_1, c_2) = (2, 1, 1)$ or $(1, 2, 2)$ (if the 3 x 3 determinant is -1) and we get $(2, 1, 2)$ or $(1, 2, 1)$ if the 3 x

3 determinant is 1, but these are all equivalent to previous actions on our list. This completes the case where precisely one of the 2 x 2 determinants is 1.

We now assume that precisely two of the 2 x 2 determinants are 1. We are still assuming without loss of generality that $a_2 = 2$ and $b_1 = 1$. Thus, we have $a_3c_1 = c_2b_3 = 0$. We break into 4 cases depending on which of the variables are 0.

If $a_3 = c_2 = 0$, then the 3 x 3 determinant condition shows $c_1b_3 = 0$ or $c_1b_3 = 1$. If $c_1b_3 = 0$, then we see that $c_1 = 0$ and b_3 is free or vice versa. These two infinite family will fit into larger infinitely families later. The case $c_1b_3 = 1$ gives a new action.

If $a_3 = b_3 = 0$, then the 3 x 3 determinant is -1 automatically. This gives our first family, with c_1 and c_2 free.

If $c_1 = c_2 = 0$, then the determinant is -1 automatically. This gives our second family, with a_3 and b_3 free.

If $c_1 = b_3 = 0$, then the 3 x 3 determinant condition forces $a_3c_2 = 0$ or 2. If it's 0, we're back in one of the 2 previous cases, so we may as well assume it's 2. This gives $(a_3, c_2) = (1, 2)$ or $(2, 1)$. This concludes the case that precisely 2 of the 2 x 2 determinants are 1.

Finally, assume all 3 2 x 2 determinants are 1. We may assume without loss of generality that $a_2 = 0$. We also have $a_3c_1 = c_2b_3 = 0$. Further, assume the 3 x 3 determinant is 1. This implies either $a_3 = 0$ or $b_1c_2 = c_1$. Assuming that $a_3 = 0$, we see that $b_3 = 0$ or $c_2 = 0$, with everything else free. It turns out that either choice gives an equivalent action, our third family.

Assume now that $a_3 \neq 0$, so $c_1 = 0$. So, $b_1 c_2 = 0$ (and we still have $c_2 b_3 = 0$), so either $c_2 = 0$ with b_1 and b_3 are free or both $b_1 = b_3 = 0$ with c_2 free. In any case, this is subsumed under the third family.

Finally, we assume the 3 x 3 determinant it -1. This automatically implies $a_3 \neq 0$ so $c_1 = 0$ and that $a_3 b_1 c_2 = -2$, which implies $c_2 \neq 0$ so $b_3 = 0$. Since we can assume without loss of generality that $a_3 > 0$, this leaves only $(a_3, b_1, c_2) = (2, 1, -1)$, $(2, -1, 1)$, $(1, 2, -1)$, $(1, -2, 1)$, $(1, 1, -2)$, or $(1, -1, 2)$, but these are all equivalent to $(a_2, a_3, b_1, b_3, c_1, c_2) = (0, 2, 1, 0, 0, -1)$.

$\square$

For computing cohomology, we must write down a corresponding biquotient action giving any of these matrices. The correct action is defined by a map

$$T^3 \to (S^3 \times S^3 \times S^3) \times (S^3 \times S^3 \times S^3)$$

and is given by sending (u, v, w) to

$$\big((u^{1+a_1} v^{a_2} w^{a_3}, u^{b_1} v^{1+b_2} w^{b_3}, u^{c_1} v^{c_2} w^{1+c_3}), (u^{a_1-1} v^{a_2} w^{a_3}, u^{b_1} v^{b_2-1} w^{b_3}, u^{c_1} v^{c_2} w^{c_3-1})\big).$$

While this embedding will be 1-1, the induced action may not be effective, so we'll use the usual trick of finding a new action with the same orbits which is effective.

Following the usual techniques for computing cohomology rings and characteristic classes in this setting, we see

Proposition 5.3.19. *For the action corresponding to* $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$*, the quotient, $G/\!\!/U$ naturally has the structure of an S^2 bundle over $\mathbb{C}P^2\sharp\mathbb{C}P^2$. The cohomology ring is $H^*(G/\!\!/U) = \mathbb{Z}[u,v,w]/u^2 = -2uv, v^2 = -vu, w^2 = -c_1uw - c_2vw$ with $|u| = |v| = |w| = 2$ and the characteristic classes are $p_1 = (2c_1c_2 - 6 - 2c_1^2 - c_2^2)uv$, $w_1 = (1+c_1)u^2 + c_2v^2$, and $w_2 = c_2u^2v^2$.*

For the action corresponding to $\begin{bmatrix} 1 & 2 & a_3 \\ 1 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix}$*, the quotient $G/\!\!/U$ naturally has the structure of an $\mathbb{C}P^2\sharp\mathbb{C}P^2$ bundle over S^2. The cohomology ring is $H^*(G/\!\!/U) = \mathbb{Z}[u,v,w]/u^2 = -2uv - a_3uw, v^2 = -vu - b_3vw, w^2 = 0$ with $|u| = |v| = |w| = 2$. The characteristic classes are $p_1 = -6uv + (4a_3 - 4b_3)vw + (2b_3 - a_3)uw$, $w_1 = (a_3 + b_3)w^2 + u^2$ and $w_2 = a_3$.*

For the action corresponding to $\begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$*, the quotient $G/\!\!/U$ naturally has two bundle structures: it is an S^2 bundle over either $S^2 \times S^2$ or $\mathbb{C}P^2\sharp-\mathbb{C}P^2$, depending on the parity of b_1 and it is a bundle over S^2 with fiber either $S^2 \times S^2$ or $\mathbb{C}P^2\sharp-\mathbb{C}P^2$, depending on the parity of c_2. The cohomology ring is $H^*(G/\!\!/U) = \mathbb{Z}[u,v,w]/u^2 = 0, v^2 = -b_1vu, w^2 = -c_1uw - c_2vw$ with $|u| = |v| = |w| = 2$. The characteristic classes are $p_1 = c_2(2c_1 - c_2b_1)uv$, $w_1 = (b_1 + c_1)u^2 + c_2v^2$, and*

$w_2 = b_1c_2u^2v^2$.

The rest of the actions give the following cohomology rings and characteristic classes. All cohomology rings are isomorphic to $\mathbb{Z}[u,v,w]/relations$ *where* $|u| = |v| = |w|$

$(a_2, a_3, b_1, b_3, c_1, c_2)$	$u^2 =$	$v^2 =$	$w^2 =$
$(2,2,1,2,1,1)$	uw	$2uv$	vw
$(2,0,1,1,1,2)$	$-2uv$	$-vu-vw$	$-wu-2vw$
$(2,0,1,2,1,1)$	$-2uv$	$-vu-2vw$	$-wu-wv$
$(2,0,1,1,2,2)$	$-2uw$	$-2vu-2vw$	$wu+wv$
$(2,4,1,2,0,1)$	$-uw$	$-4vu-2vw$	$-2uw-vw$
$(2,2,1,2,0,1)$	$-2uv-2uw$	$-vu-2vw$	$-wv$
$(2,2,1,1,1,0)$	$-2uv-2uw$	$-vu-vw$	$-wu$
$(2,2,1,2,1,0)$	$-2uv-2uw$	$-2vw-vu$	$-wu$
$(2,0,1,1,1,0)$	$-uw$	$-vu-vw$	$-2vw$
$(2,2,1,0,0,1)$	$-uw$	$-2vu-2vw$	$-wv$
$(2,1,1,0,0,2)$	$-2uw$	$-v(2w+u)$	$-wv$
$(0,2,1,0,0,-1)$	$-2uw$	$-vu$	vw

Table 5.2: The relations in the cohomology rings of the sporadic $(S^3 \times S^3 \times S^3)/\!\!/T^3$

We first handle the sporadic rings, then the families to figure out when two are isomorphic.

$(a_2, a_3, b_1, b_3, c_1, c_2)$	p_1	w_1	w_2
$(2, 2, 1, 2, 1, 1)$	$4uw + 2uv + vw$	$v^2 + w^2$	v^2w^2
$(2, 0, 1, 1, 1, 2)$	$-8uv - 10vw + uw$	w^2	u^2w^2
$(2, 0, 1, 2, 1, 1)$	$-7uv - 14vw$	$v^2 + w^2$	v^2w^2
$(2, 0, 1, 1, 2, 2)$	$10wu + 10wv$	$u^2 + v^2$	u^2v^2
$(2, 4, 1, 2, 0, 1)$	$-14uw - 7vw$	$v^2 + w^2$	v^2w^2
$(2, 2, 1, 2, 0, 1)$	$2uw - 7uv - 10vw$	$u^2 + v^2$	u^2v^2
$(2, 2, 1, 1, 1, 0)$	$-8uv + 4vw - 7wu$	w^2	u^2w^2
$(2, 2, 1, 2, 1, 0)$	$-8wu - 8uv$	0	0
$(2, 0, 1, 1, 1, 0)$	$-8vw - 4uv + uw$	u^2	w^2u^2
$(2, 2, 1, 0, 0, 1)$	$4uw - 2uv - 7vw$	$v^2 + w^2$	v^2w^2
$(2, 1, 1, 0, 0, 2)$	$-10vw + 2uw - uv$	$u^2 + v^2$	u^2v^2
$(0, 2, 1, 0, 0, -1)$	$4vw - 2uw - vu$	$u^2 + v^2$	u^2v^2

Table 5.3: The characteristic classes of the sporadic $(S^3 \times S^3 \times S^3)/\!\!/T^3$

Proposition 5.3.20. *The sporadic rings break into Stiefel-Whitney class preserving isomorphism classes as follows:* $(2, 2, 1, 2, 1, 1)$*,* $(2, 0, 1, 1, 1, 2)$*,* $(2, 2, 1, 2, 0, 1)$*,* $(2, 2, 1, 1, 1, 0)$*,* $(2, 0, 1, 1, 1, 0)$*,* $(2, 2, 1, 0, 0, 1)$*,* $(2, 1, 1, 0, 0, 2)$*, and* $(0, 2, 1, 0, 0, -1)$ *are mutually isomorphic and likewise the rings corresponding to* $(2, 0, 1, 2, 1, 1)$ *and* $(2, 4, 1, 2, 0, 1)$ *are isomorphic. These two types of rings together with the 2 remaining rings are all mutually distinct, and hence the 12 sporadic biquotients break into* *at least 4 homotopy types. All of these isomorphisms can be chosen to preserve the Pontryagin classes, so, by Jupp [17], Zubr [34], and Wall [31], the 12 sporadic examples actually separate into precisely 4 diffeomorphism types.*

Proof. For definiteness, for $(2,2,1,2,1,1)$, we use the cohomology ring

$$H^*(G/\!\!/U) = \mathbb{Z}[u,v,w]/u^2 = uv, v^2 = vw, w^2 = 2uw$$

with $p = 1 + 4uv + 2uw + vw$ and $w = 1 + (v + w) + vw$ with $|u| = |v| = |w| = 2$. This is the same as the first entry in the previous table, except we have swapped v and w.

Now, the cohomology ring corresponding to $(2,0,1,1,1,2)$ is

$$H^*(G/\!\!/U) = \mathbb{Z}[u,v,w]/u^2 = -2uv, v^2 = -vu - vw, w^2 = -wu - 2vw$$

with $p = 1 - 8vu - 10vw + uw$ and $w = 1 + w + uw$. Swapping u and w gives the new cohomology ring $H^* = \mathbb{Z}[u,v,w]/u^2 = -uw - 2vu, v^2 = -vu - vw, w^2 = -2wv$ with $p = 1 - 8vw - 10vu + uw$ and $w = 1 + u + uw$.

We now set $w' = w + 2v$ $u' = u + w'$ and use as a basis for H^2 $\{u', v, w'\}$. (This is possible since the transition matrix from $\{u,v,w\}$ to $\{u',v,w'\}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ which has determinant 1. Then we have $H^*(G/\!\!/U) = \mathbb{Z}[u,v,w']/u'^2 = u'w', v^2 = vu', w'^2 = 2w'v$ with $p = 1 + 4u'v + 2vw' + u'w'$ and $w = 1 + (u' + w') + u'w'$, which is clearly isomorphic to the ring for $(2,2,1,2,1,1)$.

For $(2,2,1,2,0,1)$, the argument is analogous - first swap u and w, then set $u' = u + v$ and $w' = 2u' + w$. For $(2,2,1,1,1,0)$, swap u and w, set $u' = u + w$, $v' = u' + v$, and then swap u' and v'. For $(2,0,1,1,1,0$, send w to $-w$ then set $u' = w - u$ and finally swap u' and v. For $(2,2,1,0,0,1)$, swap v and w, send v to $-v$ and set $u' = v - u$. For $(2,1,1,0,0,2)$, swap u and w, set $w' = 2u + w$, and then send v to $-v$. Finally, for $(0,2,1,0,0,-1)$, swap u and w and send w to $-w$.

To see $(2,0,1,2,1,1)$ and $(2,4,1,2,0,1)$ are isomorphic, we find nice forms for both. For $(2,0,1,2,1,1)$, swap u and w, set $w' = w + 2u$ and then set $v' = v + w'$. Once these changes are made we get the cohomology ring $H^*(G/\!\!/U) = \mathbb{Z}[u, v', w']/u^2 = uv', v'^2 = v'w, w'^2 = 2w'v$ with $p_1 = 7w'v'$ and $w = 1 + v'^2 + w'^2 + v'^2w'^2$. For $(2,4,1,2,0,1)$, swap v and w, set $w' = 2u + w$, and change v to $-v$ to get the same cohomology ring and characteristic classes.

Finally, we'll show these four classes are different. First, the biquotient corresponding to $(2,0,1,1,2,2)$ must be different from the others because reduction mod 2 of the ring those the (linear) squaring map $H^2(B/\!\!/U;\mathbb{Z}/2\mathbb{Z}) \to H^4(B/\!\!/U;\mathbb{Z}/2\mathbb{Z})$ has 2 dimensional kernel, while for the other examples it's one dimensional. Second, the example corresponding to $(2,2,1,2,1,0)$ has a trivial Stiefel-Whitney class, so it's different from the other examples.

This leaves only distinguishing the first type, corresponding to $(2,2,1,2,1,1)$ and $(2,0,1,2,1,1)$. Notice that in the second example, there are generators v and w of H^2 so that $v^2 = 2w^2$. I claim there are no pair of generators in the first example with this property.

So, in the first ring, consider the two elements $x = au + bv + cw$ and $y = du + ev + ew$. Such that $x^2 = 2y^2$. I claim that this implies $a \cong b \cong c \bmod 2$, so that x is not a generator. To see this, notice that equating the uv, uw and vw terms in the equation $x^2 = 2y^2$ gives

$$2ab + a^2 = 4de + 2d^2 \tag{5.3.4}$$

$$2ac + 2c^2 = 4df + 4f^2 \tag{5.3.5}$$

$$2bc + b^2 = 4ef + 2e^2 \tag{5.3.6}$$

The first and third equations, mod 2, imply a and b are both even. Reduction mod 4 of the second equation implies c is even. □

We now attempt to understand the families of biquotients.

To that end, let $R(c_1, c_2)$ denote the cohomology ring of the biquotient corresponding to $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$. Let $S(a_3, b_3)$ denote the ring corresponding to $\begin{bmatrix} 1 & 2 & a_3 \\ 1 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $T(b_1, c_1, c_2)$ correspond to $\begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}$.

Proposition 5.3.21. *Then none of the T rings are isomorphic to any of the R or S rings. An R ring is isomorphic to an S ring only in the case of $R(0,0) \cong S(0,0)$.*

Proof. We count the number of lines of elements in H^2 which square to 0. In $R(c_1, c_2)$, we see that $0 = (au+bv+cw)^2$ implies $0 = 2ab-2a^2-b^2 = -a^2-(a-b)^2$, so that clearly $a = b = 0$. But then $(cw)^2 = 0$ iff $c_1 = c_2 = 0$, so, generically, there are no nonzero elements in H^2 which square to 0 for $R(c_1, c_2)$.

Likewise, for $S(a_3, b_3)$, we find that $0 = (au + bv + cw)^2$ implies $a = b = 0$, but in this case $(cw)^2 = 0$, so we have precisely one line's worth.

Finally, for the ring $T(b_1, c_1, c_2)$, we have that $u^2 = 0$ and $(b_1u + 2v)^2 = 0$, so the line through $(b_1, 2)$ gives another set of solutions.

□

We'll now focus on each family one at a time.

Proposition 5.3.22. *Two rings $R(c_1, c_2)$ and $R(d_1, d_2)$ are isomorphic iff $c_1^2+c_2^2 = d_1^2 + d_2^2$ and $c_1 \cong d_1$ mod 2. In this case, an isomorphism can be chosen taking characteristic classes to characteristic classes.*

Proof. We begin by substituting $u' = u + v$. In this basis, we see that $u'^2 = v^2$ and $u'v = 0$, just as in the cohomology ring for $\mathbb{C}P^2\sharp\mathbb{C}P^2$. In this basis, we see that $p_1 = (6 + c_1^2 + (c_1 - c_2)^2)v^2$ and $w = 1 + (1 + c_1)u' + (1 + (c_1 - c_2))v + c_2v^2$. From here on we abuse notation and write u for u'.

Next, note that if $f : R(d_1, d_2) \to R(c_1, c_2)$ is an isomorphism, then we can define f solely in terms of the image of u, v, and w. Hence, we'll think about the isomorphism problem as finding a new basis of $H^2(G/\!\!/U)$ satisfying the same kind of relations.

I claim that in any basis $\{\alpha, \beta, \gamma\}$ of the degree 2 portion of $R(c_1, c_2)$ such that $\alpha^2 = \beta^2$ and $\alpha\beta = 1$ must be, up to changing the sign of the elements, of the form $u, v, ku + lv + w$.

To see this, we first write $\alpha = au + bv + cw$ and $\beta = du + ev + fw$. Looking at the v^2 component of $\alpha^2 = \beta^2$, we see that $a^2 + b^2 = d^2 + e^2$. Looking at the v^2 component of $\alpha\beta = 0$, we see that $ad + be = 0$.

We claim that these two facts together imply that $|a| = |e|$ and $|b| = |d|$ and that, without loss of generality, that $a, b \geq 0$ and $de \leq 0$. First note that if $e = 0$, then we see that $ad = 0$ and $a^2 + b^2 = d^2$. If $a = 0$, then $b^2 = d^2$ as claimed, and if $d = 0$, then $a^2 + b^2 = 0$ so $a = b = 0$.

Next, assuming $e \neq 0$, so $b = -ad/e$. Plugging this into $a^2 + b^2 = d^2 + e^2$ and rearranging gives $a^2 - e^2 = d^2(1 - a^2/e^2)$. Multiplying both sides by e^2 gives $e^2(a^2 - e^2) = d^2(e^2 - a^2)$. Since there is an $a^2 - e^2$ on one side but a $e^2 - a^2$ on the other, this shows that both sides must be 0, so that $a^2 = e^2$, which, together with $ad + be = 0$ shows $b^2 = d^2$.

Now, it follows from $ad + be = 0$ that a and e have the same sign iff b and d don't. Hence, by swapping α and β and replacing α with $-\alpha$, we may assume $a, b \geq 0$ and $de \leq 0$.

We now show the coefficients c and f must both be 0. As a first step, notice that by inspecting the uw and vw parts of $\alpha\beta = 0$, we see that $c = 0$ iff $f = 0$.

Now assume for a contradiction that $f \neq 0$ (and therefore that $c \neq 0$. Let p

be any prime and suppose $p^k|f$. Looking at the uw and vw components of $\alpha\beta = 0$ mod p^k shows $p^k|(cd, ce)$ so $p^k|c(d, e)$. Now, if $p|(d, e)$, then $p|(d, e, f)$ contradicting the fact that β is a basis element, so $p^k|c$. Doing this for all primes shows $f|c$. Repeating the argument the other way shows $c|f$ so $c = \pm f$.

Assuming $c = f$ and, say, $d \leq 0$, we look at the uw part of $\alpha^2 = \beta^2$ and see that $a = d$. But since $a \geq 0 \geq d$ we conclude $a = d = b = e = 0$. But this says α and β are both multiples of w, so they can't both be elements of a basis. Likewise, if $c = -f$ and, say, $d \geq 0$, we use the same uw part. This contradiction implies $f = c = 0$.

Now, a necessary condition for $\alpha = au + bv$ and $\beta = (du + ev)$ to be part of a basis is that $ae - db = \pm 1$. Notice that since $de \leq 0$ and both $a, b \geq 0$, we see that both ae and $-db$ have the same sign. But the only way to integers with the same sign can add up to ± 1 is if one is 0 and the other is ± 1. This implies that up to reordering and multiplying by -1, that $\alpha = u$ and $\beta = v$.

As this point, we must have $\gamma = ku + lv + w$. We wish to know when $\gamma^2 = \gamma(-d_1 u + (d_1 - d_2)v)$.

This in turn gives us three equations:

$$k^2 + l^2 = k(-d_1) + l(d_1 - d_2) \quad (5.3.7)$$

$$2l + (c_1 - c_2) = d_1 - d_2 \quad (5.3.8)$$

$$2k - c_1 = = -d_1 \quad (5.3.9)$$

We first note that the third equation shows that c_1 and d_1 must have the same parity. Further, assuming we have a solution, by squaring the second and third equations and adding the together and using the first equation, we see that $c_1^2 + (c_1 - c_2)^2 = d_1^2 + (d_1 - d_2)^2$, showing that this is necessary.

To see its sufficient, suppose $c_1^2+(c_1-c_2)^2 = d_1^2+(d_1-d_2)^2$ and c_1 and d_1 have the same parity. Define $k = (c_1-d_1)/2$, an integer and define $l = [(d_1-d_2)-(c_1-c_2)]/2$. So, by definition, the second and third equations are satisfies. It is easy to verify that this also solves the first equation.

It's also easy to see that the map defined by this basis takes characteristic classes to characteristic classes. □

Proposition 5.3.23. *The ring $S(a_3, b_3)$ is isomorphic to $S(a_3', b_3')$ iff we have $(a_3', b_3') \in \{(-a_3, -b_3), (a_3 - 2b_3, -b_3), (-a_3, b_3 - a_3), (a_3 - 2b_3, a_3 - b_3)\}$. In any case, the isomorphism can be chosen to take characteristic classes to characteristic classes.*

Proof. As in the previous case, we study the isomorphism problem as a problem in finding a new basis. We have already proved that w is uniquely characterized (up to multiple) by the fact that $w^2 = 0$. Hence, any basis must be of the form $\{\alpha, \beta, w\}$. We seek conditions on $\alpha = au + bv + cw$ and $\beta = du + ev + fw$ so that $\alpha^2 = -2\alpha\beta - a_3'\alpha w$ and $\beta^2 = -\alpha\beta - b_3'\beta w$.

To that end, consider the map $f : H^2(G/\!\!/U) \to \mathbb{Z}$ where we send x to $x^2 w$. Note that if $x = au + bv + cw$, then $x^2 w = [x^2]_{uv} w = (2ab - 2b^2 - a^2)uvw =$

$-(b^2+(b-a)^2)uvw$. The relations we are asking α and β to satisfy imply $f(\alpha) = f(u)$ and $f(\beta) = f(v)$. But $f(v) = -1$, so we must have $f(\beta) = -1$. If $\beta = du+ev+fw$, then we must have $d^2 + (d-e)^2 = 1$, i.e, either $d = 0$ and $d - e = \pm 1$ or $d = e$ and $d = \pm 1$. Likewise, $-2 = f(u) = f(\alpha)$ implies that $b^2 = \pm 1$ and $(b-a)^2 = \pm 1$.

All this together implies that, up to changing signs, the change of basis matrix must look like one of the 4 following matrices

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}, and \begin{bmatrix} 1 & 2 & c \\ 1 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}.$$

We'll analyze each of the 4 individually to find further restrictions on a and d.

First, consider the case where the change of basis is given as $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$.

We claim that in this case, in order to α and β to satisfy the right relations we must have $c = f = 0$.

For, if $\alpha^2 = \alpha(-2\beta - a_3' w)$, then inspecting the vw component gives $0 = -2c$, so $c = 0$. A similar argument shows $f = 0$.

In particular, this change of basis matrix must be the identity.

We now focus on the second change of basis matrix (which only gives the new basis elements up to sign), $\begin{bmatrix} 1 & 2 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$. If $\alpha = u + 2v + cw$ and $\beta = v + fw$, then we cannot have $\alpha^2 = -2\alpha\beta - a'_3\alpha w$ for any a'_3. To see this, inspect the uv term of each side. For α^2, the uv coefficient is -2, but for any a'_3, the uv coefficient is 2. This implies that we must use $\beta = -v - fw$ (changing α to $-\alpha$ will lead to the same isomorphisms).

Establishing this, it's not too hard to check that the only simultaneous solutions to $\alpha^2 = -\alpha(2\beta + a'_3 w)$ and $\beta^2 = -\beta(\alpha + b'_3 w)$ are when $f = 0$, $c = -a_3$, $a'_3 = -a_3$ and $b'_3 = a_3 - b_3$. Hence, this matrix determines an isomorphism between $S(a_3, b_3)$ and $S(-a_3, a_3 - b_3)$. It's easy to see that this isomorphism takes characteristic classes to characteristic classes.

Now, consider the third change of basis matrix, $\begin{bmatrix} 1 & 0 & c \\ 1 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$. Just as in the second case, we find that we must replace β with $-\beta$. Carrying through the similar analysis as in the second case, we learn that $S(a_3, b_3)$ is isomorphic to $S(a_3 - 2b_3, -b_3)$ by an isomorphism taking characteristic classes to characteristic classes.

This leaves us the final change of basis matrix, giving $\alpha = u + 2v + cw$ and $\beta = u + v + fw$. As in the previous two cases, we must use $-\beta$ instead of β in the basis. Doing the same analysis shows $S(a_3, b_3)$ is isomorphic to $S(a_3 - 2b_3, a_3 - b_3)$ via a characteristic class preserving isomorphism.

Thus, we have shown that $S(a_3, b_3) \cong S(a_3 - 2b_3, -b_3) \cong S(-a_3, b_3 - a_3) \cong S(a_3 - 2b_3, a_3 - b_3)$ via characteristic class preserving isomorphisms. Further, by simultaneously replacing α with $-\alpha$ and β with $-\beta$, or by replacing w with $-w$, we also see that $S(a_3, b_3) \cong S(-a_3, -b_3)$ vis a characteristic class preserving map. Further, it follows from our analysis that for a given ring $S(a_3, b_3)$, these are all the rings in the S family which it can be isomorphic to.

$\square$

Proposition 5.3.24. *In the T series of rings, the rings with $p_1 = 0$ are not isomorphic to the ones with $p_1 \neq 0$. Those with $p_2 = 0$ break into precisely 3 homotopy/diffeomorphism classes, each of which is realized by an example with $c_2 = 0$.*

Every ring of the form $T(b_1, c_1, c_3)$ is isomorphic to $T(\epsilon, c_1 - c_3\lfloor b_1/2 \rfloor, c_3)$ where $\epsilon = 0$ or 1 via a characteristic class preserving isomorphism. No ring with $\epsilon = 0$ is isomorphic with any ring with $\epsilon = 1$, except possibly in the case of $p_1 = 0$.

Further, we have $T(0, c_1, c_2) \cong T(0, d_1, d_2)$ iff $c_1c_2 = d_1d_2$ and up to reordering, $c_1 \cong d_1$ mod 2 and $c_2 \cong d_2$ mod 2. In this case, the isomorphism can be chosen to take characteristic classes to characteristic classes.

Finally, we have $T(1, c_1, c_2) \cong T(1, d_1, d_2)$ iff the first Pontryagin classes are the same multiple of uv and, up to reordering $c_1 \cong d_1$ and $c_2 \cong d_2$ mod 2.

Proof. To see that the rings with $p_1 = 0$ are not isomorphic to those with $p_1 \neq 0$, we count lines which square to 0 in H^2. Of course, all the rings share the fact that $u^2 = 0$, so we ignore that one. We claim that when $p_1 = 0$, there are precisely 3 lines in H^2 which square to 0, while when $p_1 \neq 0$, there are precisely 2.

If $p_1 = c_2(2c_1 - c_2b_1) = 0$, then either $c_2 = 0$ or $2c_1 - c_2b_1 = 0$. If $c_2 = 0$, then the lines of the form $au + bv + cw$ square to 0 with $(a, b, c) = (b_1, 2, 0)$ or $(2c_1, 0, 2)$.

If $c_2 \neq 0$ but $2c_1 - c_2 b_1 = 0$, then the lines with $(a, b, c) = (b_1, 2, 0)$ or $(b_1 c_1, 2c_1, 2b_1)$ square to 0. It remains to see that when $p_1 \neq 0$ that there is a unique line which is not $\mathbb{Z}u$ which squares to 0.

So, assume $(au + bv + cw)^2 = 0$ with at least one of b and c nonzero. In the case that $b = 0$, the vw coefficient tells us $c^2 c_2 = 0$ which implies $c = 0$, a contradiction. Hence, we assume $b \neq 0$. If $c \neq 0$, then the equations we get are $2a = bb_1$, $2b = cc_2$ and $2a = cc_1$. But by treating this as a linear system in the variables a, b, c, we see that the determinant of the defining matrix is $2c_1 - c_2 b_1$ so is nonzero. Hence, there is a unique solution to this linear system. Since $(a, b, c) = (0, 0, 0)$ is one solution, it must be the only one. Thus, we must have $c = 0$. But then the only equation we have left is $2a = bb_1$, so that the only line which squares to 0 is $(b_1, 2, 0)$.

Before we handle the case of $p_1 = 0$, we prove the second claim - that $T(b_1, c_1, c_2)$ is isomorphic to $T(\epsilon, c_1 - c_2\lfloor b_1/2 \rfloor, c_2)$. The proof is to assume $b_1 = 2k + \epsilon$ and set $v' = ku + v$. Then computation shows $v'^2 = -\epsilon uv = \epsilon uv'$ since $u^2 = 0$. It's also easy to see that this map sends c_1 to $c_1 - c_2 k$ as claimed and that this map preserves characteristic classes.

Next, we show that no ring of the form $T(0, c_1, c_2)$ is isomorphic to a ring of the form $T(1, c_1, c_2)$, at least when $p_1 \neq 0$. For in this case the isomorphism must map u and v, up to swapping and multiplying by a -1, to u and $(1, 2, 0)$. Then the matrix of this transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ & * & * \end{bmatrix},$$

which must be $\mathbb{Z}$ invertible, so have determinant ± 1. But it's easy to see that independent of the third row, such a matrix must have determinant divisible by 2.

Now we return to the $p_1 = 0$ case. Specifically, we show that there are precisely 3 homotopy and diffeomorphism types of biquotients when $c_2 = 0$.

If $c_2 = 0$, we can use the same kind of trick in showing we can assume $b_1 = \epsilon$ to show that we may assume without loss of generality $c_1 = \epsilon'$. The isomorphism types depend on the parities of both ϵ and ϵ', but when $\epsilon = 1$ and $\epsilon' = 0$ or vice versa, we get the same manifold.

Note that this is expected: when $c_2 = 0$, $G/\!\!/U$ has the structure of an $S^2 \times S^2$ bundle over S^2 with each fiber isometric to the standard product metric on $S^2 \times S^2$. Then clutching function arguments show we expect these biquotients to fall into at most 4 diffeomorphism types, depending on the element in $\pi_1(SO(3) \times SO(3)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However, in this description, the two bundles corresponding to $(1,0)$ and $(0,1)$ are clearly diffeomorphic, but not bundle isomorphic.

We now turn to the case where $c_2 \neq 0$ but we still have $2c_1 = \delta c_2$ for $\delta \in \{0,1\}$. Assume initially that $\delta = 0$ Then we have $c_1 = 0$. By making setting $c_2 = 2k + \delta'$ and setting $w' = ku + w$, we get the cohomology ring with $u^2 = 0, v^2 = 0$, and $w^2 = -\delta' uw$, which is clearly isomorphic to one with $c_2 = 0$ and $\epsilon \cong \epsilon' + 1$, in a way which preserves characteristic classes.

Hence we assume $\delta = 1$. Thus, we see $2c_1 = c_2$, so c_2 must be even. Express c_1 as $c_1 = 2n + \nu$ with $\nu \in \{0, 1\}$. Set $w' = (nu + 2nv + w)$. Then $w'^2 = 4n^2uv + 4n^2v^2 + 4nvw + 2nuw - c_1uw - 2c_1vw = -\nu uw - 2\nu vw$. Again, it is easy to verify that this isomorphism takes characteristic classes to characteristic classes. If $\nu = 0$, then the we clearly have another example which previously arose when $c_2 = 0$. Hence, assume $\nu = 1$. In this case, set $u' = u + 2v$ so $u'^2 = 0$ still, but now we have $w^2 = -u'w$. Since this isomorphism also takes characteristic classes, we see that we just get another of the previous examples with $c_2 = 0$. This concludes the case where $p_1 = 0$.

So, we now assume $p_1 \neq 0$, so, in particular, there are precisely two lines which square to 0. Since we have already seen the $b_1 = 0$ case is distinct from the case where $b_1 = 1$, we break into cases. Assume initially that $b_1 = 0$.

Since the only primitive elements which square to 0 are u and v, any ring isomorphism of $T(0, c_1, c_2)$ and $T(0, d_1, d_2)$ must send u and v to themselves, up to sign and swapping them. Hence, the isomorphism must also send w to $au + bv + w$, again up to sign. We seek conditions on a and b so that $(au + bv + w)^2 = (au + bv + cw)(-d_1u - d_2v)$. It's easy to see that this condition is equivalent to solving $2ab = ac_2 + bc_1$ in which case we'll get that $(d_1, d_2) = (c_1 - 2a, c_2 - 2b)$ are allowable and no others are.

We claim that this equation is satisfied iff $d_1 \cong c_1 \bmod 2$, $d_2 \cong c_2 \bmod 2$, and

$d_1 d_2 = c_1 c_2$. First, assuming the equation is satisfied. So, $d_1 = c_1 - 2a$ so $d_1 \cong c_1$ mod 2, and likewise for $d_2 \cong c_2$. Then note that $d_1 d_2 = (c_1 - 2a)(c_2 - 2b) = c_1 c_2 + 2(2ab - ac_2 - bc_1) = c_1 c_2$.

Conversely, Assuming $d_1 \cong c_1$ and $d_2 \cong c_2$, both mod 2, and that $d_1 d_2 = c_1 c_2$, it's easy to see that defining $a = (c_1 - d_1)/2$ and $b = (c_2 - d_2)/2$ gives a solution.

Of course, swapping u and v corresponds to swapping c_1 and c_2. It follows that in the case where $b_1 = 0$, that $T(0, c_1, c_2)$ is isomorphic to $T(0, d_1, d_2)$ via a characteristic class preserving map iff $c_1 c_2 = d_1 d_2$ and either $c_1 \cong d_1$ and $c_2 \cong d_2$ or $c_1 \cong d_2$ and $c_2 \cong d_1$, all mod 2.

We finally come to the case where $b_1 = 1$. We first change basis to $u' = (u + v)$, so the relations are $u'^2 = -v^2$ and $u'v = 0$ and $w^2 = -c_1 wu' - (c_2 - c_1)vw$. We'll abuse notation and write u for u'.

We claim that u and v are uniquely characterized (up to swapping and sign changes) by this property.

For, assume $\alpha = au + bv + cw$ and $\beta = du + ev + fw$ with $\alpha^2 + \beta^2 = \alpha\beta = 0$. Assume without loss of generality that $a \neq 0$ (if $a = d == 0$, then the v^2 coefficient of $\alpha^2 + \beta^2$ shows $b = e = 0$, so α and β would be dependent). Then, comparing the v^2 of $0 = \alpha\beta$, we get that $d = be/a$. Plugging this into the v^2 coefficient of $\alpha^2 + \beta^2 = 0$ and rearranging, we find that $a^2(b^2 - a^2) = e^2(b^2 - a^2)$ so that either $|a| = |b|$ or $|a| = |e|$. We may assume without loss of generality that $a \geq 0$ and $d \geq 0$ and hence that $be \geq 0$.

Notice that if $c = 0$ and $f \neq 0$, then by looking at the vw and uw components of $\alpha\beta = 0$, we conclude that $a = b = 0$ so $\alpha = 0$, contradicting the fact that α is in a basis. Hence $c = 0$ iff $f = 0$.

We assume for a contradiction that c and f are both nonzero. By inspecting the uw and vw components of $\alpha\beta = 0 \bmod p^k$ where $p^k || f$, we see that $f|c$ and reversing the argument, that $c|f$ so that $c = \pm f$. We now show that either choice contradicts the fact that $p_1 \neq 0$.

Assume for now that $|a| = |b|$ with both $a \geq 0$ and $b \geq 0$ and that $c = \pm f$. Setting the vw and uw coefficients of $\alpha^2 + \beta^2 = 0$ equal gives $c^2 c_1 = c^2(c_2 - c_1)$ so $2c_1 = c_2$, contradicting $p_1 \neq 0$.

If instead $a \geq 0$ but $b \leq 0$, we compare the vw and the negative of the uw coefficients of $\alpha^2 + \beta^2 = 0$ and get a similar contradiction.

Thus, we can assume $|a| = |e|$ and $|b| = |d|$. A similar argument, sometimes using the uw and vw coefficients of $\alpha\beta = 0$ gives the same contradiction to all the cases $c = \pm f$ and $b \geq 0$ or $b \leq 0$. Thus, we conclude that $c = f = 0$. Now the fact that α and β are part of a basis implies $ae - bd = \pm 1$.

Assume $|a| = |b|$ and $b \neq 0$. Then the equation $ae - bd = \pm 1$, taken mod b gives a contradiction, so either $b = 0$ or $|a| \neq |b|$. If $b = 0$ but $|a| = |b|$, then this implies $|a| = 0$, so $\alpha = 0$, a contradiction. Hence, we have $|a| \neq |b|$, so that $|a| = |e|$ and $|b| = |d|$. Thus we get $a^2 \pm b^2 = \pm 1$. If it's $a^2 + b^2 = \pm 1$, then we must have $b = d = 0$. If it's $a^2 - b^2 = \pm 1$, then we have $(a - b)(a + b) = \pm 1$

so that $a - b = \pm 1$ and $a + b = \pm 1$ so that either $a = 0$ (which can't happen by assumption) or $b = d = 0$. Of course, if $b = 0$, then $ad = \pm 1$ shows both a and d are ± 1, establishing the claim.

Thus, an isomorphism must, up to order and sign, send u' to u' and v to v. The third basis element $\gamma = au + bv + w$ satisfies the relation $\gamma^2 = -\gamma(d_1 u + d_2 v)$ iff $d_1 = c_1 - 2a$, $d_2 = c_2 - 2b$, and $b^2 - a^2 = d_1 a - d_2 b$. It's easy to see that this implies $d_1 \cong c_1$ mod 2 and $d_2 \cong c_2$ mod 2 and that $d_1^1 - d_2^2 = c_1^2 - c_2^2$. Further, this is sufficient: by setting $a = (c_1 - d_1)/2$ and $b = (c_2 - d_2)/2$, one easily sees that this choice of a and b satisfies $b^2 - a^2 = d_1 a - d_2 b$. Finally, it's easy to check that this isomorphism will send characteristic classes to characteristic classes.

Translating back into the rings $T(1, c_1, c_2$ and $T(1, d_1, d_2)$ are isomorphic iff $c_2(2c_1 - c_2) = d_2(2d_1 - d_2)$ and either $c_1 \cong d_1$ and $c_2 \cong d_2$ or $c_1 \cong d_2$ and $c_2 \cong d_1$ mod 2. □

Finally, we have

Proposition 5.3.25. *All S^2 bundles over $S^2 \times S^2$ or either of $\mathbb{C}P^2 \sharp \pm \mathbb{C}P^2$ where the structure group reduces to S^1 are biquotients.*

Proof. Dold and Whitney [8] have shown that a principal $SO(3)$ bundle $P \to M^4$ over a compact simply connected 4 manifold are classified by their second Stiefel-Whitney class $w_2 = w_2(P)$ and the first Pontryagin class $p_1 = p_1(P)$. Here, if we let $e = e(P)$ denote the Euler class, then w_2 can take on any value, but p_1 must be congruent to e^2 mod 4, where e reduced mod 2 gives w_2.

The case where the structure group reduces to an S^1 is precisely the case where $p_1 = e^2$.

Now, let $M = S^2 \times S^2$ or $\mathbb{C}P^2 \sharp \pm \mathbb{C}P^2$. In this case, the Gysin sequence for any S^2 bundle over M^4 with total space E implies that the map $\pi^* : H^2(M) \to H^2(E)$ is injective onto a summand of $H^2(E)$. Since, with the exception of the $p_1 = 0$ case of the T family of rings, we found essentially unique generators u and v so that, e.g. $u^2 = v^2$ and $uv = 0$ or $u^2 = v^2 = 0$, or $u^2 = -v^2$ and $uv = 0$, it follows that these $\mathbb{Z}$ subspaces are the image of $H^2(E)$ under π^*.

Now, we use the usual trick: given a bundle $S^2 \to E \to M$, we can write $TE \oplus \not{\mathbb{R}} = \pi^*(\mathbb{R}^3 \to \overline{E} \to M) \oplus \pi^*(\mathbb{R}^4 \to TM \to M)$ which allows us to express the characteristic classes of the bundle in terms of the characteristic classes of the tangent bundles of M^4 and E.

For example, when $M = \mathbb{C}P^2 \sharp \mathbb{C}P^2$, we are in the case of the R family. In the basis $u' = (u+v)$ and v, we have $u'v = 0$ and $u'^2 = v^2$, so this is the image of H^2 under π^*. In this basis, we have $p_1(G/\!\!/U) = (6 + c_1^2 + (c_1 - c_2)^2)v^2$ and $w_2 = u' + v + c_1 u' + (c_1 - c_2)v$. It is well known that $p_1(\mathbb{C}P^2 \sharp \mathbb{C}P^2) = 6v^2$ and $w_2(\mathbb{C}P^2 \sharp \mathbb{C}P^2) = u' + v$. Hence, we find that $p_1(S^2 \to G/\!\!/U \to \mathbb{C}P^2 \sharp \mathbb{C}P^2) = c_1^2 + (c_1 - c_2)^2$ and $w_2(S^2 \to G/\!\!/U \to \mathbb{C}P^2 \sharp \mathbb{C}P^2) = c_1 u' + (c_1 - c_2)v$.

Now, we just note that if $e = au' + bv$ and e restricts to w_2, then $p_1 = e^2 = a^2 + b^2$ and we clearly get all of these.

The same reasoning applies for S^2 bundles over $S^2 \times S^2$, coming from the T

rings with $b_1 = 0$. Recalling that $p_1(S^2 \times S^2) = 0$ and $w_2(S^2 \times S^2) = 0$, we see that $\pi^* p_1(S^2 \to G/\!\!/U \to S^2 \times S^2) = p_1(G/\!\!/U)$ and likewise for replacing w_2 with p_1. Notice that when $b_1 = 0$, we have $p_1 = 2c_1c_2$, which is precisely what the square of $(c_1u + c_2v)$ is.

The same reasoning also applies for S^2 bundles over $\mathbb{C}P^2\sharp - \mathbb{C}P^2$, coming from the T rings with $b_1 = 0$. Recalling that $p_1(\mathbb{C}P^2\sharp - \mathbb{C}P^2) = 0$ and $w_2(\mathbb{C}P^2\sharp - \mathbb{C}P^2) = u' + v$, and using the basis $u' = u + v$ and v, we find that $p_1(S^2 \to G/\!\!/U \to \mathbb{C}P^2\sharp - \mathbb{C}P^2) = 2c_1c_2 - c_2^2 = c_1^2 - (c_1 - c_2)^2$ and likewise, that $w_2(S^2 \to G/\!\!/U \to \mathbb{C}P^2\sharp - \mathbb{C}P^2) = c_1u' + (c_1 - c_2)v$. $\square$

5.4 7 Dimensional biquotients

We begin with the 2-connected biquotients.

In this section, we tabulate all the biquotients with rational homotopy groups that of S^7.

Proposition 5.4.1. *For $G = SU(4)$ and $U = SU(3)$, the only biquotient is the homogeneous space $SU(4)/SU(3) = S^7$.*

Proof. The point is that $SU(3)$ has precisely two 4-dimensional representations, $\Gamma_{1,0} \oplus \Gamma_{0,0}$ and $\Gamma_{0,1} \oplus \Gamma_{0,0}$. Here, $\Gamma_{i,j}$ denotes the unique representation of $SU(3)$ with highest weight vector....how does this notation work again? In any case, the representation $\Gamma_{0,1}$ is the complex conjugate representation to $\Gamma_{1,0}$, so it's easy to see that these two representations can't be used to create a free biquotient action. $\square$

Proposition 5.4.2. *If $G = Sp(2)$ and $U = Sp(1)$, then there are several homogeneous spaces and precisely one biquotient, the Gromoll-Meyer Sphere.*

Proof. $Sp(1)$ embeds into $Sp(2)$ in 3 ways. We have $p \to \mathrm{diag}(p, 1)$, $p \to \mathrm{diag}(p, p)$, and an embedding induced from the unique 4-dimensional irreducible representation of $Sp(1)$. (The embedding given by $Sp(1) = SU(2) \subseteq Sp(2)$ is conjugate to $p \to \mathrm{diag}(p, p)$). The weights are $(\lambda, 1), (\lambda, \lambda)$, and (λ, λ^3) respectively. Two matrices in $Sp(2)$ are conjugate iff after conjugating them to $\mathrm{diag}(e^{i\theta}, e^{i\alpha})$, they have the same eigenvalues up to order and complex conjugation.

We now check the freeness condition.

If the left action is determined by the unique irreducible 4-dimensional representation, and the right action is nontrivial, then the action will not be free. For, if the right hand side comes from the embedding $\mathrm{diag}(p, 1)$, then if λ is a nontrivial 3rd root of unity, then the two sides have eigenvalues $(\lambda, 1)$ and $(\lambda, 1)$, and hence we get a noncentral conjugacy. If, on the other hand, the right action comes from the embedding $\mathrm{diag}(p, p)$, then for $\lambda = i$, we get $\mathrm{diag}(i, -i)$ on the left and $\mathrm{diag}(i, i)$, and these give a noncentral conjugacy.

Hence, there are no nonhomogeneous biquotients which use the irreducible 4-dimensional representation of $Sp(1)$. For nonhomogeneous biquotients, this only leaves choosing the two embeddings $\mathrm{diag}(p, p)$ and $\mathrm{diag}(p, 1)$, giving the Gromoll-Meyer Sphere.

□

Proposition 5.4.3. *For $G = Spin(7)$ and $U = G_2$, the only biquotient is the homogeneous space $Spin(7)/G_2 = S^7$. For $G = Spin(8)$ and $U = Spin(7)$, there are two embeddings, but all biquotients are homogeneous and give quotient S^7.*

Proof. There is a unique embedding of G_2 into $Spin(7)$. The two embeddings of $Spin(7)$ into $Spin(8)$ are given by the lift of the block embedding of $SO(7)$ into $SO(8)$ and the spin representation of $Spin(7)$. The weights of the spin representation are $\lambda, \nu, \eta, \lambda\nu\eta$ and their complex conjugates. For the lift of the block embedding, the weights are $1, \mu, \theta, \rho$ and their complex conjugates. Given λ and ν, choose $\eta = \overline{\lambda\nu}$ and set $\mu = \lambda$, $\theta = \nu$ and $\rho = \eta$ to get a conjugacy.

Finally, for the lift of the block embedding, we naturally have $Spin(8)/Spin(7) = SO(8)/SO(7) = S^7$. For the spin representation, the triality automorphisms of $Spin(8)$ map each of the $Spin(7)$s into each other, so the triality automorphism defines a diffeomorphism between $S^7 = Spin(8)/Spin(7)$ and $Spin(8)/Spin(7)_{spin}$.

□

Proposition 5.4.4. *For $G = SU(4) \times SU(2)$ and $U = SU(3) \times SU(2)$, $G = Sp(2) \times SU(2)$ and $U = SU(2) \times SU(2)$, $G = Spin(7) \times SU(2)$ and $U = G_2 \times SU(2)$, and $G = Spin(8) \times SU(2)$ and $U = Spin(7) \times SU(2)$ all the biquotient actions are equivalent to $S^7 \times_{SU(2)} \times S^3$.*

If $G = Spin(7) \times SU(3)$ and $U = G_2 \times SU(3)$, $G = Spin(8) \times SU(3)$ and $U = Spin(7) \times SU(3)$, and $G = Spin(8) \times SU(2)$ and $U = G_2 \times Sp(2)$, there are *no biquotient actions.*

Proof. For any of the positive statements, the point is that the simple factor U' in U which is not $SU(2)$ cannot act on $SU(2)$ at all and hence must act freely on the factor of G, G', which is not $SU(2)$. In every case, by inspection, the quotient $G'/U' = S^7$. Hence, all the reduce to understanding $SU(2)$ actions on $S^7 \times S^3$ as claimed.

In order to prove the negative statements, first focus on the first two remaining pairs of groups. The issue is that in order for the either of the first two to give biquotients, we must have a free action of $SU(3)$ on $S^7 \times SU(3)$. Since we require the $SU(3)$ to act nontransitively on the $SU(3)$ factor of G, the $SU(3)$ action on itself must either be trivial, conjugation, or $A * B = ABA^t$. In either case, the every element of $SO(3)$ fixes itself. Thus, in order to have a free action, we'd need $SO(3)$ to act freely on S^7, which we've previously shown is impossible.

This leaves us in the case of $G = Spin(8) \times SU(2)$ and $U = G_2 \times Sp(2)$. As before, the G_2 and the $Sp(2)$ both cannot act on $SU(2)$. In order to have a biquotient action, we'd need $G_2 \times Sp(2)$ to act freely on $Spin(8)$. If such an action existed, it would have appeared in our list of 4-dimensional biquotients, so such an action cannot exist. $\square$

Proposition 5.4.5. *There are precisely two biquotients of the form $S^7 \times_{SU(2)} S^3$, where the $SU(2)$ is not allowed to act transitively on S^3. In both actions, the $SU(2)$ acts freely on the S^7, giving the biquotient the structure of an S^3 bundle over*

S^4. In the first action, $SU(2)$ acts trivially on S^3, while in the second, it acts by conjugation.

Proof. We want to understand $SU(2)$ actions on $S^7 \times S^3$. There are three possible actions of $SU(2)$ on itself: the trivial action, left multiplication, and conjugation. Left multiplication violates the convention that no simple factor of U acts transitively on any simple factor of G, so this leaves only the trivial action and conjugation. Since for any element $A \in SU(2)$, A fixes A when acting either trivially or via complex conjugation, we see that in order for $SU(2)$ to act freely on $S^7 \times S^3$, it must act freely on the S^7 factor, so it must be the Hopf action there.

The trivial action on $SU(2)$ gives $S^4 \times S^3$. For the other action, projection on the first factor to S^4 gives the biquotient the structure of an S^3 bundle over S^4.

Following the usual methods for computing the cohomology ring and characteristic classes, we easily see that $H^*(S^7 \times_{S^3} S^3) = H^*(S^3 \times S^4)$ and $p_1(G/\!\!/U) = \pm 4$. According to [16] this biquotient is not homotopy equivalent to $S^3 \times S^4$. $\square$

This concludes the case of 2-connected biquotients.

Proposition 5.4.6. *Suppose M is a biquotient with the same rational homotopy groups as $S^2 \times S^5$, then we must have $G = SU(3) \times SU(2)$ and $U = SU(2) \times S^1$. That is, all the other pairs (G, U) in the table do not lead to any examples. In the case where we do get a biquotient, it is either diffeomorphic to $S^5 \times_{S^1} S^3$ or $(SU(3)/SO(3)) \times_{S^1} S^3$ or has the structure of an S^3 bundle over $\mathbb{C}P^2$ where the structure group does not reduce to a circle.*

Proof. We first show that the other pairs (G, U) do not give rise to biquotients. If $G = SU(3) \times SU(3)$ and $U = SU(3) \times S^1$, then there are no free actions because the only way $SU(3)$ can act freely on $SU(3)$ is via a transitive action. If $G = SU(3) \times SU(2) \times SU(2)$ and $U = SU(2) \times SU(2) \times S^1$, then we would need $SU(2) \times SU(2)$ to act on $SU(3)$ freely, but in the classification of 2-manifolds, we ruled this out. If $G = SU(4) \times SU(2) \times SU(2)$ and $U = Sp(2) \times SU(2) \times S^1$, then the $Sp(2)$ can only act on $SU(4)$ giving quotient S^5. Hence, we'd need to find a free action of $SU(2)$ on $S^5 \times SU(2)$, but the only such free actions occur when $SU(2)$ acts transitively on itself. If $G = SU(3) \times SU(3) \times SU(2)$ and $U = SU(3) \times SU(2) \times S^1$, then we'd need a nontransitive free action of $SU(3)$ on $SU(3) \times SU(3)$, but not such actions exist. If $G = SU(3)^3$ and $U = SU(3)^2 \times S^1$, we have the same problem of there being no nontransitive free action of $SU(3)$ on $SU(3)^3$. Finally, if $G = SU(4) \times SU(3) \times SU(2)$ and $U = Sp(2) \times SU(3) \times S^1$, then again, the $Sp(2)$ can only act on $SU(4)$ giving quotient S^5, so we'd need to find a free $SU(3)$ action on $S^5 \times SU(3) \times SU(2)$, but no such action exists.

This leaves only the case of $G = SU(3) \times SU(2)$ and $U = SU(2) \times S^1$. The $SU(2)$ in U cannot act transitively on the $SU(2)$ in G, so the only possible actions of $SU(2)$ on $SU(2)$ are conjugation or the trivial action.

We now break into cases, so first assume that the $SU(2)$ in U acts trivially on the $SU(2)$ in G. In this case, the $SU(2)$ must act freely on $SU(3)$, so the quotient is either S^5 of the Wu manifold $SU(3)/SO(3)$. Adding in the circle shows the biquotient must look like either $S^5 \times_{S^1} S^3$ or $(SU(3)/SO(3) \times_{S^1} S^3$.

In the second case, the $SU(2)$ in U acts by conjugation on the $SU(2)$ in G. In this case the $SU(2)$ in U must act freely on $SU(3)$ and further, the circle can't act on $SU(2)$, so it must act on $SU(3)$. This means that we must have a free $SU(2) \times S^1$ action on $SU(3)$. Projecting onto the $SU(3)$ factor will then give $G/\!\!/U$ the structure of an S^3 bundle over $\mathbb{C}P^2$. $\square$

We will now analyze each of the three cases which can arise beginning with $SU(3) \times_{U(2)} SU(2)$, then handling the examples of the form $(SU(3)/SO(3)) \times_{S^1} SU(2)$, then handling the examples of the form $S^5 \times_{S^1} \times S^3$.

Proposition 5.4.7. *In the case of $SU(2) \times_{S^1 \times SU(2)} SU(3)$ where $SU(2)$ acts on $SU(2)$ via conjugation, we get precisely two S^3 bundles over $\mathbb{C}P^2$, whose total spaces both have euler class 0 and nontrivial second Stiefel-Whitney class, but one has first Pontryagin class $-3z^2$ while the other has first Pontryagin class $5z^2$ where z is the unique (up to sign) generator of $H^2(G/\!\!/U)$.*

Proof. We have already discussed free actions of $S^1 \times SU(2)$ on $SU(3)$ - there are precisely 2. One is the homogeneous action and the other is a two sided biquotient action (whose quotient is still $\mathbb{C}P^2$). We'll use z to denote a generator of $H^2(\mathbb{C}P^2)$.

For computing cohomology and characteristic classes, notice that if we replace $SU(2) \times S^1$ with $U(2)$ and then define the $U(2)$ action on $SU(2)$ via conjugation, we still get the same orbits. We will use this description for computing the characteristic classes.

Then we find that in either $U(2)$ action on $SU(3)$, that $G/\!\!/U = SU(3) \times_{U(2)} \times SU(2)$ has cohomology ring $H^*(G/\!\!/U) = H^*(S^3 \times \mathbb{C}P^3)$. It follows from the Gysin sequence for $S^3 \to G/\!\!/U \to \mathbb{C}P^3$, that the bundle $S^3 \to G/\!\!/U \to \mathbb{C}P^3$ has euler class 0.

We also compute that $w = 1$ for both examples, so we have $w(S^3 \to G/\!\!/U \to \mathbb{C}P^3) = 1 + z$, which uses the fact that $w(\mathbb{C}P^2) = 1 + z + z^2$.

However, for the case where $U(2)$ acts on $SU(3)$ only on one side, we have $p_1(G/\!\!/U) = 0$ so it follows that $p_1(S^3 \to G/\!\!/U \to \mathbb{C}P^3) = 1 - 3z^2$, which uses the fact that $p_1(\mathbb{C}P^2) = 3z^2$.

For the case where $U(2)$ acts on $SU(3)$ on both sides, we have $p_1(G/\!\!/U) = 8$, from which it follows that $p_1(S^3 \to G/\!\!/U \to \mathbb{C}P^3) = 5z^2$.

$\square$

Proposition 5.4.8. *Consider the map $f : SO(3) \times S^1 \to (SU(3) \times SU(2))^2$ given by $f(A, z) = (A, diag(z^a, \overline{z^a}), diag(z^b, z^c, z^d), diag(z^e, \overline{z^e})$ with $b + c + d = 0$ and $gcd(a, b, c, d, e) = 1$. Then the action is free iff $gcd(a^2 - e^2, bcd) = 1$.*

When the action is free we'll have $H^*(G/\!\!/U) = \mathbb{Z}[z_2, v_3, y_5]/(a^2 - e^2)z^2 = 0, 2v = v^2 = 0, y^2 = 0$ where the subscripts denote the degree. The first Pontryagin class is $4a^2 \in H^4 = \mathbb{Z}/(a^2 - e^2)\mathbb{Z}$.

We also have $H^*(G/\!\!/U; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[u_2, v_2, u_3]/u_3^2 = u_2^2 = v_2^2 = 0$. In this basis we have $w(G/\!\!/U) = 1 + u_2 + u_3$.

Proof. We begin by analyzing freeness. If we pick a z value so that $z^{a \pm e} = z^b = 1$

then we can always pick an element in $SO(3)$ which gives a conjugacy. In order for the action to be effectively free we require that in the case that $z^{a\pm e} = z^b = 1$ that $z^a = z^e = \pm 1$ and $z^c = z^d = 1$. This is equivalent to requiring $gcd(a \pm eb)|gcd(c, d, 2a, 2e, a - b)$. Doing this for $z^{a\pm e} = z^c = 1$ and $z^{a\pm e} = z^d = 1$, we learn that $gcd(a \pm e, c)|gcd(b, d, 2a, 2e, a - b)$ and $gcd(a \pm e, d)|gcd(b, c, 2a, 2e, a - e)$.

It's easy to see that these six gcd conditions are also sufficient to give a free action. We now turn the gcd conditions into a more suitable form.

Suppose $p > 2$ is prime and suppose that $p|(a \pm e, b)$ for some fixed choice of sign. Then $p|(c, d, a - e, 2a, 2e)$. But since $p > 2$ we have $p|2a$ iff $p|a$ and likewise for e. Hence this p divides $gcd(a, b, c, d, e)$ a contradiction.

If we also suppose $4|(a\pm e, b)$for some fixed choice of sign, then the same reasoning shows $2|gcd(a, b, c, d, e)$ giving a contradiction.

Hence we know all six of $gcd(a \pm e, b)$ $gcd(a \pm e, c)$ and $gcd(a \pm e, d)$ are either 1 or 2. Further assume one is 2. Then $2|a \pm e$ for some choice of signs so $2|a \pm e$ for all choices of signs. Further, by the freeness condition, if $2|gcd(a \pm e, b)$, then $2|c, d$. It follow that if one of the six is 2, then all 6 are.

Finally, we claim that 2 cannot occur for suppose 2 divides one, and hence all of them. It follows that b, c, and d are even, so we must have $a \cong e \cong 1$ mod 2. We now look mod 4. Since $b + c + d = 0$, we cannot have all of them congruent to 2 mod 4 so one of them, say b, is congruent to 0 mod 4. Since a and e are odd, either $a + e$ or $a - e$ is congruent to 0 mod 4. Supposing it's, say, the first, then we have $4|(a+e,b)$, a contradiction.

Thus, we have a free action iff $gcd(a \pm e, b) = gcd(a \pm e, c) = gcd(a \pm e, d) = 1$. Note that this is equivalent to $gcd(a^2 - e^2, bcd) = 1$, for if a prime p divides $gcd(a^2 - e^2, bcd)$, then it divides, say, $gcd(a - e, b)$, a contradiction and conversely, if a prime divides, say, $gcd(a-e, b)$, then the prime clearly divides $gcd(a^2 - e^2, bcd)$.

Of course, if one of a or e is 0 then the condition is automatically satisfied (since $gcd(a, b, c, d, e) = 1$). These biquotients naturally have the structure of an $SU(3)/SO(3)$ bundle over S^3. If $bcd = 0$, when we must have $a^2 - e^2 = \pm 1$, which of course implies a or e is 0. However, there are many other actions - for example, if $(a, b, c, d, e) = (2, b, c, , -b-c, 1)$ with $b \cong c \cong 1$ mod 3.

We now work out the cohomology ring and characteristic classes. With $\mathbb{Z}$ coefficients, we run into a problem: $H^*(BSO(3))$ is not a polynomial algebra, so the usual method for computing cohomology won't work as is.

Recall that we compute cohomology with a spectral sequence coming from the fibration $G \to G/\!\!/U \to BU$ which we write as the pull back of a fixed spectral sequence coming from the fibration $G \to BG \to BG \times BG$ whose differentials we know. The difficulty is in understanding what the pull back map does on the cohomology of BU and $BG \times BG$. The usual method for computing this map is to understand the map on the classifying spaces of the maximal tori of $G \times G$ and U, but in relating this information back to that of BH and $BG \times BG$, we use the assumption that $H^*(BU)$ and $H^*(BG)$ are polynomial algebras.

In our present case, the cohomology ring of BU has 2-torsion in degrees a multiple of 3, so this method can only be effectively used to understand the map between $BG \times BG$ and BU in degrees which are not a multiple of 3, but it can not be used in multiple of 3 degrees. Fortunately, we have $H^3(BU) = 0$, so we can easily compute the map there and while we cannot compute the map in degree 6, we can compute $H^6(G/\!\!/U)$ just using Poincare duality.

Here, we're using the fact that the cohomology ring of $BSO(3)$ is known and not difficult to work with: Brown [6] showed $H^*(BSO(3)) = \mathbb{Z}[v, p_1]/2v$ where $|v| = 3$ and $|p_1| = 4$. The important point is that the only degrees which have torsion are multiples of 3 and that we have $H^*(BSO(3))/Torsion \to H^*(BS^1)^W$, the Weyl group invariant elements of $H^*(BS^1)$ is an isomorphism.

If we denote $H^*(SU(3)) = \Lambda_{\mathbb{Z}}[x_3, x_5]$ and let y_3 be a generator for the cohomology of $SU(2)$ and let z generate the cohomology of BS^1, then a computation shows $d_3(x_3) = -p_1 - \sigma_2(b, c, d)z^2$, $d_3(y_3) = (a^2 - e^2)z^2$ and $d_5(x_5) = bcdz^3$.

Notice that d_3 is injective, for if $a^2 - e^2 = 0$, then we have either $a - e = 0$ or $a + e = 0$. Then the condition $gcd(a \pm e, b) = gcd(a \pm e, c) = gcd(a \pm e, d) = 1$ implies $|b| = |c| = |d| = 1$, contradicting the fact that $b + c + d = 0$.

Note however that $d_5 : E^{0,5} \to E_5^{6,0}/$ has kernel because

$$E_5^{6,0} = \langle v^2, z^3, p_1 z \rangle / \langle -p_1 z - z^2 \sigma_2(b, c, d), (a^2 - e^2)z^2 \rangle = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(a^2 - e^2)\mathbb{Z}.$$

It follows that $(a^2 - e^2)x_5$ is in the kernel of d_5.

Running through the spectral sequence machinery then gives $H^1 = 0$, $H^2 =$

$\mathbb{Z} = \langle z \rangle$, $H^3 = \mathbb{Z}/2\mathbb{Z} = \langle v \rangle$, $H^4 = \mathbb{Z}/gcd(a^2 - e^2, \sigma_2(b,c,d))\mathbb{Z} = \langle z \rangle$, and $H^5 = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \langle zv, (a^2 - e^2)x_5 \rangle$. We compute that $H^6 = H_1 = 0$ by Poincare duality. This gives us the ring structure and computing the characteristic classes follows in the routine way.

The computation of the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$ coefficients is more straight forward because Borel's method for understanding the induced map from $H^*(BG \times BG; \mathbb{Z}/2\mathbb{Z})$ to $H^*(BU; \mathbb{Z}/2\mathbb{Z})$ works. Since we already know $a \cong e + 1$ mod 2, we may assume without loss of generality that $e \cong 0$ mod 2. Likewise, we assume without loss of generality that $d \cong 0$ mod 2.

If we use as coordinates z for the maximal 2-group of S^1 and u_i as the maximal 2-group for $SO(3)$ with $i = 1, 2$, and 3 and $u_1 + u_2 + u_3 0$, then the usual techniques show $H^*(G/\!\!/U; \mathbb{Z}/2\mathbb{Z})\mathbb{Z}/2\mathbb{Z}[\sigma_2(u_i), z^2, \sigma_3(u_i)]/\sigma_2(u_i)^2 = (z^2)^2 = \sigma_3(u_i)^2 = 0$ with $|z^2| = |\sigma_2(u_i)| = 2$ and $|\sigma_3(u_i)| = 3$ and that the Stiefel-Whitney class is $1 + \sigma_2(u_i) + \sigma_3(u_i)$. $\square$

Proposition 5.4.9. *Any linear action of S^1 on $S^5 \times S^3$ is of the form*

$$z * \big((p_1, p_2, p_3), (q_1, q_2)\big) = \big((z^a p_1, z^b p_2, z^c p_3), (z^d q_1, z^e q_2)\big)$$

with $gcd(a, b, c, d, e) = 1$. Such an action is free iff $gcd(abc, de) = 1$. The cohomology ring of the quotient is $\mathbb{Z}[z, y_5]/dez^2 = z^3 = y_5^2 = 0$ when $de \neq 0$ and $\mathbb{Z}[z, y_3]/z^3 = y_3^2 = 0$ when $de = 0$. The first Pontryagin class is $p_1 = [3/4(a^2 + b^2 + c^2) + 1/2\sigma_2(a, b, c) + (d + e)^2]z^2$ when $a + b + c$ is even and $[2\sigma_2(a, b, c) + (d + e)^2 + 1]z^2$ when $a + b + c$ is odd. The Stiefel-Whitney classes are $1 + (d + e)z + \sigma_2(a, b, c)z^2$

when $a+b+c$ *is even and* $1+[1+d+e]z+(\sigma_2(a,b,c)+(d+e)^2)z^2$ *when* $a+b+c$ *is odd.*

Proof. If z fixes the point $\big((p_1,p_2,p_3),(q_1,q_2)\big)$, then if, say, $p_1 \neq 0$ and $q_1 \neq 0$, we must have that z fixes $\big((1,0,0),(1,0)\big)$. Thus, we see that if an element z of S^1 fixes any point, if fixes one of these special points. We now describe conditions which rule out fixing these special points.

Suppose $z*\big((1,0,0),(1,0)\big) = \big((1,0,0),(1,0)\big)$. Then we must have $z^a = z^d = 1$. This implies that z is a $gcd(a,d)$th root of 1. Since we want the action to be free, this requires that $gcd(a,d) = 1$. Testing out all the others points made of 0s and 1s gives the other conditions. Conversely, if $gcd(a,d) = 1$, then the only z value which fixes $\big((1,0,0),(1,0)\big)$ is $z = 1$, so these conditions are also sufficient. This shows the action is free iff $gcd(a,d) = gcd(a,e) = gcd(b,d) = gcd(b,e) = gcd(c,d) = gcd(c,e) = 1$. Notice that is a prime divides one of these, it divides $gcd(abc,de)$ and also conversely. Hence, each of the six gcd conditions are satisfied iff $gcd(abc,de) = 1$ as claimed.

To compute the cohomology and characteristic classes, we need to express this action as a biquotient.

It's easy to check that the following biquotient action gives this action on $S^5 \times S^3$: $SU(2) \times S^1 \to (SU(3) \times SU(2))^2$ sending (A,z) to

$$\big((\mathrm{diag}(z^{-a-b-c}A, z^{2a+2b+2c}), \mathrm{diag}(z^{-3d-3e}, z^{3d+3e})),$$
$$(\mathrm{diag}(z^{-4a+2b+2c}, z^{2a-4b+2c}, z^{2a+2b-4c}), \mathrm{diag}(z^{3d-3e}, z^{3e-3d}))\big)$$

and that for, say, $(a,b,c,d,e) = (11,7,5,3,2)$, this map is injective.

This action is orbit equivalent to the map sending (A,z) to

$$\big((\mathrm{diag}(z^{-a-b-c}A,1),\mathrm{diag}(z^{-2(d+e)},1)),$$
$$\mathrm{diag}(z^{-2a},z^{-2b},z^{-2c}),\mathrm{diag}(z^{-2d},z^{-2e})\big).$$

Writing $a+b+c = 2k+l$ for $l \in \{0,1\}$, we see that if $l = 0$, that $(A,z) \to \big((\mathrm{diag}(z^{-k}A,1),\mathrm{diag}(z^{d+e},1)),(\mathrm{diag}(z^{-a},z^{-b},z^{-c}),\mathrm{diag}(z^{-d},z^{-e}))\big)$ is a free action.

If $l = 1$, then we think of $\overline{z}A \in U(2)$, we get a map sending $A \in U(2)$ to

$$\big((\mathrm{diag}((detA^{k})A,1),\mathrm{diag}(detA^{d+e},1)),$$
$$(\mathrm{diag}(detA^{a},detA^{b},detA^{c}),\mathrm{diag}(detA^{d},detA^{e}))\big)$$

which is also a free action.

Computing in the case when $l = 0$, we find there are no extension problems to be worked out. Specifically, note that we cannot have $de = 0$, for in this case, we'd have $abc = 1$ so that $a+b+c$ is odd.

So, when $l = 0$,t $de \neq 0$, and so we get the cohomology ring $\mathbb{Z}[z,y_5]/dez^2 = z^3 = y_5^2 = 0$.

The usual computation shows $p_1 = (3/4(a^2+b^2+c^2)+1/2\sigma_2(a,b,c)+(d+e)^2)z^2$ and $w = (d+e)z+\sigma_2(a,b,c)z^2$

Computing in the case of l $= 1$, we see that if $de = 0$, we must work out an extension problem. That is, the cohomology groups are those of $\mathbb{C}P^2 \times S^3$ with the square of the degree generator a degree 4 generator. We claim that, via Poincare duality, this determines the ring structure. The only potential difficulty is deciding which nonzero multiple of the degree 5 generator is obtained by cupping the degree 2 generator with the degree 3 generator. But we know that the degree 3 generator cupped with the square of the degree 2 generator must be a generator of H^7, and this implies that the degree 3 generator cupped with the degree 2 generator must also be a generator. Thus, in this case $H^*(G/\!\!/U) = \mathbb{Z}[z, y_3]/z^3 = y_3^2 = 0$.

If $de \neq 0$ then $H^*(G/\!\!/U) = \mathbb{Z}[z, y_5]/dez^2 = z^3 = y_5^2 = 0$. We see that $p_1 = ((d+e)^2 + 2\sigma_2(a, b, c) + 1)z^2$ and $w = 1 + (1 + d + e)z + (\sigma_2(a, b, c) + d + e)z^2$.

□

In these examples, we can only make minimal progress towards understanding the cohomology ring structure, though we can completely determine the groups.

We first note that any linear T^2 action on $S^3 \times S^3 \times S^3$, up to equivalence has the form

$$(z, w) * \big((p_1, p_2), (q_1, q_2), (r_1, r_2)\big) =$$
$$\big((z^a p_1, z^b w^c p_2), (w^d q_1, z^e w^f q_2), (z^g w^h r_1, z^i w^j r_2)\big).$$

and we may assume without loss of generality that

$$\gcd(a, b, e, g, i) = \gcd(c, d, f, h, j) = 1.$$

We then have

Proposition 5.4.10. *An action like the above is free iff all the following gcds are 1: $gcd(a, d)$, $gcd(a, g)$, $gcd(d, h)$, $gcd(a, i)$, $gcd(d, j)$, $gcd(f, h)$, $gcd(f, j)$, $gcd(b, g)$,*

$gcd(b,i)$, $gcd(a,eh-gf)$, $gcd(a,ej-if)$, $gcd(d,bh-cg)$, $gcd(d,bj-ic)$, $gcd(bf-ec,bh-gc,eh-fg)$, $gcd(bf-ec,bj-if,ej-if)$. When the action is free we have $H^0=\mathbb{Z}$, $H^2=\mathbb{Z}+\mathbb{Z}+\mathbb{Z}/k\mathbb{Z}$ where $k=ad(bejh-bfih-bfjg+cfig)$, when $k\neq 0$.

Proof. Following the usual technique, it's enough to guarantee every (z,w) moves all points whose coordinates are all 0s and 1s. So, consider

$$(z,w)*\big((1,0),(1,0),(1,0)\big)=\big((1,0),(1,0),(1,0)\big).$$

So we see that $z^a=w^d=z^gw^h=1$. First, assume that $w=1$, so we get a nontrivial solution iff $gcd(a,g)\neq 1$. Thus, to keep freeness, we must have $gcd(a,g)=1$. A similar argument shows $gcd(d,h)=1$. Now that this is established, the map $\mathbb{Z}/a\mathbb{Z}\to\mathbb{Z}/a\mathbb{Z}$ given by $x\to gx$ is an isomorphism. Hence, the set of all outputs of z^gw^h as z and w range over $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/d\mathbb{Z}$ respectively is the same as the set of all outputs of zw. Hence, we see that there all solutions are trivial iff $gcd(a,d)=1$.

The same argument carried out on the point $\big(1,0),(1,0),(0,1)$ gives $gcd(a,i)=gcd(d,j)=1$.

Freeness at $\big((1,0),(0,1),(1,0)\big)$ comes down to simultaneously solving $z^a=1$, $z^ew^f=1$, and $z^gw^h=1$. I first claim that we must have $gcd(f,h)=1$, for by letting w be an arbitrary $gcd(f,h)$th root of 1 and setting $z=1$, we'd get a nontrivial solution. But this implies that any simultaneous solution must be an ath root of 1. For, we must have $w^f=z^{-e}\in\mu_a$ so that $w\in\mu_{af}$. Likewise, $w\in\mu_{ah}$ so we conclude $w\in\mu_{af}\cap\mu_{ah}=\mu_{gcd(f,h)a}=\mu_a$.

The rest of the ideas for the solution comes from a Math.StackExchange.com post by Qiaochu Yuan [33]

Now, by taking logs we can translate this problem to looking for the kernel of a linear map $\mathbb{Z}/a\mathbb{Z} + \mathbb{Z}/a\mathbb{Z}$ to itself with matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Such a matrix is invertible (in particular, has trivial kernel) iff its determinant is coprime to a, giving us the condition $gcd(a, eh - fg) = 1$

The same idea works for checking freeness at the points $\big((1,0),(0,1),(0,1)\big)$, $\big((0,1),(1,0),(1,0)\big)$, and $\big((0,1),(1,0),(0,1)\big)$, giving the next 6 gcd conditions.

We now focus on the point $\big((0,1),(0,1),(1,0)\big)$. Freeness there amounts to simultaneously solving $z^b w^c = 1$, $z^e w^f = 1$, $z^g w^h = 1$ and answering the question "what are necessary and sufficient conditions on b, c, e, f, g, and h to the only solution is $(z, w) = 1$.

Taking logs to turn this into a linear equation, we get the following equivalent question: "What are necessary and sufficient conditions on b, c, e, f, g, and h so that the matrix $X = \begin{bmatrix} b & c \\ e & f \\ g & h \end{bmatrix}$, thought us as a map ϕ from $\mathbb{Q}^2$ to $\mathbb{Q}^3$, satisfies $\phi^{-1}(\mathbb{Z}^3) = \mathbb{Z}^2$.

We claim that a necessary and sufficient condition is that the Smith normal form of the matrix be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. This is because if $A \in Gl_2(\mathbb{Z})$ and $B \in Gl_3(Z)$, then we have that $(B\phi A)^{-1}(\mathbb{Z}^3) = \mathbb{Z}^2$ iff $\phi^{-1}(\mathbb{Z}^3) = \mathbb{Z}^2$, which is clear precisely because A and B are $\mathbb{Z}$ invertible. By using such an A and B, we can convert X to its Smith normal form: a diagonal matrix so that $a_{11}|a_{22}|a_{33}$. Further, it is known, that the entry a_{ii} is equal to the gcd of all the determinants of all the i x i minors. Since we already know $\gcd(f, h) = 1$, this implies the gcd of all 1 x 1 minors is 1. The a_{22} entry of the Smith normal form is 1 iff the gcd of all the 2 x 2 sub determinants is 1, giving us our penultimate condition.

Finally, freeness at the point $\big((0,1),(0,1),(0,1)\big)$ works just as in the previous paragraph.

To begin computing cohomology, we convert this action into a biquotient action. It is easy to verify that the following biquotient action is orbit equivalent to the general one we've been using: map (z, w) to

$$\big(\mathrm{diag}(z^{a+b}w^c, 1), \mathrm{diag}(z^e w^{d+f}, 1), \mathrm{diag}(z^{g+i}w^{h+j})\big),$$

$$\big(\mathrm{diag}(z^b w^c, z^a), \mathrm{diag}(z^e w^f, w^d), \mathrm{diag}(z^i w^j, z^g w^h)\big)$$

Further, this map from T^2 to $(S^3)^2$ is injective since $w^f = w^h = 1$ implies $w = 1$ and $z^b = z^i = 1$ implies $z = 1$. More so, the image only trivially intersects $\Delta Z(S^3)$, so the action is free iff it's effectively free. It follows that we can use this description to compute.

If s_1, s_2, and s_3 are the generators for H^3 of each of the spheres, and if z and w are generators for $H^2(BT^2)H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ then, in the spectral sequence we have $ds_1 = abz^2 + aczw$, $ds_2 = dezw + dfw^2$ and $ds_3 = igz^2 + (ih + jg)zw + jhw^2$ which, altogether give the differential map $d_e : \mathbb{Z}^3 = \langle s_i \rangle \to \mathbb{Z}^3 = \langle z^2, w^2, zw \rangle$ which can be given in a matrix form as

$$\begin{bmatrix} ab & 0 & ig \\ ac & de & ih + jg \\ 0 & df & jh \end{bmatrix}$$

We now compute the Smith normal form of the matrix. The $a_1 1$ block of the Smith normal form is the gcd of all the elements. If a prime p divides every element, then it divides a or b. Assume initially it divides a. Since the prime divides de and $gcd(a, d) = 1$, we must have $p|e$ and likewise $p|f$. It follows that $p|gcd(a, ej - if)$, a contradiction. So, we conclude that at the beginning $p|b$. But then p cannot divide gi, giving a contradiction. Hence, no prime divides every element, so the gcd of all the elements is 1.

The $a_2 2$ block is the gcd of all the determinants of the 2 x 2 minors. We claim this is also 1.

For, if $p|det(\begin{bmatrix} ab & 0 \\ 0 & df \end{bmatrix}) = abdf$, then it divides a or b or d or e. By symmetry in z and w, the case where $p|a$ is the same as the case where $p|d$ and likewise the case of $p|b$ is the same as $p|f$.

Assume initially that $p|a$. Since $p|igdf$, we must have $p|f$. Since $p|igde$, we must have $p|e$, so $p|gcd(a, eh-gf)$, a contradiction. Next, assume p does not divide a and that $p|b$. Then $p|igac$ implies $p|c$. Since $p|igdf$, we have $p|f$. Since $p|jhde$, we have $p|e$. Putting this altogether shows $p|gcd(bf-ec, bh-gc, eh-fg)$, a contradiction.

Hence, no primes simultaneously divide all 2 x 2 determinants, so a_22 in the normal form is 1.

The term a_33 is the determinant of the 3 x 3 matrix, which turns out to be $k = ad(bejh - bfih - bfjg + cfig)$. Thus, we see that if this determinant is nonzero, then d_3 is injective, so we have $H^2(G/\!\!/U) = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/k\mathbb{Z}$ and $H^3(G/\!\!/U) = 0$. If $k = 0$, then $H^2(G/\!\!/U) = \mathbb{Z}\times\mathbb{Z}$ and $H^3 - 0$. $\square$

Chapter 6

Almost positive curvature on $Sp(3)/Sp(1) \times Sp(1)$

6.1 Curvature on Biquotients

There are two main ingredients for obtaining curvature results on biquotients: iterated Cheeger deformations and Wilking's doubling trick. The relevant information for Cheeger deformations can be found in Eschenburg's Habilitation [9] (in German), or see [19] for a good description in English. We recall it in order to establish notation.

Given a compact Lie group G and a subgroup $U \subseteq G \times G$, there is a natural action of U on G given by $(u_1, u_2) * g = u_1 g u_2^{-1}$. The action is effectively free iff whenever u_1 and u_2 are conjugate, we have $u_1 = u_2 \in Z(g)$. When the action is effectively free, the orbit space $G/\!\!/U$ naturally has the structure of a manifold such that $\pi : G \to G/\!\!/U$ is a submersion. The orbit space is called a biquotient.

By equipping G with a biinvariant metric, we see that U acts by isometries, inducing a Riemannian metric on $G/\!\!/U$ with π a Riemannian submersion. Hence, by the O'Neill formulas [22], every biquotient has a metric of nonnegative sectional curvature.

Now, consider a subgroup $K \subseteq G$. Equip G with left invariant, right K invariant metric of nonnegative sectional curvature, denoted g_0 (so the metric, when restricted to K is biinvariant). Let $\mathfrak{k} = Lie(K)$ and $\mathfrak{g} = Lie(G)$ be the corresponding Lie algebras and suppose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal splitting with respect to g_0.

We get a submersion $(G, g_0) \times (K, tg_0|_)K \to G$ given by $(g, k) \to gk^{-1}$, which induces a new left invariant metric on G, called the Cheeger deformation of g_0 in the direction of K, denoted g_1. Again, by the O'Neill formulae, this new metric also has nonnegative sectional curvature. Since G acts by isometries on the left on $G \times K$, we get an induced isometric G action on the quotient G which is easily seen to be transitive. Hence, g_1 is left invariant. By the same reasoning, K still acts by isometries on the right, so g_1 is still right K invariant. Note however that if, say, g_0 were biinvariant (i.e., right G invariant), then there is no reason to expect g_1 to be right G invariant.

We wish to understand the Riemannian submersion $\pi : G \times K \to G$. Since the metric g_1 is left invariant, we can reduce this to understanding the Riemannian submersion at (e,e). First note that by linearity, for $(X,Y) \in \mathfrak{g} \oplus \mathfrak{k}$, that $d\pi_{(e,e)}(X,Y) = X - Y$. It follows that the vertical space at (e,e) consists of all vectors of the form $(X_{\mathfrak{k}}, X_{\mathfrak{k}})$.

Hence, the horizontal space (with respect to $g_0 + tg_{0|_K}$) consists of all vectors of the form $(X_{\mathfrak{k}}, -\frac{1}{t}X_{\mathfrak{k}}$.

For $X \in \mathfrak{g}$, we write $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ for the $\mathfrak{k}$ and $\mathfrak{p}$ components. Then, we can describe $g_1(X,Y) = g_0(X_{\mathfrak{k}}, Y_{\mathfrak{k}})) + \frac{t}{1+t} g_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$. If Φ is the metric tensor relating g_1 and g_2, i.e. $g_0(\phi\cdot,\cdot) = g_1$, then we have $\phi(X) = X_{\mathfrak{p}} + \frac{t}{1+t}X_{\mathfrak{k}}$ which is clearly invertible with $\phi^{-1}(X) = X_{\mathfrak{p}} + \frac{1+t}{t}X_{\mathfrak{k}}$.

We want to understand when a plane in (G, g_1) has 0 sectional curvature. An easy necessary condition is that a horizontal lift of the plane to $G \times K$ must have 0 sectional curvature. If the plane $\sigma = \mathrm{span}\{\phi^{-1}X, \phi^{-1}Y\}$, then it's easy to check that the horizontal lift of σ is $\tilde{\sigma} = \mathrm{span}\{(X, -\frac{1}{t}X_{\mathfrak{k}}), (Y, -\frac{1}{t}Y_{\mathfrak{k}})\}$.

Hence, we see that if $sec_{g_1}(\mathrm{span}\{\Phi^{-1}X, \Phi^{-1}Y\}) = 0$, then we must have both $sec_{g_0}(\mathrm{span}\{X,Y\}) = 0$ and $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$. Since we have extra conditions, we expect to have fewer 0 curvature planes. However, we also potentially make the isometry group smaller. For example, g_1 typically only allows right multiplication by elements in K as isometries, while if g_0 is biinvariant, then right multiplication by any element of G is an isometry.

Further, this condition is also sufficient as shown by Tapp [28]. Thus, we see that $sec_{g_1}(\Phi^{-1}X, \Phi^{-1}Y)$ iff $sec_{g_0}(X,Y) = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$.

If g_0 is a biinvariant metric, then we see that $sec_{g_1}(\Phi^{-1}X, \Phi^{-1}Y) = 0$ iff $[X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$. If (G, K) is a symmetric pair, this simplifies.

Lemma 6.1.1. *$sec_{g_1}(\Phi^{-1}X, \Phi^{-1}Y) = 0$ iff $sec_{g_1}(X, Y) = 0$.*

Proof. Assume $sec_{g_1}(\Phi^{-1}X, \Phi^{-1}Y) = 0$. Then we know $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X, Y] = 0$. Expanding the second equality gives $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = 0$. Since (G, K) is a symmetric pair, we know $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Since $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ always, it follows that $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = 0$.

We now apply this knowledge to understanding $sec_{g_1}(X, Y)$ by writing $X = \Phi^{-1}\Phi X$ and $Y = \Phi^{-1}\Phi Y$. We know this is 0 iff $[\Phi X, \Phi Y] = [(\Phi X)_{\mathfrak{k}}, (\Phi Y)_{\mathfrak{k}}] = 0$. Since $(\Phi X)_{\mathfrak{k}} = \frac{t}{1+t}X_{\mathfrak{k}}$, we see the second equation is satisfied since we already know $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$. For the first we expand it, $[\Phi X, \Phi Y] = \frac{t^2}{(1+t)^2}[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] + \frac{t}{1+t}([X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}]) + [X_{\mathfrak{p}}, Y_{\mathfrak{p}}]$. However, in the previous paragraph we showed that the coefficients of powers of $\frac{t}{1+t}$ t are 0, so the whole thing is 0 as desired.

The reverse argument is analogous.

□

If one has a chain of subgroups $\{e\} = K_{n+1} \subseteq K_n \subseteq ... \subseteq K_1 \subseteq K_0 = G$, then one can iterate this process by deforming in the direction of the largest subgroup, then second largest, etc, to obtain metrics $g_1, g_2, ..., g_n$ with corresponding parameters $t_1, t_2, ..., t_n$. In the special case where each (K_i, K_{i-1}) is a symmetric pair, we obtain inductively:

Lemma 6.1.2. *$sec_{g_n}(X, Y) = 0$ iff $[X_{\mathfrak{k}_i}, Y_{\mathfrak{k}_i}] = 0$ for every i from 0 to n.*

If $\mathfrak{p}_i$ is defined by $\mathfrak{p}_i \oplus \mathfrak{k}_i = \mathfrak{k}_{i-1}$ (orthogonal with respect to a biinvariant metric), then since (K_i, K_{i-1}) is a symmetric pair, we must also have $[X_{\mathfrak{p}_i}, Y_{\mathfrak{p}_i}] = 0$ for all i.

We now describe Wilking's doubling trick. Any biquotient $G/\!\!/U$ is naturally diffeomorphic to $\Delta G \backslash G \times G/U$. However, the natural class of metrics in the right hand description is larger: we are free to choose any left invariant nonnegatively curved metric on each G factor, subject only to the constraint that U act by isometries.

Suppose g_l and g_r are two metrics as above with U acting by isometries and let g denote the induced metric on $G/\!\!/U$. We wish to understand when a 2-plane σ in $G/\!\!/U$ has 0 sectional curvature. By O'Neill's formula, the horizontal lift $\tilde{\sigma}$ of σ must have 0 sectional curvature. We want to determine when a plane is horizontal with respect to $g_l + g_r$. Let Φ_l and Φ_r denote the metric tensors relating g_l and g_r to a biinvariant metric g_0.

It's clear that every orbit of the $\Delta G \times U$ action passes through a point of the form $(g, 1)$ so it's enough to determine what it means to be horizontal at points like this. The vertical subspace V_g at $(g, e) \in G \times G$, translated to (e, e) using left translation, is $\{(Ad_{g^{-1}}X) - U_1, X - U_2 |\ X \in \mathfrak{g} \text{ and } (U_1, U_2) \in \mathfrak{u} = Lie(U)\}$.

Thus, the horizontal space H_g, again left translated to (e, e), is therefore $H_g = \{(\Phi_l^{-1}(-Ad_{g^{-1}}X), \Phi_2^{-1}X) |\ g_0(X, Ad_g U_1 - U_2) = 0 \text{ for all } (U_1, U_2) \in \mathfrak{u}\}$.

From here, since $g_l + g_r$ is a product metric of nonnegatively curved metrics, we see that a horizontal 0 curvature plane is spanned by

$$(\Phi_l^{-1}(-Ad_{g^{-1}}X), \Phi_r^{-1}X)$$

and

$$(\Phi_l^{-1}(-Ad_{g^{-1}}(Y)), \Phi_r^{-1}Y)$$

with $sec_{g_l}(\Phi_l^{-1}(-Ad_{g^{-1}}X), \Phi_r^{-1}(-Ad_{g^{-1}}(Y))) = 0$ and $sec_{g_r}(\Phi_r^{-1}X, \Phi_r^{-1}Y) = 0$.

Combining this into the form most useful to us, we have

Lemma 6.1.3. *Suppose $\{e\} = K_{n+1} \subseteq K_n \subseteq ... \subseteq K_1 \subseteq K_0 = G$ and $\{e\} = H_{m+1} \subseteq H_m \subseteq ... \subseteq H_1 \subseteq H_0 = G$ with both (K_i, K_{i-1}) and (H_j, H_{j-1}) symmetric pairs. Let g_l and g_r be the iterated Cheeger deformations of a biinvariant metric g_0 corresponding to the K_i and H_j respectively. There is a plane of 0 curvature at $\pi(g, e) \in G/\!\!/U$ iff there exists $X, Y \in \mathfrak{g}$ such that $g_0(X, Ad_g(U_1) - U_2) = g_0(Y, Ad_g(U_1) - U_2) = 0$ for all $(U_1, U_2) \in \mathfrak{u}$, and we have $[(Ad_{g^{-1}}X)_{\mathfrak{k}_i}, (Ad_{g^{-1}}Y)_{\mathfrak{k}_i}] = 0$ and $[X_{\mathfrak{h}_j}, Y_{\mathfrak{h}_j}] = 0$ for every i and j.*

We remark that since Ad_g is a Lie algebra isomorphism, the condition that $[Ad_{g^{-1}}X, Ad_{g^{-1}}Y] = 0$ is equivalent to asking $[X, Y] = 0$.

6.2 Applying this to $Sp(3)/Sp(1) \times Sp(1)$

In this section, we apply the techniques of the previous section to a specific example.

We use $Sp(n)$ to denote the n x n quaternionic unitary matrices, $Sp(n) = \{A \in M_n(\mathbb{H}) |\ A\overline{A}^t = Id\}$. The embedding of $U = Sp(1) \times Sp(1)$ into $G = Sp(3)$ is the standard block embedding sending $(p, q) \in U$ to $\begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{bmatrix}$.

Notice that $U \subseteq Sp(1) \times Sp(1) \times Sp(1) \subseteq Sp(2) \times Sp(1) \subseteq G$. Here, we're thinking of $U \subseteq Sp(2) \times Sp(1)$ by embedding the first factor of U into $Sp(2)$ via the block embedding and the second factor of U mapping surjectively onto $Sp(1)$.

Let g_0 denote the biinvariant metric on $Sp(3)$ and let g_l be the four times iterated Cheeger deformation corresponding to $\Delta Sp(1) \subseteq \Delta Sp(1) \times Sp(1) \subseteq Sp(1) \times Sp(1) \times Sp(1) \subseteq Sp(2) \times Sp(1) \subseteq Sp(3)$ and let g_r be the singly iterated Cheeger deformation corresponding to $K = Sp(2) \times Sp(1) \subseteq G$. We note that these are both deformations through symmetric pairs, so the final lemma of the previous section applies. Equip $G /\!\!/ U$ with the submersion metric on $\Delta G \backslash (G, g_l) \times (G, g_r)/e \times U$. Finally, let $\mathfrak{q}_i \oplus \mathfrak{h}_\mathfrak{i} = \mathfrak{h}_{i-1}$ and $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$.

Next, suppose $\pi((g^{-1}, e))$ has a 2-plane of 0 sectional curvature. By the last lemma of the previous section, we there must be an X and Y in $\mathfrak{g}$ which are orthogonal to U and for which $sec_{g_l}(Ad_g X, Ad_g Y) = 0 = sec_{g_r}(X, Y)$.

We first wish to investigate what it means for $sec_{g_r}(X, Y) = 0$ for X and Y orthogonal to U.

Lemma 6.2.1. *Suppose $sec_{g_r}(X, Y) = 0$ with X and Y orthogonal to U. Then we* *may assume without loss of generality that*

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ -\overline{y} & 0 & 0 \end{bmatrix}$$

with $x \in Im\mathbb{H}$ and $y \in \mathbb{H}$.

Proof. Asking that X and Y be orthogonal to U is equivalent to asking that $x_{11} = x_{33} = y_{11} = y_{33} = 0$ (here, x_{ij} denote entries of the matrix $X \in \mathfrak{sp}(3)$).

We know that $sec_{g_r}(X, Y) = 0$ iff $[X, Y] = 0$ and $[X_\mathfrak{k}, Y_\mathfrak{k}] = 0$. Since (G, K) is a symmetric pair and $G/K = \mathbb{H}P^2$ has positive sectional curvature, it follows that we may assume $X_\mathfrak{p} = 0$, so that $X = X_\mathfrak{k} = X_{\mathfrak{sp}(2)}$. Since $Sp(2)/Sp(1)$ has positive sectional curvature, we see that $[X_\mathfrak{k}, Y_\mathfrak{k}] = 0$ iff $X_\mathfrak{k}$ and $Y_\mathfrak{k}$ are dependent. Hence, we may assume without loss of generality that $Y_\mathfrak{k} = 0$.

At this point, we've shown that X has the form

$$X = \begin{bmatrix} 0 & a & 0 \\ -\overline{a} & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with a any quaternion and b purely imaginary. Likewise Y has the form

$$Y = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & d \\ -\bar{c} & -\bar{d} & 0 \end{bmatrix}$$

with c and d arbitrary quaternions.

The condition $[X, Y] = 0$ tells us that both $ad = 0$ and $-\bar{a}c + bd = 0$. Assuming $a \neq 0$, we see that $d = 0$ and hence that $c = 0$, i.e., that $Y = 0$. Hence, we may assume $a = 0$. Then we see that $bd = 0$, which implies $d = 0$. Thus, X and Y have the desired form.

□

We now wish to obtain a general understanding of when $sec_{g_l}(Z, W) = 0$.

Lemma 6.2.2. *Suppose $sec_{g_l}(Z, W) = 0$, then Z_{ij} must be proportional (over $\mathbb{R}$) to W_{ij} for $(i, j) = (1, 1)$, $(1, 2)$, and $(2, 2)$, $(3, 3)$, and we must also have $Z_{11} + Z_{22}$ is proportional to $W_{11} + W_{22}$ and that $1/2(Z_{11} + Z_{22}) + Z_{33}$ must be proportional to $1/2(W_{11} + W_{22}) + W_{33}$.*

Proof. The statement for $(1, 2)$ follows from $[X_{\mathfrak{q}_2}, Y_{\mathfrak{q}_2}] = 0$ since $Sp(2)/Sp(1) \times Sp(1) = S^4$ which has positive sectional curvature. The statement for both $(1, 1)$ and $(2, 2)$ follows from $[X_{\mathfrak{h}_2}, Y_{\mathfrak{h}_2}] = 0$, and likewise, the statement for $(3, 3)$ follows. The statement for $(1, 1) + (2, 2)$ and $1/2\big((1, 1) + (2, 2)\big) + (3, 3)$ follows from the diagonal deformations. □

Now, suppose $g^{-1} = A = (a)_{ij} \in Sp(3)$. Then we compute that

$$Ad_{g^{-1}}X = \begin{bmatrix} a_{12}x\overline{a_{12}} & a_{12}x\overline{a_{22}} & * \\ * & a_{22}x\overline{a_{22}} & * \\ * & * & a_{32}x\overline{a_{32}} \end{bmatrix}$$

and

$$Ad_{g^{-1}}Y = \begin{bmatrix} 2Im(a_{11}y\overline{a_{13}}) & a_{11}y\overline{a_{23}} - a_{13}\overline{y}\,\overline{a_{21}} & * \\ * & 2Im(a_{21}y\overline{a_{23}}) & * \\ * & * & 2Im(a_{31}y\overline{a_{33}} \end{bmatrix}$$

where an asterisks indicates the computation is irrelevant for what follows.

We will think of each of the entries of these two matrices as functions of x and y respectively, parameterized by A. So, for example, the function $X_{12}^A = X_{12} : Im\mathbb{H} \to \mathbb{H}$ given by $X_{12}(x) = a_{12}x\overline{a_{22}}$ the function $Y_{11} : \mathbb{H} \to Im\mathbb{H}$ is given by $Y_{11}(y) = 2Im(a_{11}y\overline{a_{13}})$. If we think of each of these 8 maps as matrices written with basis an ordered subset of $\{1, i, j, k\}$, then it's clear that each entry is a polynomial is the coordinates of $Sp(3)$, thinking of each a_{ij} as a 4 tuple of real numbers.

We can translate the 0 curvature plane conditions into this language:

Proposition 6.2.3. *If $\pi(g, e)$ has 0 sectional curvature for $g^{-1} = A$, then there must exist and $x \in Im\mathbb{H}$ and a $y \in \mathbb{H}$ so that*

1) $X_{ij}(x)$ is proportional to $Y_{ij}(y)$ for each pair $(i, j) = (1, 2)$, $(1, 1)$, $(2, 2)$, or $(3, 3)$ and

2) $X_{11}(x) + X_{22}(x)$ is proportional to $Y_11(y) + Y_22(y)$ and

3) $1/2(X_{11}(x)+X_{22}(x))+X_{33}(x)$ is proportional to $1/2(Y_{11}(y)+Y_{22}(y))+Y_{33}(y)$.

We now want to recall some very basic algebraic geometry over $\mathbb{R}$. We have $Sp(3) = \{A \in M_3(\mathbb{H}) | AA* = Id\}$ where $*$ denotes the Hermitian transpose. Using the natural maps $M_n(\mathbb{H}) \to M_{2n}(\mathbb{C}^2) \to M_{4n}(\mathbb{R})$, we can view $Sp(3)$ as a real algebraic variety of $\mathbb{R}^N$ for some large N. Further, $Sp(3)$ is irreducible (that is, it cannot be written as a nontrivial union of two Zariski closed sets) as it is smooth and connected.

Proposition 6.2.4. *Suppose Z is an irreducible topological space and $U \subseteq Z$ is nonempty and open. Then U is dense and irreducible*

Proof. We first show U is dense. Notice that $(Z - U) \cup \overline{U} = Z$. Since Z is irreducible, one of these two sets must be Z. But $Z - U = Z$ iff $U = \emptyset$, so we must have $\overline{U} = Z$, so that U is dense.

Now, assume we have two closed sets (in Z), F_1 and F_2 so that $(F_1 \cap U) \cup (F_2 \cap U) = U$, which implies $U \subseteq F_1 \cup F_2$. Taking closures, we find $Z \subseteq F_1 \cup F_2$, so that $Z = F_1 \cup F_2$. Since Z is irreducible, one of the two F_i, say F_1, is equal to Z. But then $F_1 \cap U = U$ so U is irreducible as well. $\square$

NOTE: Irreducible is NOT the same as connected over $\mathbb{R}$. For example, the zero set of $xy - 1$ is two hyperbolas, but this subset is irreducible since the defining polynomial is irreducible.

I claim that the linear map Y_{12}^A generically an isomorphism from $\mathbb{H}$ to itself. That is, if U_{12} denotes the subset of matrices A in $Sp(3)$ for which Y_{12}^A is an isomorphism, then U_{12} is open and dense. This is clear because the map is not an isomorphism iff it's determinant is zero, a polynomial condition in the entries a_{ij}. Hence, U_{12}^c is a closed set, and it's proper (as we'll see much later), so the complement is open (and nonempty) and thus dense. Likewise, we have that X_{ii} is generically an isomorphism for each i and that both $X_{11} + X_{22}$ and $1/2(X_{11} + X_{22}) + X_{33}$ are. Since the finite intersection of open dense sets is open and dense, there is an open dense set $U \subseteq Sp(3)$ for which all of these are simultaneously isomorphisms.

Since linear transformations preserve "is proportional to", we have the following easy corollary of the previous proposition.

Proposition 6.2.5. *If $\pi(g, e)$ has 0 curvature planes and $g^{-1} = A \in U$, then there must be a nonzero $x \in \mathbb{R}^3 = Im\mathbb{H}$ which is simultaneously an eigenvector for*

1) $X_{ii}^{-1} Y_{ii} Y_{12}^{-1} X_{12} : Im\mathbb{H} \to Im\mathbb{H}$ *for* $i = 1, 2, 3$,

2) $(X_{11} + X_{22})^{-1}(Y_11 + Y_22)Y_{12}^{-1}X_12 : Im\mathbb{H} \to Im\mathbb{H}$

3) $(1/2(X_{11} + X_{22}) + X_{33})^{-1}(1/2(Y_{11} + Y_{22}) + Y_{33})Y_{12}^{-1}X_{12} : Im\mathbb{H} \to Im\mathbb{H}$.

Proof. We begin with the fact that we know there must be a nonzero x and y so that $X_{12}(x)$ is proportional to $Y_{12}(y)$. This is equivalent to asking that $Y_{12}^{-1}X_11(x)$ be proportional to y. Since it's proportional to y, we know that $Y_{ii}Y_{12}^{-1}X_{12}(x)$ must be proportional to $X_{ii}(x)$, but then this is equivalent to asking that $X_{ii}^{-1}Y_{ii}Y_{12}^{-1}X_{12}(x)$ is proportional to x, which says exactly that x is an eigenvector for this composition for each $i = 1, 2, 3$. The proof for 2) and 3) are identical. $\square$

Using this, we define a map $f = (f_1, f_2, f_3, f_4, f_5) : U \to M_3(\mathbb{R})^5$ which takes a point in U to each of the 5 linear maps (in the basis $\{i, j, k\}$) in the previous proposition. It is clear that f is given by a polynomial in each entry of $M_3(\mathbb{R})^5 = \mathbb{R}^{45}$. It follows that the image $f(U)$ is an irreducible subset of $\mathbb{R}^{45}$, for if $f(U) = F_1 \cup F_2$ with each F_i closed, then $f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2) = U$, but each $f^{-1}(F_i)$ is closed because f is continuous. So, we have written U as a union of 2 closed sets. Since U is irreducible, one of them, say $f^{-1}(F_1) = U$. If follows that $F_1 = f(U)$, so $f(U)$ is irreducible.

Remark 6.2.6. We are not claiming that $f(U)$ is a subvariety of $\mathbb{R}^{45}$, only that it is irreducible.

Now, let $E \subseteq M_3(\mathbb{R})^5$ denote the subset of 5-tuples of matrices having a common eigenvector.

Theorem 6.2.7. *The subset E is a Zariski closed subset of $\mathbb{R}^{45}$.*

Proof. According to [2], the set $E_{\mathbb{C}} \subseteq M_3(\mathbb{C})^5$ of complex 5-tuples of matrices having a common eigenvector is Zariski closed in $\mathbb{C}^{45}$. Adding the (real) polynomial equations $x = \overline{x}$ for each coordinate in $\mathbb{C}^{45}$ gives E, so E is a real algebraic variety.

□

Since E is closed, it follows that $E \cap f(U)$ is relatively Zariski closed in $f(U)$ so that $f(U) - (E \cap f(U)) \subseteq f(U)$ is relatively Zariski open. By our previous propositions about irreducible sets, $f(U) - (E \cap f(U))$ is either empty or it is Zariski dense.

We will later show that it is nonempty (so is Zariski open and dense), but for now, we'll just assume it. Note that $f^{-1}(f(U) - (E \cap f(U)))$ is Zariski open in

U since f is continuous. Since we've already showed nonempty open subsets of irreducible spaces are dense, and since we know U is irreducible, it follows that $f^{-1}(f(U) - (E \cap f(U))$ is Zariski open and dense. Thus, we have found a Zariski open and dense subset of $U \subseteq Sp(3)$ for which all points have positive curvature.

We need another simple topological fact:

Proposition 6.2.8. *If $A \subseteq B \subseteq C$ are topological spaces so that A is open and dense in B and B is open and dense in C, then A is open dense in C.*

Proof. A is open in B means there is some open subset V of C so that $A = B \cap V$, but B is open, so A is an intersection of two open sets, hence open.

Now, choose an open set $V \subseteq C$. We wish to show $V \cap A \neq \emptyset$. Since B is dense in C, we know that $B \cap V \neq \emptyset$ and that $B \cap V$ is an open set in B. Since A is dense in B, we know that $A \cap (B \cap V) \neq \emptyset$, but $A \cap (B \cap V) \subseteq A \cap V$ so since the smaller set is nonempty, the larger must also be. $\square$

Thus, since we found a Zariski open dense subset of U consisting of positively curved points, and U is Zariski open and dense in $Sp(3)$, we have found a Zariski open and dense subset of points of positive curvature in $Sp(3)$. To finish this argument, we need two last facts:

Theorem 6.2.9. *If $X \subseteq Y \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ are both varieties, and if X is Zariski open and dense in Y, then $X \subseteq Y$ is open and dense with respect to the analytic topology.*

Proposition 6.2.10. *If $f : Z_1 \to Z_2$ is continuous and surjective and if $U \subseteq Z_1$ is dense, then so is $f(Z_1)$.*

Proof. Let $V \subseteq Z_2$ be open. Then $f^{-1}(V)$ is open and nonempty in Z_1, so intersects U. If p is a point in the intersection, then $f(p) \in V \cap f(U)$, so $V \cap f(U)$ is nonempty. $\square$

In our case, we have $Z_1 = Sp(3)$ and $Z_2 = Sp(3)/Sp(1) \times Sp(1)$ and the map is actually a (Riemannian) submersion, so is an open map. Hence, we have found an open and dense subset of points of positive curvature in $Sp(3)/Sp(1) \times Sp(1)$ (assuming we have found one point in $f(U) - (E \cap f(U))$.)

Finally, an uninspired calculation shows that there is a point in $f(U) - (E \cap f(U))$.

Proposition 6.2.11. *Consider the matrix*

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}(1+i) & \frac{1}{\sqrt{6}}(1+i) \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{60}}{30}(2+2i+j+k) & -\frac{1}{\sqrt{30}}(2i+j) & \frac{1}{\sqrt{30}}(2i+j) \end{bmatrix}.$$

Then $A \in U \subseteq Sp(3)$ and $f_4(A)$ and $f_5(A)$ have no common eigenvectors.

That is, $A \in U$ and $f(A) \in f(U) - (E \cap f(U))$.

6.3 Distinguishing the new examples

In this section, we verify that this example isn't diffeomorphic to any previously known example and that the two circle quotients $Sp(3)/Sp(1) \times Sp(1) \times S^1$ and $\Delta S^1 \backslash Sp(3)/Sp(1) \times Sp(1)$ are distinct and new examples.

The manifold $Sp(3)/Sp(1) \times Sp(1)$ is 15 dimensional. Previously, the only previously known 14 and 15 dimensional manifolds with almost positive curvature were due to Wilking [32]. In dimension 15, we have T^1S^8 and the space $U(5)/H_{kl}$ where $H_{kl} = \mathrm{diag}(B, z^k, z^l))$ for $z \in S^1$ and $B \in U(3)$. The first example is 7 connected, the second has $\pi_2 = \mathbb{Z}$ by the long exact sequence in homotopy groups associated to the fibration $H_{kl} \to U(5) \to U(5)/H_{kl}$. Our new example is 3 connected, but $\pi_4 = \mathbb{Z} \oplus \mathbb{Z}$, so it is distinct, even up to homotopy. Further, since the 15 dimensional manifold is a circle bundle over the 2 14 dimensional manifolds, each of the 14 dimensional examples also satisfies $\pi_4 = \mathbb{Z} \oplus \mathbb{Z}$.

The previously known 14 dimensional manifolds of almost positive curvature are $P_{\mathbb{C}}T^1\mathbb{C}P^4$ and $\Delta SO(2) \backslash SO(9)/SO(7)$. By using the fibrations $S^7 \to T^1\mathbb{C}P^4 \to \mathbb{C}P^4$ and $S^1 \to T^1\mathbb{C}P^4 \to P_C T^1\mathbb{C}P^4$, one can easily see $\pi_2(P_{\mathbb{C}}T^1\mathbb{C}P^4) = \mathbb{Z} \oplus \mathbb{Z}$ while $\pi_2 = \mathbb{Z}$ for each of the new 14 dimensional examples. Also, the manifold $SO(9)/SO(7)$ is a circle bundle over $\Delta SO(2) \backslash SO(9)/SO(7)$ so has the same higher homotopy groups as $SO(9)/SO(7)$. But $SO(9)/SO(7)$ is 7 connected, so is not homotopy equivalent to either of our 14 dimensional examples.

Finally, we show the two 14 dimensional examples are distinct. The cohomology ring of $M_1^{14} = Sp(3)/Sp(1) \times Sp(1) \times S^1$ is

$$H^*(M_1) = \mathbb{Z}[u^2, v^2, z]/u^2 + v^2 + z^2 = u^2v^2 + u^2z^2 + v^2z^2 = u^2v^2z^2 = 0$$

where $|z| = 2$ and $|u^2| = |v^2| = 4$. The first Pontryagin class is $p_1 = 4z^2$.

The cohomology ring of $M_2^{14} = \Delta S^1 \backslash Sp(3)/Sp(1) \times Sp(1)$ is

$$H^*(M_2) = \mathbb{Z}[u^2, v^2, z]/3z^2 - u^2 - v^2 = 3z^3 - u^2v^2 = z^6 = 0.$$

In this case, the first Pontryagin class is $p_1 = -12z^2$.

Since in both cases, $H^2 = \mathbb{Z} = \langle z \rangle$, it follows that z^2 is a well defined element in each ring. Thus, any ring isomorphism between the two rings must send z^2 to z^2, and so we can immediately see the two Pontryagin classes are distinct (even mod 28), so the two manifolds are not even homotopy equivalent.

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