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Homological Projective Duality for Gr(3,6)

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Abstract
Homological Projective Duality is a homological extension of the classical notion of projective duality. Constructing the homological projective dual of a variety allows one to describe semiorthogonal decompositions on the bounded derived category of coherent sheaves for all the complete linear sections of the initial variety. This gives a powerful method to construct decompositions for a big class of varieties, however examples for which this duality is understood are very few.
In this thesis we investigate the case of $Gr(3, 6)$ with respect to the Plucker embedding.

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HOMOLOGICAL PROJECTIVE DUALITY FOR GR(3,6)

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ABSTRACT

HOMOLOGICAL PROJECTIVE DUALITY FOR GR(3,6)

Dragos Deliu

Tony Pantev, Advisor

Homological Projective Duality is a homological extension of the classical notion of projective duality. Constructing the homological projective dual of a variety allows one to describe semiorthogonal decompositions on the bounded derived category of coherent sheaves for all the complete linear sections of the initial variety. This gives a powerful method to construct decompositions for a big class of varieties, however examples for which this duality is understood are very few.

In this thesis we investigate the case of $Gr(3,6)$ with respect to the Plucker embedding.

Contents

Chapter 1

Introduction

1.1 Introduction

Understanding derived categories of coherent sheaves on algebraic varieties has recently become an important direction of study in algebraic geometry. Among others, this is interesting in the context of Homological Mirror Symmetry ([Kon95]). For example, it is expected that semiorthogonal decompositions on a variety are given by singular fibers of the superpotential of the mirror Landau-Ginzburg model so understanding such decompositions on the derived category is useful from this point of view. Homological Projective Duality ([Kuz07]) gives the most powerful method of producing semiorthogonal decompositions, as once we know the dual variety for an algebraic variety we can describe decompositions for all (complete) linear sections of the initial variety. Moreover, in [CDH^{+}10] the authors have shown that homological projective duality plays an important role in understanding gauged linear sigma models by seeing that Kahler phases of gauged linear sigma models are related by homological projective duality.

Homological Projective Duality (HPD) is a homological extension of the classical notion of projective duality and came as an attempt to answer the question of whether having a semiorthogonal decomposition on $D^b(X)$ allows one to construct a decomposition of $D^b(X_H)$, where X_H is a hyperplane section of X. In general the answer is no, however in the context of HPD much can be said about this. One starts with a smooth (noncommutative) algebraic variety X with a map $X \to \mathbb{P}(V)$ and associates to it a smooth (noncommutative) algebraic variety Y with a map $Y \to \mathbb{P}(V^*)$ into the dual projective space (the classical projective dual variety of X will be given by the critical values of this second map). This construction will depend on a specific kind of a semiorthogonal decomposition of $D^b(X)$ called a Lefschetz decomposition. Then the main theorem of [Kuz07] gives a way to describe the derived categories of all complete linear sections of X and Y and one sees that they are related in the sense that the derived categories of dual linear sections will be generated by some exceptional bundles and a nontrivial piece that is the same for each pair of dual sections.

To make this work, one needs to first construct a Lefschetz collection on the bounded derived category of coherent sheaves on the initial variety and then find the Homological Projective Dual Variety with a dual Lefschetz collection. All the constructions depend on the choice of an ample line bundle on the variety whose dual we try to understand so essentially homological projective duality refers to pairs: a variety and an ample line bundle on it. Homological projective duality is well understood in few examples, some of them are the projective space with the double Veronese embedding and Gr(2,n) (for n lower or equal to 7) with the Plucker embedding ([Kuz08a],[Kuz06b], [Kuz06a]). In the first case the universal sheaf of even parts of Clifford algebras on $\mathbb{P}(S^2W)$ is a homologically projectively dual variety to $X = \mathbb{P}(W)$ with respect to $\mathcal{O}_X(2)$. As an application of this, one gets a proof of the theorem of Bondal and Orlov ([BO02]) about derived categories of intersections of quadrics. For the case of $Gr(2,6)$ and $Gr(2,7)$, Kuznetsov proves in ([Kuz06a]) that the homological projective dual varieties are given by noncommutative resolutions of the classically projective dual varieties, which in this case are Pfaffian varieties. When applied to linear sections of $Gr(2,6)$ and its dual, one gets that the nontrivial part in the derived category of a Pfaffian cubic fourfold is the derived category of a K3 surface, which since the Pfaffian cubic is rational leads to a very interesting conjecture relating the rationality of cubic fourfolds to their derived category containing the derived category of a commutative K3 surface as its nontrivial part. Last, but not least, considering linear sections of $Gr(2,7)$ and its dual one gets two non birational Calabi-Yau threefolds with equivalent derived categories, a result that was being predicted by physicists for a long time.

I investigate the next interesting case for homological projective duality, which is $Gr(3,6)$ with the Plucker embedding. Let W be a vector space of dimension 6. We consider $Gr(3,W) \hookrightarrow \mathbb{P}(\Lambda^3 W)$ and we would like to describe the homological projective dual for $G(3,W)$ with respect to this embedding. In this dissertation we present a program towards achieving this goal. In chapter 2, we construct a Lefschetz collection for $Gr(3,6)$ starting from the exceptional collection given by Kapranov in [Kap88] and prove it is full. In chapter 3 we give the geometric construction that should give the homological projective dual variety, some evidence towards the main conjecture and explain some of the difficulties in completing the construction. In the last chapter we construct the object that we expect will give the equivalence of categories that will prove homological projective duality and then give an outline of the t-structure argument that should help prove homological projective duality in this setup. In that section we also prove some results that are going to be important for the proof of the duality. We then give some applications of the following conjecture:

Conjecture 1. *There exists a sheaf of algebras $\mathcal{R}$ over a double cover M of $\mathbb{P}(\Lambda^3 W^*)$ ramified along a quartic hypersurface, such that the noncommutative resolution of singularities $(M, \mathcal{R})$ of M is Homologically Projective Dual to the Grassmannian $X = Gr(3,W)$.*

Chapter 2

Preliminaries

In this section we will present the context in which we are working and define the notions that we need. We will work over $\mathbb{C}$, the field of complex numbers. The bounded derived category of coherent sheaves on an algebraic variety X will be denoted by $D^b(X)$ (while $D(X)$ will be the unbounded derived category of coherent sheaves, $D^-(X)$ will be the unbounded below and $D^+(X)$ will be the unbounded above derived category of coherent sheaves). Within this setup we will denote all our derived functors as $\otimes$ (for the derived tensor product), $R\mathcal{H}om(F, G)$ (for the local $R\mathcal{H}om$ complex), $f_*, f^*, f^!$ (for the derived pushforward, pullback and twisted pullback functors).

2.1 Lefschetz decompositions

We first recall the definition of a semiorthogonal decomposition of a triangulated category.

Definition 2. [BO02] A collection $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of full triangulated subcategories in a triangulated category $\mathcal{T}$ is called a **semiorthogonal decomposition** of $\mathcal{T}$ if the following two conditions hold:

(i) $\mathsf{Hom}_T(\mathcal{A}_i, \mathcal{A}_j) = 0$ if $i > j$,

(ii) For every object $T \in \mathcal{T}$ there exists a chain of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of the morphism $T_i \to T_{i-1}$ is in $\mathcal{A}_i$ for $i = 1, \ldots, n$.

Definition 3. A full triangulated subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is called **right(left) admissible** if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{T}$ has a right (respectively left) adjoint. $\mathcal{A}$ will be called **admissible** if it is both right and left admissible.

Let now X be an algebraic variety with $O_X(1)$ an ample line bundle on it.

Definition 4. [Kuz07] A **Lefschetz decomposition** of the derived category $D^b(X)$ is a semiorthogonal decomposition of the form

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), ..., \mathcal{A}_{k-1}(k-1) \rangle,$$

where $0 \subset \mathcal{A}_{k-1} \subset \mathcal{A}_{k-2} \subset ... \subset \mathcal{A}_0 \subset D^b(X)$ is a chain of admissible subcategories of $D^b(X)$.

We will also need the notion of an exceptional object (collection).

Definition 5.

1. An object E in $D^b(X)$ is called **exceptional** if $\mathsf{Hom}(E, E) = \mathbb{C}$ and $\mathsf{Ext}^i(E, E) = 0$ for all $i \neq 0$.

2. A collection $(E_1, \dots, E_n)$ of objects in $D^b(X)$ is called **exceptional** if for each i the object E_i is exceptional and for all $i > j$ we have $\mathsf{Ext}^\bullet(E_i, E_j) = 0$.

3. A collection $(E_1, \dots, E_n)$ of objects in $D^b(X)$ is called **full** if the triangulated category generated by these objects is the whole $D^b(X)$.

2.2 Homological Projective Duality

Let X be as above, with a Lefschetz decomposition on its bounded derived category of coherent sheaves with respect to a line bundle $O_X(1)$ coming from a projective morphism $f : X \to \mathbb{P}(V)$. Let $\mathcal{X} \subset X \times \mathbb{P}(V^*)$ be the universal hyperplane section of X.

Definition 6. [Kuz07] An algebraic variety Y with a projective morphism $g : Y \to \mathbb{P}(V^*)$ is called **Homologically Projective Dual** to $f : X \to \mathbb{P}(V)$ with respect to a given Lefschetz decomposition as above, if there exists an object $\mathcal{E} \in D^b(\mathcal{X} \times_{\mathbb{P}(V^*)}$

Y) such that the kernel functor $\Phi = \Phi_{\mathcal{E}} : D^b(Y) \to D^b(\mathcal{X})$ is fully faithful and gives the following semiorthogonal decomposition

$$D^b(\mathcal{X}) = \langle \Phi(D^b(Y)), \mathcal{A}_1(1) \boxtimes D^b(\mathbb{P}(V^*)), ..., \mathcal{A}_{k-1}(k-1) \boxtimes D^b(\mathbb{P}(V^*)) \rangle$$

Now, for each linear subspace $L \subset V^*$ we consider the corresponding linear sections of X and Y, $X_L = X \times_{\mathbb{P}(V)} \mathbb{P}(L^{\perp})$ and $Y_L = Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$, where $L^{\perp} \subset V$ is the orthogonal subspace to $L \subset V$. If we take n to be the dimension of V, we have the following theorem

Theorem 7. *[Kuz07]*

If Y is Homologically Projective Dual to X then

(i) Y is smooth and $D^b(Y)$ admits a ***dual Lefschetz decomposition*** $D^b(Y) = \langle \mathcal{B}_{j-1}(1-j), ..., \mathcal{B}_1(-1), \mathcal{B}_0 \rangle$*, where* $0 \subset \mathcal{B}_{j-1} \subset \mathcal{B}_{j-2} \subset ... \subset \mathcal{B}_0 \subset D^b(Y)$

(ii) For any linear subspace $L \subset V^*$, $dim(L) = r$, such that $dimX_L = dimX - r$ and $dimY_L = dimY + r - n$, there exists a triangulated category $\mathcal{C}_L$ and semiorthogonal decompositions $D^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_r(1), ..., \mathcal{A}_{i-1}(i-r) \rangle$ and $D^b(Y_L) = \langle \mathcal{B}_{j-1}(N-r-j), ..., \mathcal{B}_{N-r}(-1), \mathcal{C}_L \rangle$.

2.3 Lefschetz decompositions and noncommutative resolutions of singularities

Here we present the main results of [Kuz08c]. The main construction in that paper gives a categorical resolution of singularities in a special situation, situation that can be applied to prove homological projective duality for $Gr(2,6)$, $Gr(2,7)$ and then hopefully $Gr(3,6)$.

First, let us define what a categorical resolution means in this context.

Definition 8. [Kuz08c] A **categorical resolution** of a triangulated category $\mathcal{D}$ is a regular triangulated category $\widetilde{\mathcal{D}}$ and a pair of functors

$$\pi_* : \widetilde{\mathcal{D}} \to \mathcal{D}, \qquad \pi^* : \mathcal{D}^{\mathsf{perf}} \to \widetilde{\mathcal{D}},$$

such that π^* is left adjoint to π_* on $\mathcal{D}^{\mathsf{perf}}$, that is

$$\mathsf{Hom}_{\widetilde{\mathcal{D}}}(\pi^* F, G) \cong \mathsf{Hom}_{\mathcal{D}}(F, \pi_* G) \qquad \text{for any } F \in \mathcal{D}^{\mathsf{perf}},\ G \in \widetilde{\mathcal{D}},$$

and the natural morphism of functors $id_{\mathcal{D}^{\mathsf{perf}}} \to \pi_*\pi^*$ is an isomorphism.

Let $\pi : \widetilde{Y} \to Y$ be a resolution of rational singularities and assume that the exceptional locus of π is an irreducible divisor $\widetilde{Z} \subset \widetilde{Y}$. Let $L = N^*_{\widetilde{Z}/\widetilde{Y}}$ be the conormal bundle, and let

$$D^b(\widetilde{Z}) = \langle \mathcal{B}_{m-1} \otimes L^{1-m}, \mathcal{B}_{m-2} \otimes L^{2-m}, \ldots, \mathcal{B}_1 \otimes L^{-1}, \mathcal{B}_0 \rangle$$

be a Lefschetz decomposition such that $p^*(\mathcal{D}^{perf}(Z)) \subset \mathcal{B}_0$ and $\mathcal{B}_0$ is stable with respect to tensoring by $p^*(\mathcal{D}^{perf}(Z))$, where $Z = \pi(\widetilde{Z}) \subset Y$ and $p = \pi_{|\widetilde{Z}} : \widetilde{Z} \to Z$. Let $i : \widetilde{Z} \to \widetilde{Y}$ be the embedding. Denote by $\widetilde{\mathcal{D}}$ the full subcategory of $\mathcal{D}^b(\widetilde{Y})$ consisting of objects $F \in \mathcal{D}^b(\widetilde{Y})$ such that $i^*F \in \mathcal{B}_0 \subset \mathcal{D}^b(\widetilde{Z})$. Then the functor $\pi^* : \mathcal{D}^{\mathsf{perf}}(Y) \to \mathcal{D}^b(\widetilde{Y})$ factors through $\mathcal{D}^{\mathsf{perf}}(Y) \to \widetilde{\mathcal{D}}$ and the restriction of the functor π_* to $\widetilde{\mathcal{D}}$ is its right adjoint.

Theorem 9. *[Kuz08c] The triangulated category $\widetilde{\mathcal{D}}$ with functors π_* and π^* is a categorical resolution of $\mathcal{D}^b(Y)$. Moreover, there exists a semiorthogonal decomposition*

$$\mathcal{D}^b(\widetilde{Y}) = \left\langle i_*(\mathcal{B}_{m-1} \otimes L^{1-m}), i_*(\mathcal{B}_{m-2} \otimes L^{2-m}), \ldots, i_*(\mathcal{B}_1 \otimes L^{-1}), \widetilde{\mathcal{D}} \right\rangle.$$

Finally, if Y is Gorenstein, $\mathcal{B}_{m-1} = \mathcal{B}_{m-2} = \cdots = \mathcal{B}_1 = \mathcal{B}_0$ and $K_{\widetilde{Y}} = \pi^*K_Y + (m-1)\widetilde{Z}$ then $\widetilde{\mathcal{D}}$ is a crepant categorical resolution of $\mathcal{D}^b(Y)$.

Another important question is when a resolution constructed this way is non-commutative in the sense of Van den Bergh.

A vector bundle E on $\widetilde{Y}$ is called **tilting** over Y if the pushforward $\pi_* \mathcal{E}nd\, E$ is a pure sheaf (i.e. $R^{>0}\pi_* \mathcal{E}nd\, E = 0$).

Theorem 10. *[Kuz08c] Assume that there exists a vector bundle E on $\widetilde{Y}$ such that the category $\mathcal{B}_0 \subset \mathcal{D}^b(\widetilde{Z})$ is generated by i^*E and E is tilting over Y. Assume also that $\mathcal{B}_0$ is admissible and $\mathcal{J}_{\widetilde{Z}} = \pi^{-1}\mathcal{J}_Z \cdot \mathcal{O}_{\widetilde{Y}}$. Then the sheaf of algebras $\mathcal{A} = \pi_* \mathcal{E}nd\, E$*

has finite homological dimension and the category $\widetilde{\mathcal{D}} \cong \mathcal{D}^b(Y, \mathcal{A})$ *is a noncommutative resolution of* $\mathcal{D}^b(Y)$*. Moreover, if* Y *is Gorenstein,* $\mathcal{B}_{m-1} = \mathcal{B}_{m-2} = \cdots = \mathcal{B}_1 = \mathcal{B}_0$ *and* $K_{\widetilde{Y}} = \pi^* K_Y + (m-1)\widetilde{Z}$ *then* $\mathcal{D}^b(Y, \mathcal{A})$ *is a noncommutative crepant resolution of* $\mathcal{D}^b(Y)$*.*

Chapter 3

A Lefschetz collection for Gr(3,6)

In this chapter we will construct a Lefschetz decomposition for $D^b(Gr(3,6))$ with respect to $\mathcal{O}_X(1) = \Lambda^3\mathcal{U}^*$ that gives the Plucker embedding into $\mathbb{P}(\Lambda^3 W^*)$.

3.1 The Borel-Weil-Bott Theorem

The Borel-Weil-Bott theorem can be used to compute the cohomology of equivariant vector bundles on Grassmannians. More generally, it helps calculate cohomology of line bundles on the flag variety of a semisimple algebraic group. However, we are only interested here in the Grassmanian case, so in particular we will only discuss the theorem for the group GL(n,C).

Let W be an n-dimensional vector space over $\mathbb{C}$. For a partition of a number d $\alpha = (a_1, \ldots, a_k)$, we can associate the Young diagram which has k rows and each row has length a_i. In [FH91], [Ful97] the authors define the Schur module $\mathbb{S}_\alpha(W)$ corresponding to α and prove that the construction is functorial in W. One way this was realized is $\mathbb{S}_\alpha(W) = W^{\otimes d} \cdot c_\alpha = W^{\otimes d} \otimes_{\mathbb{C}[S_d]} W_\alpha$, where c_α is the Young symmetrizer corresponding to α and W_α is the irreducible representation of S_d (permutation group on a set of d elements) corresponding to α. In particular, one can see that for the partitions of d, $d = (d, 0, 0, \ldots, 0) = (d)$ and $d = (1, 1, \ldots, 1) = (1^d)$, the corresponding Schur modules are $\mathbb{S}_{(d)} = S^d W$ and $\mathbb{S}_{(1^d)}(W) = \Lambda^d W$ (the d-symmetric power of W and the d-exterior power, respectively). These all give irreducible representations of $GL(W)$. The following theorem now describes all irreducible complex representations of $GL(W)$:

Theorem 11. *[FH91]*

Every irreducible representation of $GL(W)$ is isomorphic to $\mathbb{S}_\alpha(W) \otimes (\Lambda^n W)^{\otimes d}$ for some partition $\alpha = (a_1, \ldots, a_{n-1})$ and $d \in \mathbb{Z}$.

Now, we can identify the k-th fundamental weight (which is the highest weight of the representation $\Lambda^k W$) with the vector $(1, 1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ (via the standard identification of the weight lattice of the group $GL(W)$ with $\mathbb{Z}^n$). By doing this, we see that the dominant weights of $GL(W)$ are exactly those weights $(a_1, \ldots, a_n)$ given by sequences of integers for which $a_1 \geq a_2 \geq \ldots \geq a_n$. Using the theorem now, we see that the partition corresponding to an irreducible representation of $GL(W)$ with highest weight $a = (a_1, a_2, \ldots, a_n)$ is $\alpha = (a_1 - a_n, \ldots, a_{n-1} - a_n)$ and $m = a_n$.

Using this description, for any sequence of nonincreasing integers $\alpha = (a_1, \ldots, a_n)$, we denote by $\Sigma^\alpha W = \Sigma^{a_1,\ldots,a_n} W$ the corresponding representation of $GL(W)$ (so we essentially extend the definition of the Schur functor to negative integer values of the a_i's, but we follow the notation of [Kuz08b] from now on). Note that with this notation, $\Sigma^{a_1,\ldots,a_n} W^* = \Sigma^{-a_n,\ldots,-a_1} W$.

In order to formulate the Borel-Weil-Bott theorem and its applications to $Gr(3,6)$ we need some more notation. Let $\rho = (n, n-1, \ldots, 1)$ and $\alpha = (a_1, \ldots, a_n)$ a sequence of nonincreasing integers. The permutation group S_n on n letters acts naturally on the weight lattice $\mathbb{Z}^n$:

$$\sigma(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

Using the notation in [Wey03], we define the dotted action of S_n on $\mathbb{Z}^n$ by

$$\sigma \cdot (\alpha) = \sigma(\alpha + \rho) - \rho.$$

Let now F be the flag variety of $GL(W)$. It comes equipped with the collection of tautological bundles $\mathcal{U}_i$ of rank i (for $i = 1, \ldots, n-1$) and for any sequence α we define

$$L_\alpha = (\mathcal{U}_1)^{-a_1} \otimes (\mathcal{U}_2/\mathcal{U}_1)^{-a_2} \otimes \ldots \otimes (W/\mathcal{U}_{n-1})^{-a_n}.$$

The Borel-Weil-Bott theorem gives us a way to compute the cohomology of these line bundles L_α on F.

Theorem 12. *[Dem76][Wey03]*

Let $\alpha \in \mathbb{Z}^n$ be a dominant weight and let L_α be the corresponding line bundle on F. Then one of the two mutually exclusive possibilities occurs:

(1) There exists $\sigma \in S_n$, $\sigma \neq id$, such that $\sigma \cdot \alpha = \alpha$. Then $H^\bullet(F, L_\alpha) = 0$.

(2) There exists a unique $\sigma \in S_n$ such that $\sigma \cdot \alpha$ is nonincreasing. Then

$$H^k(F, L_\alpha) = \begin{cases} \Sigma^{\sigma\cdot\alpha} W^*, & \text{if } k = length(\sigma) \\ 0, & \text{otherwise} \end{cases}$$

Now let W be a n dimensional vector space over $\mathbb{C}$ and $X = Gr(k, W)$ the Grassmannian of k dimensional subspaces in W. Let $\mathcal{U} \subset W \otimes \mathcal{O}_X$ be the tautological rank k bundle on X. We will write $W/\mathcal{U}$ and $\mathcal{U}^\perp$ for the corresponding quotient bundle and, respectively, its dual. We have the following exact sequences that we will use extensively in what follows:

$$0 \to \mathcal{U} \to W \otimes \mathcal{O}_X \to W/\mathcal{U} \to 0$$

and

$$0 \to \mathcal{U}^\perp \to W^* \otimes \mathcal{O}_X \to \mathcal{U}^* \to 0.$$

Let $\pi : F \to X$ be the canonical projection from the flag variety of $GL(W)$ to $Gr(k, W)$. The following proposition gives us a nice description of the $GL(W)$-equivariant bundles on X (since every such bundle is isomorphic to $\Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp$ for β and γ nonincreasing sequences in $\mathbb{Z}^k$ and $\mathbb{Z}^{n-k}$):

Proposition 13. *[Kap88]*

Let $\beta \in \mathbb{Z}^k$ *and* $\gamma \in \mathbb{Z}^{n-k}$ *be nonincreasing sequences and* $\alpha = (\beta, \gamma) \in \mathbb{Z}^n$. *Then we have* $\pi_* L_\alpha \cong \Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp$.

Thus, together with the BWB theorem for the flag variety of $GL(W)$ we get the following:

Proposition 14. *Consider a weight* $\alpha = (a_1, \dots, a_n)$ *such that* $a_i \geq a_{i+1}$ *for all* $i \neq k$. *Then one of the following mutually exclusive cases occurs:*

(1) There exists $\sigma \in S_n$, $\sigma \neq id$, *such that* $\sigma \cdot \alpha = \alpha$. *Then*

$$H^\bullet(X, \Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp) = 0.$$

(2) There exists a unique $\sigma \in S_n$ *such that* $\sigma \cdot \alpha$ *is nonincreasing. Then*

$$H^k(X, \Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp) = \begin{cases} \Sigma^{\sigma\cdot\alpha} W^*, & \textit{if } k = \textit{length}(\sigma) \\ 0, & \textit{otherwise} \end{cases}$$

Before moving on to the construction of a Lefschetz decomposition on $Gr(3,6)$ let us mention Bott's algorithm which is useful in the actual calculations. If we start with $\alpha = (a_1, \dots, a_n) = (\beta, \gamma)$ as above, the permutation $\sigma_i = (i, i+1)$ will act on $\mathbb{Z}^n$ by $\sigma_i \cdot \alpha = (a_1, \dots, a_{i-1}, a_{i+1} - 1, a_i + 1, a_{i+2}, \dots, a_n)$. If α is not nonincreasing then we apply the σ_i's at the points where the sequence is increasing, trying to move the smaller numbers to the right. If at some point we get that $a_{i+1} = a_i + 1$

then σ_i fixes our element of $\mathbb{Z}^n$ and in this case $H^\bullet(X, \Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp) = 0$. If, after applying this process j times, we transform α into a nonincreasing sequence α' then

$$H^k(X, \Sigma^\beta \mathcal{U}^* \otimes \Sigma^\gamma \mathcal{U}^\perp) = \begin{cases} \Sigma^{\alpha'} W^*, & \text{if } k = j \\ \\ 0, & \text{otherwise} \end{cases}$$

Note that this also tells us that if α was nonincreasing to start with, the above statement holds for j=0.

3.2 The Construction of the Lefschetz collection on Gr(3,6)

As in the previous section, let $\mathcal{U}$ denote the tautological rank k subbundle on $Gr(k, W)$, the Grassmannian of k-dimensional subspaces in an n-dimensional vector space. We denote with the same letter the corresponding principal $GL(k)$-bundle on $Gr(k, W)$. We will consider a nonincreasing collection of k integers $\alpha = (a_1 \geq a_2 \geq \ldots \geq a_k)$ as a dominant weight of the group $GL(k)$ and we will denote by $\Sigma^\alpha(\mathcal{U})$ the vector bundle associated with the $GL(k)$ representation of highest weight α . In [Kap88] it was shown that the collection of vector bundles $\{\Sigma^a \mathcal{U}^* | n - k \geq a_1 \geq \ldots \geq a_k \geq 0\}$ is a full exceptional collection on $Gr(k, n)$. In particular, we have a full exceptional collection for $Gr(3, 6)$. The bundles $E_0 = O_X$, $E_1 = \mathcal{U}^*$, $E_2 = \Lambda^2 \mathcal{U}^*$, $E_3 = \Sigma^{2,1} \mathcal{U}^*$ are exceptional on $X = Gr(3, 6)$ by the above.

We define $\mathcal{A}_0 = \mathcal{A}_1 = \langle E_0, E_1, E_2, E_3\rangle$ and $\mathcal{A}_i = \langle E_0, E_1, E_2\rangle$, for $i = 2, \ldots, 5$. The main result of this chapter is:

Theorem 15. *On X, the subcategories $\mathcal{A}_5 \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ give a Lefschetz decomposition $D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5)\rangle$.*

Before we proceed to the proof of the theorem, let us note the following obvious lemma:

Lemma 16. *Let $\mathcal{A}_i$ with $i = 0, \ldots, 5$ be the subcategories of $D^b(X)$ defined above. The collection $\mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5)$ is exceptional if and only if $\mathcal{A}_0$ is an exceptional collection and $\mathsf{Ext}^\bullet(E_p, E_q(-k)) = 0$ for $1 \leq k \leq 5$, $0 \leq p \leq a_k$ (where $a_k = 3$ for $k = 1, 2$ and $a_k = 2$ for $k = 3, 4, 5$) and $1 \leq q \leq 3$.*

We can now proceed to the proof of the theorem.

Proof. There are two things that we need to do. First we will show, using some Borel-Weil-Bott calculations, that the collection we constructed is indeed exceptional. Once we do that, using the Schur-Weyl complexes as in [Wey03], we will prove that our collection is also full.

We know that $(\mathcal{O}_X, \mathcal{U}^*, \Lambda^2\mathcal{U}^*, \Sigma^{2,1}\mathcal{U}^*)$ is an exceptional collection from [Kap88]. Thus, using the lemma, we need to compute the other Ext's. For this we will use the fact that $\mathsf{Ext}^\bullet(F, G) = H^\bullet(X, F^* \otimes G)$ for locally free sheaves F and G and the Borel-Weil-Bott that we have explained in great detail in the previous section.

For $\mathcal{O}_X$:

$\mathsf{Ext}^\bullet(\mathcal{O}_X, E_i(-k)) = H^\bullet(X, E_i(-k)) = 0$ for $k = 1, \ldots, 5.$ and $i = 0, \ldots, 3$

For $\mathcal{U}^*$:

$$\mathsf{Ext}^\bullet(\mathcal{U}^*, E_i(-k)) = H^\bullet(X,\mathcal{U} \otimes E_i(-k)) = H^\bullet(X, \Sigma^{0,0,-1}\mathcal{U}^* \otimes E_i(-k)) =$$
$$= H^\bullet(X, \Sigma^{1,1,0}\mathcal{U}^* \otimes E_i(-k-1)) = 0$$

for $k = 1, \ldots, 5.$ and $i = 0, \ldots, 3.$

Here we also needed to use the Littlewood-Richardson rules for computing the tensor products:

$\Sigma^{1,1}\mathcal{U}^* \otimes \mathcal{U}^* = \Sigma^{2,1}\mathcal{U}^* \oplus \Sigma^{1,1,1}\mathcal{U}^* = \Sigma^{2,1}\mathcal{U}^* \oplus \mathcal{O}_X(1),$

$\Sigma^{1,1}\mathcal{U}^* \otimes \Lambda^2\mathcal{U}^* = \Sigma^{2,2}\mathcal{U}^* \oplus \Sigma^{2,1,1}\mathcal{U}^* = \Sigma^{2,1}\mathcal{U}^* \oplus \mathcal{U}^*(1),$

$\Sigma^{1,1}\mathcal{U}^* \otimes \Sigma^{2,1}\mathcal{U}^* = \Sigma^{3,2}\mathcal{U}^* \oplus \Sigma^{3,1,1}\mathcal{U}^* \oplus \Sigma^{2,2,1}\mathcal{U}^*$

For $\Lambda^2\mathcal{U}^*$:

$$\mathsf{Ext}^\bullet(\Lambda^2\mathcal{U}^*, E_i(-k)) = H^\bullet(X, \Lambda^2\mathcal{U} \otimes E_i(-k)) = H^\bullet(X, \Sigma^{0,-1,-1}\mathcal{U}^* \otimes E_i(-k)) =$$
$$= H^\bullet(X, \Sigma^{1,0,0}\mathcal{U}^* \otimes E_i(-k-1)) = H^\bullet(X,\mathcal{U}^* \otimes E_i(-k-1)) = 0$$

for $k = 1, \ldots, 5.$ and $i = 0, \ldots, 3$, where the Littlewood-Richardson rules gave us

$\mathcal{U}^* \otimes \mathcal{U}^* = \Sigma^{1,1}\mathcal{U}^* \oplus \Sigma^2\mathcal{U}^* = \Lambda^2\mathcal{U}^* \oplus S^2\mathcal{U}^*,$

$\mathcal{U}^* \otimes \Lambda^2\mathcal{U}^* = \Sigma^{2,1}\mathcal{U}^* \oplus \Sigma^{1,1,1}\mathcal{U}^* = \Sigma^{2,1}\mathcal{U}^* \oplus \mathcal{U}^*(1),$

$\mathcal{U}^* \otimes \Sigma^{2,1}\mathcal{U}^* = \Sigma^{3,1}\mathcal{U}^* \oplus \Sigma^{2,2}\mathcal{U}^* \oplus \Sigma^{2,1,1}\mathcal{U}^*.$

Finally, for $\Sigma^{2,1}\mathcal{U}^*$

$$\mathsf{Ext}^\bullet(\Sigma^{2,1}\mathcal{U}^*, E_i(-k)) = H^\bullet(X, \Sigma^{2,1}\mathcal{U} \otimes E_i(-k)) = H^\bullet(X, \Sigma^{0,-1,-2}\mathcal{U}^* \otimes E_i(-k))$$
$$= H^\bullet(X, \Sigma^{2,1,0}\mathcal{U}^* \otimes E_i(-k-2)) = H^\bullet(X, \Sigma^{2,1}\mathcal{U}^* \otimes E_i(-k-2)) = 0$$

for $k = 1$ and $i = 0, \ldots, 3$, where the Littlewood-Richardson rules gave us

$\Sigma^{2,1}\mathcal{U}^* \otimes \mathcal{U}^* = \Sigma^{3,1}\mathcal{U}^* \oplus \Sigma^{2,2}\mathcal{U}^* \oplus \Sigma^{2,1,1}\mathcal{U}^*$,

$\Sigma^{2,1}\mathcal{U}^* \otimes \Lambda^2\mathcal{U}^* = \Sigma^{3,1}\mathcal{U}^* \oplus \Sigma^{2,2}\mathcal{U}^* \oplus \Sigma^{2,1,1}\mathcal{U}^*$,

$\Sigma^{2,1}\mathcal{U}^* \otimes \Sigma^{2,1}\mathcal{U}^* = \Sigma^{4,2}\mathcal{U}^* \oplus \Sigma^{4,1,1}\mathcal{U}^* \oplus \Sigma^{3,3}\mathcal{U}^* \oplus (\Sigma^{3,2,1}\mathcal{U}^*)^{\oplus 2} \oplus \Sigma^{2,2,2}\mathcal{U}^*$.

Having shown that our collection is exceptional, we proceed to the proof of fullness. For that we first recall Kapranov's collection for $Gr(3,6)$ which was $\{\Sigma^a\mathcal{U}^* | 3 \geq a_1 \geq a_2 \geq a_3 \geq 0\}$.

We also note that the exceptional bundles that we have in our Lefschetz collection are given by the partitions $(0,0,0)$, $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, $(4,4,4)$, $(5,5,5)$, $(1,0,0)$, $(2,1,1)$, $(3,2,2)$, $(4,3,3)$, $(5,4,4)$, $(6,5,5)$, $(1,1,0)$, $(2,2,1)$, $(3,3,2)$, $(4,4,3)$, $(5,5,4)$, $(6,6,5)$, $(2,1,0)$, $(3,2,1)$. So, in order to prove that the collection we constructed is full, we need to check that $\Sigma^{2,0,0}\mathcal{U}^*$, $\Sigma^{2,2,0}\mathcal{U}^*$, $\Sigma^{3,0,0}\mathcal{U}^*$, $\Sigma^{3,1,0}\mathcal{U}^*$, $\Sigma^{3,1,1}\mathcal{U}^*$, $\Sigma^{3,2,0}\mathcal{U}^*$, $\Sigma^{3,3,0}\mathcal{U}^*$, $\Sigma^{3,3,1}\mathcal{U}^*$ are all generated by objects in our collection.

For that, we first recall the short exact sequence on X:

$$0 \to \mathcal{U} \to W \otimes \mathcal{O}_X \to W/\mathcal{U} \to 0.$$

For $\alpha = (a_1, \ldots, a_n)$ a sequence of nonnegative integers (a partition of, let's say, k) we can construct a Schur-Weyl complex as in [Wey03]. Since we work over $\mathbb{C}$, the

i'th term of the complex that we get can be written as $\bigoplus \Sigma^\beta \mathcal{U} \otimes \Sigma^{\alpha/\beta} W$, where the sum is taken over all β's subpartitions of α such that β is a partition of $k-i$ (here one has to define the Schur functors for a skew partition, but since we don't need to understand the objects in detail we will use just the reference above). Since $W \otimes \mathcal{O}_X$ is a trivial bundle, the same will be true for all the Schur functors $\Sigma^{\alpha/\beta} W \otimes \mathcal{O}_X$. Moreover, we will just write $\bigoplus (\Sigma^\beta \mathcal{U})^\oplus$ for the i'th terms of the complex that we get. Last, but not least, the last term of the complex will be given by $\Sigma^{\alpha'} W/\mathcal{U}$, where α' is the dual partition to α.

Let us now proceed to the proof that the remaining objects from Kapranov's collection are generated by objects in the Lefschetz collection described above. As we said, we start with the short exact sequence $0 \to \mathcal{U} \to W \otimes \mathcal{O}_X \to W/\mathcal{U} \to 0$ and form complexes out of it. To prove that $\Sigma^{a_1,a_2,a_3}\mathcal{U}^*$ is generated by our collection, we start with the complex associated to the partition (a_1-a_3, a_1-a_2), transform all the $\Sigma^\alpha \mathcal{U}$ into $\Sigma^\beta \mathcal{U}^*$, then tensor with $\mathcal{O}_X(a_1)$.

We'll first do $\Sigma^{2,2}\mathcal{U}^*$. The complex is:

$0 \to \Sigma^{2,0,0}\mathcal{U} \to \Sigma^{1,0,0}\mathcal{U}^\oplus \to \mathcal{O}_X^\oplus \to \Sigma^{1,1} W/\mathcal{U} \to 0$ which gives

$0 \to \Sigma^{0,0,-2}\mathcal{U}^* \to (\Sigma^{0,0,-1}\mathcal{U}^*)^\oplus \to \mathcal{O}_X^\oplus \to \Sigma^{1,1} W/\mathcal{U} \to 0$ which we tensor by $\mathcal{O}_X(2)$ and get

$0 \to \Sigma^{2,2,0}\mathcal{U}^* \to (\Sigma^{2,2,1}\mathcal{U}^*)^\oplus \to \mathcal{O}_X(2)^\oplus \to (\Sigma^{1,1} W/\mathcal{U})(2) \to 0.$

Since $(\Sigma^{1,1} W/\mathcal{U})(2) = \Sigma^{1,0,0}\mathcal{U}^\perp(3) = \mathcal{U}^\perp(3)$ and we have the short exact sequence $0 \to \mathcal{U}^\perp \to W^* \otimes \mathcal{O}_X \to \mathcal{U}^* \to 0$ we get that $\Sigma^{2,2}\mathcal{U}^*$ is generated by our collection.

We move on to $\Sigma^{2,0,0}\mathcal{U}^*$. The complex is:

$$0 \to \Sigma^{2,2,0}\mathcal{U} \to \Sigma^{2,1,0}\mathcal{U}^{\oplus} \to \Sigma^{2,0,0}\mathcal{U}^{\oplus} \oplus \Sigma^{1,1,0}\mathcal{U}^{\oplus} \to$$

$$\to \Sigma^{1,0,0}\mathcal{U}^{\oplus} \to \mathcal{O}_X^{\oplus} \to \Sigma^{2,2}W/\mathcal{U} \to 0$$

which gives

$$0 \to \Sigma^{0,-2,-2}\mathcal{U}^* \to (\Sigma^{0,-1,-2}\mathcal{U}^*)^{\oplus} \to (\Sigma^{0,0,-2}\mathcal{U}^*)^{\oplus} \oplus (\Sigma^{0,-1,-1}\mathcal{U}^*)^{\oplus} \to (\Sigma^{0,0,-1}\mathcal{U}^*)^{\oplus}$$

$$\to \mathcal{O}_X^{\oplus} \to \Sigma^{2,2}W/\mathcal{U} \to 0$$

which we tensor by $\mathcal{O}_X(2)$ and get $0 \to \Sigma^{2,0,0}\mathcal{U}^* \to (\Sigma^{2,1,0}\mathcal{U}^*)^{\oplus} \to (\Sigma^{2,2,0}\mathcal{U}^*)^{\oplus} \oplus (\Sigma^{2,1,1}\mathcal{U}^*)^{\oplus} \to (\Sigma^{2,2,1}\mathcal{U}^*)^{\oplus} \to (\mathcal{O}_X)(2)^{\oplus} \to (\Sigma^{2,2}W/\mathcal{U})(2) \to 0$.

Since $(\Sigma^{2,2}W/\mathcal{U})(2) = (\Sigma^{2,0,0}\mathcal{U}^{\perp})(4) = (S^2\mathcal{U}^{\perp})(4)$ and we have the short exact sequence $0 \to S^2\mathcal{U}^{\perp} \to (\mathcal{U}^{\perp})^{\oplus} \to \mathcal{O}_X^{\oplus} \to \Lambda^2\mathcal{U}^* \to 0$ we get that $\Sigma^{2,0,0}\mathcal{U}^*$ is generated by our collection too.

For $\Sigma^{3,3,1}\mathcal{U}^*$ we again start with the complex associated to $(2,0,0)$:

$0 \to \Sigma^{2,0,0}\mathcal{U} \to \Sigma^{1,0,0}\mathcal{U}^{\oplus} \to \mathcal{O}_X^{\oplus} \to \Sigma^{1,1}W/\mathcal{U} \to 0$ which gives

$0 \to \Sigma^{0,0,-2}\mathcal{U}^* \to (\Sigma^{0,0,-1}\mathcal{U}^*)^{\oplus} \to \mathcal{O}_X^{\oplus} \to \Sigma^{1,1}W/\mathcal{U} \to 0$ which we tensor by $\mathcal{O}_X(3)$ and get

$0 \to \Sigma^{3,3,1}\mathcal{U}^* \to (\Sigma^{3,3,2}\mathcal{U}^*)^{\oplus} \to \mathcal{O}_X(3)^{\oplus} \to (\Sigma^{1,1}W/\mathcal{U})(3) \to 0$.

Since $(\Sigma^{1,1}W/\mathcal{U})(3) = \Sigma^{1,0,0}\mathcal{U}^{\perp}(4) = \mathcal{U}^{\perp}(4)$ and we have the short exact sequence $0 \to \mathcal{U}^{\perp} \to W^* \otimes \mathcal{O}_X \to \mathcal{U}^* \to 0$ we get that $\Sigma^{3,3,1}\mathcal{U}^*$ is generated by our collection.

For $\Sigma^{3,3,0}\mathcal{U}^*$ we start with:

$0 \to \Sigma^{3,0,0}\mathcal{U} \to \Sigma^{2,0,0}\mathcal{U}^{\oplus} \to \Sigma^{1,0,0}\mathcal{U}\oplus \to \mathcal{O}_X^{\oplus} \to \Lambda^3(W/\mathcal{U}) \to 0$ which gives

$0 \to \Sigma^{0,0,-3}\mathcal{U}^* \to (\Sigma^{0,0,-2}\mathcal{U}^*)^{\oplus} \to (\Sigma^{0,0,-1}\mathcal{U}^*)\oplus \to \mathcal{O}_X^{\oplus} \to \Lambda^3(W/\mathcal{U}) \to 0$. Tensoring with $\mathcal{O}_X(3)$ we get:

$0 \to \Sigma^{3,3,0}\mathcal{U}^* \to (\Sigma^{3,3,1}\mathcal{U}^*)^{\oplus} \to (\Sigma^{3,3,2}\mathcal{U}^*)\oplus \to (\mathcal{O}_X)(3)^{\oplus} \to (\Lambda^3(W/\mathcal{U}))(3) \to 0$

and again this proves that $\Sigma^{3,3,0}$ is generated by our collection.

Clearly this method works for the remaining $\Sigma^{\alpha}\mathcal{U}^*$. We just have to notice that at the right of $\Sigma^{\alpha}\mathcal{U}^*$ we only get bundles given by α''s which are higher (with respect to the lexicographic order) than α and those we already know to be generated by the Lefschetz collection that we constructed (while the last term to the right can be expresses in terms of $\mathcal{U}^{\perp}$ for which we can show as above it is in our collection. $\square$

We conclude this section with another interesting question, which is what happens for $Gr(3,7)$ and other Grassmannians? For $Gr(3,7)$ we will get a rectangular Lefschetz collection (i.e. one in which all the pieces $\mathcal{A}_i$ are equal) of the form :

$D^b(Gr(3,7)) = \langle \mathcal{A}_0, \dots, \mathcal{A}_6 \rangle$, with $\mathcal{A}_i = \langle \mathcal{O}, \mathcal{U}^*, \Lambda^2(\mathcal{U}^*), S^2(\mathcal{U}^*), \Sigma^{2,1}(\mathcal{U}^*) \rangle$. This is minimal just as the one for $Gr(3,6)$ and agrees with a general construction of Lefschetz collections for $Gr(k,n)$ that has been conjectured by others. Actually it is expected that $Gr(3,n)$ with 3 and n coprime, will have a Lefschetz collection that is rectangular. However, for the proof an induction method will be easier and this is actually another way to prove fullness for the collection on $Gr(3,6)$ that we have constructed above.

Chapter 4

The geometric construction

For a smooth projective variety $X \subset \mathbb{P}(V)$ with a Lefschetz decomposition of $D^b(X)$ corresponding to $O_X(1)$, and for any hyperplane section X_H of X, we have that $D^b(X_H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), ..., \mathcal{A}_{k-1}(k-1)) \rangle$ (the composition $\mathcal{A}_i(i) \to D^b(X) \to D^b(X_H)$ is fully faithful [Kuz07]). We consider the family $\{\mathcal{C}_H\}_{H \in \mathbb{P}(V^*)}$ of triangulated categories. Finding the homological projective dual Y as above means that this family is "geometric", i.e. for $Y \to \mathbb{P}(V^*)$ and for all H we have that $\mathcal{C}_H \cong D^b(Y_H)$, where Y_H is the fiber over $H \in \mathbb{P}(V^*)$.

The way to do this is to actually look at the universal variant of this. If we let $\mathcal{X} \subset X \times \mathbb{P}(V^*)$ be the universal hyperplane section of X, we have a decomposition $D^b(\mathcal{X}) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes D^b(\mathbb{P}(V^*)), ..., \mathcal{A}_{k-1}(k-1) \boxtimes D^b(\mathbb{P}(V^*)) \rangle$ and check that $\mathcal{C}$ is equivalent to $D^b(Y)$.

4.1 Geometry

Consider now $W = \mathbb{C}^6$, $\mathcal{U}$ the tautological vector bundle on $X = Gr(3,6)$ and $X_H \subset X$ a hyperplane section of X with respect to the Plucker embedding.

We observe that the fibers of the map $\mathbb{P}_X(\mathcal{U}) \to \mathbb{P}(W)$ are $Gr(2,5)$'s. For $Gr(2,5)$ we know from [Kuz08b] that $D^b(Gr(2,5))$ has a rectangular Lefschetz decomposition with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{S}^* \rangle$ (where $\mathcal{S}$ is the tautological bundle on $Gr(2,5)$. Therefore $D^b(Gr(2,5))$ is generated by 10 exceptional bundles.

The fibers of the map $\mathbb{P}_{X_H}(\mathcal{U}) \to \mathbb{P}(W)$ are hyperplane sections of $Gr(2,5)$. From [Kuz06a] we know that the derived category of a hyperplane section has a nontrivial part only if it is singular. For smooth sections, it has 8 exceptionals coming from the restriction of the ones on $Gr(2,5)$. Therefore, on $\mathbb{P}_X(\mathcal{U})$ we have 8 exceptional bundles such that when we restrict them to $\mathbb{P}_{X_H}$ we also get 8 exceptional bundles. Using the description of the map $\mathbb{P}_{X_H}(\mathcal{U}) \to \mathbb{P}(W)$ that we have given, we see that

$$D^b(\mathbb{P}_{X_H}(\mathcal{U})) = \langle D^b(\mathbb{P}(W), \ldots, D^b(\mathbb{P}(W), \mathcal{Z}_H \rangle,$$

where we have 8 copies of $D^b(\mathbb{P}(W)$ (more precisely we have subcategories that are given by $D^b(\mathbb{P}(W)$ tensored with the 8 exceptionals that we have obtained) and $\mathcal{Z}_H$ consists of objects supported over the preimage in $\mathbb{P}_{X_H}(\mathcal{U})$ of $Z_H \subset \mathbb{P}(W)$, the discriminant of the map $\mathbb{P}_{X_H}(\mathcal{U}) \to \mathbb{P}(W)$.

We also have a description of $D^b(\mathbb{P}(W))$ which tells us that it is generated by 6 exceptional objects. Therefore, we see that $D^b(\mathbb{P}_{X_H}(\mathcal{U}))$ is generated by 48 exceptional bundles and $\mathcal{Z}_H$.

Now, we look at the map $\mathbb{P}_{X_H}(\mathcal{U}) \to X_H$. This is a $\mathbb{P}^2$ bundle over X_H and so we have a decomposition $D^b(\mathbb{P}_{X_H}(\mathcal{U})) = \langle D^b(X_H), D^b(X_H), D^b(X_H)\rangle$. However, homological projective duality tells us that there exists an admissible subcategory $\mathcal{C}_H \subset D^b(X_H)$ such that $D^b(X_H) = \langle \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4), \mathcal{A}_5(5), \mathcal{C}_H\rangle$. But $\langle \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4), \mathcal{A}_5(5)\rangle$ consists of 16 exceptional bundles, as we have seen in the previous section. Therefore using the map $\mathbb{P}_{X_H}(\mathcal{U}) \to X_H$ we see that $D^b(\mathbb{P}_{X_H})$ is generated by 48 exceptional objects and three copies of $\mathcal{C}_H$.

To sum up, we expect that there is an equivalence $\mathcal{Z}_H = \langle \mathcal{C}_H, \mathcal{C}_H, \mathcal{C}_H\rangle$, where $\mathcal{Z}_H = D^b(Z_H)$.

As we have said above, we now need to understand the picture for the full family of linear sections and thus we consider the universal discriminant $Z \subset \mathbb{P}(W) \times \mathbb{P}(\Lambda^3(W^*))$. By what we said above, it follows that the Homological Projective Dual of $X = Gr(3,6)$ should be the category $\mathcal{C}$ such that $D^b(Z) = \langle \mathcal{C}, \mathcal{C}, \mathcal{C}\rangle$. So, to understand $\mathcal{C}$ geometrically we need to find a structure of a $\mathbb{P}^2$-bundle on Z: $Z \to M$ so that $D^b(Z) = \langle D^b(M), D^b(M), D^b(M)\rangle$.

We can now give a description of Z:

Lemma 17. *Let $Z \subset \mathbb{P}(W) \times \mathbb{P}(\Lambda^3(W^*))$ be as above. Then $Z = \{(w, \lambda) \in \mathbb{P}(W) \times \mathbb{P}(\Lambda^3(W^*)) \mid rank(\lambda \lrcorner w) \leq 2\}$*

Proof. Note that Z consists of pairs $(w, \lambda) \in \mathbb{P}(W) \times \mathbb{P}(\Lambda^3(W^*))$ such that

$Gr(2,5)_w \cap H_\lambda$ is singular, where by $Gr(2,5)_w$ we mean the fiber of the map $\mathbb{P}_X(\mathcal{U}) \to \mathbb{P}(W)$ over w and by H_λ the hyperplane in $\Lambda^3 W$ corresponding to λ.

More precisely, $Gr(2,5)_w = Gr(2, W/\mathbb{C}w)$. Using this we see that the intersection $Gr(2,5)_w \cap H_\lambda$ is the hyperplane section of $Gr(2, W/\mathbb{C}w)$ corresponding to $\lambda \lrcorner w$. However, we see that $\lambda \lrcorner w \in \Lambda^2 W^*$ satisfies $(\lambda \lrcorner w)(w) = 0$ which shows that the intersection that we are interested in is given by an element in $\Lambda^2 (W/\mathbb{C}w)^*$. A generic such 2-form has rank 4 so the hyperplane section is singular if and only if $rank(\lambda \lrcorner w) \leq 2$.

$\square$

We now use that $GL(W)$ acts on $\Lambda^3(W)$ with an open orbit (for $\dim(W) = 6$). More precisely we have the following theorem:

Theorem 18. *[Don77],[SK77]*

In $\mathbb{P}(\Lambda^3 W^*)$ there exists subvarieties X_1, X_2, X_3 of dimensions 18, 14 and 9 respectively, such that the four orbits under the action of $\mathbb{P}GL(6)$ are $\mathbb{P}(\Lambda^3 W^*) \setminus X_1$, $X_1 \setminus X_2$, $X_2 \setminus X_3$ and X_3. X_1 is a degree 4 hypersurface in $\mathbb{P}(\Lambda^3 W^*)$ with singular locus X_2 which in turn has singular locus X_3, where X_3 is just $Gr(3,6)$.

Moreover, the four orbits are generated by $x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$, $x_1 \wedge x_2 \wedge x_6 + x_1 \wedge x_3 \wedge x_5 + x_2 \wedge x_3 \wedge x_4$, $x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$ and $x_1 \wedge x_2 \wedge x_3$ respectively, where $x_1, \ldots, x_6$ is a suitable chosen basis of W^*.

We now use this theorem in order to understand the fibers of the projection

$Z \to \mathbb{P}(\Lambda^3 W^*)$. Let λ be a general 3-form. By the above, there exists a basis of W^* such that $\lambda = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$. Now we can easily see that $\lambda \lrcorner w$ corresponds to a skew symmetric 6×6 block matrix $\begin{pmatrix} A & 0 \\ & \\ 0 & B \end{pmatrix}$. This tells us that $\lambda \lrcorner w$ will have rank smaller or equal to 2 if and only if one of the two 3×3 blocks A, or B, are 0. But this just means that the fiber of the map $Z \to \mathbb{P}(\Lambda^3 W^*)$ over λ is $\mathbb{P}^2 \sqcup \mathbb{P}^2 = \{x_0 = x_1 = x_2 = 0\} \sqcup \{x_3 = x_4 = x_5 = 0\}$.

Stein factorization then gives $Z \to M \to \mathbb{P}(\Lambda^3 W^*)$, where the first map is generically a $\mathbb{P}^2$-bundle and the second one is a double cover of $\mathbb{P}(\Lambda^3 W^*)$ ramified along the quartic hypersurface X_1. Now recall that the Homological Projective Dual variety that we were looking for should have given a $\mathbb{P}^2$-bundle structure on Z. Since $Z \to M$ is generically a $\mathbb{P}^2$ bundle it follows that M, appropriately resolved (by a noncommutative resolution), should give us the dual variety.

Now, the generic form of λ shows that the map

$$Y = \mathbb{P}_{Gr(3,W^*) \times Gr(3,W^*)}(\mathcal{O}(-1,0) \oplus (\mathcal{O}(0,-1))) \dashrightarrow \mathbb{P}(\Lambda^3 W^*)$$

is a rational map on the 2-fold covering M, which after blowing up the diagonally embedded $Gr(3, W^*)$ gives a regular map $p : \widetilde{Y} \to M$ and we expect the non-commutative resolution that we are looking for to come from this map, as we will explain later. We will also describe this map more explicitly, since we have to make calculations on $\widetilde{Y}$.

4.2 A Calculation

In this section we will give some more considerations regarding the construction of the homological projective dual variety. Recall from the previous section that we expect $\widetilde{M}$, the homological projective dual variety to $X = Gr(3,6)$ with respect to the Plucker embedding to be a noncommutative resolution of M. By that we mean that there exists a coherent sheaf $\mathcal{R}$ of $\mathcal{O}_M$-algebras that is a matrix algebra at the generic point of M and has finite homological dimension and $D^b(\widetilde{M}) = D^b(M, \mathcal{R})$.

Since $\widetilde{M}$ is expected to be the homological projective dual of X, $D^b(\widetilde{M})$ should come with a Lefschetz decomposition dual to the original Lefschetz decomposition of X that we constructed. If we denote by $\mathcal{B}_0 \subset D^b(\widetilde{M})$ the last piece of this dual Lefschetz collection, then we know [Kuz07] that $\mathcal{B}_0 \cong \mathcal{A}_0 \subset D^b(X)$ so $\mathcal{B}_0 = \langle F_0, F_1, F_2, F_3 \rangle$ is generated by four exceptional objects. Then $\mathcal{B}_0 \subset D^b(\widetilde{Y})$ and we get four exceptional objects on $D^b(\widetilde{Y})$ and still call them F_0, F_1, F_2 and F_3.

Moreover, to prove homological projective duality for X with respect to the Plucker embeddding, we also need to find an universal object $\mathcal{E} \in D^b(X \times \widetilde{Y})$ that will give the duality. What we know about this object is that $\mathcal{E} \in \mathcal{A}_0 \boxtimes \mathcal{B}_0$ and that $\Phi_{\mathcal{E}} : \mathcal{A}_0 \to \mathcal{B}_0$ gives an equivalence.

More precisely, we know ([Kuz07]) that the embedding $\mathcal{C} \to D^b(\mathcal{X})$ is given by a fully faithful functor $\Phi_{\mathcal{E}}$ that is also $\mathbb{P}(\Lambda^3 W^*)$-linear. The kernel $\mathcal{E}$ should be supported on the fiber product

$$\widetilde{Y} \times_{\mathbb{P}(\Lambda^3 W^*)} \mathcal{X} = I(X, \widetilde{Y}) := (X \times Y) \times_{\mathbb{P}(\Lambda^3 W) \times \mathbb{P}(\Lambda^3 W^*)} I,$$

where I is the incidence variety over $\mathbb{P}(\Lambda^3 W) \times \mathbb{P}(\Lambda^3 W^*)$. All these considerations give us a strategy to try and find the ranks of the bundles F_i that we expect to generate the dual Lefschetz collection.

Let $i : I(X, \widetilde{Y}) \to X \times \widetilde{Y}$ be the embedding. We expect that there is a resolution on $X \times \widetilde{Y}$ (or a spectral sequence)

$$\{\mathcal{O}_X \boxtimes F_0 \to \mathcal{U}^* \boxtimes F_1 \to \Lambda^2 \mathcal{U}^* \boxtimes F_2 \to \Sigma^{2,1} \mathcal{U}^* \boxtimes F_3\} \cong i_* \mathcal{E}.$$

Also, recall that we have the following decomposition on the derived category of a hyperplane section of $X = Gr(3, 6)$: $D^b(X_H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5) \rangle$.

We will now use Macaulay2 to do some calculations that will give us some more evidence regarding the construction of the kernel but also a precise result regarding the ranks of the F_i's that we want to give us $\mathcal{B}_0$ the biggest piece of the Lefschetz collection on the dual variety. We will actually use this calculation in the next section to construct what we expect to be the object that will give the homological projective duality and the bundles F_i's that will give the Lefschetz decomposition on $D^b(\widetilde{M})$. The strategy will be to find a class in $K_0(X_H)$ that is orthogonal to $(\langle \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5))$ and then express its pushforward in $K_0(X)$ as a linear combination of $(\mathcal{O}_X, \mathcal{U}^*, \Lambda^2 \mathcal{U}^*, \Sigma^{2,1} \mathcal{U}^*)$. The coefficients that we get would be the ranks of the F_i's.

More precisely, let $K_0(X)$ be the Grothendieck group of the category of coherent sheaves on X. We say that a class $a \in K_0(X)$ is numerically equivalent to 0, if it lies in the kernel of the Euler form on $K_0(X)$, $\chi : K_0(X) \otimes K_0(X) \to \mathbb{Z}$, $\chi([F],[G]) = \Sigma(-1)^i dim\, \mathsf{Ext}^i(F,G)$. We let $K_0(X)_{num} := K_0(X)/Ker(\chi)$. Recall that if $\alpha = (a_1 \geq a_2 \geq \cdots \geq a_d)$ is a partition with d parts and $a_1 \leq n-d$, then the Schubert variety in X associated to α is the subset

$$\Omega_\alpha = \{V \in \mathsf{Gr}(d, \mathbb{C}^n) \mid \dim(V \cap \mathbb{C}^{n-d+i-a_i}) \geq i \ \forall 1 \leq i \leq d\}\,. \tag{4.1}$$

Here $\mathbb{C}^k \subset \mathbb{C}^n$ denotes the subset of vectors whose last $n-k$ components are zero. The codimension of Ω_α is equal to $\sum a_i$. If we identify partitions with their Young diagrams, then a Schubert variety Ω_α is contained in Ω_β if and only α contains β. We also have the following result:

Proposition 19. *The classes of the structure sheaves of the Schubert varieties in X form a basis of $K_0(X)_{num}$.*

Using this we see that the Chern character map identifies $ch : K_0(X)_{num} \to H^\bullet(X, \mathbb{Q})$ with the lattice generated by the elements $ch(\mathcal{O}_{\Omega_\alpha})$'s. Then, by the Riemann-Roch, the Euler form can be expressed by $\chi(u,v) = \chi_0(u^* \cap v)$, where $u \to u^*$ is the involution of $H^\bullet(X,\mathbb{Q})$ given by $(-1)^k$ multiplication on $H^{2k}(X,\mathbb{Q})$ and $\chi_0(a) = (a \cdot td(X))_{top}$.

Note also that by [BMMS09] we have the following:

Lemma 20. *Assume we have a semiorthogonal decomposition $D^b(X) = \langle \mathcal{T}, E \rangle$, where E is an exceptional object in $D^b(X)$. Then*

$$K_0(\mathcal{T})_{num} \cong \{[M] \in K_0(X)_{num} \mid \chi([E],[M]) = 0\}.$$

Moreover, $K_0(X)_{num} \cong K_0(\mathcal{T})_{num} \oplus \mathbb{Z}[E]$.

Now, recall that $Gr(3,6)$ has only even cohomology and $h^i(X) = dim_{\mathbb{Q}} H^i(X,\mathbb{Q})$ can be computed by counting the Schubert varieties of the right codimension on X. By [Don77] we see that $\chi(X) = 20$ and $h^0 = h^{18} = h^2 = h^{16} = 1$, $h^4 = h^{14} = 2$ and $h^6 = h^{12} = h^8 = h^{10} = 3$. Now, for $i : X_H \to X$, the embedding of the hyperplane section into X, we know by the Lefschetz theorem on hyperplane sections ([GH94]), that the map $H^k(X,\mathbb{Q}) \to H^k(X_H,\mathbb{Q})$ induced by the inclusion i is an isomorphism for $k \leq 7$ and injective for $k = 8$. This tells us that the only new information on X_H, not coming from X is given by the cokernel of i^*, or by Poincare duality, by the kernel of i_* the map induced on homology (which can also be described via the Lefschetz hyperplane section). Moreover, in [Don77] the Euler characteristic of X_H is calculated. The result is that $\chi(X_H) = 18$ which implies that $h^8(X_H) = 4$. That is, we only have one extra class x in $H^*(X_H,\mathbb{Q})$ not coming from X, that is described explicitly in the above cited paper (actually its Poincare dual in $H_8(X_H,\mathbb{Q})$ is described).

Instead of using the description of the cohomology rings of X and X_H in terms of the Schubert varieties described, we will use the one that Macaulay2 uses in its package Schubert2. That is, we can describe the cohomology ring of X as being generated by the Chern classes of the tautological bundle $\mathcal{U}$ on X and we use this description to also calculate in the intersection ring for X_H. Note however, that we wanted to construct an object in $K_0(X_H)$ whose pushforward to $K_0(X)$ will be generated by the exceptional bundles in the first block of the Lefschetz collection for X. Therefore, we will be looking for an object orthogonal to $\langle ch(\mathcal{A}_1(1)), \ldots, ch(\mathcal{A}_5(5))\rangle$ on $H^*(X_H)$ that is generated by the restriction of the c_i's $H^*(X_H, \mathbb{Q})$. Now note that using the Lefschetz hyperplane theorem and Poincare duality, we get as extra relations for the intersection ring of X_H: $3c_1c_2c_3 = c_2^3 + 3c_3^2$ and $2c_1c_3^2 = c_2^2c_3$. Introducing all this data into Macaulay2, we can calculate $\langle ch(\mathcal{A}_1(1)), \ldots, ch(\mathcal{A}_5(5))\rangle$ in terms of the basis of the intersection ring of X_H, then calculate its orthogonal and then using Riemann-Roch (that tells us $ch(i_*\mathcal{F}) = i_*(\mathcal{F}) \cdot td(X/Y)$) we get the following element as the one which we want to describe in terms of the exceptional bundles in $\mathcal{A}_0$: $x = c_1 + \frac{3}{2}c_1^2 + (\frac{7}{6}c_1^3 - 3c_1c_2) + (-\frac{9}{8}c_1^2c_2 + \frac{11}{8}c_2^2 + \frac{5}{4}c_1c_3) + (\frac{1}{24}c_1c_2^2 + \frac{91}{120}c_1^3c_3 - \frac{21}{20}c_2c_3) + (\frac{1}{48}c_2^3 - \frac{1}{10}c_1c_2c_3 + \frac{1}{10}c_3^2) + (-\frac{1}{60}c_2^3c_3 + \frac{1}{60}c_1c_3^2) + \frac{1}{960}c_2c_3^2 - \frac{1}{4320}c_3^3$. Using Macaulay2 again, it is easy to note that the coefficients that we get are $x = 10ch(O_X) - 3ch(\mathcal{U}^*) - 3ch(\Lambda^2\mathcal{U}^*) + 1ch(\Sigma^{2,1}\mathcal{U}^*)$.

This suggests that we should get a resolution of an object supported on $I(X, \widetilde{Y})$ as follows:

$$\{\mathcal{O}_X \boxtimes F_0 \to \mathcal{U}^* \boxtimes F_1 \oplus \Lambda^2\mathcal{U}^* \boxtimes F_2 \to \Sigma^{2,1}\mathcal{U}^* \boxtimes F_3\} \cong i_*\mathcal{E}.$$

Indeed, we will construct in the next chapter bundles F_0, F_1, F_2, F_3 on $\widetilde{Y}$ of ranks 10, 3, 3 and 1, respectively, and a resolution of an object supported on $I(X, \widetilde{Y})$ as we need.

4.3 More geometry

We now come back to the construction of a resolution of M. From now on, for making the notation smoother we will write : $G = Gr(3, W^*)$, $\mathbb{P} = \mathbb{P}(\Lambda^3 W)$ and $\mathbb{P}^* = \mathbb{P}(\Lambda^3 W^*)$. On $G \times G$ we will denote by $\mathcal{A}$ and $\mathcal{B}$ the pullbacks of the tautological bundles from the first and second factor. We will also write a and b the positive generators of the Picard groups of the two G's. More precisely we have $\Lambda^3 \mathcal{A} = \mathcal{O}(-a)$ and $\Lambda^3 \mathcal{B} = \mathcal{O}(-b)$.

As before, we let $Y = \mathbb{P}_{G \times G}(\mathcal{O}(-a, 0) \oplus \mathcal{O}(0, -b))$ and $\pi : Y \to G \times G$. We let s be class of the relative $\mathcal{O}(1)$ of Y over $G \times G$ and note that we have $\pi_* \mathcal{O}(s) = \mathcal{O}(a, 0) \oplus \mathcal{O}(0, b)$. Consider now $\Delta : G \to G \times G$ to be the diagonal map. We have that $\Delta^* \mathcal{A} \cong \Delta^* \mathcal{B} = \mathcal{V}$, where $\mathcal{V}$ is the tautological bundle on G.

We now want to define a map $\iota : G \to Y$. We first note that

$$\Delta^*(\mathcal{O}(-a, 0) \oplus \mathcal{O}(0, -b)) = \mathcal{O}(-v) \oplus \mathcal{O}(-v).$$

Then the diagonal embedding

$$\mathcal{O}(-v) \to \mathcal{O}(-v) \oplus \mathcal{O}(-v) = \Delta^*(\mathcal{O}(-a, 0) \oplus \mathcal{O}(0, -b))$$

induces the map that we wanted

$$\iota : G \cong \mathbb{P}_G(\mathcal{O}(-v)) \to Y = \mathbb{P}_{G\times G}(\mathcal{O}(-a,0) \oplus \mathcal{O}(0,-b))$$

such that $\iota^*\mathcal{O}(s) = \mathcal{O}(v)$.

Let $\widetilde{Y}$ be the blowup of Y in $\iota(G)$. Let E be the exceptional divisor in $\widetilde{Y}$ and let e denote its class. Denote by $p : E \cong \mathbb{P}_G(N_{\iota(G)/Y}) \to G$ the projection. Note that $p_*\mathcal{O}_E(-e) \cong N^*_{\iota(G)/Y}$, while $p_*\mathcal{O}_E(e) \cong 0$ ([Ful98]).

The normal bundle $N_{\iota(G)/Y}$ fits into exact sequence ([Ful98])

$$0 \to \iota^* T_{Y/G\times G} \to N_{\iota(G)/Y} \to N_{G/G\times G} \to 0.$$

But

$$\iota^* T_{Y/G\times G} = \iota^*((\mathcal{O}(-a,0) \oplus \mathcal{O}(0,-b)) \otimes \mathcal{O}(s)/\mathcal{O}) = \mathcal{O}(-v)^{\oplus 2} \otimes \mathcal{O}(v)/\mathcal{O} = \mathcal{O},$$

while $N_{G/G\times G} \cong N_{\Delta(G)/G\times G} \cong T_G$. This shows that we have the following short exact sequence

$$0 \to \mathcal{O} \to N_{\iota(G)/Y} \to T_G \to 0$$

Note that the linear system $|h| = |s - e|$ gives a morphism $g : \widetilde{Y} \to \mathbb{P}(\Lambda^3 W^*)$. Indeed, it is clear that this linear system is base point free. Moreover we can calculate $H^\bullet(\widetilde{Y}, \mathcal{O}(s-e)$ as follows. First note that we have a short exact sequence on $\widetilde{Y}$:

$$0 \to \mathcal{O}(-e) \to \mathcal{O} \to \mathcal{O}_{|E} \to 0$$

which after tensoring with $\mathcal{O}(s)$ gives us

$$0 \to \mathcal{O}(s-e) \to \mathcal{O}(s) \to \mathcal{O}(s)_{|E} \to 0.$$

Since $\pi_*\mathcal{O}(s) = \mathcal{O}(a,0) \oplus \mathcal{O}(0,b)$ we can compute

$H^\bullet(\widetilde{Y}, \mathcal{O}(s)) = H^\bullet(G \times G, \mathcal{O}(a,0) \oplus \mathcal{O}(0,b)) = \Lambda^3 W \oplus \Lambda^3 W$. Similarly,

$H^\bullet(\widetilde{Y}, \mathcal{O}s_{|E}) = H^\bullet(G, \mathcal{O}(v)) = \Lambda^3 W$. This implies now that

$H^0(\widetilde{Y}, \mathcal{O}(s-e)) = \Lambda^3 W$.

We would like to also describe the fibers of the map $g : \widetilde{Y} \to \mathbb{P}^*$.

First note that $T_G \cong \mathcal{V}^* \otimes W^*/\mathcal{V}$. Tensoring $0 \to \mathcal{O} \to N_{\iota(G)/Y} \to T_G \to 0$ by $\Lambda^3\mathcal{V}$ (and observing that $\mathcal{V}^* \otimes W^*/\mathcal{V} \otimes \Lambda^3\mathcal{V} = \Lambda^2\mathcal{V} \otimes W^*/\mathcal{V}$ as in chapter 2) we get an exact sequence

$$0 \to \Lambda^3\mathcal{V} \to N_{\iota(G)/Y} \otimes \Lambda^3\mathcal{V} \to \Lambda^2\mathcal{V} \otimes W^*/\mathcal{V} \to 0.$$

On the other hand, the exact sequence $0 \to \mathcal{V} \to W^* \otimes \mathcal{O} \to W^*/\mathcal{V} \to 0$ on G induces a filtration on $\Lambda^3 W^*$ with factors being $\Lambda^3\mathcal{V}$, $\Lambda^2\mathcal{V} \otimes (W^*/\mathcal{V})$, $\mathcal{V} \otimes \Lambda^2(W^*/\mathcal{V})$, and $\Lambda^3(W^*/\mathcal{V})$. Since $\Lambda^3(\mathcal{V}) = \mathcal{O}(v)$ we see that, up to a twisting, $N_{\iota(G)/Y}$ is just the first half of this filtration. More precisely, it is the image of $W^* \otimes \Lambda^2\mathcal{V}$ in $\Lambda^3 W^*$, so we can describe E as:

$$E = \{(V, \lambda) \in G \times \mathbb{P}^* \mid \lambda \in W^* \otimes \Lambda^2 V\}.$$

At the same time, by the construction we just described we see that

$$\widetilde{Y} \setminus E = Y \setminus \iota(G) = \{(A, B, \lambda) \in G \times G \times \mathbb{P}^* \mid \lambda \in \Lambda^3 A + \Lambda^3 B\}.$$

Now recall that the four orbits of the action of $GL(6)$ on $\mathbb{P}^*$ are generated by $x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$, $x_1 \wedge x_2 \wedge x_6 + x_1 \wedge x_3 \wedge x_5 + x_2 \wedge x_3 \wedge x_4$, $x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$ and $x_1 \wedge x_2 \wedge x_3$ respectively, where $x_1, \dots, x_6$ is a suitable chosen basis of W^*.

For $\lambda = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$ we get that $g^{-1}(\lambda) \cap E = \emptyset$, $g^{-1}(\lambda) \cap (\widetilde{Y} \setminus E) = pt \sqcup pt$.

For $\lambda = x_1 \wedge x_2 \wedge x_6 + x_1 \wedge x_3 \wedge x_5 + x_2 \wedge x_3 \wedge x_4$ we see that $g^{-1}(\lambda) \cap E = pt$, where the space V is just $\langle x_1, x_2, x_3 \rangle$. At the same time $g^{-1}(\lambda) \cap (\widetilde{Y} \setminus E) = \emptyset$.

For $\lambda = x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$, to describe $g^{-1}(\lambda) \cap E$ note that V should contain x_1, should be contained in $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ and the quotient $V/\langle x_1 \rangle$ should be isotropic for the form $x_2 \wedge x_3 + x_4 \wedge x_5$. Therefore

$$g^{-1}(\lambda) \cap E = LGr(2,4) \subset \mathsf{Gr}(2,4).$$

To understand $g^{-1}(\lambda) \cap (\widetilde{Y} \setminus E)$ we need to find a pair of subspaces $A, B \subset W^*$. It is clear that both should contain x_1 and be contained in $\langle x_1, x_2, x_3, x_4, x_5 \rangle$. It also follows that $A/\langle x_1 \rangle$ should be nonisotropic for the form and the space B is uniquely determined by A. This is true as we need to consider the unique linear combination of $x_2 \wedge x_3 + x_4 \wedge x_5$ and $\Lambda^2(A/\langle x_1 \rangle)$ which is of rank 2. Therefore

$$g^{-1}(\lambda) \cap (\widetilde{Y} \setminus E) = \mathsf{Gr}(2,4) \setminus LGr(2,4).$$

For $\lambda = x_1 \wedge x_2 \wedge x_3$, to describe $g^{-1}(\lambda) \cap E$ we see that V should have intersection of dimension ≥ 2 with $\langle x_1, x_2, x_3 \rangle$. Thus $g^{-1}(\lambda) \cap E$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$ with a section contracted to a point. On the other hand, if we want to describe $g^{-1}(\lambda) \cap (\widetilde{Y} \setminus$

E) then we again need to find a pair of subspaces $A, B \subset W^*$ as above. Clearly, one of them should coincide with $\langle x_1, x_2, x_3 \rangle$ and the other should be arbitrary. Moreover, in the case when both A and B coincide with $\langle x_1, x_2, x_3 \rangle$, then we are free to choose a point in $\mathbb{P}^1 \setminus pt$ (the fiber of Y over $G \times G$). Thus

$$g^{-1}(\lambda) \cap (\widetilde{Y} \setminus E) = (\mathsf{Gr}(3,6) \setminus pt) \sqcup (\mathsf{Gr}(3,6) \setminus pt) \sqcup (\mathbb{P}^1 \setminus pt).$$

As expected (since the rank of the Picard of $\widetilde{Y}$ is 4 while the rank of the Picard of $\mathbb{P}^*$ is 1), we get that we have three divisors contracted by g. We can take D_1 to be $G \times G \cong \mathbb{P}_{G \times G}(\mathcal{O}(-a)) \subset Y$ which contracts onto $G \subset \mathbb{P}^*$ via the projection onto the first factor. Then D_2 will be $G \times G \cong \mathbb{P}_{G \times G}(\mathcal{O}(-b)) \subset Y$ that gets contracted onto $G \subset \mathbb{P}^*$ via the projection onto the second factor.

The third divisor D_3 will contract onto $X_2 \subset \mathbb{P}^*$. We don't have a good geometric description of this one, but we have a birational model for this divisor. Namely we can take the partial flag variety $F(1,3,5;W^*)$ and consider the vector bundle $\Lambda^2(\mathcal{U}_5/\mathcal{U}_1)$ on it. Then the model for D_3 will be $\mathbb{P}_{F(1,3,5;W^*)}(\Lambda^2(\mathcal{U}_5/\mathcal{U}_1))$ and it gets contracted onto $\mathbb{P}_{F(1,5;W^*)}(\Lambda^2(\mathcal{U}_5/\mathcal{U}_1))$.

We would like now to be in the context of [Kuz08c] as that would allow us to construct a noncommutative resolution of singularities for M. However, in this situation, for the resolution $g_M : \widetilde{Y} \to M$ the exceptional divisior is not irreducible. Moreover, we would like to get a Lefschetz decomposition on $D^b(\widetilde{Z})$, where $\widetilde{Z}$ is the exceptional divisor, which clearly doesn't seem possible at this point since we don't have a good geometric description of the singular D_3. Therefore, at this point we can only conjecture the following:

Conjecture 21. *There exists a sheaf of algebras $\mathcal{R}$ over the double cover M of $\mathbb{P}(\Lambda^3 W^*)$ ramified along the quartic hypersurface X_1, such that noncommutative resolution of singularities $(M, \mathcal{R})$ of M is Homologically Projective Dual to the Grassmannian $X = Gr(3, W)$.*

There are two directions in which the program to construct the homological projective dual of $Gr(3, 6)$ could go. The first one would be to just remove the $G \subset \mathbb{P}^*$. If we do that we should obtain a model for the homological projective dual variety over $\mathbb{P}^* \setminus G$. Such an approach would allow to prove a partial result that will give a description of derived categories of linear sections of $Gr(3, W)$: $Gr(3, W) \cap \mathbb{P}(A^\perp)$ for all $A \subset \mathbb{P}^*$ such that $\mathbb{P}(A) \cap G = \emptyset$ (in particular, this approach would still make the main applications to linear sections hold; however it would still not be completely satisfactory, since we would like to have the complete picture of the homological projective dual). Yet, this approach allows us to identify the sheaf of algebras $\mathcal{R}$ over $M \setminus G'$, where G' is the preimage of G. Over the smooth points of M it will just be a matrix algebra, while over the singular points of M' we can understand it by looking at D_3 and it's fibers over the singular locus to which it contracts to.

The second approach would be to apply a birational transformation to $\widetilde{Y}$ and try to be in the situation of [Kuz08c]. One way to do this is to consider $G_1 \subset G \times G$ be the locus

$$G_1 = \{(A, B) \in G \times G \; | dim(A \cap B) \geq 1\}.$$

Then $\pi^{-1}(G_1) = D_3 \cap E$. Then, we could try and define a rational map $G \times G \dashrightarrow \mathbb{P}(U)$ which is not defined at $G_2 = \{(A, B) \mid \dim(A \cap B) \geq 2\}$ whose preimage to $\widetilde{Y}$ gives the singular locus of D_3 and try to use that to construct a birational transformation that will put us in the desired context.

Chapter 5

Towards homological projective duality for Gr(3,6)

In this chapter, we first construct the object supported on $I(X, \widetilde{Y})$ that we expect to give the equivalence between the derived category of the homological projective dual variety and $\mathcal{C}$. This construction will also give us the exceptional bundles $F_i's$ that we expect to form a Lefschetz collection on the dual variety. We then present a plan for the proof of the homological projective duality, assuming we can construct the sheaf of algebras on M and the potential applications to linear sections of X.

5.1 Construction of the kernel and the dual exceptional bundles

Let $X = \mathsf{Gr}(3, W)$, let $\mathcal{U}$ be the tautological bundle on X as before and denote by u the positive generator of $\mathsf{Pic}\, X$. Let F_0 be the unique nonsplit extension $0 \to \mathcal{A} \otimes \Lambda^2\mathcal{B} \to F_0 \to \mathcal{O}_{\widetilde{Y}}(-h) \to 0$, $F_1 = \Lambda^2\mathcal{B}$, $F_2 = \mathcal{A}$ and $F_3 = \mathcal{O}_{\widetilde{Y}}$.

We will prove the following:

Proposition 22. *The complex of vector bundles*

$$\{E_0 \boxtimes F_0 \to E_1 \boxtimes F_1 \oplus E_2 \boxtimes F_2 \to E_3 \boxtimes F_3\}$$

is quasiisomorphic on $X \times \tilde{Y}$ to $i_*\mathcal{E}$, where $\mathcal{E} \in D^b(I(X, \widetilde{Y}))$.

Proof. On $X \times \widetilde{Y}$ we have two maps $\mathcal{O}_X \boxtimes \mathcal{A} \to \mathcal{U}^* \boxtimes \mathcal{O}_{\widetilde{Y}}$ and $\mathcal{O}_X \boxtimes \Lambda^2\mathcal{B} \to \Lambda^2\mathcal{U}^* \boxtimes \mathcal{O}_{\widetilde{Y}}$, coming from the composition of the following maps

$$\mathcal{O}_X \boxtimes \mathcal{A} \to W^* \otimes \mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}} \to \mathcal{U}^* \boxtimes \mathcal{O}_{\widetilde{Y}},$$

and

$$\mathcal{O}_X \boxtimes \Lambda^2\mathcal{B} \to \Lambda^2 W^* \otimes \mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}} \to \Lambda^2\mathcal{U}^* \boxtimes \mathcal{O}_{\widetilde{Y}}.$$

Both of them are injective (since they are isomorphic at the generic point and $\mathcal{O}_X \boxtimes \mathcal{A}$ and $\mathcal{O}_X \boxtimes \Lambda^2\mathcal{B}$ are both torsion free). Denote their cokernels by $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Then the tensor product of the two maps

$$\mathcal{O}_X \boxtimes (\mathcal{A} \otimes \Lambda^2\mathcal{B}) \to \mathcal{U}^* \boxtimes \Lambda^2\mathcal{B} \oplus \Lambda^2\mathcal{U}^* \boxtimes \mathcal{A} \to (\mathcal{U}^* \otimes \Lambda^2\mathcal{U}^*) \boxtimes \mathcal{O}_{\widetilde{Y}}$$

is a resolution for $\mathcal{F}_1 \otimes \mathcal{F}_2$.

We now observe that the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ is supported on $I(X, \widetilde{Y})$. We see that $\mathsf{supp}(\mathcal{F}_1)$ is the set of all (U, A, B, λ) such that $\Lambda^3 A_{|U} \equiv 0$ and $\mathsf{supp}(\mathcal{F}_2)$ is the set of all (U, A, B, λ) such that $\Lambda^3 B_{|U} \equiv 0$. Hence for any point in $\mathsf{supp}(\mathcal{F}_1 \otimes \mathcal{F}_2) = \mathsf{supp}(\mathcal{F}_1) \cap \mathsf{supp}(\mathcal{F}_2)$ we have $\lambda_{|U} \equiv 0$ as $\lambda \in \langle \Lambda^3 A, \Lambda^3 B \rangle$. But $I(X, \widetilde{Y})$ is the preimage of all (λ, U) such that $\lambda_{|U} \equiv 0$ and this proves our claim (actually it is supported scheme theoretically on $I(X, \widetilde{Y})$ by seeing it is annihilated by the ideal sheaf of our incidence variety).

Now notice that $\mathcal{U}^* \otimes \Lambda^2 \mathcal{U}^* = \Sigma^{2,1} \mathcal{U}^* \oplus \mathcal{O}_X(u)$. Thus we have a map

$\mathcal{O}_X(u) \boxtimes \mathcal{O}_{\widetilde{Y}} \to (\mathcal{U}^* \otimes \Lambda^2 \mathcal{U}^*) \boxtimes \mathcal{O}_{\widetilde{Y}}$ that induces a map $\mathcal{O}_X(u) \boxtimes \mathcal{O}_{\widetilde{Y}} \to \mathcal{F}_1 \otimes \mathcal{F}_2$. Since $\mathcal{F}_1 \otimes \mathcal{F}_2$ is supported on $I(X, \widetilde{Y})$, the composition of this map with the canonical morphism $\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h) \to \mathcal{O}_X(u) \boxtimes \mathcal{O}_{\widetilde{Y}}$ is zero. Note that this last morphism comes from the resolution (twisted by $\mathcal{O}_X(u)$) of $\mathcal{O}_{I(X,\widetilde{Y})}$:

$0 \to \mathcal{O}_X(-u) \boxtimes \mathcal{O}_{\widetilde{Y}}(-h) \to \mathcal{O}_{X \times \widetilde{Y}} \to i_* \mathcal{O}_{I(X,\widetilde{Y})}$, where $i : I(X, \widetilde{Y}) \to X \times \widetilde{Y}$ is the natural embedding.

We now apply the functor $\mathsf{Hom}(\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h), -)$ to the complex constructed above. Let us first compute

$$\mathsf{Ext}^\bullet(\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{U}^* \boxtimes \Lambda^2 \mathcal{B} \oplus \Lambda^2 \mathcal{U}^* \boxtimes \mathcal{A}) = H^\bullet(X \times \widetilde{Y}, \mathcal{U}^* \boxtimes \Lambda^2 \mathcal{B}(h) \oplus \Lambda^2 \mathcal{U}^* \boxtimes \mathcal{A}(h))$$

Let's compute $H^\bullet(Y, \mathcal{A}(h)) = H^\bullet(Y, \mathcal{A}(s-e))$. We have the short exact sequence

$$0 \to \mathcal{A}(s-e) \to \mathcal{A}(s) \to \mathcal{A}(s)_{|E} \to 0.$$

We see that $H^\bullet(\widetilde{Y}, \mathcal{A}(s)) = H^\bullet(G \times G, \mathcal{A}(a) \oplus \mathcal{A} \boxtimes \mathcal{O}(b)) = \Lambda^2 W$. Then $H^\bullet(\widetilde{Y}, \mathcal{A}(s)_{|E}) = H^\bullet(G, \mathcal{V}(v)) = \Lambda^2 W$. This gives us that $H^\bullet(\widetilde{Y}, \mathcal{A}(s-e)) = 0$. Similarly, we can compute $H^\bullet(\widetilde{Y}, \mathcal{B}(s-e)) = 0$. We thus get that

$$\mathsf{Ext}^\bullet(\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{U}^* \boxtimes \Lambda^2\mathcal{B} \oplus \Lambda^2\mathcal{U}^* \boxtimes \mathcal{A}) = 0.$$

The morphism $\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h) \to (\mathcal{U}^* \otimes \Lambda^2\mathcal{U}^*) \boxtimes \mathcal{O}_{\widetilde{Y}}$ cancels in the spectral sequence that we obtain and since the above Ext's are 0, we see that the cancelation can only come from the term $\mathsf{Ext}^1(\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{O}_X \boxtimes (\mathcal{A} \otimes \Lambda^2\mathcal{B})) = \mathbb{C}$. Let us actually check that this Ext^1 is indeed $\mathbb{C}$. For that we need to calculate $H^\bullet(\widetilde{Y}, \mathcal{A} \otimes \Lambda^2\mathcal{B} \otimes \mathcal{O}_{\widetilde{Y}}(s-e))$. As before, we use the short exact sequence

$$0 \to \mathcal{A} \otimes \Lambda^2\mathcal{B}(s-e) \to \mathcal{A} \otimes \Lambda^2\mathcal{B}(s) \to \mathcal{A} \otimes \Lambda^2\mathcal{B}(s)_{|E} \to 0.$$

We calculate now $H^\bullet(\widetilde{Y}, \mathcal{A} \otimes \Lambda^2\mathcal{B}(s)) = H^\bullet(G \times G, (\mathcal{A} \otimes \Lambda^2\mathcal{B}) \otimes (\mathcal{O}(a,0) \oplus \mathcal{O}(0,b))) = H^\bullet(G \times G, \mathcal{A}(a) \boxtimes \Lambda^2\mathcal{B} \oplus \mathcal{A} \boxtimes \Lambda^2\mathcal{B}(b) = 0$. This already gives us that

$H^0(\widetilde{Y}, (\mathcal{A} \otimes \Lambda^2\mathcal{B})(s-e)) = 0$. Now

$$H^\bullet(\widetilde{Y}, \mathcal{A} \otimes \Lambda^2\mathcal{B}(s)_{|E}) = H^\bullet(G, \mathcal{V} \otimes \Lambda^2\mathcal{V}(v)) = H^\bullet(G, (\Sigma^{2,1}\mathcal{V})(v) \oplus \mathcal{O}) = \mathbb{C}.$$

Indeed $\mathsf{Ext}^1(\mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{O}_X \boxtimes (\mathcal{A} \otimes \Lambda^2\mathcal{B})) = \mathbb{C}$.

We can conclude that we have a morphism of complexes

$$\begin{tikzcd}
& \mathcal{O}_X \boxtimes \mathcal{O}_{\widetilde{Y}}(-h) \arrow[dl, "{[1]}"'] & \\
\mathcal{O}_X \boxtimes (\mathcal{A} \otimes \Lambda^2\mathcal{B}) \arrow[r] & \mathcal{U}^* \boxtimes \Lambda^2\mathcal{B} \oplus \Lambda^2\mathcal{U}^* \boxtimes \mathcal{A} \arrow[r] & \Sigma^{2,1}\mathcal{U}^* \boxtimes \mathcal{O}_{\widetilde{Y}}
\end{tikzcd}$$

But this implies that we get the complex

$$\mathcal{O}_X \boxtimes F_0 \to \mathcal{U}^* \boxtimes F_1 \oplus \Lambda^2\mathcal{U}^* \boxtimes F_2 \to \Sigma^{2,1}\mathcal{U}^* \boxtimes F_3,$$

where F_0, F_1, F_2 and F_3 are as above, since we can think of the morphism of complexes above as a twisted complex as in [GM03] and then the complex that we wanted is just the convolution of this. Notice that this also fits perfectly with the ranks that we have obtained in the previous chapter.

The last thing we need to check is that this complex is quasiisomorphic to $i_*\mathcal{E}$ for some $\mathcal{E} \in \mathcal{D}^b(I(X, \widetilde{Y}))$. However the complex that we have is quasiisomorphic to the cone of the morphism $\mathcal{O}_{I(X,\widetilde{Y})}(h) \to \mathcal{F}_1 \otimes \mathcal{F}_2$ and both terms are supported on $I(X, \widetilde{Y})$. □

We now have the following rather straightforward lemma:

Lemma 23. *The bundles F_0 , F_1 , F_2 and F_3 are exceptional on $\widetilde{Y}$. Moreover the collection (F_0, F_1, F_2, F_3) is exceptional too.*

Proof. Indeed $\mathcal{O}_{\widetilde{Y}}$, $\mathcal{A}$ and $\Lambda^2\mathcal{B}$ are exceptional since they are exceptional on $G \times G$ and the pullback functor $\pi^* : D^b(G \times G) \to D^b(\widetilde{Y})$ is fully faithful. Moreover since $H^0(\widetilde{Y}, (\mathcal{A} \times \Lambda^2\mathcal{B})(s-e) = 0$ and $H^0(\widetilde{Y}, (\mathcal{A} \times \Lambda^2\mathcal{B})(s-e) = \mathbb{C}$ we see that F_0 is actually a mutation so it is exceptional ([Rud90]). The second part of the lemma is also clear since we can easily check that $\mathsf{Ext}^\bullet(F_i, F_j) = 0$ for $i > j$. □

This collection is expected to generate the dual Lefschetz collection on the homological projective dual of $X = Gr(3, 6)$. By [Kuz07] the dual Lefschetz collection is supposed to complete a grid which has $\mathrm{rank}(K_0(\mathcal{A}_0))$ rows and $\mathrm{rank}(K_0(\mathbb{P}(\Lambda^3 W))$ columns. Since the partition corresponding to our Lefschetz collection on X was $(4,4,3,3,3,3)$, the partition giving the dual Lefschetz collection on the homological projective dual variety will have partition $(4,\ldots,4,1,1,1,1)$, where we have 15 many 4's. Therefore we could define $\mathcal{B}_i = \langle F_0, F_1, F_2, F_3\rangle$, for $0 \leq i \leq 13$, $\mathcal{B}_j = \langle F_3\rangle$, for $14 \leq j \leq 17$ and $\mathcal{C} = \langle \mathcal{B}_{17}(-17), ..., \mathcal{B}_1(-1), \mathcal{B}_0\rangle$. This should be the $\widetilde{\mathcal{D}}$ (as in [Kuz08c]) and the checks that the collection is exceptional are calculations using the same reasoning as above (using the short exact sequence given by the exceptional divisor $E \subset \widetilde{Y}$, twisting it suitably and applying Borel-Weil-Bott on $G \times G$ and G.

5.2 t-structure argument

In this section we will give the plan of how the main conjecture can be proven, assuming we have been able to construct a sheaf of algebras $\mathcal{R}$ on M. This plan follows very closely the proofs in [Kuz06a] and [Kuz06b]. Actually, in the second paper, the author used the standard t-structure on $D^b(Y \times Y)$ (where Y was the homological projective dual) but that worked since there was no noncommutative resolution involved. So actually, the proof of the duality for $Gr(3,6)$ should follow the proof for $Gr(2,n)$, with $n = 6, 7$ where we had to noncommutatively resolve a variety in order to get the homological projective dual.

Let us recall that we have constructed $\mathcal{E} \in I(X, \widetilde{Y})$ and we would like to prove that this gives a fully faithful functor $D^b(M, \mathcal{R}) = \widetilde{\mathcal{D}} \subset D^b(\widetilde{Y}) \to \mathcal{D}^b(\mathcal{X})$. If we let $j : I(X, \widetilde{Y}) \cong \mathcal{X} \times_{\mathbb{P}^*} \widetilde{Y} \to \mathcal{X} \times \widetilde{Y}$ the hardest part of the proof is to show that the functor $\Phi_{j_*\mathcal{E}} : D^b(\widetilde{Y}) \to D^b(\mathcal{X})$ induces a fully faithful embedding of $\widetilde{\mathcal{D}}$ into $D^b(\mathcal{X})$. The way to do this is to compute $\phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}} : D^b(\widetilde{Y}) \to D^b(\widetilde{Y})$. If we let $\alpha : \mathcal{X} \to X \times \mathbb{P}^*$ be the embedding, we see that $\alpha_* \circ \Phi_{j_*\mathcal{E}} : D^b(\widetilde{Y}) \to D^b(X \times \mathbb{P}^*)$ is given by the kernel $i_*\mathcal{E}$. Now from the embedding α we get a distinguished triangle of functors from $D^b(\mathcal{X}) \to D^b(\mathcal{X})$

$$\alpha^*\alpha_* \to id \to \mathcal{O}_{\mathcal{X}}(-u-h)[2].$$

If we compose this triangle with $\Phi_{j_*\mathcal{E}}$ on the right and with $\Phi^*_{j_*\mathcal{E}}$ on the left, we get the triangle of functors from $D^b(\widetilde{Y}) \to D^b(\widetilde{Y})$

$$(\Phi^*_{j_*\mathcal{E}}\alpha^*) \circ (\alpha_*\Phi_{j_*\mathcal{E}}) \to \Phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}} \to \Phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}(-u-h)}[2],$$

which after twisting by $\mathcal{O}(tu + th)$ gives

$$(\Phi^*_{j_*\mathcal{E}}\alpha^*) \circ (\alpha_*\Phi_{j_*\mathcal{E}(tu+th)}) \to \Phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}(tu+th)} \to \Phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}((t-1)u+(t-1)h)}[2].$$

Now, in [Kuz06a] the author computes the first term of these triangles using a resolution similar to the one we described in the previous section. We will also compute the terms for our situation as this can be done without knowing the sheaf $\mathcal{R}$. Then he finds an estimate (uniform in t) of the cohomology support intervals of the kernels of the functors in the middle of the above triangle. This calculation leads to the fact that the kernel of the functor $\phi^*_{j_*\mathcal{E}} \circ \Phi_{j_*\mathcal{E}}$ is a pure object. Another calculations allows him to identify this cohomology with the object in $D^b(\widetilde{Y} \times \widetilde{Y})$ giving the projection functor to $\widetilde{\mathcal{D}}$ and this shows that $\Phi_{j_*\mathcal{E}}$ is the composition of the projection $D^b(\widetilde{Y}) \to \widetilde{\mathcal{D}}$ and a fully faithful embedding $\widetilde{\mathcal{D}} \to D^b(\mathcal{X})$. All these calculations are being made using the t-structure $(\widetilde{\mathcal{D}}_\natural^{\leq 0}, \widetilde{\mathcal{D}}_\natural^{\geq 0})$ in the category $\widetilde{\mathcal{D}}_\natural \subset \mathcal{D}^b(\widetilde{Y} \times \widetilde{Y})$ constructed as follows:

Let us first assume that our equivalence $\widetilde{\mathcal{D}} \to \mathcal{D}^b(M, \mathcal{R})$ was given by

$$\rho_* : \mathcal{D}^b(\widetilde{Y}) \to \mathcal{D}^b(M, \mathcal{R}), \qquad F \mapsto g_{Y*}(F \otimes B),$$

where B will be just the direct sum of the exceptional objects that will appear in the first block $\mathcal{C}_0$ of $D^b(\widetilde{Z})$ ([Kuz08c]). We consider

$$\begin{aligned} \widetilde{\mathcal{D}}_{\mathsf{opp}} &:= \widetilde{\mathcal{D}}^* &&= \{F \in \mathcal{D}^b(\widetilde{Y}) \mid F^* \in \widetilde{\mathcal{D}}\} = \{F \in \mathcal{D}^b(\widetilde{Y}) \mid i_{\widetilde{Y}}^* F \in (\mathcal{C}_0)^*\} \\ \widetilde{\mathcal{D}}_\natural &:= \widetilde{\mathcal{D}} \boxtimes \widetilde{\mathcal{D}}_{\mathsf{opp}} \subset \mathcal{D}^b(\widetilde{Y} \times \widetilde{Y}). \end{aligned},$$

and we get equivalences

$$\widetilde{\mathcal{D}}_{\mathsf{opp}} \cong \mathcal{D}^b(M, \mathcal{R}^{\mathsf{opp}}), \qquad \widetilde{\mathcal{D}}_\natural \cong \mathcal{D}^b(M \times M, \mathcal{R} \boxtimes \mathcal{R}^{\mathsf{opp}}).$$

The equivalences above provide the triangulated category $\widetilde{\mathcal{D}}$ with a t-structure $(\widetilde{\mathcal{D}}^{\leq 0}, \widetilde{\mathcal{D}}^{\geq 0})$ as follows:

$$\begin{aligned} \widetilde{\mathcal{D}}^{\leq 0} &= \{F \in \widetilde{\mathcal{D}} \mid g_{Y*}(F \otimes B) \in \mathcal{D}^{\leq 0}(M)\}, \\ \widetilde{\mathcal{D}}^{\geq 0} &= \{F \in \widetilde{\mathcal{D}} \mid g_{Y*}(F \otimes B) \in \mathcal{D}^{\geq 0}(M)\}, \end{aligned}.$$

For $\widetilde{\mathcal{D}}_{\mathsf{opp}}$ and $\widetilde{\mathcal{D}}_\natural$, the t-structures $(\widetilde{\mathcal{D}}_{\mathsf{opp}}^{\leq 0}, \widetilde{\mathcal{D}}_{\mathsf{opp}}^{\geq 0})$ and $(\widetilde{\mathcal{D}}_\natural^{\leq 0}, \widetilde{\mathcal{D}}_\natural^{\geq 0})$ are defined analogously and these are the ones that should be used in the calculations mentioned above.

Once fully faithfullness of $\Phi_{j_*\mathcal{E}}$ is proved, we should use theorem 2.10 in [Kuz06a] which says that to conclude that the above functor gives homological projective duality, it is enough to check that the functor $\Phi^*_{j_*\mathcal{E}} \circ \pi^* : D^b(X) \to D^b(\widetilde{Y})$ is fully faithful on $\mathcal{A}_0$ that we constructed in chapter 2 and that its image is $\mathcal{B}_0$ (constructed in the previous section). This can be done using the resolution that we constructed in the previous section, but to complete it we still need to have the complete description of the HPD.

From now on, we will forget about u and h in our notation, but it will be clear what shifts are we using in what follows. If $j : I(X, \widetilde{Y}) \to \mathcal{X} \times \widetilde{Y}$. consider the following objects on $\mathcal{X} \times \widetilde{Y}$:

$$\mathcal{E}_1 = j_*\mathcal{E}, \qquad \mathcal{E}^{\#t} = \mathcal{E}^* \otimes \mathcal{O}(t-5, 1), \qquad \mathcal{E}_1^{\#t} = j_*\mathcal{E}^{\#t} \otimes \mathcal{O}_{\mathbb{P}^*}(t)[8].$$

We now have the following useful result:

Lemma 24. *The functor $\Phi_{\mathcal{E}_1^{\#0}}$ is left adjoint to $\Phi_{\mathcal{E}_1}$.*

Proof. Recall first [Kuz06a] that a kernel functor $\Phi_K : D^b(X) \to D^b(Y)$ admits a left adjoint functor Φ^*_K which is isomorphic to a kernel functor given by the kernel $K^\# = R\mathcal{H}om(K, \omega_X[dimX])$. Thus we just need to check that $\mathcal{E}_1^{\#0} \cong R\mathcal{H}om(\mathcal{E}_1, \omega_{\mathcal{X}}[\dim \mathcal{X}])$.

To do this, let us first consider the projections $\mathcal{X} \times \tilde{Y} \to \tilde{Y}$ and $X \times \tilde{Y} \to \tilde{Y}$

and call them π_1 and π_2. We can see that $\pi_2 \circ i = \pi_1 \circ j$. We obtain the following:

$$R\mathcal{H}om(\mathcal{E}_1, \omega_{\mathcal{X}}[\dim \mathcal{X}]) \cong R\mathcal{H}om(j_*\mathcal{E}, \pi_1^! \mathcal{O}_{\tilde{Y}}) \cong$$

$$\cong j_* R\mathcal{H}om(\mathcal{E}, j^! \pi_1^! \mathcal{O}_{\tilde{Y}}) \cong j_* R\mathcal{H}om(\mathcal{E}, i^! \pi_2^! \mathcal{O}_{\tilde{Y}}) \cong$$

$$\cong j_* R\mathcal{H}om(\mathcal{E}, i^! \omega_X[\dim X]) \cong j_* R\mathcal{H}om(\mathcal{E}, \omega_X(1,1)[\dim X - 1]) \cong j_* \mathcal{E}^*(-5,1)[8].$$

For this we used the duality theorem and the functoriality of the twisted pullback along with the obvious facts that $\omega_X \cong \mathcal{O}_X(-6)$ and $\dim X = \dim Gr(3,6) = 9$. $\square$

We will now use that $i_*\mathcal{E}$ is quasiisomorphic to $\{E_0 \boxtimes F_0 \to E_1 \boxtimes F_1 \oplus E_2 \boxtimes F_2 \to E_3 \boxtimes F_3\}$ and the fact that the embedding $\alpha : \mathcal{X} \to X \times \mathbb{P}^*$ gives the short exact sequence $0 \to \mathcal{O}_X(-1) \boxtimes \mathcal{O}_{\mathbb{P}^*}(-1) \to \mathcal{O}_{X \times \mathbb{P}^*} \to \alpha_* \mathcal{O}_{\mathcal{X}} \to 0$ to prove some useful lemmas.

We now have a commutative square

$$\begin{tikzcd} I(X,\tilde{Y}) \arrow[r, "i"] \arrow[d, "j"] & X \times \tilde{Y} \arrow[d, "\beta"] \\ \mathcal{X} \times \tilde{Y} \arrow[r, "\alpha"] & X \times \mathbb{P}^* \times \tilde{Y} \end{tikzcd}.$$

Here β will be the embedding of $\tilde{Y}$ to $\mathbb{P}^* \times \tilde{Y}$ given by the graph of g. We also write α instead of $\alpha \times id_Y$ and β instead of $id_X \times \beta$. Let us now define the following object $\mathcal{E}^* = R\mathcal{H}om_{I(X,\tilde{Y})}(\mathcal{E}, \mathcal{O}_{I(X,\tilde{Y})}) \in D^b(I(X,\tilde{Y}))$.

Lemma 25. *We have $\mathcal{E}^* \in D^{[0,1]}(I(X,\tilde{Y}))$, there is a quasiisomorphism on $X \times \tilde{Y}$*

$$\{E_3^*(-1) \boxtimes F_3^*(-1) \to E_2^*(-1) \boxtimes F_2^*(-1) \oplus E_1^*(-1) \boxtimes F_1^*(-1) \to$$

$$\to E_0^*(-1) \boxtimes F_0^*(-1)\} \cong i_* \mathcal{E}^*[1]$$

and $i_*\mathcal{E}^* \in \tilde{D}_X^{[0,1]}$.

Proof. We use the resolution of $i_*\mathcal{E}$ and we apply the functor $R\mathcal{H}om(-,\mathcal{O}_X(-1)\boxtimes\mathcal{O}_{\tilde{Y}}(-1))$ to obtain a quasiisomorphism

$$\{E_3^*(-1)\boxtimes F_3^*(-1)\to E_2^*(-1)\boxtimes F_2^*(-1)\oplus E_1^*(-1)\boxtimes F_1^*(-1)\to$$
$$\to E_0^*(-1)\boxtimes F_0^*(-1)\}\cong R\mathcal{H}om(i_*\mathcal{E},\mathcal{O}_X(-1)\boxtimes\mathcal{O}_{\tilde{Y}}(-1))[2]$$

on $X\times\tilde{Y}$. On the other hand, we have $\mathcal{O}_{I(X,\tilde{Y})} = i^*\mathcal{O}_{X\times\tilde{Y}} = i^!\mathcal{O}_X(-1)\boxtimes\mathcal{O}_{\tilde{Y}}(-1)[1]$, since i is an embedding of a divisor into the ambient variety. Now we can use the duality theorem to see

$$i_*\mathcal{E}^* = i_*R\mathcal{H}om(\mathcal{E},\mathcal{O}_{I(X,Y)}) = i_*R\mathcal{H}om(\mathcal{E}, i^!\mathcal{O}_X(-1)\boxtimes\mathcal{O}_{\tilde{Y}}(-1))[1] =$$
$$= R\mathcal{H}om(i_*\mathcal{E},\mathcal{O}_X(-1)\boxtimes\mathcal{O}_{\tilde{Y}}(-1))[1].$$

and this gives the quasiisomorphism of $i_*\mathcal{E}^*[1]$ that we need.

The remaining part now follows from [Kuz06a] □

Note that using the previous lemma we can obtain a quasiisomorphism

$$\{E_3^*((t-6))\boxtimes F_3^*\to E_2^*((t-6))\boxtimes F_2^*\oplus E_1^*((t-6))\boxtimes F_1\to E_0^*((t-6)\boxtimes F_0^*\}\cong i_*\mathcal{E}^{\#t}[1].$$

and we will use this later.

We consider now the following objects in $D(\tilde{Y}\times\mathbb{P}^*\times\tilde{Y})$:

$$\mathcal{C}_t = q_*\alpha_*(p_{12}^*\mathcal{E}_1^{\#t}\otimes p_{23}^*\mathcal{E}_1), \qquad \tilde{\mathcal{C}}_t = q_*(\alpha_* p_{12}^*\mathcal{E}_1^{\#t}\otimes\alpha_* p_{23}^*\mathcal{E}_1)$$

where the maps are as in the following ([Kuz06a]):

$$\begin{tikzcd}
\tilde{Y} \times \mathcal{X} \arrow[d, "\alpha"] & \tilde{Y} \times \mathcal{X} \times \tilde{Y} \arrow[l, "p_{12}"'] \arrow[r, "p_{23}"] \arrow[d, "\alpha"] & \mathcal{X} \times \tilde{Y} \arrow[d, "\alpha"] \\
\tilde{Y} \times (X \times \mathbb{P}^*) & \tilde{Y} \times (X \times \mathbb{P}^*) \times \tilde{Y} \arrow[l, "p_{12}"'] \arrow[r, "p_{23}"] \arrow[d, "q"] & (X \times \mathbb{P}^*) \times \tilde{Y} \\
& \tilde{Y} \times \mathbb{P}^* \times \tilde{Y} \arrow[d, "\pi"] & \\
& \tilde{Y} \times \tilde{Y} &
\end{tikzcd}.$$

The following lemma is now proven in [Kuz06a]:

Lemma 26. *The convolution of kernels $\mathcal{E}_1$ and $\mathcal{E}_1^{\#t}$ is given by*

$$\mathcal{E}_1 \circ \mathcal{E}_1^{\#t} \cong \pi_* \mathcal{C}_t \in D^b(\tilde{Y} \times \tilde{Y}).$$

Using the fact that $\alpha : \mathcal{X} \to X \times \mathbb{P}^*$ is an embedding we can now note:

Lemma 27. *We have an exact triangle*

$$\tilde{\mathcal{C}}_t \to \mathcal{C}_t \to \mathcal{C}_{t-1}[2]$$

in $D^b(\tilde{Y} \times \mathbb{P}^* \times \tilde{Y})$.

Proof. $\mathcal{X}$ is the zero locus of a section of the bundle $\mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}^*}(1)$ so using the short exact sequence given by α we get a distinguished triangle

$$\alpha^* \alpha_* \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes (\mathcal{O}_X(-1) \boxtimes \mathcal{O}_{\mathbb{P}^*}(-1))[2]$$

for any object $\mathcal{F}$ on $\tilde{Y} \times \mathcal{X} \times \tilde{Y}$. We now use the projection formula $\alpha_*(p_{12}^* \mathcal{E}_1^{\#t} \otimes \alpha^* \alpha_* p_{23}^* \mathcal{E}_1) \cong \alpha_* p_{12}^* \mathcal{E}_1^{\#t} \otimes \alpha_* p_{23}^* \mathcal{E}_1$, and the definition of $\mathcal{C}_t$ and $\tilde{\mathcal{C}}_t$ to obtain our triangle (where we take as $\mathcal{F} = p_{23}^*\mathcal{E}_1$, then we tensor the triangle given by α with $p_{12}^*\mathcal{E}_1^{\#t}$ and then we apply $q_*\alpha_*$. $\square$

The next important step is to compute $\tilde{\mathcal{C}}_t$. Let $\beta : \widetilde{Y} \to \mathbb{P}^* \times \widetilde{Y}$ be the graph of the morphism $g : \widetilde{Y} \to \mathbb{P}^*$ and let $\beta' : \widetilde{Y} \to \widetilde{Y} \times \mathbb{P}^*$ the composition of β with the transposition $\mathbb{P}^* \times \widetilde{Y} \to \widetilde{Y} \times \mathbb{P}^*$. Consider now

$$\mathcal{D} := \beta'_*\mathcal{O}_{\widetilde{Y}\times\widetilde{Y}} \otimes \beta_*\mathcal{O}_{\widetilde{Y}\times\widetilde{Y}} \in D^b(\widetilde{Y} \times \mathbb{P}^* \times \widetilde{Y}),$$

where we still denoted by β and β' the maps given by $id_{\widetilde{Y}} \times \beta$ and $\beta' \times id_{\widetilde{Y}}$.

We also need the following complexes on $\widetilde{Y} \times \widetilde{Y}$:

$$\left\{\begin{array}{ccc}
F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_3 & & \\
\oplus & W^* \otimes F_2^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_3 & \\
F_2^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_2 & \oplus & \\
\oplus & (S^2W^* \oplus \Lambda^2W^*) \otimes F_1^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_3 & \\
W^* \otimes F_1^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_2 \to & \oplus & \to \Sigma^{2,1}W^* \otimes F_0^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_3 \\
\oplus & W^* \otimes F_0^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_1 & \\
F_1^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_1 & \oplus & \\
\oplus & \Lambda^2W^* \otimes F_0^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_2 & \\
F_0^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_0 & &
\end{array}\right\} \cong \mathcal{T}$$

and

$$\left\{ \Sigma^{2,1}W \otimes F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_0 \to \begin{array}{c} W^* \otimes F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_2 \\ \oplus \\ (S^2W \oplus \Lambda^2 W) \otimes F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_1 \\ \oplus \\ W \otimes F_1^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_0 \\ \oplus \\ \Lambda^2 W \otimes F_2^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_0 \end{array} \to W \otimes F_2^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_1 \begin{array}{c} F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_3 \\ \oplus \\ F_2^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_2 \\ \oplus \\ \oplus \\ F_1^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_1 \\ \oplus \\ F_0^* \boxtimes \mathcal{O}_{\mathbb{P}^*} \boxtimes F_0 \end{array} \right\} \cong \mathcal{T}^*$$

Let now $\pi : \widetilde{Y} \times \mathbb{P}^* \times \widetilde{Y} \to \widetilde{Y} \times \widetilde{Y}$ be the projection.

Proposition 28. *We have the following description of $\tilde{\mathcal{C}}_t$:*

$$\tilde{\mathcal{C}}_t = \begin{cases} \pi^*\mathcal{T}^* \otimes \mathcal{D}, & \textit{for } t = 0 \\ F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*}(3) \boxtimes F_3 \otimes \mathcal{D}, & \textit{for } t = 2 \\ F_3^* \boxtimes \mathcal{O}_{\mathbb{P}^*}(3) \boxtimes F_3 \otimes \mathcal{D}[3], & \textit{for } t = 4 \\ \pi^*\mathcal{T} \otimes \mathcal{O}_{\mathbb{P}^*}(6) \otimes \mathcal{D}[6], & \textit{for } t = 6 \\ 0, & \textit{for } t = 1, 3, 5 \end{cases}$$

Proof. We need to look at the following([Kuz06a]):

$$\begin{tikzcd}
 & \tilde{Y}\times\mathcal{X} \arrow[d, "\alpha"] & \tilde{Y}\times\mathcal{X}\times\tilde{Y} \arrow[l, "p_{12}"'] \arrow[r, "p_{23}"] \arrow[d, "\alpha"] & \mathcal{X}\times\tilde{Y} \arrow[d, "\alpha"] & \\
I(X,\tilde{Y}) \arrow[ur, "j"] \arrow[d, "i"] & \tilde{Y}\times(X\times\mathbb{P}^*) & \tilde{Y}\times(X\times\mathbb{P}^*)\times\tilde{Y} \arrow[l, "p_{12}"'] \arrow[r, "p_{23}"] \arrow[d, "q"] & (X\times\mathbb{P}^*)\times\tilde{Y} & I(X,\tilde{Y}) \arrow[ul, "j"'] \arrow[d, "i"] \\
\tilde{Y}\times X \arrow[ur, "\beta'"] & \tilde{Y}\times X\times\tilde{Y} \arrow[l, "p_{12}"'] \arrow[ur, "\beta'"] & \tilde{Y}\times\mathbb{P}^*\times\tilde{Y} \arrow[d, "\pi"] & \tilde{Y}\times X\times\tilde{Y} \arrow[ul, "\beta"'] \arrow[r, "p_{23}"] & X\times\tilde{Y} \arrow[ul, "\beta"'] \\
 & & \tilde{Y}\times\tilde{Y} & &
\end{tikzcd}$$

Since the maps p_{12} and p_{23} are flat, we have that

$$\alpha_* p_{23}^* \mathcal{E}_1 \cong p_{23}^* \alpha_* \mathcal{E}_1 = p_{23}^* \alpha_* j_* \mathcal{E} \cong p_{23}^* \beta_* i_* \mathcal{E} \cong \beta_* p_{23}^* i_* \mathcal{E}$$

and $\alpha_* p_{12}^* \mathcal{E}_1^{\#t} \cong \beta'_* p_{12}^* i_* \mathcal{E}^{\#t} \otimes \mathcal{O}_{\mathbb{P}^*}(t)[8]$. Therefore,

$$\tilde{\mathcal{C}}_t \cong q_*(\beta'_* p_{12}^* i_* \mathcal{E}^{\#t} \otimes \mathcal{O}_{\mathbb{P}^*}(t)[8] \otimes \beta_* p_{23}^* i_* \mathcal{E}).$$

We now apply $\beta_* p_{23}^*$ and $\beta'_* p_{12}^*$ to the quasiisomorphisms obtained above and we get the following quasiisomorphisms:

$$\beta'_* p_{12}^* i_* \mathcal{E}^{\#t}[1] \cong \{\beta'_*(F_3^* \boxtimes E_3^*(t-6) \boxtimes \mathcal{O}) \to$$

$$to\beta'_*(F_2^* \boxtimes E_2^*(t-6) \boxtimes \mathcal{O} \oplus F_1^* \boxtimes E_1^*(t-6) \boxtimes \mathcal{O}) \to \beta'_*(F_0^* \boxtimes E_0^*(t-6) \boxtimes \mathcal{O})\}$$

and

$$\beta_* p_{23}^* i_* \mathcal{E} \cong \{\beta_*(\mathcal{O} \boxtimes E_0 \boxtimes F_0) \to \beta_*(\mathcal{O} \boxtimes E_1 \boxtimes F_1 \oplus \mathcal{O} \boxtimes E_2 \boxtimes F_2) \to \beta_*(\mathcal{O} \boxtimes E_3 \boxtimes F_3)\}.$$

So, $\tilde{\mathcal{C}}_t$ can be represented by the complex

$$\{\tilde{\mathcal{C}}_t^{3,0} \to \tilde{\mathcal{C}}_t^{3,1} \oplus \tilde{\mathcal{C}}_t^{3,2} \oplus \tilde{\mathcal{C}}_t^{2,0} \oplus \tilde{\mathcal{C}}_t^{1,0} \to$$

$$\to \tilde{\mathcal{C}}_t^{3,3} \oplus \tilde{\mathcal{C}}_t^{2,1} \oplus \tilde{\mathcal{C}}_t^{2,2} \oplus \tilde{\mathcal{C}}_t^{1,2} \oplus \tilde{\mathcal{C}}_t^{1,1} \oplus \tilde{\mathcal{C}}_t^{0,0} \to$$

$$\to \tilde{\mathcal{C}}_t^{2,3} \oplus \tilde{\mathcal{C}}_t^{1,3} \oplus \tilde{\mathcal{C}}_t^{0,1} \oplus \tilde{\mathcal{C}}_t^{0,2} \to \tilde{\mathcal{C}}_t^{0,3}\}$$

where

$$\tilde{\mathcal{C}}_t^{k,l} := q_*(\beta'_*(F_k^* \boxtimes E_k^*(t-6) \boxtimes \mathcal{O}) \otimes \mathcal{O}_{\mathbb{P}^*}(t)[8] \otimes \beta_*(\mathcal{O} \boxtimes E_l \boxtimes F_l)) \cong$$

$$\cong q_*((F_k^* \boxtimes (E_k^*(t-6) \otimes E_l) \boxtimes \mathcal{O}_{\mathbb{P}^*}(t) \boxtimes F_l) \otimes \beta'_*\mathcal{O}_{\tilde{Y}\times X\times\tilde{Y}} \otimes \beta_*\mathcal{O}_{\tilde{Y}\times X\times\tilde{Y}})[8]$$

for $k, l = 0, 1, 2, 3$.

We now consider the diagram

$$\begin{tikzcd}
\tilde{Y} \times X \times \tilde{Y} \arrow[r, "\beta'"] \arrow[d, "q"'] & \tilde{Y} \times (X \times \mathbb{P}^*) \times \tilde{Y} \arrow[d, "q"'] & \tilde{Y} \times X \times \tilde{Y} \arrow[l, "\beta"'] \arrow[d, "q"'] \\
\tilde{Y} \times \tilde{Y} \arrow[r, "\beta'"] & \tilde{Y} \times \mathbb{P}^* \times \tilde{Y} & \tilde{Y} \times \tilde{Y} \arrow[l, "\beta"']
\end{tikzcd}$$

and note that

$$\beta'_*\mathcal{O}_{\tilde{Y}\times X\times\tilde{Y}} \otimes \beta_*\mathcal{O}_{\tilde{Y}\times X\times\tilde{Y}} \cong \beta'_*q^*\mathcal{O}_{\tilde{Y}\times\tilde{Y}} \otimes \beta_*q^*\mathcal{O}_{\tilde{Y}\times\tilde{Y}} \cong q^*\beta'_*\mathcal{O}_{\tilde{Y}\times\tilde{Y}} \otimes q^*\beta_*\mathcal{O}_{\tilde{Y}\times\tilde{Y}} \cong q^*\mathcal{D}.$$

Using this last observation we see that

$$\tilde{\mathcal{C}}_t^{k,l} \cong (F_k^* \boxtimes (H^\bullet(X, E_k^*(t-6) \otimes E_l) \otimes \mathcal{O}_{\mathbb{P}^*}(t)) \boxtimes F_l) \otimes \mathcal{D})[8].$$

However, using Borel-Weil-Bott on X we can calculate

$$H^\bullet(X, E_k^*(t-6) \otimes E_l) = \mathsf{Ext}^\bullet(E_k, E_l(t-6))$$

and we get the following ones to be nonzero:

1. Let $t = 6$. Obviously $\mathsf{Ext}^\bullet(E_k, E_k) = \mathbb{C}$. Then

$\mathsf{Ext}^\bullet(E_0, E_k) = \mathsf{Ext}^\bullet(E_0, \Sigma^\alpha \mathcal{U}^*) = \Sigma^\alpha W^*$ (where α is (1) for E_1, $(1,1)$ for E_2 and $(2,1)$ for E_3). Also $\mathsf{Ext}^\bullet(\mathcal{U}^*, \Lambda^2\mathcal{U}^*) = W^*$, $\mathsf{Ext}^\bullet(\mathcal{U}^*, \Sigma^{2,1}\mathcal{U}^*) = S^2W^* \oplus \Lambda^2 W^*$, $\mathsf{Ext}^\bullet(\Lambda^2\mathcal{U}^*, \Sigma^{2,1,1}\mathcal{U}^*) = W^*$.

2. Let $t = 0$. Then by Serre duality

$$\mathsf{Ext}^\bullet(E_k, E_l(-6)) = \mathsf{Ext}^\bullet(E_l, E_k)^*[-6]$$

and using the calculations in part 1 we again see what cohomology is.

3. There are two more nonzero $\mathsf{Ext}'\,s$, for $t = 2$ we have that

$$\mathsf{Ext}^\bullet(\Sigma^{2,1}\mathcal{U}^*, \Sigma^{2,1}\mathcal{U}^*(-4)) = \mathbb{C}[-6]$$

and for $t = 4$ we have that

$$\mathsf{Ext}^\bullet(\Sigma^{2,1}\mathcal{U}^*, \Sigma^{2,1}\mathcal{U}^*(-2) = \mathbb{C}[-3]$$

4. All the other $H^\bullet(X, E_k^*(t-6) \otimes E_l) = 0$ for $0 \leq t \leq 6$.

This proves the proposition. □

Now some arguments based on some further calculations (depending on the sheaf $\mathcal{R}$ that gives the homological projective duality) should complete the proof that $\Phi_{j_*\mathcal{E}}$ is fully faithful, as outlined at the beginning of this section.

5.3 Applications to linear sections of Gr(3,6)

In this section we explain how the construction of the dual variety will give a description of the bounded derived category of coherent sheaves on the complete linear sections. Recall our situation. We started with $X = Gr(3, W)$ embedded into $\mathbb{P}(\Lambda^3 W)$ via the Plucker embedding. We conjectured that the homological projective dual variety will be given by $(M, \mathcal{R})$, where M is the double cover of $\mathbb{P}^(\Lambda^3 W^*)$ ramified along a quartic hypersurface.

Proposition 29. *Let $L \in \Lambda^3 W^*$ be a vector subspace of dimension r. Let $L^\perp$ be its orthogonal in $\Lambda^3 W$. Assume that $X_L = X \cap \mathbb{P}(L^\perp)$ and $M_L = M \times_{\mathbb{P}(\Lambda^3 W^*)} \mathbb{P}(L)$ are of expected dimensions $9 - r$ and $r - 1$ (and actually that the intersection with the singular loci is also of the right dimension). Then, there are semiorthogonal decompositions:*

$$D^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_r(1), \ldots, \mathcal{A}_5(6 - r) \rangle$$

and

$$D^b(M_L) = \langle \mathcal{B}_{17}(2 - r), \ldots, \mathcal{B}_{20-r}(-1), \mathcal{C}_L \rangle.$$

Note now that for $r \leq 5$, $(X_2)_L = \emptyset$, which means that the algebra $\mathcal{R}$ is a matrix algebra, which implies that $D^b(M_L, \mathcal{R}) \cong D^b(M_L)$.

Among the most interesting applications we have:

$r = 1$: In this case we see that M_L is given by a zero dimensional scheme of length 2 $M_L = \{x, y\}$, corresponding to the preimage of a generic point in $\mathbb{P}(\Lambda^3 W^*)$.

Thus we get that $D^b(X_H) = \langle E_x, E_y, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5)\rangle$.

$r = 2$: In this case we see that M_L is a double cover of $\mathbb{P}^1$ ramified at four distinct points, which is just an elliptic curve. Again, $D^b(M_L) = \mathcal{C}_L$ and this implies that $X_L = \langle D^b(M_L), \mathcal{A}_2(1), \ldots, \mathcal{A}_5(4)\rangle$. Therefore the derived category of X_L, which is a Fano 7-fold of index 4, contains a copy of the derived category of an elliptic curve and 12 exceptional objects.

$r = 6$: In this case we see that X_L is a 3 dimensional Calabi-Yau variety, the last example of 3 CY coming from the Plucker embeddings whose derived category was not understood. However, in our case we have a very easy description of $D^b(X_L)$. Indeed, we first note that $D^b(X_L) = \mathcal{C}_L$. But now M_L is just a noncommutative resolution of a double cover of $\mathbb{P}^5 = \mathbb{P}(L)$ (noncommutative because the double cover will still be singular along a scheme of dimension 0). Now, the proposition above just tells us that $D^b(M_L) = \langle \mathcal{O}(-4), \ldots, \mathcal{O}(-1), \mathcal{C}_L)$.

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